

## Two-temperature Ashkin-Teller model

Sertac Cengiz<sup>✉\*</sup> and M. Cemal Yalabik<sup>✉†</sup>*Physics Department, Bilkent University, TR-06800 Bilkent, Ankara, Turkey*

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Phase diagram of the two-temperature Ashkin-Teller model with Kawasaki dynamics has been investigated. The system is placed in contact with two heat baths at different temperatures, each of which controls the dynamics of nearest-neighbor exchange in the two perpendicular directions. It was observed that the second order phase transition lines (which separate the paramagnetic, ferromagnetic, and dimer phases) in the equilibrium model (equal temperatures) bifurcate into transition surfaces at nonequilibrium. The equilibrium four-state Potts point where all three orders meet extends into a line in the nonequilibrium case, along which these phases again meet.

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### I. INTRODUCTION

The theory of equilibrium phase transitions is well developed. Renormalization Group (RG) theory [1] provides us with a universal description of the phenomena, classified in relation to the dimensionality and the symmetries of the system under consideration. The theory is based on length scale transformations of the system, which are designed to keep the partition function and the symmetries of the system invariant. Classical spin systems have played a central role in the modeling and analysis of this phenomena. The universality seen in such systems, classified based on the critical fixed point of the RG transformation, lead to invariance of the critical exponents associated with the singularities in various quantities such as  $\nu$  for the correlation length,  $\alpha$  for the specific heat, and  $\gamma$  for the susceptibility. For systems that are not in equilibrium, one no longer has an access to a partition function, but RG arguments may be extended to the master equation which controls the dynamics of the system. This leads to the identification of an additional dynamical critical exponent  $z$ , in relation to the relaxation of the order parameter. Universality classes for dynamical systems depend on the type of dynamics which drives the system, for example whether the system is driven by “spin-flip” (Glauber [2]) or “spin-exchange” (conserved or Kawasaki [3]) dynamics [4].

While the analysis of general nonequilibrium dynamical behavior is difficult, steady state phenomena in classical spin systems poses a more tractable problem. In this work, we will confine ourselves to a steady-state system with a conserved dynamics. Such models may show long range behavior of correlations while the system is in nonequilibrium steady state. Correlations can have power law slow decay with a generic

nature for conservative dynamics [5,6]. An RG analysis by Schmittmann [7] of a continuum model corresponding to a two dimensional (2D) randomly driven Ising model with conserved dynamics shows that the system is in the universality class of uniaxial ferromagnets with dipolar interactions.

A Monte Carlo study of the 2D Ising Model with diffusive dynamics, in contact with two heat baths at different temperatures was carried out by Bassler and Racz [8]. Here, the diffusive dynamics of the spins in one direction is controlled by one of the heat baths, while dynamics in the perpendicular direction is controlled by the other heat bath. They find an interesting phase diagram, where the equilibrium phase transition extends into the nonequilibrium domain, with higher transition temperatures. The shape of the bifurcation of the nonequilibrium transition lines from the equilibrium critical point is argued to be given by a crossover exponent  $\phi$  and related to susceptibility exponent  $\gamma$  of equilibrium Ising model. An RG study of this system has also been carried out [9]. The nonequilibrium transition has different symmetries, corresponding to different universality classes.

Reichl *et al.* [10] have carried out a Monte Carlo study of the two temperature, 2D XY model and have obtained a very similar phase diagram: two nonequilibrium phase transition lines bifurcating from the Kosterlitz-Thouless (KT) equilibrium transition point, and the two corresponding ordered phases separated from one another by a first order transition line. (The equilibrium system can only support a quasilong range order KT phase, while the broken symmetry nonequilibrium system allows two long range order states within the same universality class as an equilibrium model with dipole interactions.)

In the present work, we report the results of a Monte Carlo study of the two temperature 2D Ashkin-Teller (AT) model with conserved dynamics. The AT model has relevance to such diverse problems as multicritical behavior and emergent universality in fully frustrated quantum magnets [11], quantum criticality in Rydberg arrays [12,13], and the relevant conformal field theories [14]. It is also of interest due to its rich phase diagram including a phase transition line with nonuniversal critical exponents. Because the phase

\*Contact author: [sertac.cengiz@bilkent.edu.tr](mailto:sertac.cengiz@bilkent.edu.tr)†Contact author: [yalabik@fen.bilkent.edu.tr](mailto:yalabik@fen.bilkent.edu.tr)

transitions in the equilibrium AT model correspond to lines, the bifurcation of the nonequilibrium phase separators now become surfaces diverging from these lines. The nonequilibrium ferromagnetic phases with opposite temperature are again separated from one another by a first-order surface at equilibrium temperatures. We have obtained specific heat dependence as a function of energy interaction constants and system size to accurately determine transition points. Higher accuracy calculations are needed to estimate the critical exponents.

Section II describes the model and our methodology, Sec. III presents the results, and we conclude in Sec. IV.

## II. THE MODEL

The first example of a classical spin model with varying critical exponents was the Eight-Vortex (8V) model for which an exact solution was obtained by Baxter [15–17]. While the original description of the model utilize lattice statistics of directed bonds between vertices, later work incorporated equivalent descriptions using Ising-like classical spins [18]. A simplified version of this model (the “zero-field eight-vortex model”) [17,19] has the Hamiltonian

$$\begin{aligned} \mathcal{H}_{8V} = & -J_1 \sum_{i,j=1}^N \sigma_{i,j+1} \sigma_{i+1,j} - J_2 \sum_{i,j=1}^N \sigma_{i,j} \sigma_{i+1,j+1} \\ & - J_4 \sum_{i,j} \sigma_{i,j} \sigma_{i+1,j} \sigma_{i,j+1} \sigma_{i+1,j+1}, \end{aligned} \quad (1)$$

where  $\{\sigma_{ij}\}$  are classical spin variables on an  $N \times N$  square lattice with periodic boundary conditions. The spins can take values  $\pm 1$ ,  $J_1$  and  $J_2$  are the energy interaction constants among two types of diagonals in orthogonal directions, and  $J_4$  is the interaction constant among the four spins of a unit square cell. If  $J_4 = 0$ , the system uncouples into two independent 2D Ising models on two sublattices. (The original model also contains interactions among nearest neighbors.)

Scaling the interaction constants with the temperature variable  $\beta = 1/k_B T$ , we use the unitless coupling constants  $K_i = \beta J_i$ : Baxter’s solution yields criticality for the system when

$$e^{-2K_4} = \sinh(K_1 + K_2) / \cosh(K_1 - K_2) \quad (2)$$

and with critical exponents

$$\begin{aligned} \text{“correlation length”} \quad \nu &= \pi/2\mu = 1 - \alpha/2 \\ \text{“magnetic”} \quad \gamma &= 7\pi/8\mu = 2(\nu - \beta) \end{aligned} \quad (3)$$

with

$$\mu = 2 \tan^{-1}(e^{-2K_4}). \quad (4)$$

AT is a closely related model [20,21], with two types of classical spins  $\{\sigma_{ij}\}$  and  $\{\mu_{ij}\}$  on the square lattice. The Hamiltonian has the form

$$\begin{aligned} \mathcal{H}_{AT} = & - \sum_{i,j} \{ J_1 \sigma_{i,j} (\sigma_{i+1,j} + \sigma_{i,j+1}) \\ & + J_2 \mu_{i,j} (\mu_{i+1,j} + \mu_{i,j+1}) \\ & + J_4 \sigma_{i,j} \mu_{i,j} (\sigma_{i+1,j} \mu_{i+1,j} + \sigma_{i,j+1} \mu_{i,j+1}) \}. \end{aligned} \quad (5)$$

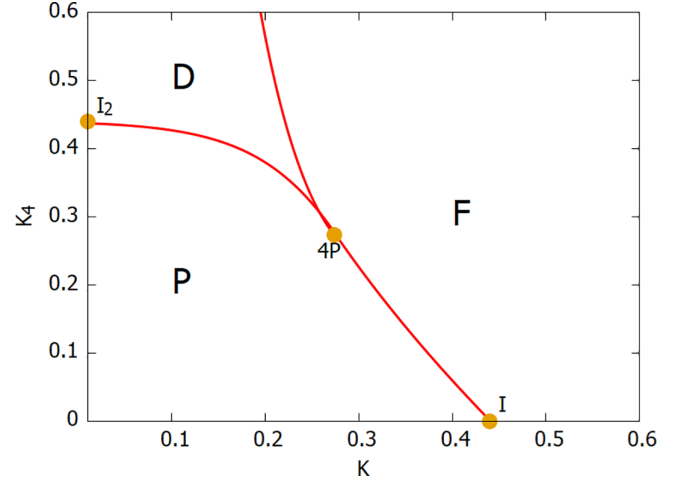


FIG. 1. The phase diagram of the isotropic AT model. At points I and I2 the model becomes equivalent to the Ising model, with the critical coupling  $K_I = \ln(1 + \sqrt{2})/2 \sim 0.44$ . Point 4P corresponds to the four-state Potts model. P, F, and D denote the paramagnetic, ferromagnetic, and dimer phases. All lines represent second-order transitions.

We will confine ourselves to the case where all interaction constants are positive. Note that the model is invariant under the exchange of the  $\sigma$  and  $\mu$  variables. It is also invariant under replacing one of the independent spin variables  $\sigma$  or  $\mu$  with the product  $\sigma\mu$  as an independent variable.

Wegner [22] showed the duality relation between the AT model and a version of the 8V model. In the isotropic case with  $\beta J_1 = \beta J_2 = K$  and  $\beta J_4 = K_4$ , the mapping is to a six-vertex model whose critical properties are known [23]. The criticality is now determined by

$$e^{-2K_4} = \sinh(2K) \quad (6)$$

with the critical exponents given in Eq. (3) but with

$$\mu = \cos^{-1} \left( \frac{1}{2} [e^{4K_4} - 1] \right) \quad (7)$$

and the constraint  $K_4 < K$ . The point for which  $K_4 = K = K_{4P} \frac{1}{4} \ln 3 \sim 0.2746$  on the critical line is an endpoint and corresponds to the case for which the model maps to a four-state Potts model. For values of  $K_4 > K_{4P}$ , the critical line bifurcates into two lines, corresponding to the appearance of an ordered phase.

The phase diagram of the equilibrium AT model [24,25] is displayed in Fig. 1. The phases of the system may be classified in relation to its order parameters  $\langle \sigma \rangle$ ,  $\langle \mu \rangle$ , and  $\langle \sigma\mu \rangle$ . The phase P which surrounds the  $K \sim K_4 \sim 0$  point is the disordered paramagnetic phase ( $\langle \sigma \rangle = \langle \mu \rangle = \langle \sigma\mu \rangle = 0$ ). The transitions are from the paramagnetic P phase to the ferromagnetic F phase ( $\langle \sigma \rangle = \pm \langle \mu \rangle \neq 0$  and  $\langle \sigma\mu \rangle \neq 0$ ). It is this transition line which displays the varying critical exponents given by Eqs. (3) and (7). Transition at  $K_4 = 0$  is Ising-like. For  $K_4 > K_{4P}$ , a phase separates the F and P phases, the “dimer” phase D, in which the ordering is such that  $\langle \sigma \rangle = \langle \mu \rangle = 0$  but  $\langle \sigma\mu \rangle \neq 0$ .

Approximate RG studies carried on the equilibrium AT model [26,27] reveal the universality classes associated with some of the transitions. None of these studies can produce a

line of fixed points required for the varying critical exponents along the line from I to 4P. This would require a marginal eigenvalue along the line, which is a delicate effect, and has not survived the approximations. Transitions from P to D and from D to F phases are governed by two distinct critical fixed points, both of which are expected to yield critical exponents consistent with the Ising universality class.

We carry out a Monte Carlo simulation of the nonequilibrium two-temperature AT model based on the Metropolis methodology [28]. The dynamics of the model corresponds to diffusive motion of the spins (Kawasaki dynamics) within the  $\sigma$  and  $\mu$  lattices, with two different temperatures controlling exchange in the two different directions.

The algorithm for the simulation is as follows: The simulation starts with a random distribution of equal number of “up” spins and “down” spins for both of the  $\sigma$  and  $\mu$  lattices. (Away from first order transition lines, we start with the final configuration of the lattices from a previous simulation of the system with similar parameters). The lattices are both  $L \times L$  with  $L = 64$  except when we look at near transitions, for which  $L$  is indicated in the appropriate figures. A spin type ( $\sigma$  or  $\mu$ ) and its location are selected at random, and a possible exchange of that spin with its neighbor at a random direction is considered. The change in energy  $\Delta E$  [based on the Hamiltonian in Eq. (5)] is calculated. If the exchange would result in a lower energy, then it is carried out. Otherwise, the exchange is carried out with a probability  $p = \exp(-\beta_i \Delta E)$  where  $\beta_i$  represents the two different temperature parameters for the two possible exchange directions. We use  $\beta_1$  for exchange in the  $x$  direction and  $\beta_2$  for exchange in the  $y$  direction. We utilize random numbers generated through a Mersenne Twister algorithm [29]. We define  $L^2$  spin exchange attempts as one Monte Carlo step (MCS). We carry out the computation for  $2 \times 10^7$  MCS. At the start of each simulation,  $2 \times 10^6$  MCS are skipped in which no data is taken. For the rest of the simulation, “samples” of the following quantities are evaluated every  $2 \times 10^3$  steps:

$$\begin{aligned} E_x &= J \sum_{i,j} (\sigma_{i,j} \sigma_{i,j+1} + \mu_{i,j} \mu_{i,j+1}) / L^2, \\ E_y &= J \sum_{i,j} (\sigma_{i,j} \sigma_{i+1,j} + \mu_{i,j} \mu_{i+1,j}) / L^2, \\ E_4 &= J_4 \sum_{i,j} (\sigma_{i,j} \sigma_{i,j+1} \mu_{i,j} \mu_{i,j+1} \\ &\quad + \sigma_{i,j} \sigma_{i+1,j} \mu_{i,j} \mu_{i+1,j}) / L^2. \end{aligned} \quad (8)$$

We then determine the fluctuations

$$\begin{aligned} C_{vx} &= \langle E_x^2 \rangle - \langle E_x \rangle^2, \\ C_{vy} &= \langle E_y^2 \rangle - \langle E_y \rangle^2, \\ C_{v4} &= \langle E_4^2 \rangle - \langle E_4 \rangle^2, \end{aligned} \quad (9)$$

where  $\langle \dots \rangle$  represents Monte Carlo averaging, which is the average value of the indicated quantities over the total number  $N$  of samples. For each quantity  $C$  we calculate in Eq. (9), we also calculate a measure of its statistical fluctuation. We do this by evaluating  $n = N/100$  subaverages  $C_i$  of these quantities using 100 consecutive samples of each. We then indicate

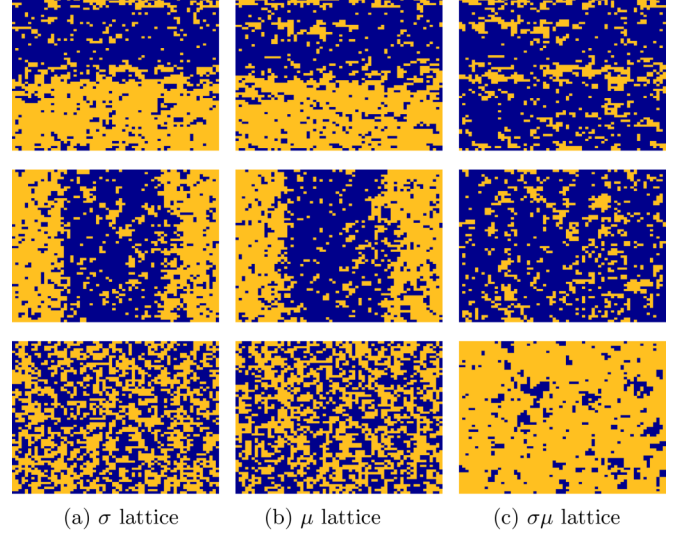


FIG. 2. First two rows display the anisotropic order in the P phase for  $(K_1, K_2, K_4) = (0.265, 0.180, 0.210)$  and  $(K_1, K_2, K_4) = (0.180, 0.265, 0.210)$ , respectively. Last row is for  $(K_1, K_2, K_4) = (0.090, 0.140, 0.450)$  and shows dimer ordering in the D phase.

the quantity

$$\epsilon_C = \frac{1}{\sqrt{n}} \sqrt{\left( \sum_{i=1}^n C_i^2 \right) / n - \left( \sum_{i=1}^n C_i \right)^2 / n^2} \quad (10)$$

as error bars while displaying these quantities in the figures.

It is apparent that the following four parameters define our problem:

$$a = \beta_1 J, \quad b = \beta_2 J, \quad c = \beta_1 J_4, \quad d = \beta_2 J_4.$$

The constraint  $a/b = c/d$  indicates that three variables will be sufficient to determine these parameters. We choose these variables as

$$K_1 = \beta_1 J, \quad K_2 = \beta_2 J, \quad K_4 = \sqrt{\beta_1 \beta_2} J_4,$$

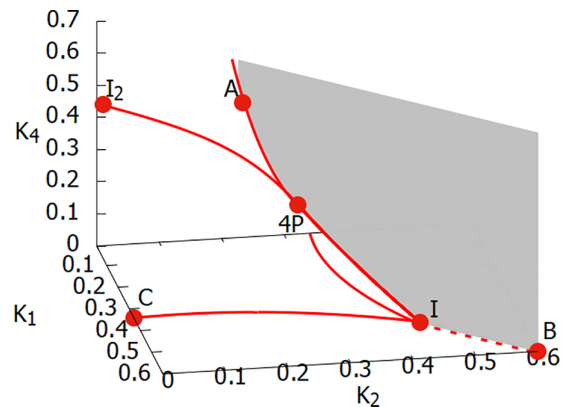


FIG. 3. First order phase separation surface for the F phase. Transitions on the  $K_4 = 0$  plane were studied in [8] and [9].

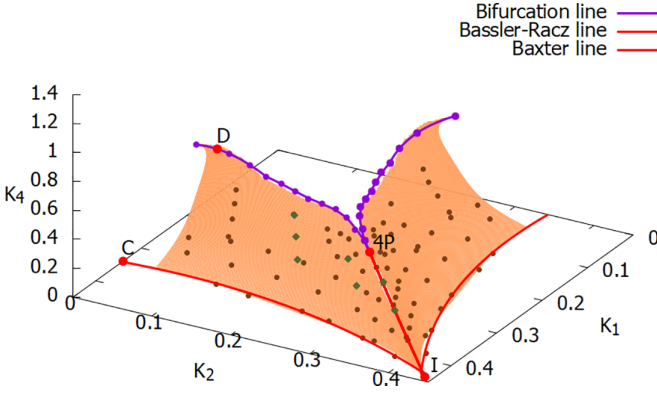


FIG. 4. Ferromagnetic-paramagnetic transition surface. The dots on the figure indicate points at which we have observed maxima in  $C_{vx} + C_{vy}$  as the critical surface is traversed.

so that

$$a = K_1 \quad b = K_2, \quad c = \sqrt{\frac{K_1}{K_2}} K_4, \quad d = \sqrt{\frac{K_1}{K_2}} K_4.$$

Unscaled  $J$  values are needed only as the overall multipliers of the quantities in Eqs. (8) and (9), for which we take  $J = 1$  and  $J_4 = K_4/\sqrt{K_1 K_2}$ .

Note that this is not equivalent to a single temperature equilibrium model with anisotropic couplings  $K_1 = \beta J_1$ ,  $K_2 = \beta J_2$ , and  $K_4 = \beta J_4$ : In our model, energy of a configuration is computed using isotropic Eq. (8), but then the  $K_i$

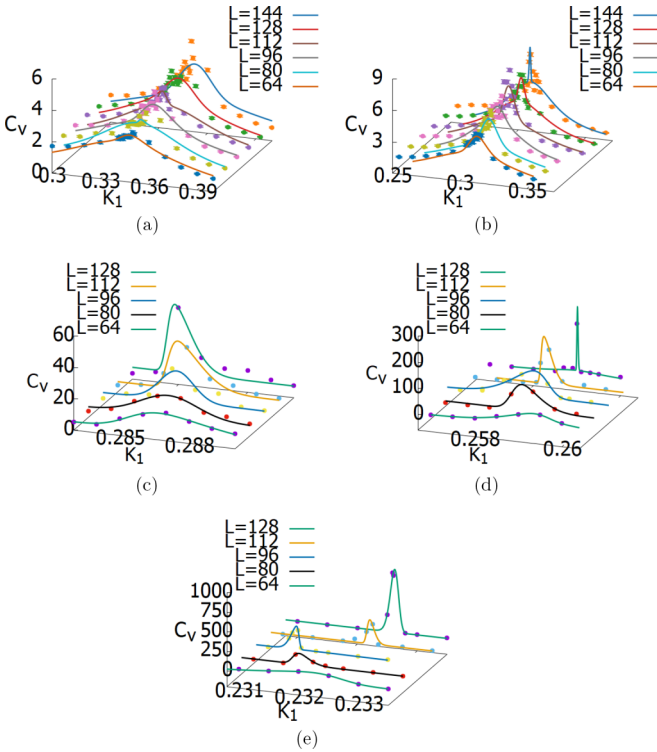


FIG. 5.  $C_{v,x}$  as a function of  $K_1$  near the F-P nonequilibrium critical surface, along a line of constant  $K_1/K_2$ . (a)  $K_4 = 0.14$  and  $K_1 = K_2$  (b)  $K_4 = 0.21$  and  $K_1 = K_2$  (c)  $K_4 = 0.14$  and  $K_1/K_2 = 0.664$  (d)  $K_4 = 0.21$  and  $K_1/K_2 = 0.664$  (e)  $K_4 = 0.28$  and  $K_1/K_2 = 0.664$ .

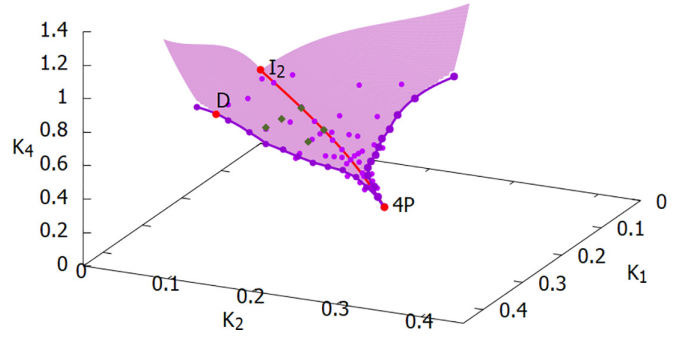


FIG. 6. Paramagnetic-dimer transition surface. The dots on the figure indicate points at which we have observed maxima in  $C_{v4}$  as the critical surface is traversed.

value appropriate for the exchange in that particular direction is used in the decision on whether the exchange is going to be carried out or not. (Our model is an adaptation of the study by Bassler and Racz [8] on the Ising model to the AT model.)

### III. NUMERICAL RESULTS

To simplify our description of the phase diagram of the two-temperature AT model, we will display various phase separation surfaces sequentially. A feature of the ferromagnetic phase in the presence of anisotropic nonequilibrium drive is to produce two anisotropically ordered phases that meet at a first-order transition surface when  $K_1 = K_2$ . The dimer phase has no such discontinuity. Figure 2 displays the anisotropic order for various values of  $K_1$ ,  $K_2$ , and  $K_4$ . Note that since the exchange dynamics conserves the total number  $\sigma$  and  $\mu$  spins with  $+1$  and  $-1$  values, the ordering in the P phase results in stripes of  $+1$  and  $-1$  spins along the direction with the lower temperature exchange. No such order exists in the D phase,

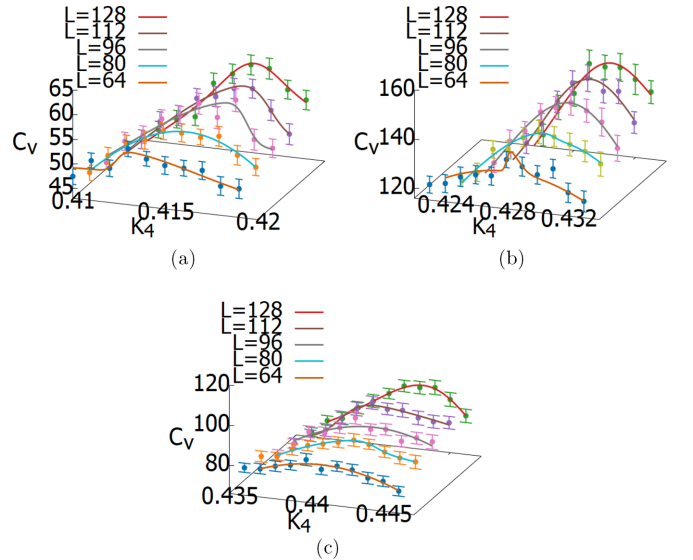


FIG. 7.  $C_{v4}$  versus  $K_4$  for different system sizes. (a)  $K_1 = 0.14$  and  $K_2 = 0.14$  (b)  $K_1 = 0.09$  and  $K_2 = 0.09$  (c)  $K_1 = 0.09$  and  $K_2 = 0.14$ .

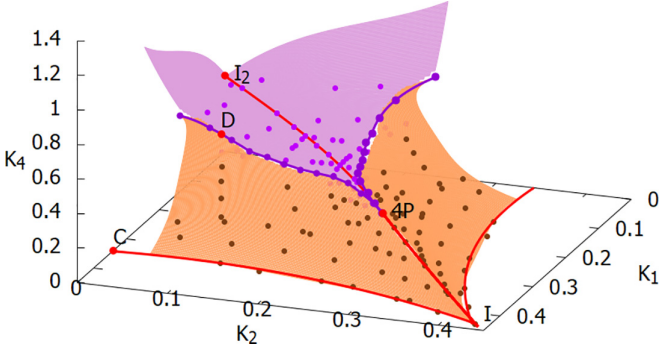


FIG. 8. This figure is a superposition of the plots in Figs. 4 and 6.

but the dimer product  $\sigma\mu$  at each site orders with a positive or negative average value.

Figure 3 displays the first order phase separation surface for the ferromagnetic phase across equilibrium. The surface separates two phases with different anisotropic order. The nonequilibrium phase diagram reduces to the equilibrium version for  $K_1 = K_2$ . For  $K_1 \neq K_2$  and  $K_4 = 0$  we have the two-temperature Ising model studied by Bassler and Racz [8]. The second order phase transition line between the F and P phases in the two-temperature Ising model (containing points I and C) now extends to a surface along the equilibrium transition line from I to 4P as shown in Fig. 4. The surface was constructed by joining points which yield maxima in  $C_v$  as it is traversed. This behavior at some of the points is displayed in Fig. 5.

Figure 5 shows plots of  $C_{vx} + C_{vy}$  as a function of  $K_1$ , for various values of  $K_4$ , near the nonequilibrium critical surface, along a line of constant  $K_1/K_2$ . The variation of the peak amplitude of  $C_v$  allows us to estimate the transition point. Combination of skewed Gaussian and Lorentz function fits to the  $C_v$  are also indicated on the figures. We used lattices of various sizes for these quantities to see the trend. Finite size effects are very pronounced in lattices with conserved order, and larger lattices need to be utilized for obtaining critical exponents using finite size scaling.

The equilibrium phase transition line between the P and the D phases (which extends between the four-state Potts point 4P and I2 in Fig. 1) expands into the surface in Fig. 6 when

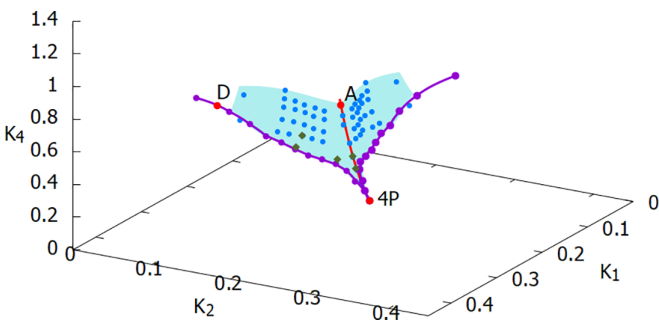


FIG. 9. Ferromagnetic-dimer transition surface. The dots on the figure indicate points at which we have observed maxima in  $C_{vx} + C_{vy}$  as the critical surface is traversed.

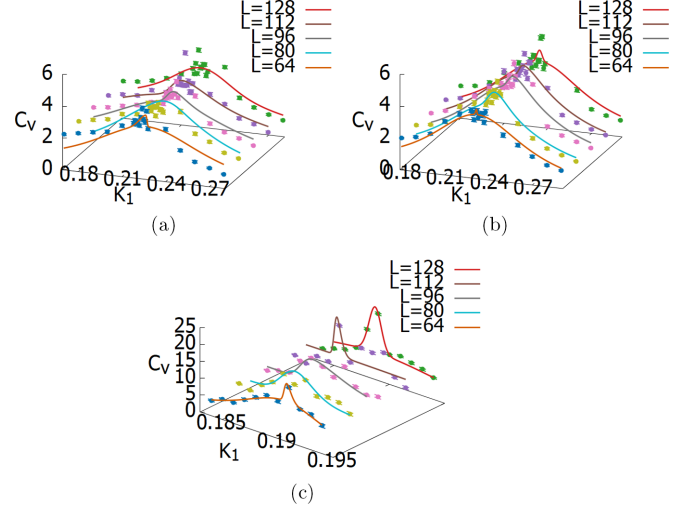


FIG. 10.  $C_{vx}$  versus  $K_1$  for different system sizes at different points on the ferromagnetic to dimer transition surface. (a)  $K_4 = 0.42$  and  $K_1/K_2 = 1.00$  (b)  $K_4 = 0.37$  and  $K_1/K_2 = 1.00$  (c)  $K_4 = 0.42$  and  $K_2 = 0.14$ .

$K_1 \neq K_2$ . The surface is delineated by a critical end line which also ends at 4P.

The variation of  $C_{v4}$  across this surface is displayed in Fig. 7. Note that because of the way we scale the temperature variable, as Eqs. (8) and (9) indicate,  $J_4 = K_4/\sqrt{K_1 K_2}$  which multiplies  $C_{v4}$  can reach large values for smaller values of  $K_1$  or  $K_2$ . This results in the relatively large quantities displayed in Fig. 7. The quantities are plotted against  $K_4$ , which is the parameter that drives the dimer order.

Fig. 8 displays the superposition of the F-P and P-D phase boundaries.

Finally, the second order phase transition between D and F is depicted in Fig. 9. When the two temperatures differ, the

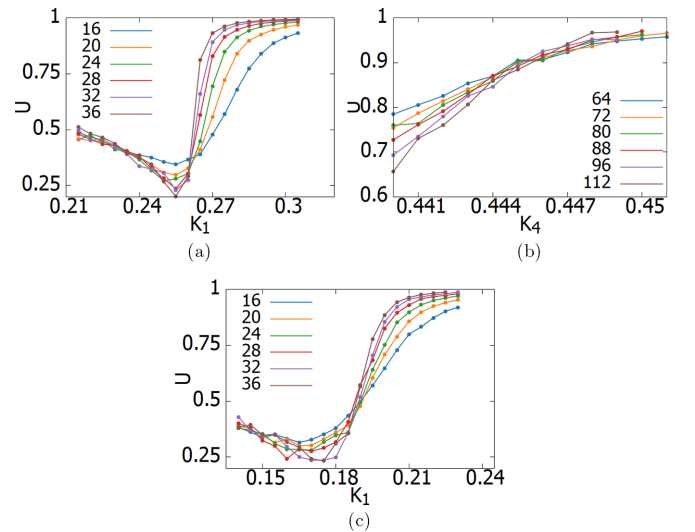


FIG. 11. (a)–(c) display the behavior of the Binder cumulant across the phase transitions associated with Figs. 5(d), 7(c), and 10(c), respectively. (a)  $K_4 = 0.21$  and  $K_1/K_2 = 0.664$  (b)  $K_1 = 0.09$  and  $K_2 = 0.14$  (c)  $K_4 = 0.42$  and  $K_2 = 0.14$ .

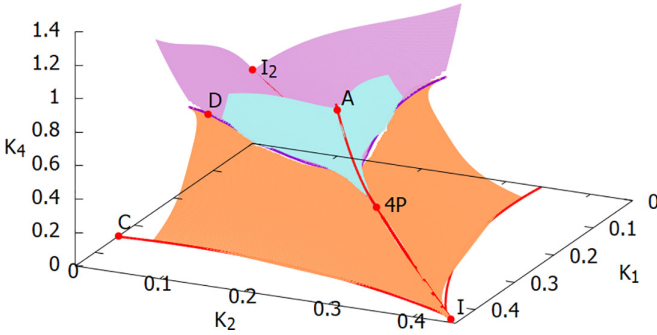


FIG. 12. Phase diagram of the two-temperature 2D AT model. (This figure is a superposition of the plots in Figs. 4, 6, and 9).

equilibrium phase transition line between 4P and A in Fig. 1 extends into a surface along the critical end line from 4P to D. Figure 10 illustrates the variation of  $C_v$ s across the D-F transition surface as a function of  $K_1$ .

#### IV. CONCLUSION

We investigated the phase diagram of the two-temperature Ashkin-Teller model with conserved dynamics using Monte Carlo simulations. We identify phase transitions via peaks in specific-heat type correlations, which should scale with finite size length. To ensure that the transitions are indeed second order, we looked at the relevant Binder cumulants across the three types of transitions (P-F, P-D, D-F). We used Binder cumulant  $U = \langle \Phi^2 \rangle^2 / \langle \Phi^4 \rangle$  with order parameter  $\Phi$  [10,30]. The results are displayed in Fig. 11, which display behavior consistent with second order transitions. At this stage, the

whole phase diagram of the two temperature AT model is illustrated in Fig. 12. It is evident that each of the three transition surfaces terminates along the critical end line from 4P to D, which represents the region where the surfaces meet. The main nonequilibrium feature is the extension of the equilibrium second order transition lines into second order transition surfaces, and the extension of the four-state Potts (4P) point into a critical end line (extending to point D, but probably with a different universality class) which delimits the nonequilibrium transition surface which separates the Ferromagnetic and Dimer phases.

A number of features of the model remain to be explored: A more detailed Monte Carlo analysis with larger lattices and larger MCS values would reveal values for critical exponents. The bifurcation of the nonequilibrium transition surfaces from the equilibrium transition lines incorporates a crossover exponent  $\phi$ . This was related to the susceptibility exponent  $\gamma$  of equilibrium Ising model in the two-temperature Ising model [8] but is not known for this model. A dynamical RG analysis could also reveal the universality classes associated with the transitions, although it would involve difficulties as RG for even the equilibrium AT model would need to incorporate the varying critical exponents.

#### ACKNOWLEDGMENT

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#### DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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