

Souvenir collector's walk: The distribution of the number of steps of a continuous-time random walk ending at a given position

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We consider a random walker performing a continuous-time random walk (CTRW) with a symmetric step lengths' distribution possessing a finite second moment and with a power-law waiting time distribution with finite or diverging first moment. The problem we pose concerns the distribution of the number of steps of the corresponding CTRW conditioned on the final position of the walker at some long time t . For positions within the scaling domain of the probability density function (PDF) of final displacements, the distributions of the number of steps show a considerable amount of universality, and are different in the cases when the corresponding CTRW corresponds to subdiffusion and to normal diffusion. We moreover note that the mean value of the number of steps can be obtained independently and follows from the solution of the Poisson equation whose right-hand side depends on the PDF of displacements only. This approach works not only in the scaling domain but also in the large deviation domain of the corresponding PDF, where the behavior of the mean number of steps is very sensitive to the details of the waiting time distribution beyond its power-law asymptotics.

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I. INTRODUCTION: THE SOUVENIRS COLLECTOR'S WALK

Let us consider the following problem: A random walker starts at the origin and moves between different bars, from one to another one, spending a random amount of time in each of them. From each bar the walker takes a souvenir (say, a beer mat), and at the end of a very long day is found at distance x from the starting point. What is the distribution of the number of collected souvenirs conditioned on x ? What is the mean number of such souvenirs?

The mathematical problem position is as follows: We consider a continuous-time random walk (CTRW) with a symmetric step length distribution with probability density function (PDF) $\lambda(x)$, and with waiting time density $\psi(t)$ which may possess finite or infinite mean. For the case of the δ distribution of waiting times, $\psi(t) = \delta(t - t_0)$, the number of steps at any given time is given by the whole part of t/t_0 , $n = \lfloor t/t_0 \rfloor$ and does not fluctuate. Therefore, the number of steps for a walker found at different positions is the same. This changes if the walker does not move at a constant pace. In this case, the conditional probability $p(n|x, t)$ may show a nontrivial dependence on x . In the present work, we consider only an unbiased, decoupled random walk with $\lambda(x)$ possessing finite variance σ^2 , but similar problems may be posed for other

CTRW variants like biased walks, Lévy flights, Lévy walks, etc., and also for random walks in quenched landscapes.

There are several realistic problems which lead to the mathematical formulations akin to (but much more complicated than) our toy problem above. CTRW with power-law waiting times density (WTD) was formulated as a model for transport properties of disordered semiconductors [1,2]. It was proved equally applicable in such diverse disciplines as economics, hydrology, movement ecology, etc. (see Ref. [3] for a review, and references therein). While the charge carrier in a semiconductor cannot keep track on the history of its movement, a biological entity definitely does. The problem at hand arose as an oversimplification of one of such biological problems: Imagine organisms whose genome undergoes random changes (due to recombination), changing at random their fitness. What is the population's diversity conditioned on fitness? In spite of maybe limited usefulness of the solution of the toy problem for getting insights into realistic biological ones (containing many details making these really complex), the results for the toy problem are very beautiful and interesting on their own and seem not to be discussed in the present literature. This motivated the present work.

In what follows, we will consider a simple decoupled CTRW with an asymptotically power-law WTD,

$$\psi(t) \sim t^{-1-\alpha},$$

and with a symmetric PDF of displacement in a single step $\lambda(x)$ possessing the second moment σ^2 . The results for $0 < \alpha < 1$, corresponding to subdiffusion of the walker, show a nontrivial behavior with a considerable amount of universality. On the other hand, for $\alpha > 1$ [including the Poisson case $\psi(t) = t_0^{-1} \exp(-t/t_0)$ corresponding to $\alpha \rightarrow \infty$] leading to normal diffusion, the distribution of n is narrow and independent on the position x within in the body of its distribution.

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The special case $\alpha = 1$ corresponding to transition between these two regimes is singular but shows a behavior which is in some respects similar to the one for $\alpha > 1$.

In addition to the asymptotic forms of $p(n|x, t)$ for t sufficiently large, we will derive the expression for the conditional mean $\langle n|x, t \rangle$, which will be investigated both in the scaling domain, and in the domain of large deviations. We will concentrate on one-dimensional walks but make some notes on higher dimensions in Sec. VI.

Here a terminological note is in place. For long time $t \gg t_0$, with t_0 being the typical time of the step, the shape of the displacements' PDF in CTRW tends to a scaling form, which, in one dimension, is given by a Mainardi function for $\alpha < 1$ [4–8] and by a Gaussian for $\alpha > 1$. Both correspond to PDFs with a single mode and with finite width (defined by its second moment). This width is proportional to the mean number of steps made by the walker up to time t : $w^2 = \sigma^2 \langle n(t) \rangle$, which scales as $\langle n(t) \rangle \propto t^\alpha$ for $\alpha < 1$ and as $\langle n(t) \rangle \propto t$ for all $\alpha > 1$. The “body” of the distribution includes the absolute displacements $|x|$ in the central part of the distribution $|x| \sim w = \sigma \langle n(t) \rangle^{1/2}$, and in its moderately far tails, corresponding to typical trajectories of the walk with many directional changes. The extremely stretched trajectories with only few directional changes lead to the possibility to reach points at distances of the order of $|x| \sim \sigma \langle n(t) \rangle$ in the far tails. This domain of $|x|$ will be called the large deviation domain in what follows.

II. GENERAL FORMULA

A continuous-time random walk is a process subordinated to a simple random walk under operational time (counting process) being the number of steps performed up to the time t . In the simplest decoupled CTRW consisting of waiting periods on a site followed by instantaneous jumps between different positions, the distribution of displacements in particular steps does not depend on the waiting times. We use here an almost standard notation following Ref. [9], which, however, causes the need to make some comments on the nature of the mathematical objects involved. Let the distribution of the number of steps performed up to the physical time t be given by $\chi_n(t)$. This quantity is essentially a conditional probability $P(n|t)$ to make exactly n steps up to time t . Then the joint distribution $p(n, x|t)$ of the number of steps and position of a particle at time t is given by $p(n, x|t) = p_n(x)\chi_n(t)$, where $p_n(x)$ is the PDF of displacements of a simple random walk after n steps, i.e., the conditional probability density $p(x|n)$. The function $p(n, x|t)$ is then a *probability* in n and a *density* in x . The PDF of the walker's position at time t is now given by

$$p(x|t) = \sum_{n=0}^{\infty} p_n(x)\chi_n(t). \quad (1)$$

The joint distribution $p(n, x|t)$ can be expressed as

$$p(n, x|t) = p(n|x, t)p(x|t),$$

from which it follows that

$$p(n|x, t) = \frac{p_n(x)\chi_n(t)}{p(x|t)}. \quad (2)$$

This leads to the final expression for $p(n|x, t)$:

$$p(n|x, t) = \frac{p_n(x)\chi_n(t)}{\sum_{n=0}^{\infty} p_n(x)\chi_n(t)}, \quad (3)$$

which makes evident the overall normalization. In what follows we will also often use the means: $\langle n(t) \rangle = \sum_{n=0}^{\infty} n\chi_n(t)$ and $\langle n|x, t \rangle = \sum_{n=0}^{\infty} np(n|x, t)$.

We note that Eq. (3) is valid for all values of n, x , and t involved. The large- t behavior is, however, of particular interest due to its remarkable universality connected with the scaling of the corresponding properties. In this domain, the number of steps n is typically large, and n can be well interpolated by a continuous variable.

III. GENERAL PROPERTIES IN THE SCALING DOMAIN

Let us discuss the scaling properties of $\chi_n(t)$ and $p(x|t)$ in the body of the distribution for t sufficiently large. The typical number of steps [say, the mean number of steps $\langle n(t) \rangle$] grows with t , see, e.g., Eq. (22) below. On the other hand, for large n the distribution $p_n(x)$ universally tends to a Gaussian,

$$p_n(x) \simeq \frac{1}{\sqrt{2\pi n\sigma}} \exp\left(-\frac{x^2}{2\sigma^2 n}\right), \quad (4)$$

and is exactly Gaussian for any n if $\lambda(x)$ is Gaussian. Moreover, for t large, both $\chi_n(t)$ and $p(x|t)$ tend to universal forms. Here the cases $\alpha < 1$ and $\alpha > 1$ (i.e., the distribution whose first moment is infinite and the ones for which this moment is finite) show different behavior. The case $\alpha = 1$ corresponds to a singular limit (when approaching from the both sides). We will make some notes on this case in Appendix A but will not go into much detail of this very special case.

The distribution $\chi_n(t)$ has a simple expression in the Laplace domain:

$$\chi_n(s) = \frac{1 - \psi(s)}{s} \psi^n(s), \quad (5)$$

and the density $p(x|t)$ has a simple expression in Fourier-Laplace domain,

$$p(k, s) = \frac{1 - \psi(s)}{s} \frac{1}{1 - \lambda(k)\psi(s)} \quad (6)$$

(Montroll-Weiss formula), see, e.g., Ref. [9].

For the future use we give here the expressions for the mean number of steps and for the mean-squared number of steps as following from Eq. (5). To calculate the mean number of steps in the Laplace domain, $\langle n \rangle(s) = \hat{L}\{\langle n(t) \rangle\}(s)$, with $\hat{L}$ denoting the operator of the Laplace transform, we note that

$$\begin{aligned} \langle n \rangle(s) &= \sum_{n=0}^{\infty} n\chi_n(s) = \frac{1 - \psi(s)}{s} \sum_{n=0}^{\infty} n\psi^n(s) \\ &= \frac{1 - \psi(s)}{s} \psi(s) \frac{d}{d\psi} \sum_{n=0}^{\infty} n\psi^n(s) \Big|_{\psi=\psi(s)} \\ &= \frac{\psi(s)}{s[1 - \psi(s)]}. \end{aligned} \quad (7)$$

To calculate the second moment in the Laplace domain $\langle n^2 \rangle(s) = \hat{L}\{\langle n^2(t) \rangle\}(s)$ we proceed similarly and first define

$$\begin{aligned} \langle [n(n-1)] \rangle(s) &= \langle n^2 \rangle(s) - \langle n \rangle(s) \\ &= \sum_{n=0}^{\infty} n(n-1) \frac{1 - \psi(s)}{s} \psi^n(s) \\ &= \frac{1 - \psi(s)}{s} \psi^2(s) \frac{d^2}{d\psi^2} \sum_{n=0}^{\infty} \psi^n \bigg|_{\psi=\psi(s)} \\ &= \frac{2\psi^2(s)}{s[1 - \psi(s)]^2}. \end{aligned} \quad (8)$$

Knowing that $\langle n \rangle(s)$ is given by Eq. (7), we get

$$\langle n^2 \rangle(s) = \frac{\psi(s)[1 + \psi(s)]}{s[1 - \psi(s)]^2}. \quad (9)$$

A somewhat awkward notation is due to the fact that for a nonlinear property $\langle n^2(t) \rangle$ the averaging procedure and the Laplace transform do not commute.

A. Waiting times densities

For power-law WTD asymptotically following

$$\psi(t) \sim \frac{t_0^\alpha}{t^{1+\alpha}} \quad (10)$$

for t sufficiently large, with t_0 being the characteristic time of a single jump, the cases $\alpha < 1$ (diverging first moment of the waiting time) and $\alpha > 1$ (finite first moment) have to be considered separately.

For $\alpha < 1$ we introduce t_0 such that

$$\psi(t) \simeq \frac{\alpha}{\Gamma(1-\alpha)} \frac{t_0^\alpha}{t^{1+\alpha}}. \quad (11)$$

This is done in order to have the simplest form for the Laplace transform:

$$\psi(s) \simeq 1 - (t_0 s)^\alpha + o(s^\alpha). \quad (12)$$

For $\alpha > 1$, the first moment of the waiting times exists, and we will take t_0 equal to this first moment. For $\alpha < 2$ we will then have

$$\psi(s) \simeq 1 - t_0 s + (t_1 s)^\alpha + o(s^\alpha), \quad (13)$$

where t_1 is an additional parameter of the dimension of the time characterizing the “width” of the distribution around its mean. For distributions possessing the second moment, one universally has

$$\psi(s) \simeq 1 - t_0 s + \frac{1}{2} \langle t^2 \rangle s^2 + o(s^2),$$

which simply corresponds to the previous expression where one now formally takes $\alpha = 2$ and $t_1 = [\frac{1}{2} \langle t^2 \rangle]^{1/2}$. As we proceed to show, all cases of distributions possessing the second moment (i.e., for $\alpha > 2$, including the exponential WTD with $\alpha \rightarrow \infty$) behave in the asymptotic domain of long times similarly to the ones with $1 < \alpha < 2$ and differ only in the values of some prefactors, and in the speed of convergence to final expressions, i.e., in answering the question, what times can be considered as sufficiently long. This, however, will not be a part of our present discussion.

The singular case $\alpha = 1$ leads to logarithmic corrections of the type

$$\psi(s) \simeq 1 + t_0 s \ln(t_0 s) + t_2 s + \dots \quad (14)$$

with parameters, t_0 and t_2 having the dimension of time. The form of Eq. (14) gets evident when considering special cases of Pareto type I or type II distributions with $\alpha = 1$; see Appendix A. Note that t_2 might be of either sign and that at difference with the case $\alpha > 1$ the linear term is always subleading. This case can be considered along similar lines but will be only discussed in Appendix A and only briefly mentioned in the main text.

We start our discussion from the case $\alpha < 1$, and then turn to $\alpha > 1$.

B. Scaling of the step number distribution

For α strictly smaller than unity, the asymptotic form of the distribution of the number of steps n done up to the time t , for both n and t large, is given by

$$\chi_n(t) \simeq \frac{1}{\alpha} \frac{t}{t_0} n^{-\frac{1}{\alpha}-1} \mathcal{L}_\alpha \left(\frac{t}{t_0} n^{-\frac{1}{\alpha}} \right), \quad (15)$$

where $\mathcal{L}_\alpha(y)$ is the PDF of a one-sided Lévy law of index α , i.e., the inverse Laplace transform of $\exp(-s^\alpha)$, $\mathcal{L}_\alpha(y) = \hat{L}^{-1}[\exp(-s^\alpha)](y)$; see Exercise 3.8 of Ref. [9]. Since we will need the derivation for further discussions, we will reproduce it in the following paragraph. The form, Eq. (15), together with Eqs. (4) and (2), shows that the function $p(n|x, t)$ depends on two dimensionless variables, $\xi = x/\sigma$ and $\tau = t/t_0$.

A short derivation of Eq. (15)

For $\alpha < 1$ we use the form, Eq. (12), and put Eq. (5) for s small as

$$\begin{aligned} \chi_n(s) &= \frac{1 - \psi(s)}{s} \exp[n \ln \psi(s)] \\ &\simeq t_0^\alpha s^{\alpha-1} \exp(-nt_0^\alpha s^\alpha). \end{aligned} \quad (16)$$

Therefore, we note that $\chi_n(s) = \frac{1}{s} \frac{d}{dn} \exp[-nt_0^\alpha s^\alpha]$. Note that the formal possibility to take a derivative with respect to n assumes that it is large and can be considered as a continuous variable. Knowing that the inverse Laplace transform of $\exp(-s^\alpha)$ is $\mathcal{L}_\alpha(y)$, we can return to time domain using the scaling theorem for the Laplace transform:

$$\begin{aligned} \chi_n(t) &\simeq \frac{d}{dn} \int_0^t \frac{1}{t_0 n^{1/\alpha}} \mathcal{L}_\alpha \left(\frac{t'}{t_0 n^{1/\alpha}} \right) dt' \\ &= \frac{d}{dn} \int_0^{\frac{t}{t_0 n^{1/\alpha}}} \mathcal{L}_\alpha(\eta) d\eta, \end{aligned}$$

from which Eq. (15) immediately follows. An alternative derivation of Eq. (15), as well as short discussion of the cases with $\alpha \geq 1$ are given in Appendix A.

We note that the same approach (namely, keeping only the leading asymptotics in s when expanding the logarithm) can be applied also for $\alpha > 1$, which leads in this case to

$$\chi_n(s) \simeq t_0 \exp(-nt_0 s).$$

The Laplace inversion now immediately gives

$$\chi_n(t) = t_0 \delta(nt_0 - t) \equiv \delta\left(n - \frac{t}{t_0}\right). \quad (17)$$

Here we make several remarks. First, while this result looks like simply taking $\alpha = 1$ in Eq. (16), it corresponds to the whole domain of $\alpha > 1$ and remarkably does not include the case $\alpha = 1$ for which $\psi(s)$ has a different form. Moreover, neglecting higher terms does not allow us to say in what sense the convergence to a δ function takes place, i.e., in what quantity does really converge in distribution to a δ function. This quantity is *not* n , as we will discuss somewhat later [see Eqs. (40) and (41)].

Let us return to the case $\alpha < 1$. The Lévy density $\mathcal{L}_\alpha(y)$ for $0 < \alpha < 1$ can be represented via the Mainardi function $M_\alpha(y)$ (sometimes called the auxiliary function in the theory of Wright functions), see Refs. [4–7]:

$$\mathcal{L}_\alpha(y) = \frac{\alpha}{y^{\alpha+1}} M_\alpha\left(\frac{1}{y^\alpha}\right), \quad (18)$$

where the Mainardi function is given by its series expansion,

$$\begin{aligned} M_\alpha(y) &= W_{-\alpha, 1-\alpha}(-y) = \sum_{n=0}^{\infty} \frac{(-y)^n}{n! \Gamma[-\alpha n + (1-\alpha)]} \\ &\equiv \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \Gamma(\alpha n) \sin(\pi \alpha n)}{(n-1)!} y^{n-1}, \end{aligned} \quad (19)$$

with $W_{a,b}(y)$ being the Wright function. The asymptotic behavior of the Mainardi function for large values of its arguments is given by

$$M_\alpha(y) \simeq \frac{1}{\sqrt{2\pi(1-\alpha)}} (\alpha y)^{\frac{\alpha-1/2}{1-\alpha}} \exp\left[-\frac{1-\alpha}{\alpha} (\alpha y)^{\frac{1}{1-\alpha}}\right]. \quad (20)$$

We thus have:

$$\chi_n(\tau) = \frac{1}{\tau^\alpha} M_\alpha\left(\frac{n}{\tau^\alpha}\right). \quad (21)$$

For $\alpha = 1$, Eq. (15) needs only for minor corrections, see Appendix A, but the relation between the Lévy density and the Mainardi function breaks down fully, as can be seen from the fact that the series expansion of the Mainardi function, Eq. (19), vanishes identically. The limit $\alpha \rightarrow 1_-$ is therefore a singular one.

We note for the following use that the mean number of steps in the CTRW with $\alpha < 1$, as given by Eq. (7) for $\psi(s)$ given by Eq. (12), is

$$\langle n(t) \rangle = \frac{1}{\Gamma(1+\alpha)} \tau^\alpha \quad (22)$$

in the asymptotics of long times (see Eq. (3.22) of Ref. [9]), while for $\alpha > 1$ it is simply

$$\langle n(t) \rangle = \tau. \quad (23)$$

Although it may seem that the results, Eq. (22) and Eq. (23) seamlessly merge for $\alpha = 1$, this is not true. Using Eqs. (7) and (14) we get in the leading order:

$$\langle n(s) \rangle \simeq -s^{-2} \frac{1}{t_0 \ln(t_0 s)} = s^{-2} L(s),$$

where the function

$$L(s) = -\frac{1}{t_0 \ln(t_0 s)}$$

is slowly varying at zero. The inversion of this expression follows by the Tauberian theorem [10] and gives

$$\langle n(t) \rangle \simeq t L\left(\frac{1}{t}\right) = -\frac{t}{t_0 \ln(t_0/t)}.$$

Introducing $\tau = t/t_0$ we get:

$$\langle n(t) \rangle \simeq \frac{\tau}{\ln \tau}.$$

C. Scaling of the PDF of positions

The PDF in the Fourier-Laplace domain is given by the Montroll-Weiss formula, Eq. (6). For large time- and space scales, $k \ll \sigma^{-1}$ and $s \ll t_0^{-1}$, Eq. (6) can be approximated by

$$p(k, s) \simeq t_0^\alpha s^{\alpha-1} \frac{1}{\frac{1}{2} \sigma^2 k^2 + (t_0 s)^\alpha} = \frac{s^{\alpha-1}}{\frac{\sigma^2}{2 t_0^\alpha} k^2 + s^\alpha}. \quad (24)$$

In this approximation it is necessary to assume that $\sigma^2 k^2$ and $(t_0 s)^\alpha$ are of the same order of magnitude, which corresponds to the “body” of the distribution, i.e., in the original domain to the region $x^2 \sim \sigma^2 \langle n \rangle$, essentially $|x| \ll \sigma \langle n \rangle$. The prefactor of k^2 , $K = \frac{\sigma^2}{2 t_0^\alpha}$, is related to the coefficient of the anomalous diffusion. The domain of large deviations, $|x| \sim \sigma \langle n \rangle$, follows a different scaling [11–13], which will be discussed later.

The inverse transform of Eq. (24) gives a scaling form of the PDF; see Ref. [4] and references therein:

$$p(x|t) = \frac{1}{2\sqrt{K}t^\alpha} M_{\alpha/2}\left(\frac{|x|}{\sqrt{K}t^\alpha}\right), \quad (25)$$

again involving the Mainardi function but of a different index $\beta = \alpha/2$. To see that this is indeed the case, it is enough to perform the inverse Fourier transform of Eq. (24) to get

$$p(x, s) = \frac{s^{\frac{\alpha}{2}-1}}{2\sqrt{K}} \exp\left(-\frac{|x|}{\sqrt{K}} s^{\frac{\alpha}{2}}\right),$$

recognize in it a structure similar to the one in Eq. (16) and perform the inverse Laplace transform following the same steps as above.

The scaling form, Eq. (25), shows that the distribution of $\xi = x/\sigma$ is a universal function of $\tau = t/t_0$. The asymptotic behavior of this function is known. The Mainardi function $M(y)$ is defined on the whole real axis [8]. Considered as a function of the absolute value of the argument, i.e., on the non-negative half-axis, $M_\beta(|y|)$ for $\beta \leq 1/2$ possesses a maximum at $y = 0$. For all $\beta < 1/2$ (i.e., for $\alpha < 1$) this maximum corresponds to a cusp at zero, which disappears only in the case $\beta = 1/2$ when all terms of odd orders disappear due to the divergence of the Gamma functions of whole nonpositive arguments. In this case the Mainardi function on the non-negative half-axis is a “half-Gaussian,”

$$M_{1/2}(y) = \pi^{-1/2} \exp(-y^2/4) \quad (26)$$

($y \geq 0$). The height of the Mainardi function at the mode is $M_{\alpha/2}(0) = \frac{1}{\Gamma(1-\alpha/2)}$. Here the limit $\alpha \rightarrow 1$ is a regular one

for the Mainardi function, but the approximation as a whole breaks down because of the presence of the logarithmic corrections in $\psi(s)$, i.e., due to the inaccuracy of the initial Eq. (24).

In the dimensionless variables ξ and τ we have therefore

$$p(\xi|\tau) = \frac{1}{\sqrt{2}\sqrt{\tau^\alpha}} M_{\alpha/2} \left(\frac{\sqrt{2}|\xi|}{\tau^{\frac{\alpha}{2}}} \right). \quad (27)$$

In what follows, it will be convenient to express the PDF in term of the relative position κ of the walker within the body of the distribution,

$$\kappa^2 = \frac{x^2}{\langle x^2 \rangle} = \frac{x^2}{\sigma^2 \langle n(t) \rangle} = \frac{\xi^2 \Gamma(\alpha + 1)}{\tau^\alpha}. \quad (28)$$

For long times, the distribution of the relative positions gets universal,

$$p(\kappa) = \frac{1}{\sqrt{2\alpha\Gamma(\alpha)}} M_{\alpha/2} \left(\sqrt{\frac{2}{\alpha\Gamma(\alpha)}} |\kappa| \right). \quad (29)$$

This expression will be very convenient for the discussion of far asymptotics of $p(n|x, t)$ in the body of the distribution since the dimensionless variable κ has a clear physical meaning: It is the distance from the origin measured in units of the standard deviation of the final position of the random walker at given time from its initial position.

D. Scaling of the relative number of steps conditioned on relative position

1. The case $\alpha < 1$

For $\alpha < 1$, our Eqs. (2), (21), and (27) lead us to the final expression for $p(n|x, t)$ for x within the body of the distribution. This can be the most convenient expressed via dimensionless parameters ξ and τ :

$$p(n|\xi, \tau) = \frac{\frac{1}{\sqrt{2\pi n}} \exp\left(-\frac{\xi^2}{2n}\right) \frac{1}{\tau^\alpha} M_\alpha\left(\frac{n}{\tau^\alpha}\right)}{\frac{1}{\sqrt{2}\sqrt{\tau^\alpha}} M_{\alpha/2}\left(\frac{\sqrt{2}|\xi|}{\tau^{\frac{\alpha}{2}}}\right)}. \quad (30)$$

The correct normalization of this expression (for the case that n is so large that it can be approximated by a continuous variable) is guaranteed by virtue of Eq. (2.13) of Ref. [5] leading in our notation to:

$$\begin{aligned} & \int_0^\infty \frac{1}{\sqrt{2\pi n}} \exp\left(-\frac{\xi^2}{2n}\right) \frac{1}{\tau^\alpha} M_\alpha\left(\frac{n}{\tau^\alpha}\right) dn \\ &= \frac{1}{\sqrt{2}\sqrt{\tau^\alpha}} M_{\alpha/2}\left(\frac{\sqrt{2}|\xi|}{\tau^{\frac{\alpha}{2}}}\right). \end{aligned}$$

Instead of looking for the distribution of n , one can look for the distribution of the *relative* number of steps,

$$v = \frac{n}{\langle n \rangle} = \Gamma(1 + \alpha) n \tau^{-\alpha}. \quad (31)$$

Now, for $\alpha < 1$, we have:

$$p(v|\xi, \tau) = \frac{1}{\sqrt{\pi v \Gamma(1 + \alpha)}} \frac{\exp\left(-\frac{\kappa^2}{2v}\right) M_\alpha\left(\frac{v}{\Gamma(1 + \alpha)}\right)}{M_{\alpha/2}(\sqrt{2/\Gamma(1 + \alpha)} |\kappa|)} \quad (32)$$

being the universal function of the relative position κ , introduced by Eq. (28), only. In general it holds

$$p(v|\kappa) = \mathcal{N}(\kappa) \frac{1}{\sqrt{\pi v}} \exp\left(-\frac{\kappa^2}{2v}\right) p_\alpha(v), \quad (33)$$

with $p_\alpha(v)$ being the PDF of the relative number of steps as derived from $\chi_n(t)$. The normalization factor is given by

$$[\mathcal{N}(\kappa)]^{-1} = \int_0^\infty \frac{1}{\sqrt{\pi v}} \exp\left(-\frac{\kappa^2}{2v}\right) p_\alpha(v) dv.$$

For $\alpha < 1$ we have $p_\alpha(v) = M_\alpha\left(\frac{v}{\Gamma(1 + \alpha)}\right)$ and $\mathcal{N}(\kappa) = 1/\sqrt{\Gamma(1 + \alpha) M_{\alpha/2}(\sqrt{2/\Gamma(1 + \alpha)} |\kappa|)}$.

Close to the origin we have $\kappa = 0$, and therefore

$$p(v|\kappa = 0) = \frac{1}{\sqrt{\pi v \Gamma(1 + \alpha)}} M_\alpha\left(\frac{v}{\Gamma(1 + \alpha)}\right) [M_{\alpha/2}(0)]^{-1}.$$

This PDF has a square-root singularity at $v \rightarrow 0$ and a squeezed-exponential tail following from the large- y asymptotics of the Mainardi function.

For κ large, small values of v are strongly suppressed by the inverse-exponential behavior stemming from the Gaussian, so that the large argument asymptotics of the Mainardi function, Eq. (20), can be used universally. Since the denominator in our equation only guarantees for the correct normalization of the overall expression, we can omit it in the following discussion and normalize *a posteriori*. Absorbing all constant prefactors into the new normalization constant $\mathcal{N}_1(\kappa)$ we may write:

$$p(v|\kappa) = \mathcal{N}_1(\kappa) v^{\frac{3\alpha-2}{2(1-\alpha)}} \exp\left[-\frac{\kappa^2}{2v} - C v^{\frac{1}{1-\alpha}}\right] \quad (34)$$

with the constant

$$C = \frac{1 - \alpha}{\alpha} [\Gamma(\alpha)]^{-\frac{1}{1-\alpha}}. \quad (35)$$

For κ large, the exponential term $\exp[S(v, \kappa)]$ in Eq. (34) gets a pronounced maximum at

$$\begin{aligned} v_m &= \left(\frac{2C}{1 - \alpha} \right)^{-\frac{1-\alpha}{2-\alpha}} (\kappa^2)^{\frac{1-\alpha}{2-\alpha}} \\ &= \left(\frac{2}{\alpha} \right)^{-\frac{1-\alpha}{2-\alpha}} [\Gamma(\alpha)]^{\frac{1}{2-\alpha}} |\kappa|^{2\frac{1-\alpha}{2-\alpha}}, \end{aligned} \quad (36)$$

which moves to larger values of v when κ grows, and the second derivative at the maximum is

$$\begin{aligned} S''(v_m, \kappa) &= \frac{d^2}{dv^2} S(v, \kappa) \Big|_{v=v_m} \\ &= \frac{2 - \alpha}{2(1 - \alpha)} \left(\frac{2C}{1 - \alpha} \right)^{\frac{3}{2-\alpha}} (\kappa^2)^{\frac{1-2\alpha}{2-\alpha}}. \end{aligned}$$

Note that for $\alpha = 1/2$ this second derivative is equal to $-6C$ and does not depend on κ ; for $\alpha < 1/2$ it grows with κ , and for $\alpha > 1/2$ it decays with κ .

The existence of the preexponential shifts the maximum and changes the second derivative, but both changes vanish for κ large: Let us put the power-law prefactor inside the

exponential and look at the maximum of the expression

$$f(v) = \frac{3\alpha - 2}{2(1 - \alpha)} \ln v - \frac{\kappa^2}{2v} - C v^{\frac{1}{1-\alpha}}.$$

The position of this maximum, corresponding to the most probable value $v_{m.p.}$ of v , is given by the solution of the algebraic equation

$$\kappa^2 = -\frac{3\alpha - 2}{1 - \alpha} v + \frac{C}{1 - \alpha} v^{\frac{2-\alpha}{1-\alpha}},$$

where the power of v in the second term in the right-hand side is always larger than 2. Independent on the sign of the prefactor of the first term on the right-hand side, if the value of the right-hand side is large, then it is dominated by the second term, and therefore for $|\kappa|$ large, $v_{m.p.}$ coincides with v_m . Therefore, for κ large, the distribution of v , close to its mode, resembles a Gaussian, with the mean $\mu = v_m$ and variance $\sigma_v^2 = S''(v_m, \kappa)$. As we proceed to show, for κ large, the mean of this distribution indeed coincides with its mode.

For $\alpha > 1/2$, this distribution narrows, so that the fluctuations of v around its mean get the smaller the further in the tail of the distribution of positions the final position of the walker lies. For $\alpha < 1/2$ the distribution gets broader, but the coefficient of variation $C_v = \sigma_v/\mu$ of this distribution tends to

$$C_v = \sqrt{\frac{2 - \alpha}{2(1 - \alpha)}} \left(\frac{2C}{1 - \alpha} \right)^{\frac{5}{2} \frac{1-\alpha}{2-\alpha}} (\kappa^2)^{-\frac{1}{2(2-\alpha)}},$$

and decays monotonically with κ for all $\alpha < 1$, indicating that the spread of the number of souvenirs with respect to its mean gets always smaller in the tails of the distribution of positions: The further the walker is found from the initial position, the smaller is the *relative* fluctuation of the number of souvenirs collected. Note that our expression for C_v does not possess any regular limit for $\alpha \rightarrow 1_-$ due to the breakdown of our initial approximation.

In Figs. 1–3 we plot the corresponding distributions for $\kappa = 0, 1, 3$, and 5 for the values of $\alpha = 0.5, 0.4$, and 0.65 where one can evaluate the Mainardi functions by summing up 100 to 200 terms of the corresponding series in Mathematica. In Fig. 1 we show in the upper panel the PDF $p(\kappa)$ to indicate that the curves at a lower panel correspond to the behavior or the distribution of the relative number of steps at the mode and at distances equal to 1, 3, and 5 standard deviations; the last one already corresponds to a relatively far tail of the distribution of displacements. The PDFs of displacements for $\alpha = 0.4$ and $\alpha = 0.65$ look essentially very similar to the one for $\alpha = 0.5$ with the only difference that the peak is somewhat steeper for $\alpha = 0.4$ and somewhat lower for $\alpha = 0.65$, so that we refrain from plotting these.

Figures 2 and 3 show that the corresponding asymptotic behavior of the width of $p(v|\kappa)$ sets up already at moderate values of κ : Although for $\alpha = 0.65$ the peaks of the distribution go up when κ grows (indicating that the normalized distribution should narrow), for $\alpha = 0.4$ the peaks go down, indicating that the normalized distribution should broaden. The PDFs for $\kappa = 5$ and $\alpha = 0.5$ and 0.65 are already not

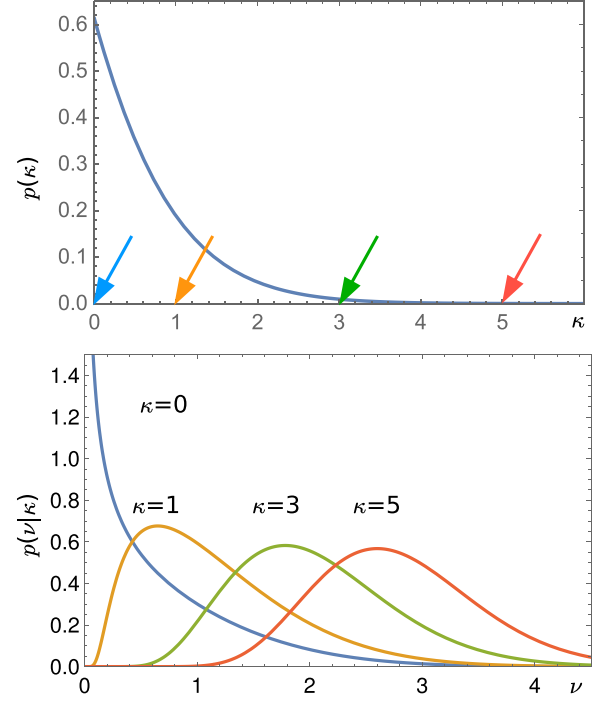


FIG. 1. Upper panel: The PDF of scaled positions for a CTRW with $\alpha = 0.5$ right from its mode. The arrows indicate the positions for which the PDFs of the relative number of steps are shown in Figs. 2–4. Lower panel: The PDF of the relative numbers of steps $p(v|\kappa)$ evaluated for positions indicated above.

much different from Gaussian ones, while the skewness is still perceptible for smaller $\alpha = 0.4$.

2. The case $\alpha > 1$

Let us discuss the properties of $\chi_n(t)$ and $p_\alpha(v)$ for the case $\alpha > 1$, using the Laplace transform $\psi(s) \simeq 1 - t_0 s + (t_1 s)^\alpha + \dots$ as given by Eq. (13). We use Eqs. (7) and (9) and obtain the first and the second moments of the number of steps n in the leading and lowest subleading orders in s .

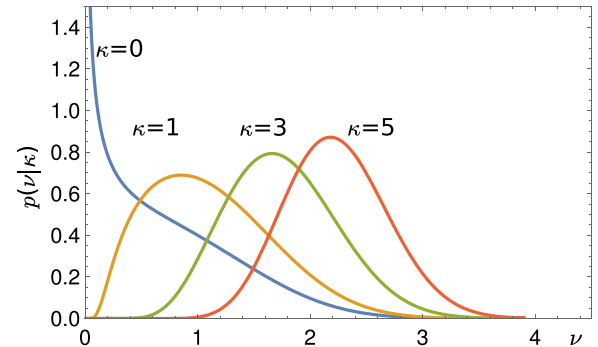


FIG. 2. The PDF of the relative numbers of steps $p(v|\kappa)$ evaluated for a CTRW with $\alpha = 0.65$ at the same relative positions as used in Fig. 1.

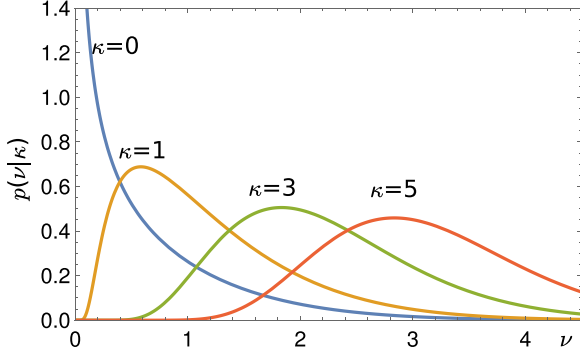


FIG. 3. The PDF of the relative numbers of steps $p(v|\kappa)$ evaluated for a CTRW with $\alpha = 0.4$ at the same relative positions as used in Fig. 1.

We get:

$$\langle n \rangle(s) = \frac{1}{t_0} s^{-2} - s^{-1} + \frac{t_1^\alpha}{t_0^\alpha} s^{\alpha-3} \quad (37)$$

and

$$\langle n^2 \rangle(s) = \frac{2}{t_0^2} s^{-3} - \frac{3}{t_0^2} s^{-2} + \frac{4t_1^\alpha}{t_0^3} s^{\alpha-4}. \quad (38)$$

The reasons to keep the second terms in both equations above is that for the case $\alpha > 2$, corresponding to formally substituting the $\alpha = 2$ into previous equations, their second and third terms get to be of the same order in s , and we will need both in the following expressions. For $1 < \alpha < 2$ the second terms in both expansions can be safely neglected.

Therefore, for t long, and $1 < \alpha < 2$ we get

$$\langle n(t) \rangle = \frac{t}{t_0} + \frac{t_1^\alpha}{t_0^2 \Gamma(3-\alpha)} t^{2-\alpha}. \quad (39)$$

For $\langle n(t) \rangle^2$ in the leading and first subleading order, we get

$$\langle n(t) \rangle^2 \simeq \left(\frac{t}{t_0} \right)^2 + \frac{2t_1^\alpha}{t_0^3 \Gamma(3-\alpha)} t^{3-\alpha}.$$

For $\langle n^2(t) \rangle$ we find

$$\langle n^2(t) \rangle \simeq \left(\frac{t}{t_0} \right)^2 + \frac{4}{\Gamma(4-\alpha)} \frac{t_1^\alpha}{t_0^3} t^{3-\alpha}.$$

The variance of the number of steps in the leading order in t for $1 < \alpha < 2$ is therefore

$$\begin{aligned} \langle \Delta n^2(t) \rangle &= \langle n^2(t) \rangle - \langle n(t) \rangle^2 \\ &\simeq \left[\frac{4}{\Gamma(4-\alpha)} - \frac{2}{\Gamma(3-\alpha)} \right] \frac{t_1^\alpha}{t_0^3} t^{3-\alpha} \\ &= \frac{2(\alpha-1)}{\Gamma(4-\alpha)} \frac{t_1^\alpha}{t_0^3} t^{3-\alpha}, \end{aligned} \quad (40)$$

where in the last line we used that $\Gamma(3-\alpha) = \Gamma(4-\alpha)/(3-\alpha)$.

For $\alpha > 2$ (where the expressions follow by formally putting $\alpha = 2$) the second and the third terms in Eqs. (37) and

(38) are of the same order in s . We thus get:

$$\langle n \rangle(s) \simeq \frac{1}{t_0} s^{-2} + \left(\frac{t_1^2}{t_0^2} - 1 \right) s^{-1}.$$

We remind that in this case $t_1 = (\frac{t^2}{2})^{\frac{1}{2}}$. Moreover,

$$\langle n^2 \rangle(s) \simeq \frac{2}{t_0^2} s^{-3} + \frac{4t_1^2 - 3t_0^2}{t_0^3} s^{-2}.$$

Turning to the time domain we get:

$$\langle n(t) \rangle \simeq \frac{t}{t_0} + \left(\frac{t_1^2}{t_0^2} - 1 \right),$$

so that

$$\langle n(t) \rangle^2 \simeq \left(\frac{t}{t_0} \right)^2 + 2 \left(\frac{t_1^2}{t_0^2} - 1 \right) \left(\frac{t}{t_0} \right)$$

(note that for the exponential WTD with $t_1 = t_0$ the subleading terms vanish identically), and

$$\langle n^2(t) \rangle \simeq \left(\frac{t}{t_0} \right)^2 + \frac{4t_1^2 - 3t_0^2}{t_0^2} \left(\frac{t}{t_0} \right).$$

For $\alpha > 2$ we thus universally have

$$\langle \Delta n^2(t) \rangle \simeq \left(\frac{2t_1^2}{t_0^2} - 1 \right) \left(\frac{t}{t_0} \right). \quad (41)$$

The first bracket in the last expression is always non-negative since $2t_1^2 = \langle t^2 \rangle$ while $t_0^2 = \langle t \rangle^2$ and $\langle t^2 \rangle \geq \langle t \rangle^2$. The equality can only be attained if $\psi(t)$ is a δ function, i.e., for the walk at a constant pace. For the exponential distribution of waiting times $t_1 = t_0$, and the first bracket is equal to unity. We note that the result, Eq. (40), does not coincide with Eq. (41) in the limit $\alpha \rightarrow 2_-$, where Eq. (40) gives a prefactor being $2t_1^2/t_0^2$, and not $(2t_1^2/t_0^2) - 1$: The limit $\alpha = 2$ is singular, since it corresponds to merging of two subleading contributions.

Thus, the distribution of n itself therefore does not tend to a δ distribution but broadens. Its coefficient of variation

$$\frac{\langle \Delta n^2(t) \rangle^{\frac{1}{2}}}{\langle n(t) \rangle} \simeq \begin{cases} \sqrt{\frac{2(\alpha-1)}{\Gamma(4-\alpha)}} \left(\frac{t_1}{t_0} \right)^{\frac{\alpha}{2}} \left(\frac{t}{t_0} \right)^{\frac{1-\alpha}{2}}, & 1 < \alpha < 2 \\ \sqrt{\frac{2t_1^2}{t_0^2} - 1} \left(\frac{t}{t_0} \right)^{-\frac{1}{2}}, & 2 < \alpha, \end{cases}$$

however, decays to zero. Therefore, when the distribution of n moves to the right in course of the time, its width grows slower than its mean. What tends to a δ function is then a distribution of a the number of steps relative to its mean,

$$v = \frac{n}{\langle n(t) \rangle} = n \left(\frac{t}{t_0} \right)^{-1}.$$

Thus, for $\alpha > 1$, $\chi_n(t)$ does not tend to a δ function, but $p_\alpha(v)$ in Eq. (33) does. Using Eq. (33) and noting that for $\alpha > 2$ the normalization factor in the denominator of this equation is universally equal to the exponential form in the numerator, we get

$$p(v|\kappa) = \delta(v-1).$$

The distribution of the *relative* number of steps tends to a δ function, in spite of the fact the the true number of steps leading to a given position fluctuates. The character of these

fluctuations is the same as in thermodynamic extensive variables because for $\alpha > 1$ the number of steps n is extensive in time.

As we show in Appendix A, the behavior for the case $\alpha = 1$ is essentially the same as for $\alpha > 1$, so that the distribution of the relative number of steps,

$$\nu = \frac{n}{\langle n(t) \rangle} \simeq \frac{n \ln \tau}{\tau}, \quad (42)$$

narrows to a δ function, but the convergence to this limit is very slow (logarithmic in time), so that at any “reasonable” time the distribution of the number of steps is nonuniversal. The convergence to a δ function can be anticipated when calculating $\langle n^2(t) \rangle$ using Eq. (9) and performing exactly the same steps as we used at the end of Sec. III B to calculate $\langle n(t) \rangle$. Even including the slowly varying functions we would get $\langle n^2(t) \rangle = \langle n(t) \rangle^2$: The accuracy of our initial approximations is not enough to see the difference.

IV. THE MEAN NUMBER OF STEPS LEADING TO A GIVEN POSITION

Now we turn to the calculation of the mean number of steps which our walker has to make in order to come to the position x at a fixed time t . We may do so by using the asymptotic expressions derived above (and returning to initial, dimensional coordinates), but for the case of symmetric CTRW with finite second moment of the jump length, another, much more elegant method of calculations exists, which gives a relation which is exact. This expression is asymptotically exact for the body of the distribution for any step lengths’ PDF and is exact for the case of Gaussian steps length distribution even in the large deviations’ domain, where our previous results do not apply. We will again concentrate our discussion at the one-dimensional case but later will make some notes on higher dimensions as well.

We again start from Eq. (3) and look at

$$\langle n|x, t \rangle = \frac{\sum_{n=0}^{\infty} n p_n(x) \chi_n(t)}{p(x|t)} = \frac{\Phi(x, t)}{p(x|t)}, \quad (43)$$

where we note that $p(x|t)$ is given by the inverse Fourier-Laplace transform of Eq. (6). The sum in the numerator of Eq. (43), $\Phi(x, t)$, can be obtained via its Fourier-Laplace transform,

$$\Phi(k, s) = \sum_{n=0}^{\infty} n \lambda^n(k) \chi_n(s) = \frac{1 - \psi(s)}{s} \sum_{n=0}^{\infty} n \lambda^n(k) \psi^n(s).$$

One can get the expression for $\Phi(k, s)$ by formally manipulating the one for $p(k, s)$:

$$\begin{aligned} \Phi(k, s) &= \lambda \sum_{n=0}^{\infty} \frac{1 - \psi(s)}{s} \psi^n(s) \frac{d}{d\lambda} \lambda^n(k) \Big|_{\lambda=\lambda(k)} \\ &= \lambda \frac{1 - \psi(s)}{s} \frac{d}{d\lambda} \frac{1}{1 - \lambda \psi(s)} \Big|_{\lambda=\lambda(k)} \\ &= \psi(s) \lambda(k) \frac{1 - \psi}{s} \frac{1}{[1 - \lambda(k) \psi(s)]^2}. \end{aligned} \quad (44)$$

Let us assume that instead of manipulating our formal expression we start from the dependence of $\lambda(x)$ on the formal parameter σ (which defines the spatial scale, i.e., the unit of measurements for the coordinate):

$$\lambda(x) = \frac{1}{\sigma} q\left(\frac{x}{\sigma}\right),$$

so that $\lambda(k)$ is $q(\sigma k)$, and

$$p(k, s; \sigma) = \frac{1 - \psi(s)}{s} \frac{1}{1 - q(\sigma k) \psi(s)}.$$

Moreover,

$$\Phi(k, s; \sigma) = \psi(s) q(\sigma k) \frac{1 - \psi(s)}{s} \frac{1}{[1 - q(\sigma k) \psi(s)]^2}. \quad (45)$$

Now we note that

$$\frac{d}{d\sigma} p(k, s; \sigma) = \frac{1 - \psi(s)}{s} \frac{1}{[1 - q(\sigma k) \psi(s)]^2} \psi(s) \frac{d}{d\sigma} q(\sigma k). \quad (46)$$

Comparing Eqs. (45) and (46) we see that

$$\frac{d}{d\sigma} p(k, s; \sigma) = \Phi(k, s; \sigma) \frac{d}{d\sigma} \ln q(\sigma k).$$

Therefore, for the Gaussian step-length PDF with $q(\sigma k) = \exp(-\frac{1}{2}\sigma^2 k^2)$ we have exactly

$$\frac{d}{d\sigma} p(k, s; \sigma) = -\sigma k^2 \Phi(k, s; \sigma). \quad (47)$$

For any other symmetric distribution possessing the second moment, this expression is valid asymptotically, for small k . We note that in a rotationally symmetric case in d dimensions the expression $q(\sigma k) = \exp(-\frac{1}{2d}\sigma^2 k^2)$ holds, so that Eq. (47) holds for this case in any dimension as well, up to an additional prefactor $1/d$ on the right-hand side.

For $d = 1$ one can obtain the solution which is absolutely independent from the previous discussion of the PDF $p(x|t)$. We note that Eq. (47) can be formally solved in the Fourier domain giving

$$\Phi(k, s; \sigma) = -\frac{1}{\sigma} \frac{1}{k^2} \frac{d}{d\sigma} p(k, s; \sigma).$$

Turning to the original domain, we see that

$$\Phi(x, t; \sigma) = \frac{1}{\sigma} \int_{-\infty}^x dx' \int_{-\infty}^{x'} dx'' \frac{d}{d\sigma} p(x''|t; \sigma).$$

Now we bring the σ derivative in between the integrals and see that

$$\Phi(x, t; \sigma) = \frac{1}{\sigma} \int_{-\infty}^x dx' \frac{d}{d\sigma} F(x'|t; \sigma),$$

with $F(x|t; \sigma) = \int_{-\infty}^x p(x''|t; \sigma) dx''$ being the cumulative distribution function of x . Now we note that σ essentially only sets the length scale of our problem, so that rescaling only changes the x dependence of F according to

$$F(x|t; \sigma) = G\left(\frac{x}{\sigma}, t\right).$$

We therefore get:

$$\begin{aligned}\Phi(x, t; \sigma) &= \frac{1}{\sigma} \int_{-\infty}^x dx' \frac{d}{d\sigma} G\left(\frac{x'}{\sigma}, t\right) \\ &= -\frac{1}{\sigma} \int_{-\infty}^x dx' \frac{x'}{\sigma^2} G\left(\frac{x'}{\sigma}, t\right).\end{aligned}$$

We moreover note that

$$p(x|t; \sigma) = \frac{d}{dx} F(x|t; \sigma) = \frac{1}{\sigma} G\left(\frac{x}{\sigma}, t\right),$$

so that

$$\Phi(x, t; \sigma) = -\frac{1}{\sigma^2} \int_{-\infty}^x x' p(x'|t; \sigma) dx'. \quad (48)$$

Now we note that for a symmetric walk the mean position at any time is at the origin,

$$\langle x|t; \sigma \rangle = \int_{-\infty}^{\infty} x' p(x'|t; \sigma) dx' = 0, \quad (49)$$

and rewrite our final expression as

$$\Phi(x, t; \sigma) = \frac{1}{\sigma^2} \int_x^{\infty} x' p(x'|t; \sigma) dx'.$$

From this we obtain the final expression

$$\langle n|x, t \rangle = \frac{1}{\sigma^2} \frac{\int_x^{\infty} x' p(x'|t; \sigma) dx'}{p(x|t; \sigma)} \quad (50)$$

right from the mode. The result to the left from the mode follows by symmetry. Turning to scaled variables, we get for $\kappa > 0$:

$$\langle v|\kappa \rangle = \frac{\int_{\kappa}^{\infty} \kappa' p(\kappa') d\kappa'}{p(\kappa)}. \quad (51)$$

Let us now consider specific examples. For Gaussian case corresponding to all $\alpha > 1$ we have

$$p(\kappa) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\kappa^2}{2}\right),$$

so that

$$\langle v|\kappa \rangle = \frac{\int_{\kappa}^{\infty} \kappa' \exp\left(-\frac{\kappa'^2}{2}\right) d\kappa'}{\exp\left(-\frac{\kappa^2}{2}\right)} = 1.$$

The mean rescaled number of steps on a way leading to any point in the body of the distribution is independent on the position of this point and is equal to the mean number of steps done up to time t .

The scaling form of $p(x|t)$ for CTRW with $\alpha < 1$ is given by Mainardi function. Using Eq. (51) and Eq. (29), we may write

$$\langle v|\kappa \rangle = \frac{\int_{\kappa}^{\infty} \kappa' M_{\alpha/2}\left(\sqrt{\frac{2}{\alpha\Gamma(\alpha)}} \kappa'\right) d\kappa'}{M_{\alpha/2}\left(\sqrt{\frac{2}{\alpha\Gamma(\alpha)}} \kappa\right)}. \quad (52)$$

Let us first discuss the asymptotic behavior of this function. In what follows we denote the integral in numerator by

$$I = \int_{\kappa}^{\infty} \kappa' M_{\alpha/2}\left(\sqrt{\frac{2}{\alpha\Gamma(\alpha)}} \kappa'\right) d\kappa'. \quad (53)$$

For $\kappa \gg 1$ the behavior of the Mainardi function is given by its asymptotic expansion, Eq. (20), and the integral I can be calculated using the Laplace method. The asymptotics of the Mainardi function for large values of its argument has a squeezed-exponential form $M_{\alpha/2}(y) \simeq Ay^a \exp(-by^c)$, where all constants A, a, b , and c are positive for $\alpha < 1$. We will first perform the calculation with this general form and then substitute only the values of the constants that we really need. Denoting $y = \sqrt{\frac{2}{\alpha\Gamma(\alpha)}} \kappa$ we write the integral I as

$$I = \frac{\alpha\Gamma(\alpha)A}{2} \int_z^{\infty} y^{a+1} \exp(-by^c) dy$$

with $z = \sqrt{\frac{2}{\alpha\Gamma(\alpha)}} \kappa$. The application of the Laplace method now gives:

$$I = \frac{\alpha\Gamma(\alpha)A}{2} \frac{z^{a+1} \exp(-bz^c)}{bcz^{c-1}},$$

so that the whole expression, Eq. (52) reads

$$\begin{aligned}\langle v|\kappa \rangle &= \frac{\alpha\Gamma(\alpha)A}{2} \frac{z^{a+1} \exp(-bz^c)}{bcz^{c-1}} \frac{1}{Az^a \exp(-bz^c)} \\ &= \frac{\alpha\Gamma(\alpha)}{2} \frac{z^{2-c}}{bc}.\end{aligned}$$

The constants b and c follow from Eq. (20): $b = \frac{2-\alpha}{\alpha} \left(\frac{\alpha}{2}\right)^{\frac{2}{2-\alpha}}$ and $c = \frac{2}{2-\alpha}$. Substituting these we get:

$$\langle v|\kappa \rangle \simeq \left(\frac{2}{\alpha}\right)^{-\frac{1-\alpha}{2-\alpha}} [\Gamma(\alpha)]^{\frac{1}{2-\alpha}} |\kappa|^{\frac{2}{2-\alpha}}, \quad (54)$$

and is equal to $v_{m.p.}$ as given by Eq. (36). Therefore, for $|\kappa|$ large, the mean of the corresponding distribution coincides with its mode.

We can also consider what happens for $\kappa \rightarrow 0$, in the center of the distribution of displacements. Here the integral we have to evaluate is

$$I = \frac{\alpha\Gamma(\alpha)}{2} \int_0^{\infty} y M_{\alpha/2}(y) dy.$$

We change the variable of integration to $z = y^{-2/\alpha}$ so that $y = z^{-\frac{\alpha}{2}}$ and rewrite the integral as

$$I = \frac{\alpha\Gamma(\alpha)}{2} \int_0^{\infty} z^{-\frac{\alpha}{2}} M_{\alpha/2}\left(z^{-\frac{\alpha}{2}}\right) \frac{\alpha}{2} z^{-\frac{\alpha}{2}-1} dz.$$

Now we recognize that according to Eq. (18) $M_{\alpha/2}(z^{-\frac{\alpha}{2}}) z^{\frac{\alpha}{2}-1} = \mathcal{L}_{\alpha/2}(z)$, so that

$$I = \frac{\alpha\Gamma(\alpha)}{2\Gamma(1+\frac{\alpha}{2})} = \frac{\Gamma(\alpha)}{\Gamma(\frac{\alpha}{2})},$$

where we used the relation

$$\int_0^{\infty} z^{\eta} \mathcal{L}_{\beta}(z) dz = \frac{\Gamma(1-\eta/\beta)}{\Gamma(1-\eta)}$$

holding for all $-\infty < \eta < \beta$; see Ref. [14] (in our case $\beta = \alpha/2$, $\eta = -\alpha/2$). On the other hand, the value of $M_{\alpha/2}(0)$ follows from its series expansion, and is equal to

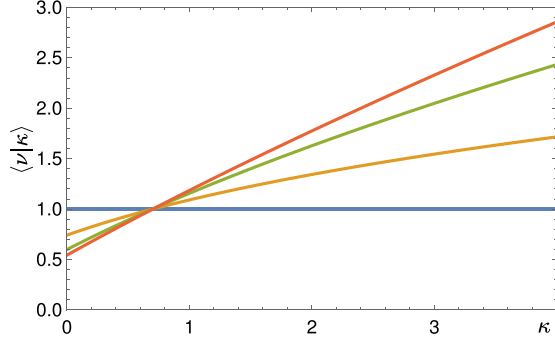


FIG. 4. The mean relative number of steps $\langle \nu | \kappa \rangle$ conditioned on position for CTRW with $\alpha = \frac{1}{3}, \frac{1}{2}, \frac{3}{4}$, and for $\alpha > 1$. The local slope of the curve decreases with increasing α .

$M_{\alpha/2}(0) = \frac{1}{\Gamma(1-\frac{\alpha}{2})}$, so that

$$\langle \nu | \kappa = 0 \rangle = \frac{\Gamma(\alpha)\Gamma(1-\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})}. \quad (55)$$

Note that having calculated $I = \frac{\alpha\Gamma(\alpha)}{2} \int_0^\infty y M_{\alpha/2}(y) dy$ we can also obtain the series expansion of $\langle \nu | \kappa \rangle$ for $\kappa \neq 0$ by using the series expansion of $M_{\frac{\alpha}{2}}$ in denominator and by noting that

$$\begin{aligned} & \int_\kappa^\infty \kappa' M_{\alpha/2} \left(\sqrt{\frac{2}{\alpha\Gamma(\alpha)}} \kappa' \right) d\kappa' \\ &= I - \int_0^\kappa \kappa' M_{\alpha/2} \left(\sqrt{\frac{2}{\alpha\Gamma(\alpha)}} \kappa' \right) d\kappa' \\ &= \frac{\Gamma(\alpha)\Gamma(1-\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})} - \frac{\alpha\Gamma(\alpha)}{2} \int_0^{\sqrt{\frac{2}{\alpha\Gamma(\alpha)}}\kappa} y M_{\alpha/2}(y) dy, \end{aligned}$$

where the last integral follows by term-by-term integration of the series expansion of the Mainardi function, Eq. (19):

$$\begin{aligned} I_{1,\alpha}(z) &= \int_0^z y M_{\alpha/2}(y) dy \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \Gamma(\alpha n) \sin(\pi \alpha n)}{(n-1)!} \frac{z^{n+1}}{n+1}. \end{aligned}$$

These expansions are used to plot $\langle \nu | \kappa \rangle$ close to the mode (up to $\kappa = 4$) where both the Mainardi function and the integral $I_{1,\alpha}(z)$ can be evaluated by Mathematica by summing up the first 100 terms of the corresponding series without additional precautions. The plots are given in Fig. 4.

V. LARGE DEVIATIONS DOMAIN

Equation (50) is written in the natural variables and holds independently on the type of scaling which is used.

The scaling domain in the PDF corresponds to the typical displacements which may be large but are considerably smaller than a typical contour length of a walk with the typical number of steps, i.e., $x \ll \sigma \langle n \rangle$. For x of the order of $\sigma \langle n \rangle$ and larger we enter the large deviation domain of

the corresponding process. The large deviation behavior of CTRW depends on the probability densities of large jumps and very short waiting times and is nonuniversal. However, the method outlined in Sec. IV gives us the possibility to calculate the mean number of steps at positions belonging to the large deviation domain in the case when the probability distribution of steps is *exactly* Gaussian, i.e., when Eq. (50) is exact. Other step lengths distributions require for an independent treatment. Although this Gaussian case is special, it gives considerable intuition about the generic behavior. We will still stick to dimensionless variables ξ and τ but refrain from using the rescaled number of steps ν and from measuring the distance in the standard deviations from the origin, as we did it in the scaling domain, since in the large deviation domain the width of the distribution loses its importance as the characteristic scale. The large deviation domain therefore corresponds to $\xi \gg \tau^\alpha$. The final results presented below give the scaling in the “very large deviations domain” $\xi \gg \tau$ for τ by itself large (otherwise, one should keep subleading terms in the corresponding equations).

For large x and t (i.e., large ξ and τ) in this domain

$$p(\xi | \tau) \sim \exp \left[-\tau \mathcal{X} \left(\frac{\xi}{\tau} \right) \right],$$

where $\mathcal{X}(z)$ is the large deviation rate for our process. The function $\mathcal{X}(z)$ is nonuniversal and depends on the details of WTD for very short times. It is an even function of z , and for $z > 0$ it is a monotonically growing function of its argument, see Ref. [13], where also explicit forms of the rate functions for some particular WTDs were calculated.

Now, for both ξ and τ large, the integral in the numerator of Eq. (50)

$$\begin{aligned} I_1 &= \int_\xi^\infty \xi' \exp \left[-\tau \mathcal{X} \left(\frac{\xi'}{\tau} \right) \right] d\xi' \\ &= \tau^2 \int_{\xi/\tau}^\infty z \exp [-\tau \mathcal{X}(z)] dz \end{aligned}$$

can be evaluated by the Laplace method:

$$I_1 = \frac{\tau^2 \frac{\xi}{\tau} \exp \left[-\tau \mathcal{X} \left(\frac{\xi}{\tau} \right) \right]}{\tau \mathcal{X}' \left(\frac{\xi}{\tau} \right)} = \frac{\xi \exp \left[-\tau \mathcal{X} \left(\frac{\xi}{\tau} \right) \right]}{\mathcal{X}' \left(\frac{\xi}{\tau} \right)}$$

[here $\mathcal{X}'(z)$ is the derivative of the rate function with respect to its argument] so that

$$\langle n | \xi, \tau \rangle = \frac{\xi}{\mathcal{X}' \left(\frac{\xi}{\tau} \right)}. \quad (56)$$

Equation (56) is our main result for the large deviation domain.

We will consider three cases, where the results of Ref. [13] can be immediately used. Due to the symmetry of the problem, only the behavior for $\xi > 0$ (i.e., for $\xi \gg 1$) will be considered.

(i) The WTD is a one-sided Lévy law, $\psi(\tau) = \mathcal{L}_\alpha(\tau)$. Then

$$\mathcal{X}(z) \simeq C z^{\frac{2}{2-\alpha}},$$

with the constant $C = (2 - \alpha)2^{-\frac{1}{2-\alpha}}\alpha^{\frac{\alpha}{2-\alpha}}\Gamma(\alpha + 1)^{-\frac{1}{2-\alpha}}$ calculated in Ref. [13]. We thus get

$$\langle n|\xi, \tau \rangle = \frac{2 - \alpha}{2C} \xi^{\frac{1-\alpha}{2-\alpha}} \tau^{\frac{\alpha}{2-\alpha}}.$$

We can compare this with the previous result in the “moderate tail,” where we reexpress Eq. (54) in terms of n :

$$\begin{aligned} \langle n|\xi, \tau \rangle &= \langle n(\tau) \rangle \langle v|\kappa \rangle \\ &= \frac{\tau^\alpha}{\Gamma(1 + \alpha)} \text{const} \cdot \left(\frac{\xi^2 \Gamma(1 + \alpha)}{\tau^\alpha} \right)^{\frac{1-\alpha}{2-\alpha}} \\ &\propto \xi^{2\frac{1-\alpha}{2-\alpha}} \tau^{\frac{\alpha}{2-\alpha}}. \end{aligned}$$

Thus, in the large deviation domain $\langle n|\xi, \tau \rangle$ shows the same τ dependence, but the dependence on ξ gets “flatter,” with a twice smaller power. The formal transition $\alpha \rightarrow 1$ gives the behavior where the dependence on ξ vanishes, and $\langle n|\xi, \tau \rangle \sim \tau$. This behavior is similar to the one for the Pareto type I distribution discussed below, with the only difference that the prefactor of τ is given by a different constant.

(ii) For Pareto type I waiting time PDF

$$\psi(\tau) = \begin{cases} 0, & 0 \leq \tau < \tau_m \\ \frac{\alpha \tau_m^\alpha}{\tau^{\alpha+1}}, & \tau \geq \tau_m \end{cases}$$

one gets $\mathcal{X}(z) \simeq \frac{\tau_m}{2} z^2 + \frac{2}{\alpha} \ln z$, so that

$$\langle n|\xi, \tau \rangle \simeq \frac{\tau}{\tau_m}.$$

for $\xi \gg \tau \gg 1$ very large. In this case, the mean number of steps in the large deviation domain ceases to depend on position, and grows proportionally to the time. This behavior does not depend on α at all.

(iii) For Pareto type II (Lomax) distribution of the type

$$\psi(\tau) = \frac{\alpha C_{II}^\alpha}{(C_{II} + \tau)^{\alpha+1}}$$

[with $C_{II} = (\frac{\sin \pi \alpha}{\pi \alpha})^{1/\alpha}$ for $\alpha < 1$ and $C_{II} = (1 - 1/\alpha)$ for $\alpha > 1$], and for an exponential distribution one has $\mathcal{X}(z) \simeq z\sqrt{2 \ln z}$. In this case, in the very large deviations domain $\xi \gg \tau$ one has

$$\langle n|\xi, \tau \rangle \simeq \frac{\xi}{\sqrt{2 \ln \xi}},$$

showing a practically linear dependence on ξ (with a logarithmic correction) and no dependence on τ .

The difference between the two Pareto situations can be discussed based on the findings of Ref. [13]. For the Pareto type I distribution the number of steps at any time is bounded from above by the value of τ/τ_m , where τ_m is the shortest possible waiting time. Reaching the position ξ in less steps is highly improbable since it would require many extremely long steps outside of the body of the Gaussian distribution.

On the contrary, for the Pareto type II and exponential distributions, steps may follow arbitrarily often, and therefore the temporal aspect only poses a mild restriction. A typical number of steps to reach position x then corresponds to $n \sim x/\sigma \gg 1$ steps following in the same direction. The logarithmic correction ensues due to the fact that when very many

steps are made, some of these steps could be considerably longer than σ .

The Lévy situation for $\alpha < 1$ with a softer cutoff of the probability to have very short waiting time is intermediate between these two cases, for $\alpha \rightarrow 1_-$ the cutoff gets sharper, and in this limit merges with the Pareto type I case with a sharp cutoff.

The main lesson we learn here is that the behavior in the large deviation domain is nonuniversal.

VI. NOTES ON BEHAVIOR IN HIGHER DIMENSIONS

Our previous discussion applies also to CTRW with isotropic step length distributions in higher dimensions, but several remarks should be made.

A. Distribution of the relative number of steps

Our Eq. (2) is still valid, and the only difference is that the PDF of displacements $p_n(\mathbf{x})$ now has a different form. Asymptotically, for large n we will have

$$p_n(\mathbf{r}) = \left(\frac{2\pi n \sigma^2}{d} \right)^{-\frac{d}{2}} \exp\left(-\frac{d}{2} \frac{|\mathbf{r}|^2}{\sigma^2 n}\right), \quad (57)$$

with d being the spacial dimension and $\mathbf{r}$ being $(x, y)^T$ or $(x, y, z)^T$ in two or three dimensions, respectively. Note that this form is exact if $\lambda(\mathbf{x})$ is Gaussian, and asymptotically exact for large n in other cases. The corresponding form is simply the product of the marginal distributions of particular coordinates. Note that this function is the function of the distance $r = |\mathbf{r}|$ only: $p_n(\mathbf{r}) = p_n(r)$. Now we have:

$$p(\mathbf{r}|\tau) = \int_0^\infty \left(\frac{2\pi n \sigma^2}{d} \right)^{-\frac{d}{2}} e^{-\frac{dr^2}{2\sigma^2 n}} \tau^{-\alpha} M_\alpha\left(\frac{n}{\tau^\alpha}\right) dn. \quad (58)$$

The function $p(\mathbf{r}|\tau)$ is also a function of the distance $r = |\mathbf{r}|$ only (and this dependence is given via the dimensionless distance $\xi = |\mathbf{r}|/\sigma$) but is no more represented by the Mainardi function. The argument of the Mainardi function in Eq. (58) can be expressed via the dimensionless number of steps ν given by Eq. (31), and then the function $p(\mathbf{r}|\tau)$ may be rewritten as a function of a single dimensionless argument $\kappa > 0$ given by

$$\kappa^2 = d \frac{r^2}{\langle r^2 \rangle} = d \frac{\xi^2 \Gamma(\alpha + 1)}{\tau^\alpha}. \quad (59)$$

The further steps follow the ones around Eq. (34).

Although, since the expression for $p(\nu|\kappa)$ can be normalized *a posteriori*, we essentially do not need explicit expressions for $p(\mathbf{x}|\tau)$, we discuss the properties of $p(\mathbf{x}|\tau)$ in Appendix B. The aim of the discussion there is to define the type of a singularity these distributions show at the origin:

$$p(\mathbf{r}|t) \sim \begin{cases} -\log r & \text{for } d = 2 \\ r^{\frac{1-d}{2}} & \text{for } d \geq 3 \end{cases}. \quad (60)$$

We will need this to explain one of the interesting effects seen in the behavior $p(\nu|\kappa)$ for κ small.

Returning to our Eq. (34) we note that the only difference in higher dimensions is the prefactor of $\exp(-\kappa^2/2\nu)$ which is now not $\nu^{-\frac{1}{2}}$ but $\nu^{-\frac{d}{2}}$. Noting the the Mainardi function with

index $\alpha < 1$ tends to a constant for $\nu \rightarrow 0$ and is given by its asymptotics with superexponential decay for ν large, we see that $p(\nu|\kappa) \simeq \mathcal{N}(\kappa)f(\nu, \kappa)$ with

$$f(\nu, \kappa) = \left(\frac{1}{\pi\nu}\right)^{\frac{d}{2}} \exp\left(-\frac{\kappa^2}{2\nu}\right) M_\alpha\left(\frac{\nu}{\Gamma(1+\alpha)}\right) \quad (61)$$

with κ now defined via Eq. (59). The function $f(\nu, \kappa)$ behaves as

$$f(\nu, \kappa) \simeq \begin{cases} \nu^{-\frac{d}{2}} \exp\left(-\frac{\kappa^2}{2\nu}\right) M_\alpha(0) & \text{for } \nu \ll 1 \\ \nu^{\frac{(d+2)\alpha-d-1}{2(1-\alpha)}} \exp\left[-\frac{\kappa^2}{2\nu} - C\nu^{\frac{1}{1-\alpha}}\right] & \text{for } \nu \gg 1 \end{cases}, \quad (62)$$

with the constant C given by Eq. (35). For $\kappa^2 \gg 1$, the inverse-exponential term in Eq. (62) cuts off the too-small values of ν , and the overall behavior is very similar to the one in one dimension except for the value of the preexponential, which ceases to play a role for κ in the tails.

A more interesting thing happens, however, in the center of the distribution of x (or κ), close to its mode. We note that $p(\nu|\kappa)$ prior to normalization, i.e., the function $f(\nu, \kappa)$ in Eq. (61), is integrable for all $\kappa > 0$, but loses the integrability at the lower limit of integration (at zero) for $\kappa = 0$. Let us consider what happens for $\kappa \rightarrow 0$ for $d \geq 2$. We now have

$$f(\nu, \kappa) = \pi^{-\frac{d}{2}} \exp\left[-\frac{d}{2} \ln \nu - \frac{\kappa^2}{2\nu} + \ln M_\alpha\left(\frac{\nu}{\Gamma(1+\alpha)}\right)\right]. \quad (63)$$

The function $g(\nu) = \ln M_\alpha\left(\frac{\nu}{\Gamma(1+\alpha)}\right)$ behaves as

$$g(\nu) = \begin{cases} \ln M_\alpha(0) + \frac{M'_\alpha(0)}{M(0)\Gamma(1+\alpha)}\nu & \text{for } \nu \ll 1 \\ \frac{\alpha-1/2}{1-\alpha} \ln \nu - C\nu^{\frac{1}{1-\alpha}} & \text{for } \nu \gg 1. \end{cases}$$

Here the constants $M_\alpha(0)$ and $M'_\alpha(0)$ are the first two Taylor coefficients of expansion of the Mainardi function of the corresponding argument. The last one can be negative or positive in dependence on whether $0 < \alpha < 1/2$ or $1/2 < \alpha < 1$. At $\alpha = 1/2$ the derivative $M'_\alpha(0)$ vanishes.

Looking at the extremum of the exponent in Eq. (63), we see that its position is given by the solution of the equation

$$-\frac{d}{2}\nu^{-1} + \frac{\kappa^2}{2}\nu^{-2} + g'(\nu) = 0,$$

so that

$$\kappa^2 = d\nu - 2\nu^2 g'(\nu).$$

Now we note that $g'(\nu)$ stays finite for $\nu \rightarrow 0$. Therefore, for κ small, we get $\nu_{\text{m.p.}} \simeq \kappa^2/d$. The peak moves to the left when κ decays. The value of the function $f(\nu, \kappa)$ at its maximum then dominated by the first term in the argument of the exponential in Eq. (63) and grows as κ^{-d} : The peak gets higher. The overall normalization factor is given by $p(\kappa)$ and behaves as $-\log \kappa$ in $d = 2$ and as $\kappa^{\frac{1-d}{2}}$ in higher dimensions. Therefore, even after normalization, the height of the peak grows. This growth corresponds to $p_{\text{max}}(\nu|\kappa) \sim -\kappa^{-2}/\log \kappa$ in $d = 2$ and to $p_{\text{max}}(\nu|\kappa) \sim -\kappa^{-d-\frac{1-d}{2}} = \kappa^{-\frac{d+1}{2}}$ in higher dimensions. Note that since the function is normalized as a whole and represents a probability density, the corresponding

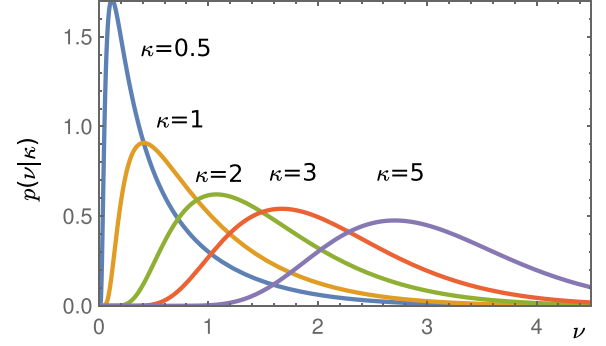


FIG. 5. The mean relative number of steps $\langle \nu|\kappa \rangle$ conditioned on position for CTRW in two dimensions with $\alpha = 0.4$ for different values of κ . The smaller κ the higher and narrower is the peak of the distribution. As in 1d, for $\alpha < \frac{1}{2}$ the distribution of the number of steps leading to a given position broadens with increasing distance to the origin.

peak has to narrow, and the sequence of functions $p(\nu|\kappa)$ for $\kappa \rightarrow 0$ builds a δ sequence. Therefore, for $\alpha < 1$, $p(\nu|\kappa \rightarrow 0) \rightarrow \delta(\nu)$. Thus, for $\alpha < \frac{1}{2}$ the number n of souvenirs for a walker found at the origin is small, and n/t^α tends to zero when the time of the walk grows.

This property is connected with the following circumstance. As already discussed, the probability density of finding a walker at the distance r from the origin diverges for $r \rightarrow 0$ as $p(r) \sim r^{\frac{1-d}{2}}$ for $d \geq 3$ and as $p(r) \sim -\log r$ in $d = 2$ [although the total amount of walkers found in the vicinity of the origin is essentially small, so that $\int_0^\infty p(r)r^{d-1}dr$ stays finite]. This divergence is connected with the presence of the walkers which made very few steps, since the domain available for walkers after n steps grows with n as $n^{d\alpha/2}$, and walkers which made a considerable amount of steps are underrepresented in the vicinity of the origin. Therefore the divergence of $p(\nu|\kappa \rightarrow 0)$ is an interesting consequence of conditioning and is not immediately connected with the recurrence property of the walk.

The PDFs of the relative number of collected souvenirs in a two-dimensional walk are presented in Figs. 5–7. Here the normalization factor $\mathcal{N}(\kappa)$ is obtained by numerical integration of $f(\nu, \kappa)$, Eq. (61). At difference from the one-dimensional case, we do not plot the PDF for $\kappa = 0$ (which is

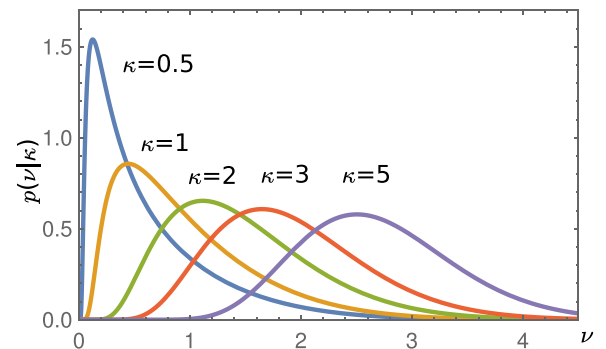


FIG. 6. Same as in Fig. 5 but now for $\alpha = 0.5$, which is the marginal situation.

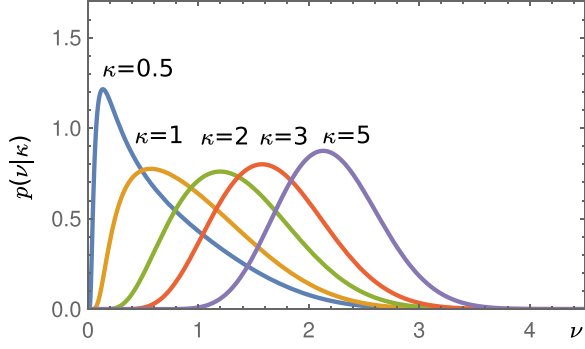


FIG. 7. Same as in Fig. 5 but now for $\alpha = 0.65$. Here the peak first broadens when κ grows and then starts to narrow again.

a δ function) but show what happens at a relatively small value of $\kappa = 0.5$, when a sharp and narrow peak in the PDF starts to develop. Note the nonmonotonic feature of the peak's height for $\alpha = 0.65$, where the distribution is the broadest (and peak the lowest) for intermediate values of κ .

The δ -functional behavior of $p(v|\kappa)$ for $\kappa \rightarrow 0$ is reflected in the behavior of the mean relative number of steps $\langle v|\kappa \rangle$ in the same limit. This asymptotic behavior will be discussed in the next section. Figure 8 shows $\langle v|\kappa \rangle$ as obtained by numerical integration of Eq. (61): $\langle v|\kappa \rangle = \int_0^\infty v f(v, \kappa) dv / \int_0^\infty f(v, \kappa) dv$. The corresponding behavior is shown in Fig. 8 for an exemplary value $\alpha = 1/2$ for which $M_\alpha(y)$ has an analytic expression given by Eq. (26) which allows for obtaining highly precise results. The most important feature of $\langle v|\kappa \rangle$ illustrated by the figure is that it vanishes for $\kappa \rightarrow 0$ for all $d > 1$, but the behavior in the vicinity of zero is different for $d = 2, 3$, and 4, which illustrates our discussion in the next section.

For $\alpha > 1$ instead of Eq. (33) we have

$$p(v|\kappa) = \mathcal{N}(\kappa) \left(\frac{1}{\pi v} \right)^{\frac{d}{2}} \exp \left(-\frac{\kappa^2}{2v} \right) p_\alpha(v)$$

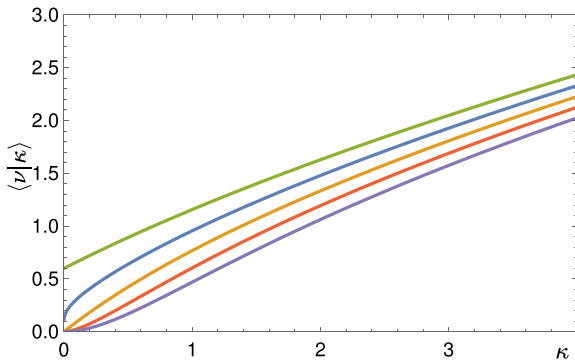


FIG. 8. The mean relative number of steps as a function of relative distance. The curves correspond to $d = 1, 2, 3, 4$, and 5, from top to bottom, and are obtained by the direct numerical integration of Eq. (61). The upper curve, although obtained via a different approach, coincides with the curve for $\alpha = 1/2$ in Fig. 4 obtained by evaluation of Eq. (52).

with the normalization factor is given by

$$[\mathcal{N}(\kappa)]^{-1} = \int_0^\infty \left(\frac{1}{\pi v} \right)^{\frac{d}{2}} \exp \left(-\frac{\kappa^2}{2v} \right) p_\alpha(v) dv,$$

and $p_\alpha(v) = \delta(v - 1)$, so that $p(v|\kappa) = \delta(v - 1)$ independently on κ . Note that for continuous distributions of step lengths the behavior of $p(v|0) = \delta(v - 1)$ is pertinent to finding a particle in an arbitrarily small environment of the origin, but not *exactly* at $\mathbf{r} = 0$, since the only possibility to be exactly at the origin is to make no steps. This shows how the transition from $\alpha < 1$ (with $p(v|0) = \delta(v)$) to $\alpha > 1$ takes place, and indicates the noncommutativity of the limits $t \rightarrow \infty$ and $\mathbf{r} \rightarrow 0$ for this case.

B. Mean number of steps leading to a given position

Let us now turn to the mean number of steps leading to a given position in $d \geq 2$. In space-time domain, Eq. (47) with the additional prefactor corresponds to a Poisson equation for the function $\Phi(\mathbf{r}, t) = \langle n|\mathbf{r}, t \rangle p(\mathbf{r}|t)$:

$$\Delta \Phi(\mathbf{r}, t) = \frac{d}{\sigma} \frac{d}{d\sigma} p(\mathbf{r}|t; \sigma),$$

where the value of the right-hand side is taken at the actual value of σ . The problem is therefore analog to finding an electrostatic potential $\Phi(\mathbf{r})$ of a given rotationally symmetric charge density $\rho(r) = \rho(\mathbf{r}) = -\frac{d}{\sigma} \frac{d}{d\sigma} p(\mathbf{r}|t; \sigma)$:

$$\Delta \Phi(\mathbf{r}) = -\rho(\mathbf{r})$$

(we omit other arguments of Φ and ρ for the time being, because we are solving an *electrostatic* problem). If $\rho(\mathbf{r})$ is known, then the solution follows either via the Green's function of the Poisson equation or by applying the Gauss's law, which is very simple, due to rotational (i.e., cylindrical or spherical) symmetry of the problem in $d = 2$ and $d = 3$ (and also in higher dimensions).

Let us first try to follow the way used in undergraduate courses in electrostatics. Let S_d be the surface of the unit d -dimensional sphere [$S_d = 2\pi^{\frac{d+1}{2}}/\Gamma(\frac{d+1}{2})$], so that $S_2 = 2\pi$, $S_3 = 4\pi$, etc.]. Interpreting $\rho(\mathbf{r})$ as the charge density, we get the charge inside the (hyper)-spherical region of radius r around the origin to be $Q(r) = S_d \int_0^r \rho(r') r'^{d-1} dr'$. This charge defines the flux of the “electric field” $E = -\nabla \Phi$ through the region's boundary and then gives the value of the electric field to be

$$E(r) = \frac{1}{r^{d-1}} \int_0^r \rho(r') r'^{d-1} dr'.$$

The potential $\Phi(r)$ follows by the direct integration:

$$\Phi(r) = \Phi_0 - \int_0^r E(r') dr',$$

with Φ_0 being the boundary condition posed at zero, so that

$$\Phi(r) = \Phi_0 - \int_0^r dr' \frac{1}{r'^{d-1}} \int_0^{r'} \rho(r'') r''^{d-1} dr''.$$

Now we insert our expression for $\rho(r) = -\frac{d}{\sigma} \frac{d}{d\sigma} p(r|t; \sigma)$ (where we now indicate σ as an additional parameter of the

function), and split the operator $\sigma^{-1} \frac{d}{d\sigma}$:

$$\Phi(r, t) = \Phi_0 + \frac{d}{\sigma} \int_0^r dr' \frac{1}{r'^{d-1}} \frac{d}{d\sigma} \int_0^{r'} p(r''|t, \sigma) r''^{d-1} dr''.$$

Now we use the scaling property of $p(r''|t, \sigma)$ as following from Eq. (58): $p(r''|t, \sigma) = \frac{1}{\sigma^d} q(\frac{r''}{\sigma})$, with q being a function of a single nontemporal variable $\xi = r/\sigma$ (the temporal argument t or τ is omitted). Therefore, changing the variable of integration in the internal integral we get

$$\begin{aligned} \Phi(r, t) &= \Phi_0 + \frac{d}{\sigma} \int_0^r dr' \frac{1}{r'^{d-1}} \frac{d}{d\sigma} \int_0^{\frac{r'}{\sigma}} q(\xi) \xi^{d-1} d\xi \\ &= \Phi_0 - \frac{d}{\sigma} \int_0^r dr' \frac{1}{r'^{d-1}} \frac{r'}{\sigma^2} q\left(\frac{r'}{\sigma}\right) \left(\frac{r'}{\sigma}\right)^{d-1} \\ &= \Phi_0 - \frac{d}{\sigma^2} \int_0^r dr' r' \frac{1}{\sigma^d} q\left(\frac{r'}{\sigma}\right) \\ &= \Phi_0 - \frac{d}{\sigma^2} \int_0^r dr' r' p(r'|t; \sigma). \end{aligned} \quad (64)$$

This approach works in $d = 2$ and in $d = 3$, where the integral converges, in spite of the divergence of $p(r|t)$ at the origin, given by Eq. (60). However, also in this case this approach does not lead to a unequivocal value of $\langle n|r, t \rangle$ since the boundary value Φ_0 is not known. In dimensions higher than 3, the integral diverges, which means that Φ_0 is infinite, and the approach leads nowhere.

One can, however, circumvent this difficulty by putting the boundary condition at infinity and integrating inwards: Repeating the steps, we get

$$\Phi(r, t) = \Phi_\infty + \frac{d}{\sigma^2} \int_r^\infty dr' r' p(r'|t; \sigma).$$

Since $p(r|t)$ has a squeezed-exponential tail, the last integral converges for any finite r . It is important to note that although the boundary condition Φ_∞ might depend on t , it has to be independent of r (and therefore of σ). The boundary condition Φ_∞ has to follow from an independent discussion involving the behavior of $p(r|t)$ in its tail (either at the far tail of the scaling domain of the distribution or in its large deviation domain). We can now carry out the following qualitative discussion. In the far tail of the distribution (both in the still scaling domain, where the asymptotic shape of the PDF is known, and in the large deviations domain, where it is known for some special cases), the mean number of steps does not grow with r faster than a power law (or may even get independent of r).

Thus,

$$\langle n|r, t \rangle = \frac{\Phi(r, t)}{p(r|t)} = \frac{\Phi_\infty}{p(r|t)} + \frac{d}{\sigma^2} \frac{\int_r^\infty dr' r' p(r'|t)}{p(r|t)}$$

should behave as a power of r for $r \rightarrow \infty$. On the other hand, since the far tail of $p(r|t)$ decays faster than a power law (exponentially, or faster) any nonzero value of Φ_∞ will correspond to an exponential or even faster growth of $\langle n|r, t \rangle$. This forces us to set the boundary condition at infinity as

$\Phi_\infty = 0$. Doing so, we obtain

$$\langle n|r, t \rangle = \frac{d}{\sigma^2} \frac{\int_r^\infty dr' r' p(r'|t)}{p(r|t)}. \quad (65)$$

It is instructive to compare the situation described by Eq. (64) [or Eq. (65)] with the one-dimensional one given by Eq. (48). Comparing the equations, we see that in one dimension Φ_0 is given by

$$\Phi_0 = -\frac{1}{\sigma^2} \int_{-\infty}^0 x' p(x'|t) dx'$$

and is positive and finite, and

$$\Phi(x, t) = -\frac{1}{\sigma^2} \int_{-\infty}^x x' p(x'|t) dx' = \Phi_0 - \frac{1}{\sigma^2} \int_0^x x' p(x'|t) dx' \quad (66)$$

for positive x . Instead of the boundary condition defining Φ_0 we have now, however, an additional condition following from the left-right symmetry, namely Eq. (49). Note that Eq. (49) [or Eq. (66)] essentially sets Φ_∞ to zero automatically and makes knowing Φ_0 superfluous. This left-right symmetry in a one-dimensional case is therefore a stronger condition than the rotational one in higher dimensions, in which case the boundary condition at infinity follows from an additional discussion.

Discussion of Eq. (65) allows us to understand the asymptotic behavior of $\langle v|\kappa \rangle$ for $\kappa \rightarrow 0$ as shown in Fig. 8. In dimensionless, relative units, Eq. (65) reads:

$$\langle v|\kappa \rangle = \frac{\int_\kappa^\infty d\kappa' \kappa' p(\kappa')}{p(\kappa)}.$$

For large κ , the function $p(\kappa)$ shows a superexponential decay stemming from the corresponding Mainardi function which guarantees the convergence of the integral in numerator at its upper limit. For small κ , the asymptotic behavior of $p(\kappa)$ corresponds to $p(\kappa) \sim -\log \kappa$ in $d = 2$ and $p(\kappa) \sim \kappa^{(1-d)/2}$ for $d \geq 3$, as discussed at the end of the previous section. Thus, three distinct types of behavior arise for $d = 2, 3$ and $d \geq 4$. In $d = 2$ and 3 the numerator stays finite for $\kappa \rightarrow 0$, and the behavior is dominated by the divergence of the denominator leading to $\langle v|\kappa \rangle \sim -1/\log \kappa$ in $d = 2$ and $\langle v|\kappa \rangle \sim \kappa$ in $d = 3$. For $d \geq 4$ both the denominator, and the integral in the numerator diverge for $\kappa \rightarrow 0$ as $\kappa^{(1-d)/2}$ and $\kappa^{(5-d)/2}$, respectively, leading to the overall quadratic behavior of $\langle v|\kappa \rangle$ for κ small.

VII. SUMMARY

In the present work we considered statistics of the number n of steps of a random walker performing a continuous-time random walk with power-law-distributed waiting times conditioned on its final position provided the total time of the walk is sufficiently long. We mainly concentrate on a one-dimensional situation and provide general expressions for the distribution of the relative number of steps $v = n/\langle n(t) \rangle$ for different final positions in the “body,” i.e., in the scaling domain of the corresponding distribution of displacements. This distribution depends asymptotically only on a single parameter $\kappa = x/\langle x^2(t) \rangle^{1/2}$ indicating the position of the final

point within the distribution. For subdiffusion, the distribution of the relative position has a finite width and tends to a Gaussian for κ large. In the regime corresponding to the normal diffusion, the distribution of ν tends to a δ distribution centered at unity, although the distribution of n itself shows finite fluctuations.

The behavior in higher dimension is essentially similar except for the fact that for $d > 1$ the distribution of ν at the origin tends to a δ function for $\alpha < 1$.

The mean number of steps of a walk ending at x can be calculated independently, and follows from the solution of a Poisson equation. Such a solution can be used also in the large deviation domain corresponding to the far tails of the distribution of displacements. The behavior in this domain is nonuniversal and depends on the details of the distribution of waiting times.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data are not publicly available. The data are available from the author upon reasonable request.

APPENDIX A: ADDITIONAL DISCUSSION OF EQ. (15)

We have seen that Eq. (16) covers a continuum of different asymptotic behaviors, where for $\alpha > 1$ one should formally put $\alpha = 1$ in this equation. We have moreover seen that for $\alpha > 1$ the equation suggests that the distribution of the number of steps tends to a δ function, while in reality this is true only for the distribution of the relative number of steps.

Let us go in somewhat more detail and present the alternative derivation of Eq. (15) for arbitrary α , following the approach outlined in Ref. [15]; see also Refs. [16,17]. This starts from mentioning that since $n(t)$ in each realization is a monotonically nondecaying function of t ,

$$\text{Prob}(n > N | t = T) = \text{Prob}(t < T | N).$$

The function $\text{Prob}(t < T | N)$ is the cumulative distribution function of the time necessary to make exactly N steps, and $\text{Prob}(n > N | t = T)$ is the survival probability, i.e., the probability to make more than N steps up to time T . In our discussion the typical number of steps is so large that the variable n can be considered as a continuous one. Let $X(n | t)$ be the continuous approximation (interpolation) for $\chi_n(t)$ and represent the probability density of n . Then the probabilities above can be approximated by integrals

$$\int_N^\infty X(n | t) dn = \int_0^T p_t(t | N) dt,$$

where $p_t(t | N)$ is the probability density of the time t necessary to complete N steps.

From the equation above it follows then that

$$X(N, t) = -\frac{d}{dN} \int_0^T p_t(t | N) dt.$$

For N large, $p_t(t | N)$ being a density of a sum of N independent, identically distributed non-negative random variables with a power-law tail of the distribution tends to one of the asymmetric Lévy stable laws $\mathcal{L}_{\alpha,\beta}(y)$ with $\beta = 1$, see, e.g., Ref. [9], and the integral tends to its cumulative distribution function. Here the situations for $\alpha < 1$, $\alpha = 1$, $1 < \alpha < 2$ differ; see Ref. [10]. The situation with $\alpha = 2$ is nongeneric but not much different from the last one, and all situations with $\alpha > 2$ belong to the domain of attraction of the normal law.

The situation with $\alpha < 1$ is exactly the one described in the main text. In this case $p_t(t | N)$ depends on N via a single parameter of the dimension of time, the width of the distribution $w(N) = c_\alpha t_0 N^{\frac{1}{\alpha}}$ with c_α being a number constant:

$$p_t(t | N) = \frac{1}{w(N)} f\left[\frac{t}{w(N)}\right],$$

so that

$$\begin{aligned} X(N, T) &= -\frac{d}{dN} \int_0^T \frac{1}{w(N)} f\left[\frac{t}{w(N)}\right] dt \\ &= -\frac{d}{dN} \int_0^{T/w(N)} f(z) dz \\ &= \frac{T w'(N)}{w(N)} \frac{1}{w(N)} f\left(\frac{T}{w(N)}\right). \end{aligned}$$

In the case we discussed in the main text, the parameters were chosen such that $w \propto N^{1/\alpha}$, and $f(y) = \mathcal{L}_\alpha(y)$, and we obtain our Eq. (15). Note that for $\alpha < 1$ the corresponding Lévy distribution vanishes identically on the negative half-axis, and no additional problems arise. Note, moreover, that the asymptotic distribution does not depend on the centering of initial variables, since the width of the distribution grows faster than N , so that the results are the same if the initial waiting time PDF is taken, say, as a Pareto type I and or a Pareto type II distributions with the same far tails.

If we pass to the case $\alpha > 1$ with the first moment $\mu(N) = t_0 N$, then we can formally pursue a similar program as for $\alpha < 1$, with the only difference that centering now plays a role. For example, for $1 < \alpha < 2$ we may write

$$p_t(t | N) = \frac{1}{w(N)} f\left[\frac{t - \mu(N)}{w(N)}\right] = \frac{1}{w(N)} f\left[\frac{t - t_0 N}{w(N)}\right], \quad (\text{A1})$$

where f describes the corresponding extremely skewed Lévy PDF $\mathcal{L}_{\alpha,1}(y)$ with zero mean and unit scale parameter. The width of the distribution (proportional to its scaling parameter) is now $w(N) = c_\alpha t_1 N^{\frac{1}{\alpha}}$ with some positive constant c_α for $1 < \alpha < 2$ and $w(N) = c t_1 N^{\frac{1}{2}}$ (i.e., formally $\alpha = 2$) for all $\alpha > 2$.

Now we may write

$$X(N, T) = -\frac{d}{dN} \frac{1}{c_\alpha t_1 N^{\frac{1}{\alpha}}} \int_0^T f\left(\frac{t - t_0 N}{c_\alpha t_1 N^{\frac{1}{\alpha}}}\right) dt. \quad (\text{A2})$$

Changing the variable of integration to $z = \frac{t - t_0 N}{c_\alpha t_1 N^{\frac{1}{\alpha}}}$ we get

$$X(N, T) = -\frac{d}{dN} \int_{-t_0 N^{1-\frac{1}{\alpha}}/(c_\alpha t_1)}^{(T-t_0 N)N^{-\frac{1}{\alpha}}/(c_\alpha t_1)} f(z) dz.$$

At difference with the case $\alpha < 1$, for $\alpha > 1$ the function $f(z)$ is defined on the whole real axis; on the negative part it is rapidly decaying (faster than a Gaussian for $1 < \alpha < 2$ and as a Gaussian for $\alpha > 2$), and has a power-law tail on the positive part for $\alpha < 2$. The function $f(z)$ is integrable to unity. Now we see that

$$\begin{aligned} X(N, T) &= -\frac{t_0}{c_\alpha t_1} \left(1 - \frac{1}{\alpha}\right) N^{-\frac{1}{\alpha}} f\left(-\frac{t_0}{c_\alpha t_1} N^{1-\frac{1}{\alpha}}\right) + \frac{1}{c_\alpha t_1} \\ &\quad \times \left[\frac{1}{\alpha} T N^{-\frac{1}{\alpha}-1} + t_0 \left(1 - \frac{1}{\alpha}\right) N^{-\frac{1}{\alpha}}\right] f\left(\frac{T - t_0 N}{c_\alpha t_1 N^{\frac{1}{\alpha}}}\right). \end{aligned} \quad (\text{A3})$$

For $\alpha > 1$, we can introduce a scaled number of steps $\nu = N t_0 / T$ and show that the distribution of this variable, $p_\alpha(\nu; T) = \frac{T}{t_0} X(\nu \frac{T}{t_0}, T)$ tends to a δ function when T grows. This is simply another way to confirm our previous considerations. For $T \rightarrow \infty$, the first term vanishes for each finite ν because the argument of f gets large and negative, and in the second term the argument of the f function gets to be

$$\frac{T - \nu T}{c_\alpha t_1 (\nu T / t_0)^{\frac{1}{\alpha}}} = \frac{t_0^{\frac{1}{\alpha}}}{c_\alpha t_1} T^{1-\frac{1}{\alpha}} \frac{1 - \nu}{\nu^{\frac{1}{\alpha}}}.$$

For $T \rightarrow \infty$ the absolute value of this argument gets very large except for the narrow vicinity of $\nu = 1$ of width of the order of $c_\alpha t_1 t_0^{-\frac{1}{\alpha}} T^{\frac{1}{\alpha}-1}$ which tends to zero when T grows. Outside of this region the f function gets small. The corresponding prefactor (taken then at $\nu = 1$) guarantees for the overall normalization. The fast decay of f for large negative values of arguments leads to a superexponential cutoff for large ν or N , guaranteeing the finiteness of the moments thereof. The discussion of Eqs. (A1) and (A2) shows that when the center of the distribution of T at given N moves to the right faster than the width of this distribution grows, the behavior of the distribution of N for given T is similar: The mean value of N grows faster with T than the spread of N values. The whole discussion shows moreover that both for $\alpha < 1$ and for $\alpha > 1$ one can use the asymptotic forms, the Lévy stable densities, to obtain the distributions $p_\alpha(\nu)$ which, for long times, get independent on time.

The situation for $\alpha = 1$ is nongeneric and shows a behavior which is different from the ones in both adjacent domains, with features being a bizarre mixture of behaviors on the two sides.

Let us remind of the origin of Eq. (12). The expression for $\psi(s)$ in this case follows from the discussion of the survival probability $\Psi(t) = \int_t^\infty \psi(t) dt$ [9] which for large t follows a power law $\Psi(t) \sim t^{-\alpha}$. The expression for $\Psi(s)$ follows asymptotically by the application of the Tauberian theorem: From $\Psi(t) \sim t^{-\alpha}$ it follows that $\Psi(s) \sim \Gamma(1 - \alpha) s^{\alpha-1}$. Then we note that $\psi(t) = -\frac{d}{dt} \Psi(t)$, so that $\psi(s) = \Psi(t=0) - s\Psi(s) = 1 - s\Psi(s)$, and get the final result, Eq. (12).

The case $\alpha = 1$ needs for a separate discussion. For $\alpha = 1$, the divergence of a Gamma function $\Gamma(0)$ prevents for the application of Tauberian theorem, and the explicit calculation leads to an additional correction in form of a slowly varying function (logarithm).

To understand Eq. (14) it is instructive to start from Pareto distributions. The Laplace transforms of these distributions readily follow in terms of incomplete Gamma functions, see, e.g., the calculations in Ref. [13]. Thus, for a Pareto type I distribution,

$$\psi(\tau) = \begin{cases} 0, & 0 \leq t < t_0 \\ t_0 t^{-2}, & t \geq t_0 \end{cases},$$

we have $\psi(s) = 1 + (\gamma + \ln(st_0))st_0 + \dots$ with γ being the Euler's constant, so that $t_2 = \gamma t_0$. For a generic Pareto type II distribution (a two-parametric one, not necessary of the Lomax type)

$$\psi(\tau) = \begin{cases} 0, & 0 \leq t < t_m \\ \frac{1}{t_0} [1 + \frac{t-t_m}{t_0}]^{-2}, & t \geq t_m \end{cases},$$

we get $\psi(s) = 1 + (\gamma - t_m/t_0)t_0 s + t_0 s \ln(t_0 s) + \dots$, so that t_m contributes only to t_2 . The Lomax distribution corresponds to the case $t_m = 0$, and in this case $\psi(s)$ is asymptotically the same as for the Pareto type I distribution.

This situation is quite general. To see this, let us start from the survival probability $\Psi(t) = \int_t^\infty \psi(t) dt$ which for large t follows a hyperbolic law. We note that $\psi(t) = -\frac{d}{dt} \Psi(t)$, so that $\psi(s) = \Psi(t=0) - s\Psi(s) = 1 - s\Psi(s)$.

Since $\Psi(t=0) = 1$, the power law $\Psi(t) = t_0/t$ cannot be valid for small values of t but will be followed sufficiently well starting from some value t_l , separating the long-time domain from the short time one. We start from the case that the power law is followed *exactly*, starting at t_l . The Laplace transform of $\Psi(t)$ now reads:

$$\begin{aligned} \Psi(s) &= \int_0^\infty \Psi(t) e^{-st} dt \\ &\simeq \int_0^{t_l} \Psi(t) e^{-st} dt + \int_{t_l}^\infty \frac{t_0}{t} e^{-st} dt \\ &= g_1(s, t_l) + t_0 \Gamma(0, st_l) \simeq g_1(s, t_l) - t_0 \ln st_l. \end{aligned}$$

Now we note that since $\Psi(t)$ is positive and bounded, the first integral allows for a regular series expansion in s starting from the constant term depending on t_l only: $g_1(s, t_l) = a_0(t_l) + a_1(t_l)s + \dots$. Now we take $t_l = t_0 c$ with some sufficiently large positive constant c and see that

$$\Psi(s) \simeq a_0(t_0 c) - a_1(t_0 c)s - (t_0 \ln c)s - t_0 \ln st_0,$$

so that the exact value of c is important only for subleading terms. Then in the leading order $\psi(s) = 1 + t_0 s \ln(t_0 s)$. The same situation occurs if, from t_l on, $\Psi(t)$ does not follow a power law exactly, but approaches it fast enough, so that the integral

$$\int_{t_l}^\infty \left[\Psi(t) - \frac{t_0}{t} \right] dt$$

converges. The correction then contributes only to the subleading terms (i.e., to the value of t_2) as well.

Now we discuss the difference between the cases $\alpha \neq 1$ and $\alpha = 1$ with respect to the corresponding distributions. While the Lévy laws for $\alpha \neq 1$ are strictly stable, the asymptotic law for $\alpha = 1$ is stable only in a broader sense [10]. If

we consider the corresponding centered and nondimensionalized α -stable variables, $z_i = (x_i - \mu)/w(1)$, with μ being the position parameter of the distribution of x , and with $\alpha \neq 1$, then the sum of N such variables has the same distribution as $N^{\frac{1}{\alpha}} z_i$ (and is also centered at zero). This property governed the scaling of the width, as considered before. For $\alpha = 1$ the distribution of the sum differs in its centering from the distribution of a single z variable by an amount of $\ln N$, as it follows from the property of the corresponding characteristic function, see Chap. XVII of Ref. [10]. Thus,

$$p_t(t|N) = \frac{1}{w(N)} f\left[\frac{t - t_c N}{w(N)} - \ln N\right].$$

Here t_c corresponds to the centering of the distribution of a single Lévy random variable (the correct definition of centering of a distribution lacking the mean is discussed in Ref. [10] but is of no importance in further considerations). The change in centering given by the $\ln N$ term is of importance, since if neglecting it and performing the same program as before, we would end with $X(N, T)$ which is neither properly normalized nor possessing any moments. With this centering, however, similarly to the case $\alpha > 1$, the center of the distribution of T at given N moves to the right faster than the width of this distribution grows, and the distribution of v narrows.

Choosing the characteristic time t_w such that $w(N) = t_w N$, we get

$$p_t(t|N) = \frac{1}{t_w N} f\left(\frac{t}{t_w N} - \frac{t_c}{t_w} - \ln N\right).$$

Taking $z = t/(t_w N) - t_c/t_w - \ln N$ we obtain

$$X(N, T) = -\frac{d}{dN} \int_{t_c/t_w - \ln N}^{\frac{T}{t_w N} - \frac{t_c}{t_w} - \ln N} f(z) dz,$$

so that

$$X(N, T) = -\frac{1}{N} f\left(\frac{t_c}{t_w} - \ln N\right) + \left(\frac{T}{t_c N^2} + \frac{1}{N}\right) f\left(\frac{T}{t_w N} - \frac{t_c}{t_w} - \ln N\right).$$

Now we change the variable to

$$v = \frac{N \ln(T/t_0)}{(T/t_0)},$$

denote $T/t_0 = \tau$, $t_c/t_0 = \tau_c$, and $t_w/t_0 = \tau_w$, and note that $p_1(v; \tau) = \frac{\tau}{\ln \tau} X(v\tau/\ln \tau, \tau)$. We get:

$$p_1(v, T) = -\frac{1}{v} f\left(\frac{\tau_c}{\tau_w} - \ln v + \ln \ln \tau - \ln \tau\right) + \left(\frac{\ln \tau}{v^2} + \frac{1}{v}\right) \times f\left[\left(\frac{1}{v\tau_w} - 1\right) \ln \tau + \frac{\tau_c}{\tau_w} - \ln v + \ln \ln \tau\right].$$

Now we see that for $\tau \rightarrow \infty$ the distribution of v gets a sharp maximum in vicinity of $v = \tau_w^{-1}$, and essentially converges to a δ function, but the convergence is extremely slow (logarithmic). Moreover, we see that the correct normalization is only guaranteed if we take $\tau_w = 1$, i.e., $t_w = t_0$. This can also be seen when comparing the position of the corresponding δ function with the mean value of $v = 1$ as following from the discussion in the main text.

Note that since the weight of the maximum grows only logarithmically in time, at any reasonable time T strong fluctuations and explicit T dependence persist.

The fact that $t_w = t_0$ could essentially be anticipated from comparison of Eq. (14) and Eq. (13). Writing Eq. (14) as $\psi(s) = 1 + t_0 s \ln s + (t_0 \ln t_0 + t_2)s$ (forgetting for the time being the fact that a logarithm of a dimensional argument lacks any precise meaning), and comparing it with $\psi(s) = 1 - t_0 s + (t_1 s)^\alpha$ with $\alpha = 1 + \epsilon$ for $\epsilon \rightarrow 0$, we see that t_0 in Eq. (14) plays essentially the role of t_1 in Eq. (13), while the role of t_0 (the mean being at the same time centering) in Eq. (13) is played by an ill-defined combination $t_0 \ln t_0 + t_2$. The last observation is connected to the fact that for $\alpha = 1$ the mean diverges, and the centering position is given by a different rule.

APPENDIX B: THE DISTRIBUTION OF THE DISTANCE TO THE ORIGIN IN HIGHER DIMENSIONS

We start from our Eq. (58). Introducing a new dimensionless non-negative variable $\xi = \sqrt{d} \frac{|r|^2}{\sigma^2} \geq 0$, we rewrite Eq. (58) as

$$p_d(\xi|\tau) = \int_0^\infty \left(\frac{1}{2\pi n}\right)^{\frac{d}{2}} \exp\left(-\frac{\xi^2}{2n}\right) \frac{1}{\tau^\alpha} M_\alpha\left(\frac{n}{\tau^\alpha}\right) dn, \quad (\text{B1})$$

which for $d = 1$ coincides with the previous expression given by the Mainardi function:

$$p_1(\xi|\tau) = \int_0^\infty \frac{1}{\sqrt{2\pi n}} \exp\left(-\frac{\xi^2}{2n}\right) \frac{1}{\tau^\alpha} M_\alpha\left(\frac{n}{\tau^\alpha}\right) dn = \frac{1}{\sqrt{2}\tau^{\frac{\alpha}{2}}} M_{\alpha/2}\left(\frac{\sqrt{2}\xi}{\tau^{\frac{\alpha}{2}}}\right),$$

($\xi \geq 0$), see Eq. (27). From Eq. (B1) it follows that

$$p_{d+2}(\xi|\tau) = \int_0^\infty \left(\frac{1}{2\pi n}\right)^{\frac{d}{2}} \frac{1}{2\pi n} \exp\left(-\frac{\xi^2}{2n}\right) \frac{1}{\tau^\alpha} M_\alpha\left(\frac{n}{\tau^\alpha}\right) dn = -\frac{1}{2\pi\xi} \frac{d}{d\xi} p_d(\xi|\tau). \quad (\text{B2})$$

Therefore, there exist two series of distributions: one for odd d , starting at the one for $d = 1$, given by the Mainardi function, and another one for even d , starting with $d = 2$.

For example, for $d = 3$ we have

$$p_3(\xi|\tau) = \frac{1}{2\pi\xi} M'_{\alpha/2}\left(\frac{\sqrt{2}\xi}{\tau^{\frac{\alpha}{2}}}\right)$$

with $M'_{\alpha/2}(z)$ being the derivative of the Mainardi function. Note that since for all $\alpha < 1$ the Mainardi function $M_{\alpha/2}(z)$ has a cusp with a finite and negative right derivative at zero, the series expansion of the function $p_3(r|\tau)$ starts from a term proportional to r^{-1} and has a hyperbolic divergence at zero. This is not a fault since the normalization condition for this function is $4\pi \int_0^\infty p_3(r)r^2 dr = 1$, and this integral converges. In general, $p_d(r|\tau)$ has at the origin a singularity of the type $r^{\frac{1-d}{2}}$, which is integrable with the corresponding weight being proportional to r^{d-1} .

The series for even d starting with $d = 2$ can be expressed not via the derivatives of Mainardi functions but via the inverse Abel transformations thereof.

Let us start from calculating $p_2(r|\tau)$. We note that, according to Eqs. (57) and (58) the distribution $p_2(\mathbf{r}|\tau) = p_2(x, y|\tau)$ is rotationally symmetric, and its marginal distribution is given by $p_1(x|\tau)$:

$$\int_{-\infty}^{\infty} p_2(x, y|\tau) dy = p_1(x|\tau). \quad (\text{B3})$$

Now, for the time being, we omit the temporal variable of p_2 and p_1 functions and pass to polar coordinates by making the corresponding transformation of the PDF:

$$p(r, \theta) = p(x, y) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right|$$

with $\frac{\partial(x, y)}{\partial(r, \theta)}$ being the Jakobian of the transformation. Now we take $x = r \cos \theta$ and write

$$p_1(x) = \int p_2(r, \theta) \delta(x - r \cos \theta) r dr d\theta. \quad (\text{B4})$$

Using the symmetry, we can restrict θ to the interval from $-\pi$ to π and double the result. Then we perform integration in θ using the rule of variable transformation for a δ function: If the function $g(z)$ has a single root z_0 on the interval of integration (a, b) , then one takes $\delta(g(z)) = \delta(z - z_0) |g'(z_0)|^{-1}$. In our

case $g(z) = x - r \cos z$ and $z_0 = \arccos \frac{x}{r}$. We therefore get

$$\int_{-\pi}^{\pi} d\theta \delta(x - r \cos \theta) = \frac{1}{\sqrt{r^2 - x^2}} \Theta(r - x),$$

so that the integral over r reads:

$$\begin{aligned} p_1(x) &= 2 \int_0^{\infty} dr r p_2(r) \frac{1}{\sqrt{r^2 - x^2}} \Theta(r - x) \\ &= 2 \int_x^{\infty} \frac{r}{\sqrt{r^2 - x^2}} p_2(r) dr. \end{aligned}$$

Therefore, $p_1(x)$ is now given by the Abel transform of $p_2(r)$. The function $p_2(r)$ now follows by the inverse Abel transform of the $p_1(z)$ (represented by a Mainardi function):

$$p_2(r) = -\frac{1}{\pi} \int_r^{\infty} p'_1(y) \frac{dy}{\sqrt{y^2 - r^2}}.$$

This equation can be readily used for obtaining the asymptotic behavior of $p_2(r)$ for r large but does not readily lead to a series expansion. For our further discussion it will be of importance that since the derivative of a Mainardi function at zero is finite, the function $p_2(r)$ is singular at zero showing there a logarithmic divergence. The distributions for even $d = 4, 6, \dots$ following from Eq. (B2) then show power-law divergence at the origin, of the same type as in the odd-dimension series of distributions.

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