

Biased continuous-time random walks with asymmetric waiting-time distributions as a model of anomalous diffusion in the presence of drift

Guohua Li*

School of Mathematical Sciences, *Chengdu University of Technology*, Chengdu 610059, China
and Institute of Physics and Astronomy, *University of Potsdam*, 14476 Potsdam-Golm, Germany

Andrey G. Cherstvy†

Institute of Physics and Astronomy, *University of Potsdam*, 14476 Potsdam-Golm, Germany

Wenya Guo, Hong Zhang, Ting Liu, and Wendong Pu

School of Mathematical Sciences, *Chengdu University of Technology*, Chengdu 610059, China

Ralf Metzler‡

Institute of Physics and Astronomy, *University of Potsdam*, 14476 Potsdam-Golm, Germany
and *Asia Pacific Center for Theoretical Physics*, Pohang 37673, Republic of Korea



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Mathematical models of diffusion based on stochastic processes of the type of continuous-time random walks (CTRWs) are widely used for the description of various transport characteristics in complex media. An external or internal bias in a given CTRW model is often attributed to an asymmetric distribution of jump lengths, while the waiting times are assumed to possess identical distributions in all directions. However, for a number of real-world transport phenomena the latter assumption is violated. To account for such situations, we examine the properties of such biased CTRWs in one dimension. We propose a CTRW-based model of diffusion with the waiting-time distributions (WTDs) of the tracers chosen *unequal* for right- and left-directed jumps. Below we name CTRWs with such asymmetric WTDs (AWTDs) as A_{ℓ} -CTRWs. We derive the corresponding generalized Montroll-Weiss and master equations in Laplace and Z space. We consider both exponential and power-law WTDs as the standard and most useful functional choices for the purposes of data analysis. For these two WTD classes, we analyze the average displacement (generally nonzero) and the mean-squared displacement of the diffusing particles and analytically compute the variance as effective statistical measures. Our analytical predictions favorably agree with the results of the performed computer simulations. Specifically, for a CTRW with power-law WTDs we find that at long times the emerging anomalous diffusion depends not only on the smallest exponent of the left- and right-directed WTDs. Such diffusion models can, e.g., be used to describe the anomalous diffusion in various anisotropic media in the presence of a drift. One hydrological example is a gravity-affected spreading of contaminants in porous rocks, a system where the CTRW-based models are widely known to be applicable. A biological example of a biased-CTRW-based transport is a regulated driven bidirectional transport of cargoes between the center of a living biological cell and its periphery taking place along the microtubule filaments.

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I. INTRODUCTION

The mathematical models of anomalous diffusion [1–3] employing the concept of the continuous-time random walk (CTRW) were established in the 1970s [2–19]. Since then, these models have received a lot of attention and have proved

themselves to be useful to describe various processes in physics [1,20], geophysics [21–23], chemistry [24,25], biology [26,27], and other disciplines, including some recent studies [3,18,23]. The CTRW-based models [17,28] can describe certain macroscopically complicated dynamics, such as the subdiffusion and localization in some one-dimensional (1D) trap-containing models [29], anomalous transport in porous and heterogeneous media [21,23,30–39], the transport characteristics in supercooled liquids [40], and the epidemic spreading in heterogeneous networks [41], to mention a few examples.

Conventional left-versus-right symmetric 1D CTRWs feature random waiting times between any two consecutive jump events and also often random jump lengths [42–45]. Namely, after arriving at a new site, a particle can jump right or left

*Contact author: liguohua13@cdut.edu.cn†Contact author: a.cherstvy@gmail.com‡Contact author: rmetzler@uni-potsdam.de

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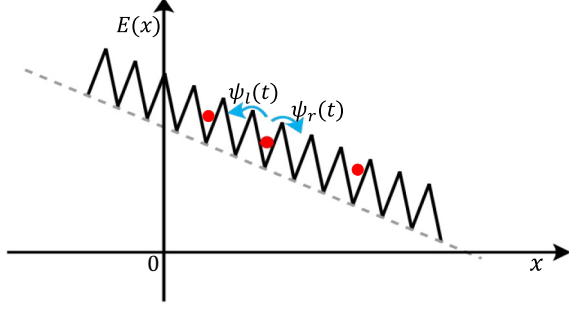


FIG. 1. Example of a jump-based system with a nonzero drift: an asymmetric motion in a tilted ratchet-type potential.

with equal probabilities, while waiting for a random time prior to making such a jump. The diffusion on comblike structures often takes place in a similar fashion [46].

From the mathematical perspective, the statistical properties of aging [47,48], noisy [49], doubly stochastic [50], confined [51,52], reset [53], waiting-time to jump-length coupled [54], and biased [9,45,47,55–58] CTRWs were considered in the literature. Note that the heterogeneous diffusion processes (HDPs) [59–63] have several very similar features of the CTRW-based dynamics (regarding the spreading dynamics of the particles and the overall nonergodicity [3,59]) making these two processes potentially hard to distinguish if these quantifiers are compared.

One important characteristic feature of the biased-CTRW-based models is the existence of drift in the macroscopic limit [64–70]. In many CTRW models, a bias is often attributed to an asymmetry of the jump-length distributions. This asymmetry is often realized in 1D via (i) assigning nonequal probabilities of jumps to the right and to the left [57], (ii) introducing a convection velocity [71–73], or (iii) imposing a self-reinforcing directionality [74,75]. For biased 1D CTRWs, the waiting times typically follow *the same* waiting-time distributions (WTDs) in both directions.

However, for a number of real-world transport phenomena this assumption is violated. For instance, upon a global migration in a biological population [76]—e.g., upon the motion of (i) a flock of birds or (ii) a school of bait fish as a whole—some asymmetric tendencies of the movement of individuals do exist. These trends lead to essential consequences for determining the mechanisms behind the functioning principles of such systems.

On a model level, for the diffusion in a 1D energy landscape, see Fig. 1, the potential wells may be separated by energy barriers of unequal height as the particle moves in the opposite directions, leading to a left-versus-right asymmetric WTD (AWTD). A spatial asymmetry of the mathematical diffusion kernels is a large-scale consequence of a locally asymmetric distribution of probabilities of random jump-lengths of the particles. The impact of external fields in certain transport models often follows similar mathematical principles [45,77,78].

Here we consider a biased 1D CTRW with AWTDs having the same functional forms. We quantify the spreading characteristics of the diffusing particles in the presence of a global drift. We introduce a model of AWTD-CTRWs (A_t -CTRWs)

on a lattice and derive the generalized master equation for the particle-position distribution at time t . We study both the exponential and power-law probability-density functions (PDFs) of the associated WTDs by analyzing the average displacement (the first moment), the second moment, and the corresponding variance of the diffusion process. We discuss the emerging anomalous-diffusion regimes, the regimes absent for CTRWs with *symmetric* exponential WTDs [17]. We also examine the variability of the anomalous-diffusion regimes for CTRWs with power-law WTDs.

The rest of the paper is organized as follows. We start in Sec. II A with a short overview of the models of conventional CTRWs. In Sec. II B we introduce the model of biased 1D A_t -CTRWs. In Secs. III A and III B, respectively, we present the results of this model for exponential and power-law WTDs. The discussion of the main theoretical results, their comparison with the results of computer simulations, and the final conclusions are provided in Sec. IV. The properties of the Laplace and Z transformations as well as the relations for the special functions used in the main text are given in Appendix A.

II. MODEL AND MAIN EQUATIONS

A. Symmetric CTRWs: Definitions and brief summary

Here we list some properties of standard unbiased CTRWs, see, i.e., Refs. [1–3,42,45,79]. Assuming that the jumps of lengths $\{\xi_1, \xi_2, \dots\}$ are independent and identically distributed random variables and that the jumps occur at random times $\{T_1, T_2, \dots\}$, the waiting intervals between the jumps, denoted as

$$\theta_n = T_n - T_{n-1}, \quad (1)$$

can also be treated as independent and identically distributed variables with $T_0 = 0$. With the position of a particle at time t denoted as $X(t)$ and with the initial position $X(0) = 0$, one has

$$X(t) = \sum_{i=1}^{N(t)} \xi_i, \quad (2)$$

where $N(t)$ is the number of jumps performed up to time t . The sequence of positions $X(t)$ forms a CTRW.

In the case when the waiting-time intervals θ_n and the jump lengths ξ_n are *not* mutually independent, with the function $\phi(\xi, t)$ denoting their *joint* PDF, the resulting PDF of the jump lengths becomes [13,79,80]

$$\Lambda(\xi) = \int_0^\infty \phi(\xi, t) dt, \quad (3)$$

while the PDF of the waiting times turns into [79]

$$\psi(t) = \int_{-\infty}^\infty \phi(\xi, t) d\xi. \quad (4)$$

Let $\Psi(t)$ be the probability that no jump has occurred up to t , that is, the respective survival probability of the particle is [1,17]

$$\Psi(t) = 1 - \int_0^t \psi(\tau) d\tau. \quad (5)$$

We denote below $\rho(x, t)$ being the PDF of the particles at position x and at time t . The evolution equation of this PDF can be written as [79]

$$\rho(x, t) = \rho_0(x)\Psi(t) + \int_0^t d\tau \int_{-\infty}^{\infty} \rho(x - \xi, \tau)\phi(\xi, t - \tau)d\xi, \quad (6)$$

where $\rho_0(x) \equiv \rho(x, t = 0)$ is the initial PDF at $t = 0$. In Fourier-Laplace space (see Appendix A for the definitions) the well known Montroll-Weiss equation follows [4,10,14,45,79]

$$\rho(k, u) = \frac{\rho_0(k)[1 - \psi(u)]}{u[1 - \phi(k, u)]}. \quad (7)$$

For $\rho(x, 0) = \delta(x)$ this simplifies to $\rho(k, u) = \frac{1 - \psi(u)}{u[1 - \phi(k, u)]}$.

B. Biased and A_r -CTRWs: Generalized master equation

1. Problem formulation and conditional probabilities

We first consider a general case when the waiting-time intervals and the jump-lengths have the joint PDF $\phi(\xi, t|x)$ depending on the position x of a given particle. We suppose that $\phi(\xi, t|x)$ can be written in the product form [17]

$$\phi(\xi, t|x) = \psi(t|\xi) \times \Lambda(\xi|x), \quad (8)$$

with

$$\Lambda(\xi|x) = g(x)\delta(\xi - 1) + (1 - g(x))\delta(\xi + 1), \quad (9)$$

the Ansatz commonly used in the random-walk literature [28], and with the expression

$$\psi(t|\xi) = \psi_r(t)\delta(\xi - 1) + \psi_l(t)\delta(\xi + 1) \quad (10)$$

coupling the jump-WTDs for the neighboring jumps on the lattice, the fact reflected by the Dirac delta functions $\delta(\xi \pm 1)$ in Eq. (9). Here $\Lambda(\xi|x)$ is the x -dependent distribution function of the jump lengths ξ_n , see also Ref. [1].

Note that a mathematically not well-defined product of the delta-functions emerges in Eq. (10). A better approach is to avoid such mathematical issues and to redefine the conditional PDF (10) using a piecewise function, i.e., Ref. [81],

$$\psi(t|\xi) = \begin{cases} \psi_r(t), & \xi = 1, \\ \psi_l(t), & \xi = -1. \end{cases} \quad (11)$$

This has a minimal impact on the subsequent derivations and understanding of the obtained formulas.

Physically, in Eq. (9) a particle has a probability $g(x)$ to move from the current site x to the right with $\xi = 1$ and a probability $(1 - g(x))$ to move left from the current site x with $\xi = -1$ [given by the first and second term in Eq. (9), respectively]. For the two-point distribution (9) the position space of the particle reduces to a 1D lattice. Note that the variable ξ hereafter is discrete, but the application of the generalized functions $\delta(\xi \pm 1)$ converts ξ into a continuous variable; thus, below we use a standard continuous integration over ξ , instead of a representation via a discrete sum.

We further denote the *conditional* PDFs of the WTDs by $\psi_r(t)$ and $\psi_l(t)$ for a particle to go right and left, respectively. Hereafter, the subscripts r and l also denote these respective directions for all other quantities. The PDF $\phi(\xi, t|x)$ is normalized and the expressions $\int_0^t \psi_{r,l}(\tau)d\tau$ give the conditional

probabilities for a particle to jump in a respective direction, starting from $t = 0$ up to time t . With the definition (4), we find from Eqs. (8)–(10) that the survival probability (5) satisfies the relation

$$\begin{aligned} \Psi(t|x) &= 1 - \int_0^t d\tau \int_{-\infty}^{\infty} \phi(\xi, \tau|x)d\xi \\ &= 1 - g(x) \int_0^t \psi_r(\tau)d\tau - (1 - g(x)) \int_0^t \psi_l(\tau)d\tau. \end{aligned} \quad (12)$$

We consider below a simple situation of a constant, position-independent bias, i.e.,

$$g(x) = \gamma. \quad (13)$$

In this case, with the help of the balance-equations method [17,79], we can obtain the generalized Montroll-Weiss [4,10] equation in Laplace and Z space for A_r -CTRWs (see Sec. II B 2 below). A more general scenario of a *position-dependent* drift $g(x)$ naturally leads to some heterogeneous biased diffusion processes, in analogy to the recently developed HDPs [60–62] describing diffusion in the presence of a position-dependent diffusion coefficient.

2. Evolution equation for the PDF

We use the Z -transform technique according to the balance equations [17,79] in discrete space. Let $I(x, t)$ be the density of particles that reach a position x at a time t . The evolution equation of the PDF (see Eqs. (3.39) and (3.40) of Chapter 3 in Ref. [79]) can be written as

$$\rho(x, t) = \rho_0(x)\Psi(t) + \int_0^t I(x, t - \tau)\Psi(\tau)d\tau. \quad (14)$$

The first term on the right-hand side of Eq. (14) is the PDF of the particles staying at point x when no jump has occurred up to time t . The second term represents the contribution of the particles jumping to the point x at time $(t - \tau)$, provided that no additional jumps occur in the remaining time τ . Considering the simplified constant-drift scenario (13) the survival probability does not contain a dependence on the position x ,

$$\Psi(t) = 1 - \gamma \int_0^t \psi_r(\tau)d\tau - (1 - \gamma) \int_0^t \psi_l(\tau)d\tau, \quad (15)$$

that in Laplace space gives

$$\Psi(u) = [1 - \gamma\psi_r(u) - (1 - \gamma)\psi_l(u)]/u. \quad (16)$$

Substituting (15) into (14) we obtain

$$\begin{aligned} \rho(x, t) &= \rho_0(x) \left[1 - \gamma \int_0^t \psi_r(\tau)d\tau - (1 - \gamma) \int_0^t \psi_l(\tau)d\tau \right] \\ &\quad + \int_0^t I(x, t - \tau) \left[1 - \gamma \int_0^\tau \psi_r(\tau')d\tau' \right. \\ &\quad \left. - (1 - \gamma) \int_0^\tau \psi_l(\tau')d\tau' \right] d\tau. \end{aligned} \quad (17)$$

Taking the Laplace transform $t \rightarrow u$ and Z transform $x \rightarrow z$ (see Appendix A) of Eq. (17) yields our *main new*

equation for the PDF in the form

$$\rho(z, u) = [1 - \gamma\psi_r(u) - (1 - \gamma)\psi_l(u)]\rho_0(z)/u + [1 - \gamma\psi_r(u) - (1 - \gamma)\psi_l(u)]I(z, u)/u. \quad (18)$$

The PDF of the particles arriving at position x exactly at time t can be generally assumed to obey the balance equation [13,79],

$$I(x, t) = \int_{-\infty}^{\infty} \rho_0(x - \xi)\phi(\xi, t)d\xi + \int_0^t d\tau \int_{-\infty}^{\infty} I(x - \xi, t - \tau)\phi(\xi, \tau)d\xi. \quad (19)$$

The first term on the right-hand side of Eq. (19) is the contribution of particles located at position $(x - \xi)$ at time $t = 0$ which make their first jump to position x . The second term in the right-hand side stems from those particles which arrive at position $(x - \xi)$ at time $(t - \tau)$ and jump from the old position $(x - \xi)$ to a new position x after waiting for the time τ .

Considering the simplest drift scenario (13), we substitute Eqs. (8)–(10) into expression (19) and find

$$I(x, t) = \gamma\rho_0(x - 1)\psi_r(t) + (1 - \gamma)\rho_0(x + 1)\psi_l(t) + \gamma \int_0^t I(x - 1, t - \tau)\psi_r(\tau)d\tau + (1 - \gamma) \int_0^t I(x + 1, t - \tau)\psi_l(\tau)d\tau. \quad (20)$$

The Laplace-Z transform of this function is given by

$$I(z, u) = z^{-1}\gamma\rho_0(z)\psi_r(u) + z\rho_0(z)\psi_l(u) - z\gamma\rho_0(z)\psi_l(u) + z^{-1}\gamma I(z, u)\psi_r(u) + zI(z, u)\psi_l(u) - z\gamma I(z, u)\psi_l(u). \quad (21)$$

Combining this expression with Eqs. (18) and (21), after some algebra, we obtain the generalized evolution equation for A_r -CTRWs in the form

$$[1 - z^{-1}\gamma\psi_r(u) - z\psi_l(u) + z\gamma\psi_l(u)]\rho(z, u) = [u^{-1} - \gamma u^{-1}\psi_r(u) - (1 - \gamma)u^{-1}\psi_l(u)]\rho_0(z). \quad (22)$$

With this we straightforwardly arrive at the following generalized Montroll-Weiss equation for $\rho(z, u)$:

$$\rho(z, u) = \frac{\rho_0(z)[1 - \gamma\psi_r(u) - (1 - \gamma)\psi_l(u)]}{u[1 - z^{-1}\gamma\psi_r(u) - z(1 - \gamma)\psi_l(u)]}. \quad (23)$$

Here $\psi_{r,l}(u)$ are the Laplace transforms of $\psi_{r,l}(t)$, while $\rho(z, u)$ and $\rho_0(z)$ are the Laplace and Z transforms of $\rho(x, t)$ and $\rho_0(x)$, respectively.

One can easily check the normalization of the PDF $\rho(x, t)$ via the result $\rho(z, u)|_{z=1} = 1/u$ in Laplace-Z space. We use Eq. (23) with the specific Laplace transforms of the exponential WTDs in Sec. III A and of the power-law WTDs in Sec. III B in order to compute the first two moments of A_r -CTRWs in these two scenarios. The general result (23) can be used for arbitrary $\psi_{l,r}(u)$, thus representing our *main PDF-related* result.

3. Inversion of the PDF

From Eq. (23) with the help of Eq. (16) we find

$$\rho(z, u) = \rho_0(z)\Psi(u) + \gamma z^{-1}\rho(z, u)\psi_r(u) + (1 - \gamma)z\rho(z, u)\psi_l(u) \quad (24)$$

and thus the space-time evolution equation follows from (24) via applying the inverse Laplace-Z transform as

$$\rho(x, t) = \rho_0(x)\Psi(t) + \gamma \int_0^t \rho(x - 1, t - \tau)\psi_r(\tau)d\tau + (1 - \gamma) \int_0^t \rho(x + 1, t - \tau)\psi_l(\tau)d\tau. \quad (25)$$

Note that Eq. (23) can also be rewritten as

$$u\rho(z, u) - \rho_0(z) = \gamma M_r(u)(z^{-1} - 1)\rho(z, u) + (1 - \gamma)M_l(u)(z - 1)\rho(z, u), \quad (26)$$

where the left-versus-right symmetric functions were introduced,

$$M_{l,r}(u) = \frac{u\psi_{l,r}(u)}{1 - \gamma\psi_r(u) - (1 - \gamma)\psi_l(u)}. \quad (27)$$

With the inverse Laplace-Z transform of Eq. (26), we find the generalized master equation in real space-time as

$$\frac{\partial \rho(x, t)}{\partial t} = \gamma \int_0^t M_r(t - \tau)[\rho(x - 1, \tau) - \rho(x, \tau)]d\tau + (1 - \gamma) \int_0^t M_l(t - \tau)[\rho(x + 1, \tau) - \rho(x, \tau)]d\tau. \quad (28)$$

The functions $M_{r,l}(t)$ are the inverse Laplace transforms of the respective expressions (27).

4. Solution for the PDF in Laplace space

To derive the explicit result for the PDF, we take the inverse Z transform but still stay in Laplace space. From Eq. (23) for the case $\gamma \neq \{0, 1\}$ and for the standard initial condition,

$$\rho_0(x) = \delta(x), \quad (29)$$

which defines the Green's function of the problem and also simplifies the inversion to real space, we get an alternative form of expression (23) via substituting Eq. (16), namely

$$\rho(z, u) = -\frac{\Psi(u)z/[(1 - \gamma)\psi_l(u)]}{z^2 - \frac{z}{(1 - \gamma)\psi_l(u)} + \frac{\gamma\psi_r(u)}{(1 - \gamma)\psi_l(u)}}. \quad (30)$$

Note that $\rho_0(z) = 1$ in Z space. Therefore, we get

$$\rho(z, u) = \frac{\Psi(u)\left(\frac{z}{z - z_2} - \frac{z}{z - z_1}\right)}{(1 - \gamma)\psi_l(u)(z_1 - z_2)}, \quad (31)$$

where

$$z_{1,2} = \frac{1 \pm \sqrt{1 - 4\gamma(1 - \gamma)\psi_r(u)\psi_l(u)}}{2(1 - \gamma)\psi_l(u)} \quad (32)$$

are the two roots of the quadratic equation

$$z^2 - \frac{z}{(1 - \gamma)\psi_l(u)} + \frac{\gamma\psi_r(u)}{(1 - \gamma)\psi_l(u)} = 0 \quad (33)$$

following from the determinant of Eq. (30). The method of Z transform is therefore useful not only for computing the moments but also for inverting the PDFs of the type (23) from Z to real space.

We now take the inverse Z transform of expression (31) under the condition $z_2 < z < z_1$ with the help of relations [82]

$$Z^{-1}(z/(z-a)) = a^n \Theta(n) \quad (34)$$

for $|z| > |a|$ and

$$Z^{-1}(z/(z-a)) = -a^n \Theta(-n-1) \quad (35)$$

for $|z| < |a|$. Here $\Theta(n)$ is the Heaviside step function,

$$\Theta(n) = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \quad (36)$$

We thus obtain for $x \geq 0$ and $x < 0$ the solutions $\rho_+(x, u)$ and $\rho_-(x, u)$, respectively, so that after using Eqs. (16) and (32) we find

$$\rho_{\pm}(x, u) = \left(\frac{1 \mp \sqrt{1 - 4\gamma(1-\gamma)\psi_r(u)\psi_l(u)}}{2(1-\gamma)\psi_l(u)} \right)^x \times \frac{1 - \gamma\psi_r(u) - (1-\gamma)\psi_l(u)}{u\sqrt{1 - 4\gamma(1-\gamma)\psi_r(u)\psi_l(u)}}. \quad (37)$$

This expression agrees with the former results of Refs. [83,84] for a similar system, but considered without any spatial asymmetry of the jumps.

For the case $\gamma = 1$ with the initial condition (29) we get from Eq. (23) under the condition $|z| > \psi_r(u)$ that

$$\rho(z, u) = \frac{1 - \psi_r(u)}{u} \frac{1}{1 - z^{-1}\psi_r(u)} = \sum_{n=0}^{\infty} \frac{1 - \psi_r(u)}{u} \psi_r^n(u) z^{-n} \quad (38)$$

and thus for $x \geq 0$ [according to the definition of the Z transform given by (A6)]

$$\rho(x, u) = \frac{1 - \psi_r(u)}{u} (\psi_r(u))^x. \quad (39)$$

Analogously, for the case $\gamma = 0$ we find from Eq. (23) with the condition $|z| < (\psi_l(u))^{-1}$ that

$$\begin{aligned} \rho(z, u) &= \frac{1 - \psi_l(u)}{u} \frac{1}{1 - z\psi_l(u)} = \sum_{n=0}^{\infty} \frac{1 - \psi_l(u)}{u} \psi_l^n(u) z^n \\ &= \sum_{n=0}^{\infty} \frac{1 - \psi_l(u)}{u} \psi_l^{-n}(u) z^{-n} \end{aligned} \quad (40)$$

and thus for $x < 0$ the result is

$$\rho(x, u) = \frac{1 - \psi_l(u)}{u} (\psi_l(u))^{-x}. \quad (41)$$

5. Moments in Laplace space for arbitrary WTDs

From Eq. (28) with the initial PDF (29) the solution for $\rho(z, u)$ follows from Eq. (23) for arbitrary PDFs of the WTDs as

$$\rho(z, u) = \frac{1 - \gamma\psi_r(u) - (1-\gamma)\psi_l(u)}{u[1 - z^{-1}\gamma\psi_r(u) - z(1-\gamma)\psi_l(u)]}. \quad (42)$$

Its first two derivatives with respect to z are

$$\rho'_z(z, u)|_{z=1} = \frac{(1-\gamma)\psi_l(u) - \gamma\psi_r(u)}{u[1 - \gamma\psi_r(u) - (1-\gamma)\psi_l(u)]}, \quad (43)$$

$$\begin{aligned} \rho''_{zz}(z, u)|_{z=1} &= \frac{2\gamma\psi_r(u)}{u[1 - \gamma\psi_r(u) - (1-\gamma)\psi_l(u)]} \\ &+ \frac{2[\gamma\psi_r(u) - (1-\gamma)\psi_l(u)]^2}{u[1 - \gamma\psi_r(u) - (1-\gamma)\psi_l(u)]^2}. \end{aligned} \quad (44)$$

Using the relations (A9a) and (A9b), we find from this PDF the first two moments as

$$\langle x(u) \rangle = \frac{\gamma\psi_r(u) - (1-\gamma)\psi_l(u)}{u[1 - \gamma\psi_r(u) - (1-\gamma)\psi_l(u)]}, \quad (45a)$$

$$\begin{aligned} \langle x^2(u) \rangle &= \frac{\gamma\psi_r(u) + (1-\gamma)\psi_l(u)}{u[1 - \gamma\psi_r(u) - (1-\gamma)\psi_l(u)]} \\ &+ \frac{2[\gamma\psi_r(u) - (1-\gamma)\psi_l(u)]^2}{u[1 - \gamma\psi_r(u) - (1-\gamma)\psi_l(u)]^2}. \end{aligned} \quad (45b)$$

The same results follow from the PDF via a different method [83] from Eqs. (37), (39), and (41), i.e.,

$$\langle x(u) \rangle = \sum_{x=-\infty}^{\infty} x \rho(x, u), \quad \langle x^2(u) \rangle = \sum_{x=-\infty}^{\infty} x^2 \rho(x, u). \quad (46)$$

III. MAIN RESULTS

A. Exponential WTDs

1. Equation for the PDF

We now apply Eq. (28) to the particular case of waiting intervals which obey the exponential WTDs, with the PDFs [85]

$$\psi_{l,r}(t) = \lambda_{l,r} \exp(-\lambda_{l,r}t) \quad (47)$$

and expressions in Laplace space

$$\psi_{l,r}(u) = \frac{\lambda_{l,r}}{\lambda_{l,r} + u}. \quad (48)$$

Here $\lambda_{r,l}$ are the average rates of left- and right-directed jumps. The inverse Laplace transforms of Eq. (27) read

$$M_r(t) = \lambda_r \delta(t) + (1-\gamma)\lambda_r(\lambda_l - \lambda_r) e^{-[(1-\gamma)\lambda_r + \gamma\lambda_l]t}, \quad (49a)$$

$$M_l(t) = \lambda_l \delta(t) + \gamma\lambda_l(\lambda_r - \lambda_l) e^{-[(1-\gamma)\lambda_r + \gamma\lambda_l]t}. \quad (49b)$$

These relations are substituted into Eq. (28) to yield

$$\begin{aligned} \frac{\partial \rho(x, t)}{\partial t} &= \gamma \int_0^t [\lambda_r \delta(t-\tau) + (1-\gamma)\lambda_r(\lambda_l - \lambda_r) \\ &\times e^{-[(1-\gamma)\lambda_r + \gamma\lambda_l](t-\tau)}] [\rho(x-1, \tau) - \rho(x, \tau)] d\tau \\ &+ (1-\gamma) \int_0^t [\lambda_l \delta(t-\tau) + \gamma\lambda_l(\lambda_r - \lambda_l) \\ &\times e^{-[(1-\gamma)\lambda_r + \gamma\lambda_l](t-\tau)}] [\rho(x+1, \tau) - \rho(x, \tau)] d\tau. \end{aligned} \quad (50)$$

When $\lambda_l = \lambda_r = \lambda$ —i.e., when the WTDs for the left- and right-directed jumps are identical—Eq. (50) naturally reduces

to a conventional memoryless master equation [17],

$$\frac{\partial \rho(x, t)}{\partial t} = \gamma \lambda_r \rho(x-1, t) - \lambda_r \rho(x, t) + (1-\gamma) \lambda_l \rho(x+1, t). \quad (51)$$

From Eq. (50) we can also get an approximate generalized advection-diffusion equation using the Taylor expansions of $\rho(x \pm \Delta x, t)$. For the particles going right or left by one step-length Δx we get (ignoring all higher-order terms)

$$\begin{aligned} \frac{\partial \rho(x, t)}{\partial t} \approx & [-\gamma \lambda_r + (1-\gamma) \lambda_l] \frac{\partial \rho(x, t)}{\partial x} (\Delta x)^1 \\ & + \frac{1}{2} [\gamma \lambda_r + (1-\gamma) \lambda_l] \frac{\partial^2 \rho(x, t)}{\partial x^2} (\Delta x)^2 \\ & + \gamma(1-\gamma) (\lambda_r^2 - \lambda_l^2) \int_0^t e^{-\alpha(t-\tau)} \frac{\partial \rho(x, \tau)}{\partial x} d\tau (\Delta x)^1 \\ & - \frac{1}{2} \gamma(1-\gamma) (\lambda_l - \lambda_r)^2 \int_0^t e^{-\alpha(t-\tau)} \frac{\partial^2 \rho(x, \tau)}{\partial x^2} d\tau (\Delta x)^2. \end{aligned} \quad (52)$$

Here the asymmetry parameter α was introduced

$$\alpha = (1-\gamma) \lambda_r + \gamma \lambda_l. \quad (53)$$

Note that for the case $\lambda_r = \lambda_l = \lambda$ we recover from Eq. (52) the standard advection-diffusion equation [79]

$$\frac{\partial \rho(x, t)}{\partial t} = (1-2\gamma) \lambda \frac{\partial \rho(x, t)}{\partial x} (\Delta x)^1 + \frac{\lambda}{2} \frac{\partial^2 \rho(x, t)}{\partial x^2} (\Delta x)^2. \quad (54)$$

For the general case $\lambda_r \neq \lambda_l$, however, both the advection and the diffusion terms change [the first and the second term on the right-hand side of Eq. (52), respectively]. Additionally, two new memory terms emerge in Eq. (52) which ultimately lead to the anomalous-diffusion behavior, see below.

2. Special cases and approximations

We split the procedure of deriving the general solution of Eq. (51) into three special cases. Note that an explicit analytical solution for the PDF in real space can only be obtained for the scenario $\lambda_l = \lambda_r$; see Eq. (57) below.

Case (i). At $\lambda_l = \lambda_r$, from Eq. (37) for $\lambda_l = \lambda_r = \lambda$ and for $\gamma \neq \{0, 1\}$ from Eq. (51) with the initial-state PDF given by Eq. (29) we get

$$\begin{aligned} \rho(x, u) = & \left(\frac{\gamma}{1-\gamma} \right)^{x/2} (2\lambda \sqrt{\gamma(1-\gamma)})^{|x|} \\ & \times \frac{((u+\lambda) + \sqrt{(u+\lambda)^2 - 4\gamma(1-\gamma)\lambda^2})^{-|x|}}{\sqrt{(u+\lambda)^2 - 4\gamma(1-\gamma)\lambda^2}}. \end{aligned} \quad (55)$$

We then take its inverse Laplace transform with the help of the relation (see Eq. (30) on p. 240 of Ref. [86])

$$\mathcal{L}^{-1} \left(\frac{a^\nu}{\sqrt{u^2 - a^2}(u + \sqrt{u^2 - a^2})^\nu} \right) = I_\nu(at). \quad (56)$$

Here $I_\nu(z)$ is the modified Bessel function of the first kind of the ν th order, with $\text{Re}(\nu) > -1$. In real space-time this yields

the PDF of the form

$$\rho(x, t) = \left(\frac{\gamma}{1-\gamma} \right)^{x/2} e^{-\lambda t} I_{|x|}(2\lambda \sqrt{\gamma(1-\gamma)}t), \quad (57)$$

similarly to the result of Ref. [84]. Note that the WTDs in Refs. [83,84] were splitted into parts to describe these features.

Case (ii). At $\gamma = 1$, i.e., when the probability of jumps to the *right* is equal to unity, Eq. (50) reduces to

$$\frac{\partial \rho(x, t)}{\partial t} = \lambda_r \rho(x-1, t) - \lambda_r \rho(x, t), \quad (58)$$

with Laplace-Z transform

$$\rho(z, u) = \frac{\rho_0(z)}{u - \lambda_r z^{-1} + \lambda_r}. \quad (59)$$

With the help of Eq. (34) and for the initial PDF (29) for $x \in Z_+$ the solution for the PDF becomes a Poisson process, namely [85]

$$\rho(x, t) = e^{-\lambda_r t} \frac{(\lambda_r t)^x}{\Gamma(x+1)}. \quad (60)$$

Case (iii). At $\gamma = 0$, in contrast, with all jumps taking place to the *left*, with the initial condition (29) for particle positions $x \in Z_-$ we get

$$\rho(x, t) = e^{-\lambda_l t} \frac{(\lambda_l t)^{-x}}{\Gamma(-x+1)}. \quad (61)$$

Note that as the variable x takes only discrete values on an infinite 1D lattice (positive and negative for the cases (ii) and (iii) above), the PDFs (60) and (61) are normalized.

3. Moments and variance

To obtain the moments of this stochastic process, we start with Eq. (43) and apply the Laplace transform (48) to get the first derivative of the PDF as

$$\rho'_z(z, u)|_{z=1} = -\frac{\beta u + \theta}{u^2(u + \alpha)}, \quad (62)$$

where for compactness we defined two new parameters

$$\beta = \gamma \lambda_r - (1-\gamma) \lambda_l, \quad (63a)$$

$$\theta = (2\gamma - 1) \lambda_r \lambda_l. \quad (63b)$$

The inverse Laplace transform of (62) gives

$$\langle x(t) \rangle = -\mathcal{L}^{-1}(\rho'_z(z, u)|_{z=1}) \quad (64)$$

that finally yields the first moment in the form

$$\langle x(t) \rangle = \alpha^{-2}(\alpha\beta - \theta + (\theta - \alpha\beta)e^{-\alpha t} + \alpha\theta t). \quad (65)$$

By analogy, from Eq. (44) the second derivative $\rho''_{zz}(z, u)$ can be obtained with the WTD (48) as

$$\begin{aligned} \rho''_{zz}(z, u)|_{z=1} = & \frac{2\gamma \lambda_r}{u(u + \alpha)} + \frac{2\gamma \lambda_r \lambda_l}{u^2(u + \alpha)} + \frac{2\beta^2}{u(u + \alpha)^2} \\ & + \frac{4\beta\theta}{u^2(u + \alpha)^2} + \frac{2\theta^2}{u^3(u + \alpha)^2}. \end{aligned} \quad (66)$$

The inverse Laplace transform yields—with the help of Eq. (A8b)—the second moment as

$$\langle x^2(t) \rangle = \mathcal{L}^{-1}(\rho''_{zz}(z, u)|_{z=1}) - \langle x(t) \rangle, \quad (67)$$

that can be expressed explicitly as

$$\begin{aligned} \langle x^2(t) \rangle = & \frac{1}{\alpha} \left[\left(\beta - 2\gamma\lambda_r + \frac{2\gamma\lambda_r\lambda_l - 2\beta^2 - \theta}{\alpha} \right. \right. \\ & + \frac{8\beta\theta}{\alpha^2} - \frac{6\theta^2}{\alpha^3} \Big) (e^{-\alpha t} - 1) \\ & - \left(2\beta^2 - \frac{4\beta\theta}{\alpha} + \frac{2\theta^2}{\alpha^2} \right) t e^{-\alpha t} \\ & \left. + \left(2\gamma\lambda_r\lambda_l - \theta + \frac{4\beta\theta}{\alpha} - \frac{4\theta^2}{\alpha^2} \right) t + \frac{\theta^2}{\alpha} t^2 \right]. \quad (68) \end{aligned}$$

We ordered the terms in expressions (65) and (68) in their progressively growing “impact” on or order with respect to the particle displacement at long times. Namely, each of these results is a combinations of a constant time-independent factor, an exponentially decreasing function of time, a function scaling as time multiplied by an exponentially decaying function of time, a linearly increasing function of time (intuitively expected for the first moment in our system with a drift), and, finally, a term scaling quadratically in time. Note that the ballistic term is, naturally, present only in the full expression for the second moment (68).

$$\text{Var}[x(t)] \approx \frac{\lambda_r\lambda_l[\lambda_r^2(1 - 4\gamma + 7\gamma^2 - 4\gamma^3) + 2\gamma\lambda_r\lambda_l(1 - \gamma) + \lambda_l^2(2\gamma - 5\gamma^2 + 4\gamma^3)]t}{[(1 - \gamma)\lambda_r + \gamma\lambda_l]^3}. \quad (72)$$

In Eqs. (71) and (72) the exponentially decaying and the constant-in-time terms of the respective contributing moments (65) and (68) were neglected at $t \rightarrow \infty$. Therefore, despite of a drift in the system, the spreading of the particles—considered in the sense of diffusion measure (69)—features the law of normal diffusion and a linear variance for all values of the model parameters.

Finally, we consider two special cases: (i) $\gamma = 1$ and (ii) $\gamma = 0$, corresponding to the situations when all particles jump exclusively to the *right* or to the *left*, respectively.

Case (i). It is easy to check that for $\gamma = 1$ one gets

$$\langle x(t) \rangle = \lambda_r t, \quad (73a)$$

$$\langle x^2(t) \rangle = \lambda_r^2 t^2 + \lambda_r t, \quad (73b)$$

$$\text{Var}[x(t)] = \lambda_r t. \quad (73c)$$

Case (ii). For $\gamma = 0$ we arrive at

$$\langle x(t) \rangle = -\lambda_l t, \quad (74a)$$

$$\langle x^2(t) \rangle = \lambda_l^2 t^2 + \lambda_l t, \quad (74b)$$

$$\text{Var}[x(t)] = \lambda_l t. \quad (74c)$$

Clearly, the linear-in-time behaviors of the variance given by Eqs. (73c) and (74c) follow from the general expression (72). Therefore, as expected, the results of an effective Poisson process [85] are obtained here, namely the first moment is

The parameter α yields the reciprocal time $1/\alpha$ which in turn defines the domain of intermediate times: Here multiple terms in (65) and (68) contribute to the magnitude of the computed moments. At times $t \gg \alpha^{-1}$ the impact of all constant-in-time or exponentially decaying terms become negligible and the power-law terms with the highest power dominate the moments.

As the first moment is generally nonzero for a biased stochastic process such as A_r -CTRW, see Eq. (64), the variance $\text{Var}[x(t)]$ —as the effective statistical measure of diffusion—is defined via the first two moments as

$$\text{Var}[x(t)] = \langle (x(t) - \langle x(t) \rangle)^2 \rangle \equiv \langle x^2(t) \rangle - \langle x(t) \rangle^2. \quad (69)$$

To save space, here we only analyze the asymptotic behaviors of the first moment and of the variance in the limit $t \rightarrow \infty$. Namely, from the general expression (65) it follows that in this limit the first moment grows linearly with time, i.e.,

$$\langle x(t) \rangle \approx (\theta/\alpha)t. \quad (70)$$

Furthermore, one can check that in Eq. (69) the terms from the first and second moments (65) and (68) yielding a ballistic dependence on time *cancel* each other in the variance. Thus, the dominant long-time behavior of the variance is linear in time,

$$\text{Var}[x(t)] \approx \frac{\alpha^2\lambda_r\lambda_l + 2\theta(\alpha\beta - \theta)}{\alpha^3}t, \quad (71)$$

or—explicitly in terms of the parameters λ_r, l and γ —we get

equal to the variance. The PDFs for the cases $\gamma = 1$ and $\gamma = 0$ are given by Eqs. (60) and (61), correspondingly.

4. Comparison with simulations

To verify these analytical results, we implemented lattice-based Monte Carlo computer simulations with the particles going one step to the right and to the left with the probabilities γ and $(1 - \gamma)$, respectively. This mimics the asymmetry of the underlying displacement process, see Eq. (9). A sequence of random waiting times is generated from the distributions (47) and then used to sample the displacement statistics of the particles.

For the PDF, in Fig. 2 we show the agreement of the results of the simulations with the analytical solutions. Note that the bias of the green-triangle data set for the PDF in Fig. 2 is solely due to unequal probabilities of jump to the right and to the left, quantified by $\gamma = 0.55$, whereas the WTDs are the same in both directions.

For the moments and variance, the results of the simulations up to the total time $t = 10^2$ are illustrated in Figs. 3(a)–3(c) for the first moment, the second moment, and the variance, respectively. These results are in close agreement with the theoretical predictions [Eqs. (65), (68), and (69), correspondingly]. In Figs. 3(b) and 3(c) we include the long-time [87] slopes $\propto t^2$ and $\propto t^1$ to indicate that at $\gamma \neq 1/2$

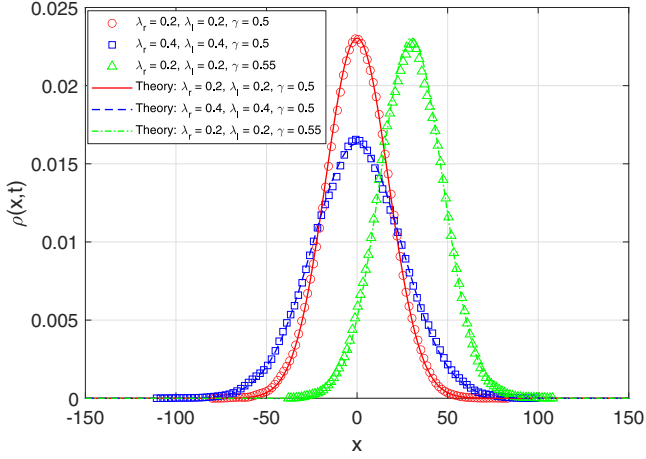


FIG. 2. Results of Monte Carlo simulations (symbols) versus the derived analytical solutions (solid curves) for the PDF $\rho(x, t)$ (57) for A_t -CTRWs with exponential WTDs (47). The parameters are $\lambda_l = \lambda_r$ (when the analytical PDF result in real space exists). The results are plotted at the diffusion time $t = 1500$ and averaged over $N = 10^4$ trajectories of the particles. The values of relevant model parameters are provided in the legend.

the second moment grows ballistically, while the variance increases linearly with time at long times.

The results for the variance of A_t -CTRWs in Fig. 3(c) at short-to-intermediate times reveal a region of anomalous diffusion (i.e., a nonlinear scaling with time), in contrast to the normal diffusion for standard symmetric exponential-WTD CTRWs. The time interval of this anomalous-diffusion behavior depends on the actual values of the model parameters, see the legend in Fig. 3. At long times, for all values of the parameters considered, we find the linear-in-time asymptotics (72) for the variance.

B. Power-law WTDs

1. Equation for the PDF

For A_t -CTRWs with power-law WTDs with a characteristic timescale τ_0 in the long-time limit (corresponding to $u \rightarrow 0$) considered below one gets

$$\psi_{r,l}(u) = \frac{1}{1 + \gamma_{r,l}(\tau_0 u)^{\beta_{r,l}}} \approx 1 - \gamma_{r,l}(\tau_0 u)^{\beta_{r,l}}, \quad (75)$$

with the definitions (for compactness)

$$\gamma_{r,l} = |\Gamma(1 - \beta_{r,l})|. \quad (76)$$

For all situations with $\gamma \neq \{0, 1\}$ one more additional parameter is defined here as

$$\gamma_{rl} = \frac{\gamma_l}{\gamma_r} \frac{1 - \gamma}{\gamma}. \quad (77)$$

In the limit $u \rightarrow 0$, for the general expression (27) we obtain

$$M_r(u) \approx \frac{(u\tau_0)^{1-\beta_r}}{\tau_0 \gamma \gamma_r (1 + \gamma_{rl}(u\tau_0)^{\beta_l - \beta_r})}, \quad (78a)$$

$$M_l(u) \approx \frac{(u\tau_0)^{1-\beta_l}}{\tau_0 (1 - \gamma) \gamma_l (1 + \gamma_{rl}^{-1}(u\tau_0)^{\beta_r - \beta_l})}, \quad (78b)$$

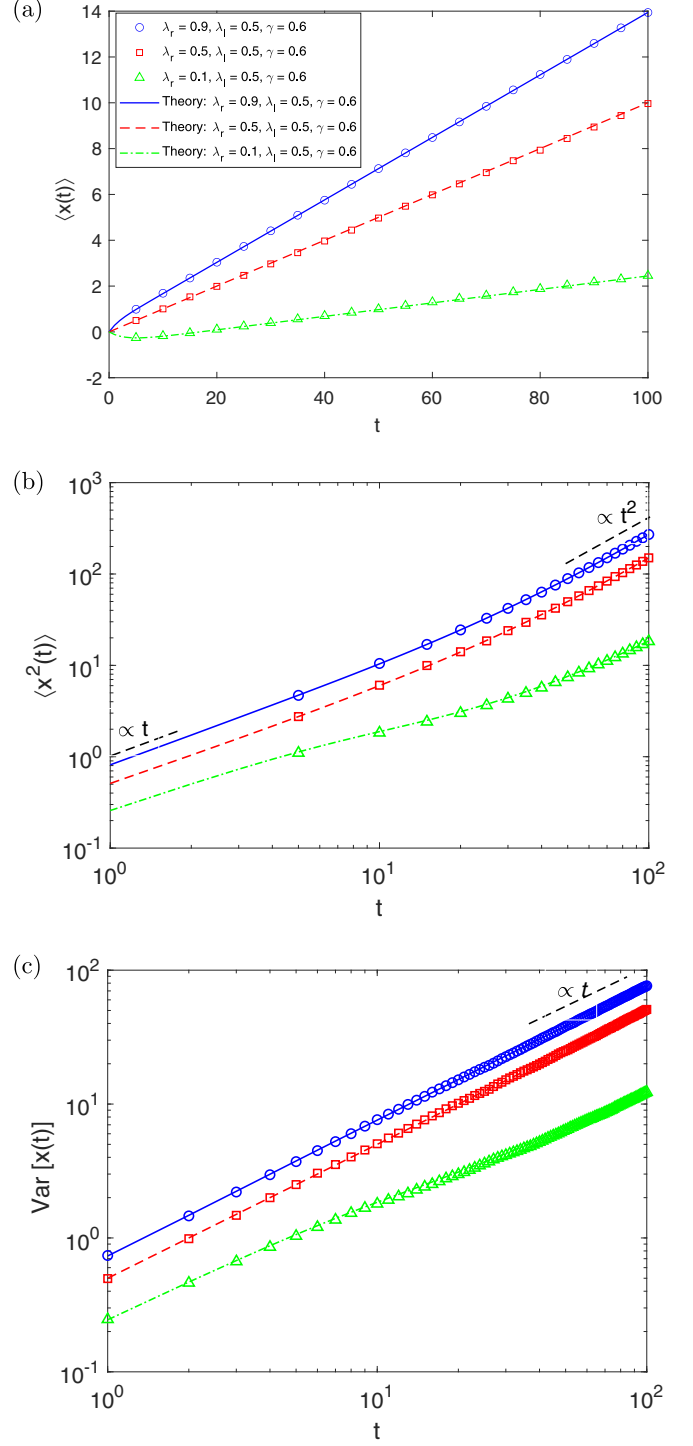


FIG. 3. First two moments and variance of A_t -CTRWs with exponential WTDs, for the parameters of Fig. 2 [as listed in the legend of panel (a)]. The results of the simulations for $\langle x(t) \rangle$, $\langle x^2(t) \rangle$, and $\text{Var}[x(t)]$ and the respective analytical predictions (65), (68), and (69) are presented in panels (a), (b), and (c), correspondingly. In panels (b) and (c) log-log scale is used and the long-time slopes are indicated.

and after substituting these results into Eq. (26)—with the help of the inverse Z transformation—we get at long times that

$$u\rho(x, u) - \rho_0(x) = \frac{(\tau_0 u)^{1-\beta_r}}{\tau_0 \gamma_r} \frac{[\rho(x-1, u) - \rho(x, u)]}{1 + \gamma_{rl}(u\tau_0)^{\beta_l - \beta_r}} + \frac{(\tau_0 u)^{1-\beta_l}}{\tau_0 \gamma_l} \frac{[\rho(x+1, u) - \rho(x, u)]}{1 + \gamma_{rl}^{-1}(u\tau_0)^{\beta_r - \beta_l}}. \quad (79)$$

Inverting Eq. (79), in real space and real time we obtain the fractional master equations for an A_t -CTRW with the WTDs of a power-law form. Namely, for $\beta_l > \beta_r$ we get

$$\frac{\partial \rho(x, t)}{\partial t} = \frac{{}_0D_t^{1-\beta_r} \left\{ [t^{\beta_l - \beta_r - 1} E_{\beta_l - \beta_r, \beta_l - \beta_r} \left(-\frac{1}{\gamma_{rl}(t/\tau_0)^{\beta_r - \beta_l}} \right)] * \left(\rho(x-1, t) + \frac{1-\gamma}{\gamma} \rho(x+1, t) - \frac{1}{\gamma} \rho(x, t) \right) \right\}}{\gamma_r \tau_0^{\beta_r}} \quad (80)$$

whereas for $\beta_l < \beta_r$ we arrive at

$$\frac{\partial \rho(x, t)}{\partial t} = \frac{{}_0D_t^{1-\beta_l} \left\{ [t^{\beta_r - \beta_l - 1} E_{\beta_r - \beta_l, \beta_r - \beta_l} (-\gamma_{rl}(t/\tau_0)^{\beta_r - \beta_l})] * \left(\frac{\gamma}{1-\gamma} \rho(x-1, t) + \rho(x+1, t) - \frac{1}{1-\gamma} \rho(x, t) \right) \right\}}{\gamma_l \tau_0^{\beta_l}} \quad (81)$$

Here ${}_0D_t^{1-\alpha}$ is the Riemann-Liouville fractional operator defined as [1,42,73]

$${}_0D_t^{1-\alpha} \rho(x, t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t dt' \frac{\rho(x, t')}{(t-t')^{1-\alpha}} \quad (82)$$

and the asterisk (*) denotes the convolution (see Eq. (2.237) in Ref. [88])

$$f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau. \quad (83)$$

In the constant-drift scenario with $\gamma = 1$, substituting

$$M_r(u) \approx (\tau_0 \gamma_r)^{-1} (u\tau_0)^{1-\beta_r} \quad (84)$$

into Eq. (26) and after taking the inverse Z transform, we get

$$u\rho(x, u) - \rho_0(x) = \frac{(\tau_0 u)^{1-\beta_r}}{\tau_0 \gamma_r} (\rho(x-1, u) - \rho(x, u)), \quad (85)$$

with inverse Laplace transform being

$$\frac{\partial \rho(x, t)}{\partial t} = \frac{{}_0D_t^{1-\beta_r} (\rho(x-1, t) - \rho(x, t))}{\gamma_r \tau_0^{\beta_r}}. \quad (86)$$

Analogously, when $\gamma = 0$ we obtain a simplified equation

$$\frac{\partial \rho(x, t)}{\partial t} = \frac{{}_0D_t^{1-\beta_l} (\rho(x+1, t) - \rho(x, t))}{\gamma_l \tau_0^{\beta_l}}, \quad (87)$$

while when $\beta_r = \beta_l = \beta$ from Eq. (26) we obtain

$$\frac{\partial \rho(x, t)}{\partial t} = \frac{\gamma_0 {}_0D_t^{1-\beta} (\rho(x-1, t) - \rho(x, t))}{\gamma_r \tau_0^\beta} + \frac{(1-\gamma) {}_0D_t^{1-\beta} (\rho(x+1, t) - \rho(x, t))}{\gamma_r \tau_0^\beta}. \quad (88)$$

Taking the Taylor expansion of Eq. (88) and ignoring all higher-order terms, we obtain the *approximate* fractional convection-diffusion equation in the form

$$\frac{\partial \rho(x, t)}{\partial t} \approx \frac{{}_0D_t^{1-\beta} [(1-2\gamma) \frac{\partial \rho(x, t)}{\partial x} (\Delta x)^1 + \frac{1}{2} \frac{\partial^2 \rho(x, t)}{\partial x^2} (\Delta x)^2]}{\gamma_r \tau_0^\beta}. \quad (89)$$

Note that, in contrast to A_t -CTRWs with the exponential WTDs (considered in Sec. III A) yielding the analytical PDF solution (57), for A_t -CTRWs with the power-law WTDs the PDF results in real space-time cannot be derived analytically. In certain special cases, however, some results are obtainable. For instance, for the case (i) with $\gamma = 1$ from Eq. (39) it follows that

$$\rho(x, u) = \frac{u^{-1} \gamma_r (\tau_0 u)^{\beta_r}}{(1 + \gamma_r (\tau_0 u)^{\beta_r})^{x+1}} \quad (90)$$

and thus in real space-time in the long-time limit for the situations $x \geq 0$ we get

$$\rho(x, t) = \gamma_r^{-x} (t/\tau_0)^{x\beta_r} E_{\beta_r, x\beta_r+1, x+1} (-\gamma_r^{-1} (t/\tau_0)^{\beta_r}) \approx \frac{\gamma_r (t/\tau_0)^{-\beta_r}}{\Gamma(1-\beta_r)} - \frac{(x+1) \gamma_r^2 (t/\tau_0)^{-2\beta_r}}{\Gamma(1-2\beta_r)}. \quad (91)$$

For the case (ii) with $\gamma = 0$ from Eq. (41) with $x < 0$ it follows that

$$\rho(x, t) = \gamma_l^x (t/\tau_0)^{-x\beta_l} E_{\beta_l, -x\beta_l+1, -x+1} (-\gamma_l^{-1} (t/\tau_0)^{\beta_l}) \approx \frac{\gamma_l (t/\tau_0)^{-\beta_l}}{\Gamma(1-\beta_l)} - \frac{(-x+1) \gamma_l^2 (t/\tau_0)^{-2\beta_l}}{\Gamma(1-2\beta_l)}. \quad (92)$$

Note that the x dependence in Eqs. (91) and (92) is absent in the leading-order expansion of the respective Mittag-Leffler functions (see Appendix A); the next-order curly underlined terms in these expressions do, however, feature a coupled dependence on both x and t variables.

2. Moments and variance

From Eq. (26), for A_t -CTRWs with the PDF in the initial state described by (29), the Laplace- Z -space propagator has

the form

$$\mathcal{G}(z, u) = \frac{1}{u - \gamma M_r(u)(z^{-1} - 1) - (1 - \gamma)M_l(u)(z - 1)}. \quad (93)$$

Its first and second derivatives with respect to z taken at $z = 1$ are, respectively,

$$\mathcal{G}'_z(z, u)|_{z=1} = \frac{-\gamma M_r(u) + (1 - \gamma)M_l(u)}{u^2}, \quad (94a)$$

$$\mathcal{G}''_{zz}(z, u)|_{z=1} = \frac{2(\gamma u M_r(u) + (-\gamma M_r(u) + (1 - \gamma)M_l(u))^2)}{u^3}. \quad (94b)$$

Thus, after rearranging the terms, we get

$$\mathcal{G}''_{zz}(z, u)|_{z=1} = 2(\gamma u^{-2} M_r(u) + u(\mathcal{G}'_z(z, u)|_{z=1})^2). \quad (95)$$

Taking the inverse Laplace transform of Eqs. (94a) and (95)—with the help of Eqs. (64) and (67)—the first two moments of such A_t -CTRWs in real time can be computed. The obtained results are consistent with the results in Laplace space presented in Ref. [83]. Note that the method of Z transform enables us to compute the third and fourth moments and thus to analytically evaluate the skewness and kurtosis of the PDF as well.

Note that in Eq. (93) we utilized the small- u expansion (75) to get the leading-order terms (78a) and (78b). If, on the contrary, we use the *exact* expressions for $\psi_{r,l}(u)$ and proceed in Eq. (93) instead with the expressions

$$M_r(u) = \frac{\frac{u}{1+\gamma_r(\tau_0 u)^{\beta_r}}}{1 - \frac{\gamma}{1+\gamma_r(\tau_0 u)^{\beta_r}} - \frac{1-\gamma}{1+\gamma_l(\tau_0 u)^{\beta_l}}}, \quad (96a)$$

$$M_l(u) = \frac{\frac{u}{1+\gamma_l(\tau_0 u)^{\beta_l}}}{1 - \frac{\gamma}{1+\gamma_r(\tau_0 u)^{\beta_r}} - \frac{1-\gamma}{1+\gamma_l(\tau_0 u)^{\beta_l}}}, \quad (96b)$$

then, via performing the Taylor expansion more accurately, the leading long-time scaling relations for the first two moments remain *unaltered*, as we have checked (results not shown).

Case (i). For the situation

$$\beta_r = \beta_l \equiv \beta \quad (97)$$

with Eqs. (75) and (94a) after defining the parameter

$$K = \frac{1 - 2\gamma}{\gamma_r} \quad (98)$$

we find

$$\mathcal{G}'_z(z, u)|_{z=1} = \tau_0 K (u\tau_0)^{-1-\beta}, \quad (99)$$

and with the help of Eq. (A12) we get the first moment as

$$\langle x(t) \rangle = -\frac{K(t/\tau_0)^\beta}{\Gamma(\beta + 1)}. \quad (100)$$

Analogously, from Eq. (95) we arrive at

$$\mathcal{G}''_{zz}(z, u)|_{z=1} = 2\tau_0(\gamma_r^{-1}\gamma(u\tau_0)^{-1-\beta} + K^2(u\tau_0)^{-1-2\beta}) \quad (101)$$

that with the help of (67) yields the second moment as

$$\langle x^2(t) \rangle = \frac{(t/\tau_0)^\beta}{\gamma_r \Gamma(\beta + 1)} + \frac{2K^2(t/\tau_0)^{2\beta}}{\Gamma(2\beta + 1)}. \quad (102)$$

In this special case the situation is the simplest: the second moment is just the sums of two respective power-law terms.

From the analytical expressions (100) and (102) at long times ($t \gg \tau_0$) for the situations when $\gamma \neq 1/2$ one gets the scaling relations

$$\langle x(t) \rangle \propto t^\beta, \quad (103a)$$

$$\langle x^2(t) \rangle \propto t^{2\beta}, \quad (103b)$$

$$\text{Var}[x(t)] \propto t^{2\beta}. \quad (103c)$$

Note that at $\gamma = 1/2$ we arrive at $K = 0$ in Eq. (98). This corresponds to a situation with no drift (no left-versus-right symmetry in the jump probabilities). Thus, all drift-related terms with the “double” power which are proportional to K in the second moment (102) disappear and at $\gamma = 1/2$ one gets

$$\langle x(t) \rangle = 0, \quad (104a)$$

$$\langle x^2(t) \rangle \propto t^\beta, \quad (104b)$$

$$\text{Var}[x(t)] \propto t^\beta. \quad (104c)$$

Case (ii). With the help of expressions (78a) and (78b), from Eq. (94a) for the case

$$\beta_l > \beta_r \quad (105)$$

and with the additional parameters

$$C = \frac{1 - 2\gamma}{\gamma_l(1 - \gamma)}, \quad (106a)$$

$$a_{rl} = \gamma_{rl}^{-1} \tau_0^{-\beta_l + \beta_r}, \quad (106b)$$

we find

$$\mathcal{G}'_z(z, u)|_{z=1} \approx \frac{C\tau_0^{-\beta_l} u^{-1-\beta_r}}{a_{rl} + u^{\beta_l - \beta_r}}, \quad (107)$$

that ultimately yields

$$\langle x(t) \rangle \approx -C(t/\tau_0)^{\beta_l} E_{\beta_l - \beta_r, 1 + \beta_l}(-\gamma_{rl}^{-1}(t/\tau_0)^{\beta_l - \beta_r}). \quad (108)$$

After using the series of the Mittag-Leffler functions given by Eq. (A14), the long-time expansion of the first moment becomes

$$\langle x(t) \rangle \approx -\frac{K(t/\tau_0)^{\beta_r}}{\gamma \Gamma(\beta_r + 1)} + \frac{\gamma_{rl} K(t/\tau_0)^{2\beta_r - \beta_l}}{\gamma \Gamma(2\beta_r - \beta_l + 1)}. \quad (109)$$

In a similar fashion, from Eq. (95) we arrive at

$$\mathcal{G}''_{zz}(z, u)|_{z=1} \approx 2 \left(\frac{\tau_0^{-\beta_l} \gamma_{rl}^{-1} u^{-1-\beta_r}}{\gamma_r(a_{rl} + u^{\beta_l - \beta_r})} + \frac{\tau_0^{-2\beta_l} C^2 u^{-1-2\beta_r}}{(a_{rl} + u^{\beta_l - \beta_r})^2} \right) \quad (110)$$

that yields the second moment at long times in the form

$$\begin{aligned} \langle x^2(t) \rangle \approx & \frac{(t/\tau_0)^{\beta_l}}{\gamma_l(1 - \gamma)} E_{\beta_l - \beta_r, 1 + \beta_l}(-\gamma_{rl}^{-1}(t/\tau_0)^{\beta_l - \beta_r}) \\ & + 2C^2(t/\tau_0)^{2\beta_l} E_{\beta_l - \beta_r, 1 + 2\beta_l, 2}(-\gamma_{rl}^{-1}(t/\tau_0)^{\beta_l - \beta_r}). \end{aligned} \quad (111)$$

After using the power-law expansion (A14), we arrive at

$$\begin{aligned} \langle x^2(t) \rangle \approx & \frac{(t/\tau_0)^{\beta_r}}{\gamma\gamma_r\Gamma(\beta_r+1)} + \frac{2K^2(t/\tau_0)^{2\beta_r}}{\gamma^2\Gamma(2\beta_r+1)} \\ & - \frac{\gamma_{rl}(t/\tau_0)^{2\beta_r-\beta_l}}{\gamma\gamma_r\Gamma(2\beta_r-\beta_l+1)} - \frac{2\gamma_{rl}K^2(t/\tau_0)^{3\beta_r-\beta_l}}{\gamma^2\Gamma(3\beta_r-\beta_l+1)}. \end{aligned} \quad (112)$$

In contrast to the simplest special case (i), in the more general special cases (ii) and (iii) described here and below, not only the timescale τ_0 , but also the model parameter γ_{rl} given by Eq. (77), controls the value of the argument of the Mittag-Leffler functions describing the full analytical solutions for the first and second moments. The corresponding domains of intermediate times will therefore, in turn, be the function of this model parameter. We emphasize that the parameter γ_{rl} is a function of both the drift parameter γ and of the two scaling exponents $\beta_{r,l}$ of the contributing WTDs.

Note that, for completeness—both in Eq. (112) above and in Eq. (121) below—we included both the leading- and the subsequent-order terms of the expansion (the latter are annotated as the underlined expressions). These next-order terms can be important for the short-time dynamics, but for the asymptotic behavior in the long-time limit (we mainly focus on hereafter) they make no effect.

Note also that the conditional random waiting times control the properties of diffusion on 1D lattices, as compared to other models of biased CTRWs [56,57]. It implies that even if the probability of particle jumps left and right is equal (i.e., $\gamma = 1/2$), a bias arises in a statistical sense because of different left-versus-right WTDs in our current model of A_r -CTRWs.

From Eqs. (109) and (112) we obtain that at long times (again, at $\gamma \neq 1/2$) that—in virtue of the relations $(2\beta_r - \beta_l) < \beta_r$ and $(3\beta_r - \beta_l) < 2\beta_r$ valid in this special case—the following scaling behaviors emerge:

$$\langle x(t) \rangle \propto t^{\beta_r}, \quad (113a)$$

$$\langle x^2(t) \rangle \propto t^{2\beta_r}, \quad (113b)$$

$$\text{Var}[x(t)] \propto t^{2\beta_r}. \quad (113c)$$

For comparison, at $\gamma = 1/2$ we arrive [as $K = 0$, defined in Eq. (98)] at the following results:

$$\langle x(t) \rangle = 0, \quad (114a)$$

$$\langle x^2(t) \rangle \propto t^{\beta_r}, \quad (114b)$$

$$\text{Var}[x(t)] \propto t^{\beta_r}. \quad (114c)$$

Case (iii). For the situation

$$\beta_r > \beta_l, \quad (115)$$

following the same procedure, we derive

$$\mathcal{G}'_z(z, u)|_{z=1} \approx \frac{\tau_0^{-\beta_r}(K/\gamma)u^{-1-\beta_l}}{1/a_{rl} + u^{\beta_r-\beta_l}} \quad (116)$$

that yields

$$\langle x(t) \rangle \approx -(K/\gamma)(t/\tau_0)^{\beta_r} E_{\beta_r-\beta_l, 1+\beta_r}(-\gamma_{rl}(t/\tau_0)^{\beta_r-\beta_l}). \quad (117)$$

The long-time expansion for the first moment therefore (for the relation (115) between β_l and β_r) reads

$$\langle x(t) \rangle \approx -\frac{C(t/\tau_0)^{\beta_l}}{\Gamma(\beta_l+1)} + \frac{C(t/\tau_0)^{2\beta_l-\beta_r}}{\gamma_{rl}\Gamma(2\beta_l-\beta_r+1)}. \quad (118)$$

From the derivative

$$\mathcal{G}''_{zz}(z, u)|_{z=1} \approx 2 \left(\frac{\tau_0^{-\beta_r} u^{-1-\beta_l}}{\gamma_r(a_{rl}^{-1} + u^{\beta_r-\beta_l})} + \frac{\tau_0^{-2\beta_r} (K/\gamma)^2 u^{-1-2\beta_l}}{(a_{rl}^{-1} + u^{\beta_r-\beta_l})^2} \right) \quad (119)$$

one arrives at the result

$$\begin{aligned} \langle x^2(t) \rangle \approx & \frac{(t/\tau_0)^{\beta_r}}{\gamma_r\gamma} E_{\beta_r-\beta_l, 1+\beta_r}(-\gamma_{rl}(t/\tau_0)^{\beta_r-\beta_l}) \\ & + 2(K/\gamma)^2(t/\tau_0)^{2\beta_r} E_{\beta_r-\beta_l, 1+2\beta_r, 2}(-\gamma_{rl}(t/\tau_0)^{\beta_r-\beta_l}) \end{aligned} \quad (120)$$

and thus the second moment at long times is given by

$$\begin{aligned} \langle x^2(t) \rangle \approx & \frac{(t/\tau_0)^{\beta_l}}{(1-\gamma)\gamma_l\Gamma(\beta_l+1)} + \frac{2C^2(t/\tau_0)^{2\beta_l}}{\Gamma(2\beta_l+1)} \\ & - \frac{(t/\tau_0)^{2\beta_l-\beta_r}}{(1-\gamma)\gamma_l\gamma_{rl}\Gamma(2\beta_l-\beta_r+1)} - \frac{2C^2(t/\tau_0)^{3\beta_l-\beta_r}}{\gamma_{rl}\Gamma(3\beta_l-\beta_r+1)}. \end{aligned} \quad (121)$$

The scaling relations following from Eqs. (117) and (121) at long times—due to the relations $(2\beta_l - \beta_r) < \beta_l$ and $(3\beta_l - \beta_r) < 2\beta_l$ valid for this special case—are

$$\langle x(t) \rangle \propto t^{\beta_l}, \quad (122a)$$

$$\langle x^2(t) \rangle \propto t^{2\beta_l}, \quad (122b)$$

$$\text{Var}[x(t)] \propto t^{2\beta_l}. \quad (122c)$$

For comparison, in the absence of drift (i.e., at $\gamma = 1/2$) we arrive at [as $C = 0$, defined in Eq. (106a)] at

$$\langle x(t) \rangle = 0, \quad (123a)$$

$$\langle x^2(t) \rangle \propto t^{\beta_l}, \quad (123b)$$

$$\text{Var}[x(t)] \propto t^{\beta_l}. \quad (123c)$$

From these results for A_r -CTRWs with power-law WTDs in the situations of *nonzero* drift achieved at $\gamma \neq 1/2$, the coefficient of variation (COV)—the ratio of the standard deviation to the mean, characterizing the relative “broadness” of a distributed stochastic variable—can be computed. We get from Eqs. (103a) and (103c), from Eqs. (113a) and (113c), and from Eqs. (122a) and (122c) that—similarly to biased Scher-Montroll CTRWs—the COV at long timescales is

$$\text{COV}(t) \propto t^0. \quad (124)$$

In contrast, for A_r -CTRWs with *exponential* WTDs considered in Sec. III A the COV is evidently *not a constant* in the long-time limit: both the first moment (70) and the variance (71) of that process are growing linearly with diffusion time.

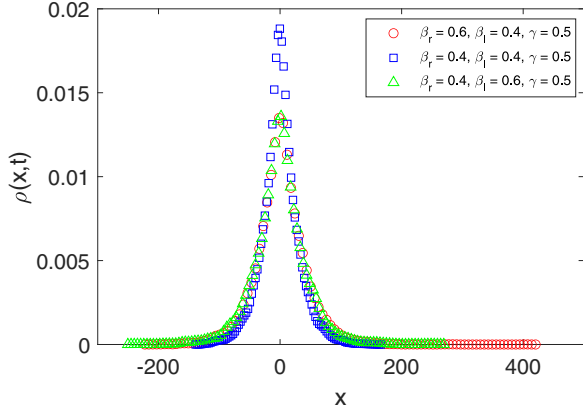


FIG. 4. The same quantities as in Fig. 2, but for A_t -CTRWs with power-law WTDs (125) for the case of *symmetric* jumps, $\gamma = 1/2$, plotted at time $t = 10^5$ and for the WTD-parameter in Eq. (125) having the value $\tau_0 = 2/1000$.

3. Comparison with simulations

To generate the power-law WTDs (75), in Monte Carlo simulations we use a standard Pareto-type long-time (for $t > \tau_0$) WTD [57]

$$\psi(t) = \beta \tau_0^\beta / t^{1+\beta}. \quad (125)$$

The results of the simulations for the PDF are presented in Figs. 4 and 5, correspondingly, for CTRWs with symmetric WTDs and with AWTDs. We find that, expectedly, for the conditions of symmetric jumps the obtained PDFs are symmetric; see Fig. 4. Note again that no rigorous analytical results are possible for the case $\gamma \neq 1/2$ (see Fig. 5 for the results of simulations).

For the moments and variance of A_t -CTRWs driven by nonequal power-law WTDs, we observe—as for the corresponding quantities in the scenario of A_t -CTRWs with exponential WTDs—a quantitative agreement of the numerical results with the full analytical results as well as with the dominant long-time power-law predictions derived from these results. The long-time tails are given in this situation by

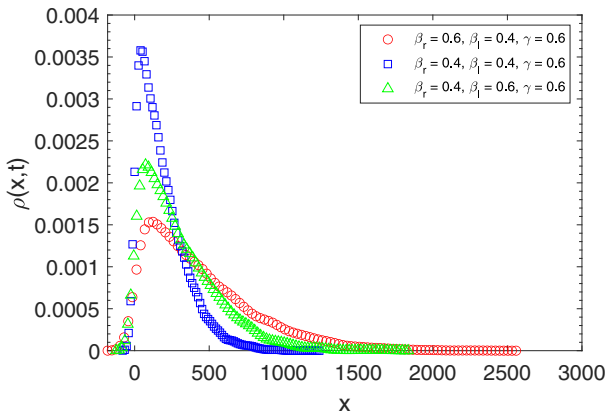


FIG. 5. The same quantities as in Fig. 4, but for A_t -CTRWs with *asymmetric* distributions of jumps $\gamma \neq 1/2$ (see the legend for the values of relevant parameters).

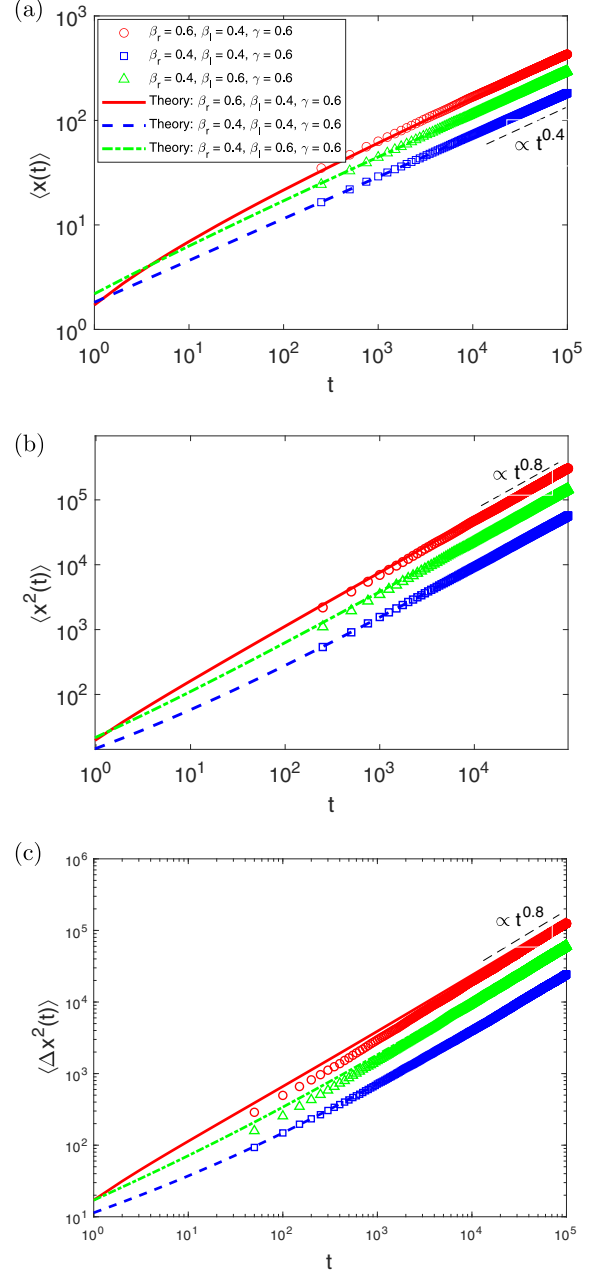


FIG. 6. The same quantities as in Fig. 3, but for A_t -CTRWs with power-law WTDs (125). The full analytical results for the first moment (100), (109), and (118) valid at all times are shown with the results of simulations in panel (a) as the solid curves. The theoretical results for the second moment (102), (112), and (121) valid at all times are presented in panel (b) as the solid curves, together with the numerical results. The results for the variance expressed via these expressions for the moments via (69) and the predictions from the simulations are shown in panel (c). The long-time asymptotes are shown as the dashed lines. The relevant parameters are the same as in Fig. 5 (see also the legend); $N = 10^4$.

Eqs. (103c), (113c), and (122c), see the results illustrated in Figs. 6(a)–6(c), correspondingly.

The behaviors of the moments of A_t -CTRWs at extremely short times (at $t \ll \tau_0$) are usually expected to be dominated by the *smallest* exponent of the power-law functions of time

which enters into the corresponding analytical expressions (the region not considered here). At intermediate times—i.e., in the region of a crossover from the short- to the long-time behavior—a contribution of several terms and powers control the final result. At long times the *largest* exponents of the respective power-law expansions of the Mittag-Leffler functions in the results for the moments dominate. The respective exponents in the power-law expansions for the cases (ii) and (iii) considered in Sec. III B 2 are the *smallest* of the two exponents of the contributing WTD β_r and β_l . Specifically, for the special case (ii) with $\beta_l > \beta_r$ the first and second moments are $\propto t^{\beta_r}$ and $\propto t^{2\beta_r}$, respectively [see Eqs. (113a) and (113b)], while for the case (iii) featuring $\beta_l < \beta_r$ the same two moments scale as $\propto t^{\beta_l}$ and $\propto t^{2\beta_l}$ [see Eqs. (122a) and (122b)].

At intermediate times, the next-order terms of the long-time expansions for the first and second moments of A_l -CTRWs—for the case (i) given by Eq. (102), for the case (ii) expressed by Eqs. (109) and (112), and for the case (iii) given by Eqs. (118) and (121)—improve the quantitative agreement of the analytical predictions with the results of numerical simulations. As a natural consequence, the actual diffusion time above which the theoretical scalings and the simulation results coincide becomes shorter. Therefore, these “improved” asymptotic long-time relations are used as the theoretical curves shown in Fig. 6.

IV. DISCUSSION AND CONCLUSIONS

In a number of previous studies of biased 1D CTRWs, different jump probabilities to the left and to the right were often considered as the main mechanism of bias, see, i.e., Refs. [56,57]. Some specific features in the WTDs were predicted when considering the properties of biased diffusion [89–91]. However, particularly in the chemical-physics literature [90] often different waiting times—in magnitude and possibly in distribution—are used to describe the variability of the transition rates. A full theory of random walks with unequal WTDs in different directions is yet not developed.

To bridge this gap, in this study we introduced the concept of A_l -CTRWs and studied the particle-spreading properties obeying such random walks. We derived the corresponding generalized Montroll-Weiss equation in Laplace- and Z-transform space, as well as obtained the generalized master equations. The left-versus-right jumping probabilities play evidently a decisive role in determining the final direction of drift. A number of the obtained results are valid for arbitrary functional forms of the WTDs. As widely used examples (both in the theory and to describe experimental data), we here used exponential and power-law WTDs. We computed the first two moments as well as the variance (with the help of the Z transform) of such A_l -CTRWs. Via computer simulations, we also revealed a quantitative theory-versus-simulations agreement of the long-time diffusion behaviors for both choices of the WTDs. The enumerated PDFs were also demonstrated to be in agreement with the theory, at least in the cases when explicit theoretical solutions were amenable.

Moreover, we unveiled the anomalous-diffusion regimes for A_l -CTRWs with exponential-WTDs at short-to-

intermediate times, in contrast to the normal-diffusion behavior of standard exponential-WTD CTRWs. For A_l -CTRWs with exponential WTDs we also found a linear long-time scaling of the variance, irrespective of particular values of the model parameters. For A_l -CTRWs with power-law WTDs, we found that the anomalous diffusion at long times depends not only on the right- and left-jump probabilities, but that the scaling is dominated by the smaller of the two exponents of the respective WTDs.

The main differences of this study from Ref. [84] are as follows. First, for an arbitrary initial distribution of the particles, we derived a generalized Montroll-Weiss equation (23) in Laplace and Z space. Second, we investigated the anomalous-diffusion properties of CTRWs with the power-law asymmetric WTDs (in addition to the exponential WTDs) and we found that anomalous-dynamics scaling at long times is dominated by the smaller of the two contributing exponents of the respective WTDs. Last, a novel Z-transform-based approach in discrete space for studying anomalous-diffusion phenomena was proposed here. Its applicability and effectiveness were demonstrated, providing a promising methodology for investigating more complex diffusion problems. For example, next-nearest-neighbor jumps can be coupled in such models: namely, instead of (9) one can use $\Lambda(\xi) = \gamma_{-1}\delta(\xi + 1) + \gamma_1\delta(\xi - 1) + \gamma_2\delta(\xi - 2)$ and $\gamma_{-1} + \gamma_1 + \gamma_2 = 1$. Note that Refs. [80,84] do not consider a spatial asymmetry of the jumps.

A number of other statistical quantifiers for such biased CTRWs with birandom WTDs can also be computed. For instance, in addition to the *ensemble-averaged* moment-based characteristics considered here, the computation of the *time-averaged* moments and variance can be conducted. Comparing the ensemble- and time-averaged results for the variance—the main particle-spreading characteristics which plays the role of the mean-squared displacement for zero-mean stochastic processes—the investigation of nonergodicity is possible (a matter of future studies). Such an investigation was, e.g., performed in Ref. [57] for biased CTRWs with unequal left-versus-right jump probabilities, but with identical power-law WTDs of the respective jumps (in contrast to our current model). For the biased CTRWs, the survival probability of the particles can be computed too, according to the total-probability formula with conditional random waiting times. The situation of independent random waiting times as an alternative diffusion model is also interesting [89] and to study how these two models differ with regard to their statistical properties can be worth exploring in the future.

Last, the proposed scheme for the biased CTRWs can provide a way to understand some anomalous-diffusion behaviors emerging due to different WTDs for left- and right-directed jumps. It can also help to model bias-featuring phenomena of biological migration (for example, the cellular transport [92]), of muscle contraction, of directed biochemical reactions, of human mobility and of animal motion [93,94], to mention a few possible applications.

As such a versatile model, biased and A_l -CTRWs should thus be included in classification schemes of various anomalous stochastic processes potentially underlying the dynamics of particles in experimentally recorded single-particle-tracking data sets, such as the schemes, e.g., in the

(Bayesian) deep learning [95–99] or the Bayesian maximum-likelihood [100,101] approaches.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: MATHEMATICAL INSTRUMENTARIUM: TRANSFORMATIONS AND SPECIAL FUNCTIONS

1. Fourier and Laplace transforms

The Fourier transform of $\rho_0(x)$ is given by

$$\rho_0(k) = \int_{-\infty}^{\infty} e^{ikx} \rho_0(x) dx, \quad (\text{A1})$$

while the inverse Fourier transform is defined as

$$\rho_0(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \rho_0(k) dk. \quad (\text{A2})$$

The Laplace transform of a function $\psi(t)$ is

$$\psi(u) \equiv \mathcal{L}(\psi(t)) = \int_0^{\infty} e^{-ut} \psi(t) dt, \quad (\text{A3})$$

whereas the inverse Laplace transform is defined via the following contour integral in the complex plane

$$\psi(t) = \mathcal{L}^{-1}(\psi(u)) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{\gamma-iT}^{\gamma+iT} e^{ut} \psi(u) du. \quad (\text{A4})$$

The Fourier-Laplace transformed function of $\rho(x, t)$ is then

$$\rho(k, u) = \int_0^{\infty} dt \int_{-\infty}^{\infty} e^{-ut+ikx} \rho(x, t) dx. \quad (\text{A5})$$

2. Z transform

Let us consider the PDF $f(m)$ of a random variable x taking a set of *discrete* values m on a lattice. Its Z transform is given by [102]

$$\mathcal{F}(z) = \sum_{m=-\infty}^{+\infty} z^{-m} f(m). \quad (\text{A6})$$

Differentiating Eq. (A6) over z once and twice, we get, respectively,

$$\mathcal{F}'_z(z) = \sum_{m=-\infty}^{+\infty} (-m) z^{-m-1} f(m), \quad (\text{A7a})$$

$$\mathcal{F}''_{zz}(z) = \sum_{m=-\infty}^{+\infty} (-m)(-m-1) z^{-m-2} f(m), \quad (\text{A7b})$$

where via setting $z = 1$ we arrive at

$$\mathcal{F}'_z(z)|_{z=1} = - \sum_{m=-\infty}^{+\infty} m f(m), \quad (\text{A8a})$$

$$\mathcal{F}''_{zz}(z)|_{z=1} = \sum_{m=-\infty}^{+\infty} m^2 f(m) + \sum_{m=-\infty}^{+\infty} m f(m). \quad (\text{A8b})$$

Therefore, with the standard definitions of the moments, one gets $\mathcal{F}'_z(z)|_{z=1} = -\langle x \rangle$ and $\mathcal{F}''_{zz}(z)|_{z=1} = \langle x^2 \rangle + \langle x \rangle$, so that for the first two moments of x we have

$$\langle x \rangle = -\mathcal{F}'_z(z)|_{z=1}, \quad (\text{A9a})$$

$$\langle x^2 \rangle = \mathcal{F}''_{zz}(z)|_{z=1} + \mathcal{F}'_z(z)|_{z=1}. \quad (\text{A9b})$$

The higher moments—used, e.g., to compute the statistical characteristics such as the skewness and kurtosis—can be evaluated similarly. Note here that for drift-free subdiffusive CTRWs the skewness and kurtosis were calculated in Ref. [103].

3. Mittag-Leffler functions

We use in the main text the generalized two- and three-parametric Mittag-Leffler functions [88,104–107] defined, respectively, as (see Chapter XVIII in Ref. [105])

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (\text{A10a})$$

$$E_{\alpha,\beta,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n z^n}{\Gamma(\alpha n + \beta) n!}. \quad (\text{A10b})$$

Here $(\delta)_n = \Gamma(\delta + n)/\Gamma(\delta)$ is the Pochhammer symbol. For clarity, in contrast to the standard mathematical notation, we put the index δ as a *subscript* of $E_{\alpha,\beta,\delta}(z)$, for this index not to be misinterpreted as a δ power of $E_{\alpha,\beta}(z)$. Note that $E_{\alpha,\beta,\delta}$ at $\delta = 1$ reduces to $E_{\alpha,\beta}$ and $E_{\alpha,\beta}$ at $\beta = 1$ simplifies to the single-parameter Mittag-Leffler function (a “natural” generalization of the exponential function).

The Laplace transforms of the functions (A10a) and (A10b) for their parameters in the range $\alpha > 0$ and $\beta > 0$ are [108] (see Chapter 2.2.7 in Ref. [109])

$$\mathcal{L}[t^{\beta-1} E_{\alpha,\beta}(-at^\alpha)] = \frac{u^{\alpha-\beta}}{u^\alpha + a}, \quad (\text{A11a})$$

$$\mathcal{L}[t^{\beta-1} E_{\alpha,\beta,\delta}(-at^\alpha)] = \frac{u^{\alpha\delta-\beta}}{(u^\alpha + a)^\delta}, \quad (\text{A11b})$$

respectively. The inverse Laplace transform of a power-law function (to be used later) for $\beta > 0$ is

$$\mathcal{L}^{-1}(u^{-1-\beta}) = \frac{t^\beta}{\Gamma(1+\beta)}. \quad (\text{A12})$$

The short-time behavior of (A10b) is of a stretched exponential form [1,108,110], with the leading term

$$E_{\alpha,\beta,\delta}(-t^\alpha) \stackrel{t \rightarrow 0}{\approx} \frac{1}{\Gamma(\beta)} \exp\left(-\frac{\delta \Gamma(\beta) t^\alpha}{\Gamma(\alpha + \beta)}\right), \quad (\text{A13})$$

while the long-time asymptotics of the same function has a power-law form, with the first two terms being

$$E_{\alpha,\beta,\delta}(-t^\alpha) \stackrel{t \rightarrow \infty}{\approx} \frac{t^{-\alpha\delta}}{\Gamma(\beta - \alpha\delta)} - \frac{\delta t^{-\alpha(\delta+1)}}{\Gamma(\beta - \alpha(\delta+1))}. \quad (\text{A14})$$

Here $\Gamma(x)$ is the standard Euler Gamma-function defined for $\alpha > 0$ via $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.

APPENDIX B: ABBREVIATIONS

one-dimensional: 1D;
 continuous-time random walk: CTRW;
 heterogeneous diffusion process: HDP;
 waiting-time distribution: WTD;
 asymmetric WTD: AWTD;
 AWTD-CTRWs: A_t -CTRWs;
 independent identically distributed: i.i.d.;
 probability-density function: PDF;
 right-hand side: RHS.

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