

Continuous-time random walks with asymmetric waiting-time and jump-length distributions as a model of biased anomalous transport

Guohua Li ^{*}

*School of Mathematical Sciences, [Chengdu University of Technology](#), Chengdu 610059, China
and [University of Potsdam](#), Institute of Physics and Astronomy, Potsdam-Golm 14476, Germany*

Andrey G. Cherstvy [†]

[University of Potsdam](#), Institute of Physics and Astronomy, Potsdam-Golm 14476, Germany

Hong Zhang [‡] and Zeyu Tu

School of Mathematical Sciences, [Chengdu University of Technology](#), Chengdu 610059, China

Ralf Metzler [§]

*[University of Potsdam](#), Institute of Physics and Astronomy, Potsdam-Golm 14476, Germany
and [Asia Pacific Center for Theoretical Physics](#), Pohang 37673, Republic of Korea*



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Continuous-time random walks (CTRWs) are highly appreciated as a model of anomalous diffusion (AD) in complex media such as structured and heterogeneous environments. These walks are used to describe AD phenomena characterized by a nonlinear functional dependence of the mean-squared displacement (MSD) of particles with time. The introduction of asymmetric waiting-time distributions (WTDs) depending on the diffusion directions leads to biased CTRWs and to AD phenomena. Here, we examine the general formulation of such CTRWs with asymmetric WTDs in continuous space. From the balance equations, we derive the corresponding generalized Montroll-Weiss equation in Fourier-Laplace space as well as the generalized master equations with asymmetric waiting times. For random jump lengths obeying an asymmetric modified Gaussian distribution (AMGD), we analytically obtain the probability-density function of the particles as well as the first two moments in Laplace space. We unveil that CTRWs with exponential WTDs and jump lengths following the AMGD exhibit a ballistic diffusion, whereas in a symmetric case for the jump lengths both the MSD and variance at long time have a linear scaling with time. The AD properties of the MSD and of the variance at long times depend on a *smaller* exponent of the power-law WTDs, while the AMGD-distributed jump lengths *double* the AD scaling exponents for such CTRWs. In this continuous-space model, the possibility of length-distributed jumps—in contrast to the case of only neighboring-site jumps allowed in the discrete version of a similar model we proposed recently [G. Li *et al.*, [Phys. Rev. E](#) **113**, 064116 (2026)]—leads to different resulting scaling properties of the MSD. The analytical results are in close agreement with the results of computer simulations, both for the case of asymmetric exponential and of power-law WTDs. The current theoretical results are extendable to more elaborate CTRW models. The latter are applicable for the description of more complicated AD-related phenomena taking place in the presence of external fields.

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I. INTRODUCTION

Anomalous diffusion (AD) of different nonlinear functional forms [1–10] has attracted a great deal of attention

among researchers across many disciplines, including mathematics [11,12], physics [13], biology [14], material science [15–17], and artificial intelligence [18]. One widely used mathematical model applied to the description of a number of physical AD phenomena is the model of continuous-time random walks (CTRWs) [19–28], which in the continuous limit can lead to the fractional diffusion equations with memory effects [3,26,29]. Certain ergodic [7] characteristics of CTRWs are reminiscent of those computed for the processes of heterogeneous diffusion [30–32].

To describe AD phenomena in the presence of external fields, a biased-CTRW model with velocity and force fields is often employed [23,33–39]. One more possibility is asymmetric Levy flights, as recently considered in Ref. [40].

^{*}Contact author: liguohua13@cdut.edu.cn

[†]Contact author: a.cherstvy@gmail.com

[‡]Contact author: zhanghong13@cdut.edu.cn

[§]Contact author: rmetzler@uni-potsdam.de

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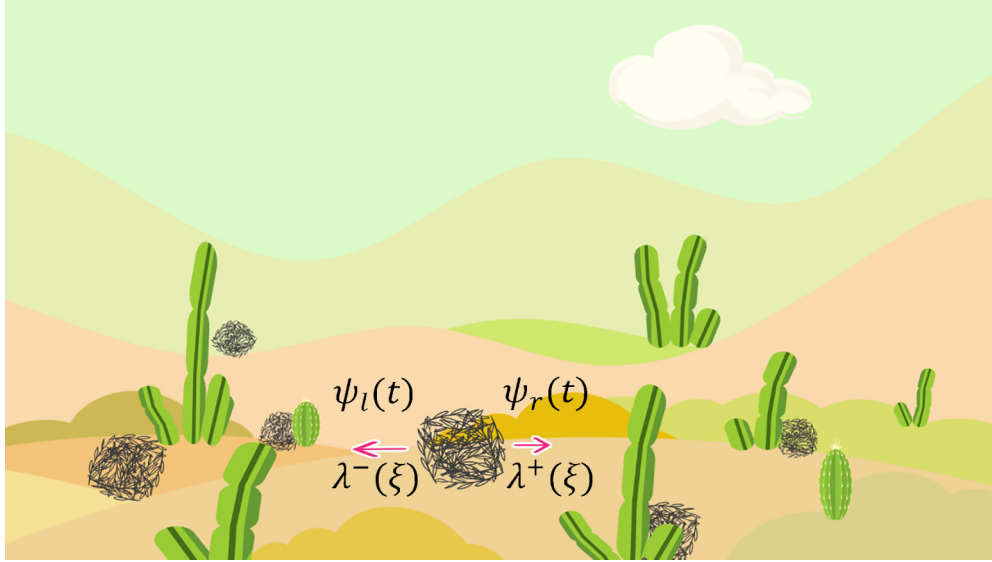


FIG. 1. A tumbleweed employing the chamaechoric strategy and drifting in the wind, continuously changing its direction on “complex” terrain containing many obstacles, is a real-world example of a mathematical model of biased generally AD-containing transport.

The CTRW model with asymmetric jump-length distributions (JLDs) can be utilized, e.g., to solve the problem of AD-driven transport under the action of a constant force [41]. As an example, the CTRW model [42,43] was used to describe both diffusion and dissociation of particles.

Multiples issues of AD in *biased* environments were addressed in a number of studies via applying the concepts of CTRWs and the master-equation approach [44]. We also point out here Refs. [45–47] in which constant-velocity and space-dependent velocity fields were used to describe a bias and the corresponding fractional convection-diffusion equations were derived. In Ref. [41], the asymmetric modified Gaussian distributions (AMGDs) for random jump lengths in the CTRW model were proposed to capture the bias. In these models of biased CTRWs, the waiting-time distributions (WTDs) were assumed to be *the same* in all directions.

Here, we study biased CTRWs [23,33,34,48] with *different* WTDs in *different* diffusion directions, thus extending the model of CTRWs to both asymmetric WTDs (AWTDs) and asymmetric JLDs (AJLDs) in *continuous* space. For such AJLDs, (i) there are nonequal probabilities for a particle to jump right or left; (ii) even with equal total probabilities to jump right and left, they can still generate bias; and (iii) a bias also naturally arises via AWTDs in different directions.

Therefore, as we considered recently, the main difference between CTRWs with AWTDs (abbreviated as A_t -CTRWs) and AJLDs ($A_{t,l}$ -CTRWs) in *continuous* space and A_t -CTRWs in *discrete* space [48] is a special bias arising due to different JLDs in the opposite directions; see point (ii) above. Similar walks were studied for a constant-force problem for AD transport in Ref. [41]. The abbreviation A_t -CTRW is consistent with that previously used in Ref. [48], while the acronym $A_{t,l}$ -CTRW is its logical extension. The biases stemming from points (i) and (iii) above are also important, as examined below. Compared to the *discrete* lattice space—where the random-jump lengths follow a two-point

probability-density function (PDF) for a particle jumping solely to its neighboring points—random jump lengths in CTRWs constructed in *continuous* space can follow *any* continuous distribution. This allows the particles to make larger jumps and thereby better reflect the reality governing the properties of AD in a given complex medium.

This AD model can be used to describe the stochastic motion of particles in an energy-potential field with high-frequency fluctuations, where the WTDs in each direction depend on the field properties. The motions of electrons and holes in variable external electric and magnetic fields is one suitable physical system. From a macroscopic perspective, a tumbleweed “turning around” and drifting in the wind, which erratically changes direction on rough terrain, can also be a relevant phenomenon to apply such $A_{t,l}$ -CTRWs; see Fig. 1. As jump lengths in such walks follow *continuous* JLDs, the particles are capable of making large jumps. This fact makes such a biased-CTRW model in a certain sense superior and more flexible as compared to our similarly formulated biased-CTRW model in *discrete* space [48], where only jumps to the neighboring sites on a periodic lattice were allowed.

Note that asymmetric JLDs play a major role in the emergence of biased diffusion, whereas an inherent bias caused by asymmetric WTDs (with a symmetric JLD) is the result of ensemble averaging over all the particles. However, the presence of *both* asymmetric JLDs and asymmetric WTDs exerts a non-negligible “regulatory” effect on the amplitude of bias [48]. Asymmetric WTDs result in particle diffusion in certain directions taking a longer amount of time. This directional preference can have significant applications in practice. For example, when herds of animals migrate to the heart of a grassland area in their search for food, distinct directional features of such “diffusion” emerge in the duration of feeding pauses [49]. Lastly, asymmetric WTDs can give rise to AD, with nontrivial statistical characteristics of diffusion at intermediate-to-long times. Therefore,

investigating the influence of asymmetric WTDs on the transport characteristics in complex environments is a valuable research topic.

The rest of the paper is organized as follows. In Sec. II we introduce a model of $A_{t,l}$ -CTRWs in *continuous* space and derive the corresponding generalized Montroll-Weiss equation in Fourier-Laplace space; see Appendix A. We establish the generalized master equations for a system with AWTDs in Appendix B and consider an example of AMGDs for the JLD in Sec. II B. In Secs. II C and II D we show that the AMGD-distributed jump length leads to ballistic diffusion; however, in the symmetric case the MSD and the variance at long time both have linear time scaling for CTRWs with exponential WTDs and AD at long times depends on the smaller exponent of the power-law WTDs. In Sec. III we present a short discussion and outline the main conclusions of the current study.

Here, to clarify the structure of the paper, we provide the reader with a detailed list of the Appendixes. For the detailed derivations in terms of the Fourier transforms and master equations, we refer the reader to Appendixes A and B, respectively. Furthermore, the derivations of the PDFs in Fourier-Laplace and in real space used in Sec. II B are provided in Appendix C, the first two real-space moments for the exponential WTDs used in Sec. II C are derived in Appendix D, the properties of the Mittag-Leffler functions used in Sec. II D are listed in Appendix E, the first and second moments in real space for the power-law WTDs examined in Sec. II C are obtained in Appendix F, whereas the exact solutions for the memory kernels mentioned in Sec. II D are presented in Appendix G.

II. MODEL AND MAIN RESULTS

A. Model formulation

We present here the main equations of the model of $A_{t,l}$ -CTRWs. We denote the density of particles located at a position x at a time t by $\rho(x, t)$, the density of particles having reached position x exactly at time t by $I(x, t)$, and the joint PDF of waiting-time intervals γ_n and of jump lengths by $\phi(\xi, t)$. We suppose that $\phi(\xi, t)$ can be written as (as detailed in our recent study [48])

$$\phi(\xi, t) = \psi(t|\xi) \cdot \lambda(\xi), \quad (1)$$

with the conditional PDF $\psi(t|\xi)$ here (ξ is measured in unit lengths below) being

$$\psi(t|\xi) = \psi_r(t)\Theta(+\xi) + \psi_l(t)\Theta(-\xi), \quad (2)$$

while random jumps follow the relation

$$\lambda(\xi) = \lambda(\xi)\Theta(+\xi) + \lambda(\xi)\Theta(-\xi) \equiv \lambda^+(\xi) + \lambda^-(\xi), \quad (3)$$

where $\Theta(\xi)$ is the standard Heaviside unit-step function. Hereafter, the subscripts l (r) denote the quantities responsible for the jumps to the left (right). If the PDF of random jump-lengths satisfies a two-point distribution, i.e.

$$\lambda(\xi) = \gamma\delta(\xi - 1) + (1 - \gamma)\delta(\xi + 1), \quad (4)$$

with the probability γ to move to the right and $(1 - \gamma)$ to move to the left site, the current CTRW model is reducible to

our recent discrete-space model [48]. Here, $\psi_r(t)$ and $\psi_l(t)$ are the PDFs of random waiting intervals for a particle going right and left, respectively.

B. AMGDs for JLDs

We assume that random jump lengths follow the so-called AMGD with the PDF [41],

$$\begin{aligned} \lambda(\xi) &= \lambda^+(\xi) + \lambda^-(\xi) \\ &= \sqrt{a/\pi} \exp\{-a\xi^2\}\Theta(\xi) + \sqrt{b/\pi} \exp\{-b\xi^2\}\Theta(-\xi). \end{aligned} \quad (5)$$

Its Fourier transform (up to the k^2 -order) is [41]

$$\begin{aligned} \lambda(k) &= [\lambda_{\cos}^+(k) + \lambda_{\cos}^-(k)] - i[\lambda_{\sin}^+(k) - \lambda_{\sin}^-(k)] \\ &\approx [1 - k^2\sigma_+/8] + ik\sigma_-(2\sqrt{\pi}), \end{aligned} \quad (6)$$

where we define the parameters

$$\sigma_- = 1/\sqrt{b} - 1/\sqrt{a}, \quad \sigma_+ = 1/a + 1/b. \quad (7)$$

The PDF (5) is clearly normalized, but it is not continuous at $\xi = 0$. Also note that

$$\int_0^\infty \lambda(\xi)d\xi = \int_{-\infty}^0 \lambda(\xi)d\xi = \frac{1}{2}, \quad (8)$$

meaning *equal* total probabilities for a particle to jump to the right and to the left. As we show below, however, $A_{t,l}$ -CTRWs can still have a *bias* in this case.

According to the Fourier-transform definition (A7), the first two moments of the PDF $\rho(x, u)$ given by Eqs. (C11) and (C12) can be calculated directly from $\rho(k, u)$ given by Eq. (C4) via the relations [41]

$$\langle x(u) \rangle = i \frac{\partial \rho(k, u)}{\partial k} \Big|_{k=0} = - \frac{\frac{1}{\sqrt{b}}\psi_l(u) - \frac{1}{\sqrt{a}}\psi_r(u)}{\sqrt{\pi}u[2 - \psi_r(u) - \psi_l(u)]}, \quad (9a)$$

$$\begin{aligned} \langle x^2(u) \rangle &= i^2 \frac{\partial^2 \rho(k, u)}{\partial k^2} \Big|_{k=0} = \frac{2}{u} \left[\frac{\frac{1}{a}\psi_r(u) + \frac{1}{b}\psi_l(u)}{4[2 - \psi_r(u) - \psi_l(u)]} \right. \\ &\quad \left. + \frac{(\frac{1}{\sqrt{a}}\psi_r(u) - \frac{1}{\sqrt{b}}\psi_l(u))^2}{\pi[2 - \psi_r(u) - \psi_l(u)]^2} \right]. \end{aligned} \quad (9b)$$

As the first moment is generally nonzero, the variance is the effective statistical measure of dispersion of the particles. It is defined as

$$\text{Var}[x(t)] = \langle [x(t) - \langle x(t) \rangle]^2 \rangle = \langle x^2(t) \rangle - \langle x(t) \rangle^2, \quad (10)$$

while the bare MSD (used for zero-mean stochastic processes) is

$$\text{MSD}[x(t)] = \langle [x(t) - x(0)]^2 \rangle. \quad (11)$$

With the definition by Eq. (11) we get $\text{MSD}[x(t)] = \langle x^2(t) \rangle$ for the initial condition (C3) of the particles starting at the origin.

C. Exponential WTDs

We now apply the general diffusion equation (C2) of $A_{r,l}$ -CTRWs to a particular case of exponential WTDs [50] with

$$\psi_{l,r}(t) = \lambda_{l,r} e^{-\lambda_{l,r} t}. \quad (12)$$

Here, $\lambda_{r,l}$ are the respective frequencies of jump events to the right and to the left. We use here in continuous space the same type of AWTs as the ones for recently considered A_r -CTRWs on discrete lattices [48]; in the current analysis, we additionally equip such AWTs with AJLDs to study more complicated—but at the same time tunable—random walks of the type of $A_{r,l}$ -CTRWs.

Note that some WTDs of form (12) might cease to produce time reversibility [51] and, therefore, might not provide an adequate description of certain natural phenomena or processes featuring a time-reversal symmetry. Specifically, for CTRW-based models, the properties of semi-Markovian processes come into play and the detailed-balance condition is not sufficient to guarantee microscopic reversibility. Here, an additional condition is required, namely, that the transition direction and the waiting time for each state are statistically independent [51,52]. In our current model of biased CTRWs, however, the waiting times depend on the direction of diffusion, so that the microscopic reversibility no longer holds.

Some differences of our approach compared to some other models of biased CTRWs are as follows. The model of CTRWs with nonindependent and correlated waiting times considered, e.g., in Ref. [53] had no dependence in the JLD or diffusion direction. In a coupled-CTRWs model proposed in Ref. [54], the asymmetric direction-dependent JLD was implemented via a Gaussian-like function with the variance being a function of waiting times, but the latter were *independent* of the diffusion direction. In our model, the asymmetry of WTDs and of JLDs is independent and can be tuned individually.

We refer the reader to the detailed Appendix D for the derivation of the first two moments in Laplace and real space for the exponential WTDs, while here only the special cases and limiting behaviors of the obtained results will be considered. Note that in the special case of *symmetric* WTDs and *asymmetric* JLDs, i.e., for

$$\lambda_r = \lambda_l \equiv \lambda, \quad (13)$$

the first two moments given by (D7a) and (D7b) reduce to (as expressed here for mathematical clarity and for overall model-applicability purposes via the parameters of the *original* WTDs and JLDs)

$$\langle x(t) \rangle = -\frac{\lambda(1/\sqrt{b} - 1/\sqrt{a})t}{2\sqrt{\pi}}, \quad (14a)$$

$$\langle x^2(t) \rangle = \frac{[(1/\sqrt{b} - 1/\sqrt{a})\lambda t]^2 + \pi(1/a + 1/b)\lambda t}{4\pi}. \quad (14b)$$

In the case of *asymmetric* WTDs and *symmetric* JLDs, i.e., for $b = a$, the first two moments (D7a) and (D7b) become

$$\langle x(t) \rangle = -\frac{(\lambda_l - \lambda_r)(1 - e^{-(\lambda_r + \lambda_l)t/2})}{\sqrt{a\pi}(\lambda_r + \lambda_l)}, \quad (15a)$$

$$\begin{aligned} \langle x^2(t) \rangle = & \frac{\lambda_r \lambda_l t}{a(\lambda_r + \lambda_l)} + \frac{(4 + \pi)(\lambda_r - \lambda_l)^2 (1 - e^{-(\lambda_r + \lambda_l)t/2})}{2a\pi(\lambda_r + \lambda_l)^2} \\ & + \frac{((4 + \frac{\pi}{2})(\lambda_r^2 + \lambda_l^2) + (7\pi - 8)\lambda_r \lambda_l) t e^{-(\lambda_r + \lambda_l)t/2}}{2a\pi(\lambda_r + \lambda_l)}. \end{aligned} \quad (15b)$$

At *short* times, for $(\lambda_r + \lambda_l)t/2 \ll 1$, the moments (D7a) and (D7b) reveal a linear scaling with time, i.e.,

$$\langle x(t) \rangle \approx \frac{(\lambda_r/\sqrt{a} - \lambda_l/\sqrt{b})t}{2\sqrt{\pi}}, \quad (16a)$$

$$\langle x^2(t) \rangle \approx \frac{(\lambda_r/a + \lambda_l/b)t}{4}. \quad (16b)$$

The same solutions at *long* times for the case of $a \neq b$ give a linear first-moment, a quadratic second-moment, and a linear variance scaling relation as functions of time t , i.e.,

$$\langle x(t) \rangle \propto t, \quad (17a)$$

$$\langle x^2(t) \rangle \propto t^2, \quad (17b)$$

$$\text{Var}[x(t)] \propto t. \quad (17c)$$

Note that for the case $a = b$ from relations (15a) and (15b), we find different dominant dependencies in the limit of *long* times, namely

$$\langle x(t) \rangle \approx \frac{\lambda_r - \lambda_l}{\sqrt{a\pi}(\lambda_r + \lambda_l)}, \quad (18a)$$

$$\langle x^2(t) \rangle \propto t, \quad (18b)$$

$$\text{Var}[x(t)] \propto t. \quad (18c)$$

Thus, a drift at long times stems from the difference of the WTD-parameters $\lambda_{r,l}$ for diffusion in the opposite directions. Moreover, we find that for the AJLDs described by the AMGDs, a *ballistic* diffusion for the MSD and a *linear* diffusion for the variance are observed [see Eqs. (17b) and (17c), respectively], while in the case of *symmetric* JLDs at $a = b$ both the MSD and the variance features a *linear* scaling at long times [see Eqs. (18b) and (18c), respectively].

To get the PDF of the particles, by using the WTDs (D1) for the situation

$$\lambda_r = \lambda_l \equiv \lambda, \quad (19)$$

from expression (C8) we get the two roots as

$$d_{\pm}(u) = \frac{-2\lambda\sigma_{\pm} \pm 2\sqrt{(\lambda\sigma_{\pm})^2 + 2\pi\lambda\sigma_{\pm}u}}{\lambda\sigma_{\pm}\sqrt{\pi}}. \quad (20)$$

After substituting (D1) and (20) into Eqs. (C11) and (C12), we obtain

$$\rho(x, u) = \frac{\exp\left[-\frac{2\sigma_{-}x}{\sqrt{\pi}\sigma_{+}}\right] \exp\left[-\frac{2\sqrt{2}|x|}{\sqrt{\lambda}\sigma_{+}}\sqrt{u + \frac{\lambda\sigma_{-}^2}{2\pi\sigma_{+}}}\right]}{\sqrt{\frac{\lambda\sigma_{+}}{2}} \sqrt{u + \frac{\lambda\sigma_{-}^2}{2\pi\sigma_{+}}}}. \quad (21)$$

Taking its inverse Laplace transform, the real-space PDF of the particles becomes

$$\rho(x, t) = \frac{\sqrt{2}}{\sqrt{\pi\lambda(1/a + 1/b)t}} \exp \left[-\frac{1}{(1/a + 1/b)} \left(\frac{2x^2}{\lambda t} + \frac{2(1/\sqrt{b} - 1/\sqrt{a})x}{\sqrt{\pi}} + \frac{\lambda(1/\sqrt{b} - 1/\sqrt{a})^2 t}{2\pi} \right) \right]. \quad (22)$$

If, in addition to the condition (19), we set in expression (22) also $a = b$, the PDF naturally becomes purely Gaussian, namely

$$\rho(x, t) = e^{-x^2/(\lambda t/a)} / \sqrt{\pi(\lambda t/a)}. \quad (23)$$

D. Power-law WTDs

Below, we study another useful choice of the so-called power-law WTDs and compute the moments of respective $A_{r,l}$ -CTRWs. For *symmetric* power-law WTDs, the A_l -CTRWs reduce to a constant-force problem for the AD transport [41], which can be treated based on the fractional diffusion equation in a field [41,55] and solved via a separation of variables [3,56]. For *asymmetric* power-law WTDs in both directions, with $0 < \alpha_{l,r} < 1$, we use the representation

$$\psi_{r,l}(t) = \gamma_{r,l}^{-1} \tau_0^{-1} \left(\frac{t}{\tau_0} \right)^{\alpha_{r,l}-1} E_{\alpha_{r,l}, \alpha_{r,l}} \left(-\frac{(t/\tau_0)^{\alpha_{r,l}}}{\gamma_{r,l}} \right). \quad (24)$$

The main reason for this “approximation” [57] is that at long times (i.e., at $u \rightarrow 0$) the standard Pareto-type WTD is

$$\langle x(t) \rangle = -\frac{\sigma_- (\frac{t}{\tau_0})^\alpha}{2\gamma_r \sqrt{\pi}} \left\{ \gamma_r^{-1} \left(\frac{t}{\tau_0} \right)^\alpha E_{\alpha, 2\alpha+1, 1} \left[-\gamma_r^{-1} \left(\frac{t}{\tau_0} \right)^\alpha \right] + E_{\alpha, \alpha+1, 1} \left[-\gamma_r^{-1} \left(\frac{t}{\tau_0} \right)^\alpha \right] \right\} \approx -\frac{\sigma_- (\frac{t}{\tau_0})^\alpha}{2\sqrt{\pi} \gamma_r \Gamma(\alpha+1)}. \quad (27)$$

For this condition, expression (F7) reduces to

$$\langle x^2(u) \rangle = \frac{\sigma_+}{4\gamma_r u} \frac{\gamma_r^{-1} (\tau_0 u)^{-\alpha} + 1}{(\tau_0 u)^\alpha + \gamma_r^{-1}} + \frac{\sigma_-^2}{2\pi \gamma_r^2 u} \frac{\gamma_r^{-2} (\tau_0 u)^{-2\alpha} + 2\gamma_r^{-1} (\tau_0 u)^{-\alpha} + 1}{[(\tau_0 u)^\alpha + \gamma_r^{-1}]^2}, \quad (28)$$

which, after Laplace inversion with the help of (E2a) and (E2b), gives

$$\begin{aligned} \langle x^2(t) \rangle &= \frac{\sigma_+}{4\gamma_r^2} \left(\frac{t}{\tau_0} \right)^{2\alpha} E_{\alpha, 2\alpha+1, 1} \left(-\gamma_r^{-1} \left(\frac{t}{\tau_0} \right)^\alpha \right) + \frac{\sigma_+}{4\gamma_r} \left(\frac{t}{\tau_0} \right)^\alpha E_{\alpha, \alpha+1, 1} \left(-\gamma_r^{-1} \left(\frac{t}{\tau_0} \right)^\alpha \right) \\ &+ \frac{\sigma_-^2}{2\pi \gamma_r^4} \left(\frac{t}{\tau_0} \right)^{4\alpha} E_{\alpha, 4\alpha+1, 2} \left(-\gamma_r^{-1} \left(\frac{t}{\tau_0} \right)^\alpha \right) + \frac{\sigma_-^2}{2\pi \gamma_r^2} \left(\frac{t}{\tau_0} \right)^{2\alpha} E_{\alpha, 2\alpha+1, 2} \left(-\gamma_r^{-1} \left(\frac{t}{\tau_0} \right)^\alpha \right) \\ &+ \frac{\sigma_-^2}{\pi \gamma_r^3} \left(\frac{t}{\tau_0} \right)^{3\alpha} E_{\alpha, 3\alpha+1, 2} \left(-\gamma_r^{-1} \left(\frac{t}{\tau_0} \right)^\alpha \right) \approx \frac{\sigma_+}{4\gamma_r \Gamma(\alpha+1)} \left(\frac{t}{\tau_0} \right)^\alpha + \frac{\sigma_-^2}{2\pi \gamma_r^2 \Gamma(2\alpha+1)} \left(\frac{t}{\tau_0} \right)^{2\alpha} - \frac{3\sigma_-^2}{2\pi}. \end{aligned} \quad (29)$$

In the *long-time* limit, for this situation from analytical expressions (27) and (29) at $a \neq b$ in Eq. (5) we get the leading-order scaling relations for the first two moments and variance as

$$\langle x(t) \rangle \propto t^\alpha, \quad (30a)$$

$$\langle x^2(t) \rangle \propto t^{2\alpha}, \quad (30b)$$

$$\text{Var}[x(t)] \propto t^{2\alpha}. \quad (30c)$$

Note that for the situation $a = b$ —when the AJLD (5) becomes symmetric and Gaussian—we find

recovered

$$\psi(t) = \alpha \tau_0^\alpha / t^{1+\alpha}, \quad (25)$$

with the exponents $0 < \alpha < 1$ defined for $t > \tau_0$ and the Mittag-Leffler functions feature the same expansions in Laplace space [33]; see Appendix E for the properties of these special functions. Note that the exponent regime $0 < \alpha < 1$ also applies to our recent discrete-space model in Ref. [48], whereas the power-law WTDs of Eq. (25) with $\alpha > 1$ as in Ref. [33] remain a topic of future research. The Laplace-space first and second moments are derived for the power-law WTDs in Appendix F, while here we only examine their special cases after back-transformation to real-time domain.

1. Case A: $\alpha_r = \alpha_l \equiv \alpha$

For the *symmetric* case, for $\alpha_r = \alpha_l \equiv \alpha$, expression (F6) reduces to

$$\langle x(u) \rangle = -\frac{\sigma_-}{2\sqrt{\pi} \gamma_r u} \frac{\gamma_r^{-1} (\tau_0 u)^{-\alpha} + 1}{(\tau_0 u)^\alpha + \gamma_r^{-1}} \quad (26)$$

and the inverse Laplace transform of the first moment with the help of relations (E2a) and (E2b) yields

instead

$$\langle x(t) \rangle = 0, \quad (31a)$$

$$\langle x^2(t) \rangle \propto t^\alpha, \quad (31b)$$

$$\text{Var}[x(t)] \propto t^\alpha. \quad (31c)$$

2. Case B: $\alpha_r < \alpha_l$

For $\alpha_r < \alpha_l$, expression (F6) at $u \rightarrow 0$ has the following asymptotic expansion:

$$\langle x(u) \rangle \approx -\frac{\frac{\sigma_-}{\sqrt{\pi} \gamma_l} (\tau_0 u)^{-\alpha_r} + \frac{\gamma_{rl}}{\sqrt{b\pi}} - \frac{(\tau_0 u)^{\alpha_l - \alpha_r}}{\sqrt{a\pi}}}{u[(\tau_0 u)^{\alpha_l - \alpha_r} + \gamma_{rl}]}. \quad (32)$$

This can be Laplace-inverted to give

$$\begin{aligned} \langle x(t) \rangle &\approx -\frac{\sigma_-\left(\frac{t}{\tau_0}\right)^{\alpha_l} E_{\alpha_l-\alpha_r, \alpha_l+1, 1}\left(-\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r}\right)}{\sqrt{\pi} \gamma_l} - \frac{\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r} E_{\alpha_l-\alpha_r, \alpha_l-\alpha_r+1, 1}\left(-\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r}\right)}{\sqrt{b\pi}} \\ &+ \frac{E_{\alpha_l-\alpha_r, 1, 1}\left(-\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r}\right)}{\sqrt{a\pi}} \approx \frac{1}{\sqrt{\pi}} \left[-\frac{\sigma_-\left(\frac{t}{\tau_0}\right)^{\alpha_r}}{\gamma_r \Gamma(\alpha_r+1)} + \frac{\sigma_-\left(\frac{t}{\tau_0}\right)^{2\alpha_r-\alpha_l}}{\gamma_r \gamma_{rl} \Gamma(2\alpha_r-\alpha_l+1)} - \frac{1}{\sqrt{b}} \right]. \end{aligned} \quad (33)$$

For this case, the *long-time* asymptotic expansion of Eq. (F7) gives

$$\langle x^2(u) \rangle \approx \frac{\frac{\sigma_+}{2\gamma_l}(\tau_0 u)^{-\alpha_r} + \frac{1}{2b}\gamma_{rl} + \frac{1}{2a}(\tau_0 u)^{\alpha_l-\alpha_r}}{u(\tau_0 u)^{\alpha_l-\alpha_r} + \gamma_{rl}u} + \frac{2}{u} \left(\frac{\frac{\sigma_-}{\sqrt{\pi}\gamma_l}(\tau_0 u)^{-\alpha_r} + \frac{1}{\sqrt{b\pi}}\gamma_{rl} - \frac{1}{\sqrt{a\pi}}(\tau_0 u)^{\alpha_l-\alpha_r}}{(\tau_0 u)^{\alpha_l-\alpha_r} + \gamma_{rl}} \right)^2, \quad (34)$$

so that after inverting, we get

$$\begin{aligned} \langle x^2(t) \rangle &\approx \frac{\sigma_+}{2\gamma_l} \left(\frac{t}{\tau_0}\right)^{\alpha_l} E_{\alpha_l-\alpha_r, \alpha_l+1, 1}\left(-\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r}\right) + \frac{\gamma_{rl}}{2b} \left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r} E_{\alpha_l-\alpha_r, \alpha_l-\alpha_r+1, 1}\left(-\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r}\right) \\ &+ \frac{1}{2a} E_{\alpha_l-\alpha_r, 1, 1}\left(-\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r}\right) + \frac{2\sigma_-^2}{\pi \gamma_l^2} \left(\frac{t}{\tau_0}\right)^{2\alpha_l} E_{\alpha_l-\alpha_r, 2\alpha_l+1, 2}\left(-\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r}\right) \\ &+ \frac{2\gamma_{rl}^2}{b\pi} \left(\frac{t}{\tau_0}\right)^{2(\alpha_l-\alpha_r)} E_{\alpha_l-\alpha_r, 2(\alpha_l-\alpha_r)+1, 2}\left(-\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r}\right) + \frac{2}{a\pi} E_{\alpha_l-\alpha_r, 1, 2}\left(-\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r}\right) \\ &+ \frac{4\sigma_-\gamma_{rl}}{\pi \sqrt{b}\gamma_l} \left(\frac{t}{\tau_0}\right)^{2\alpha_l-\alpha_r} E_{\alpha_l-\alpha_r, 2\alpha_l-\alpha_r+1, 2}\left(-\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r}\right) - \frac{4\sigma_-}{\pi \sqrt{a}\gamma_l} \left(\frac{t}{\tau_0}\right)^{\alpha_l} E_{\alpha_l-\alpha_r, \alpha_l+1, 2}\left(-\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r}\right) \\ &- \frac{4\gamma_{rl}}{\sqrt{ab}\pi} \left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r} E_{\alpha_l-\alpha_r, \alpha_l-\alpha_r+1, 2}\left(-\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{\alpha_l-\alpha_r}\right) \approx \frac{\left(\frac{\sigma_+}{2} + \frac{4\sigma_-}{\sqrt{b\pi}}\right)\left(\frac{t}{\tau_0}\right)^{\alpha_r}}{\gamma_r \Gamma(\alpha_r+1)} + \frac{2\sigma_-^2\left(\frac{t}{\tau_0}\right)^{2\alpha_r}}{\pi \gamma_r^2 \Gamma(2\alpha_r+1)} \\ &- \frac{4\sigma_-^2\left(\frac{t}{\tau_0}\right)^{3\alpha_r-\alpha_l}}{\pi \gamma_r^2 \gamma_{rl} \Gamma(3\alpha_r-\alpha_l+1)} + \frac{8\sigma_-\left(\frac{t}{\tau_0}\right)^{3\alpha_r-2\alpha_l}}{\pi \sqrt{a}\gamma_r \gamma_{rl}^2 \Gamma(1+3\alpha_r-2\alpha_l)} + \frac{\left(\frac{8-\pi}{2a} - \frac{16+\pi}{2b} + \frac{4}{\sqrt{ab}}\right)\left(\frac{t}{\tau_0}\right)^{2\alpha_r-\alpha_l}}{\pi \gamma_r \gamma_{rl} \Gamma(2\alpha_r-\alpha_l+1)} + \frac{4+\pi}{2b\pi}. \end{aligned} \quad (35)$$

At *long* times, for this condition on the model parameters, from Eqs. (33) and (35) we analogously obtain scalings of AD transport

$$\langle x(t) \rangle \propto t^{\alpha_r}, \quad (36a)$$

$$\langle x^2(t) \rangle \propto t^{2\alpha_r}, \quad (36b)$$

$$\text{Var}[x(t)] \propto t^{2\alpha_r}, \quad (36c)$$

while for the case of symmetric JLDs with $a = b$ we get

$$\langle x(t) \rangle = 0, \quad (37a)$$

$$\langle x^2(t) \rangle \propto t^{\alpha_r}, \quad (37b)$$

$$\text{Var}[x(t)] \propto t^{\alpha_r}. \quad (37c)$$

3. Case C: $\alpha_r > \alpha_l$

Finally, for the model parameters satisfying the condition $\alpha_r > \alpha_l$, we get

$$\langle x(u) \rangle \approx -\frac{\frac{\sigma_-}{\sqrt{\pi}\gamma_r}(\tau_0 u)^{-\alpha_l} + \frac{1}{\sqrt{b\pi}}(\tau_0 u)^{\alpha_r-\alpha_l} - \frac{1}{\sqrt{a\pi}}\gamma_{rl}^{-1}}{u[(\tau_0 u)^{\alpha_r-\alpha_l} + \gamma_{rl}^{-1}]}, \quad (38)$$

which leads to

$$\begin{aligned} \langle x(t) \rangle &\approx -\frac{\sigma_-\left(\frac{t}{\tau_0}\right)^{\alpha_r} E_{\alpha_r-\alpha_l, \alpha_r+1, 1}\left(-\gamma_{rl}^{-1}\left(\frac{t}{\tau_0}\right)^{\alpha_r-\alpha_l}\right)}{\sqrt{\pi} \gamma_r} + \frac{\left(\frac{t}{\tau_0}\right)^{\alpha_r-\alpha_l} E_{\alpha_r-\alpha_l, \alpha_r-\alpha_l+1, 1}\left(-\gamma_{rl}^{-1}\left(\frac{t}{\tau_0}\right)^{\alpha_r-\alpha_l}\right)}{\gamma_{rl} \sqrt{a\pi}} \\ &- \frac{E_{\alpha_r-\alpha_l, 1, 1}\left(-\gamma_{rl}^{-1}\left(\frac{t}{\tau_0}\right)^{\alpha_r-\alpha_l}\right)}{\sqrt{b\pi}} \approx \frac{\sigma_-}{\gamma_l \sqrt{\pi}} \left[-\frac{\left(\frac{t}{\tau_0}\right)^{\alpha_l}}{\Gamma(\alpha_l+1)} + \frac{\gamma_{rl}\left(\frac{t}{\tau_0}\right)^{2\alpha_l-\alpha_r}}{\Gamma(2\alpha_l-\alpha_r+1)} + \frac{\gamma_l}{\sigma_- \sqrt{a}} \right]. \end{aligned} \quad (39)$$

For this case, from the general expression for the second moment (F7) we get

$$\langle x^2(u) \rangle \approx \frac{\frac{\sigma_+}{2\gamma_r}(\tau_0 u)^{-\alpha_l} + \frac{1}{2b}(\tau_0 u)^{\alpha_r - \alpha_l} + \frac{1}{2a\gamma_{rl}}}{u(\tau_0 u)^{\alpha_r - \alpha_l} + \gamma_{rl}^{-1}u} + \frac{2}{u} \left(\frac{\frac{\sigma_-}{\sqrt{\pi}\gamma_r}(\tau_0 u)^{-\alpha_l} + \frac{1}{\sqrt{b\pi}}(\tau_0 u)^{\alpha_r - \alpha_l} - \frac{1}{\sqrt{a\pi}\gamma_{rl}}}{(\tau_0 u)^{\alpha_r - \alpha_l} + \gamma_{rl}^{-1}} \right)^2 \quad (40)$$

so that in real space the second moment becomes

$$\begin{aligned} \langle x^2(t) \rangle &\approx \frac{\sigma_+}{2\gamma_r} \left(\frac{t}{\tau_0} \right)^{\alpha_r} E_{\alpha_r - \alpha_l, \alpha_r + 1, 1} \left(-\gamma_{rl}^{-1} \left(\frac{t}{\tau_0} \right)^{\alpha_r - \alpha_l} \right) + \frac{1}{\gamma_{rl} 2a} \left(\frac{t}{\tau_0} \right)^{\alpha_r - \alpha_l} E_{\alpha_r - \alpha_l, \alpha_r - \alpha_l + 1, 1} \left(-\gamma_{rl}^{-1} \left(\frac{t}{\tau_0} \right)^{\alpha_r - \alpha_l} \right) \\ &+ \frac{1}{2b} E_{\alpha_r - \alpha_l, 1, 1} \left(-\gamma_{rl}^{-1} \left(\frac{t}{\tau_0} \right)^{\alpha_r - \alpha_l} \right) + \frac{2\sigma_-^2}{\pi \gamma_r^2} \left(\frac{t}{\tau_0} \right)^{2\alpha_r} E_{\alpha_r - \alpha_l, 2\alpha_r + 1, 2} \left(-\gamma_{rl}^{-1} \left(\frac{t}{\tau_0} \right)^{\alpha_r - \alpha_l} \right) \\ &+ \frac{2}{a\pi \gamma_{rl}^2} \left(\frac{t}{\tau_0} \right)^{2(\alpha_r - \alpha_l)} E_{\alpha_r - \alpha_l, 2(\alpha_r - \alpha_l) + 1, 2} \left(-\gamma_{rl}^{-1} \left(\frac{t}{\tau_0} \right)^{\alpha_r - \alpha_l} \right) + \frac{2}{b\pi} E_{\alpha_r - \alpha_l, 1, 2} \left(-\gamma_{rl}^{-1} \left(\frac{t}{\tau_0} \right)^{\alpha_r - \alpha_l} \right) \\ &- \frac{4\sigma_-}{\pi \sqrt{a}\gamma_r \gamma_{rl}} \left(\frac{t}{\tau_0} \right)^{2\alpha_r - \alpha_l} E_{\alpha_r - \alpha_l, 2\alpha_r - \alpha_l + 1, 2} \left(-\gamma_{rl}^{-1} \left(\frac{t}{\tau_0} \right)^{\alpha_r - \alpha_l} \right) + \frac{4\sigma_-}{\pi \sqrt{b}\gamma_r} \left(\frac{t}{\tau_0} \right)^{\alpha_r} E_{\alpha_r - \alpha_l, \alpha_r + 1, 2} \left(-\gamma_{rl}^{-1} \left(\frac{t}{\tau_0} \right)^{\alpha_r - \alpha_l} \right) \\ &- \frac{4}{\sqrt{ab}\pi \gamma_{rl}} \left(\frac{t}{\tau_0} \right)^{\alpha_r - \alpha_l} E_{\alpha_r - \alpha_l, \alpha_r - \alpha_l + 1, 2} \left(-\gamma_{rl}^{-1} \left(\frac{t}{\tau_0} \right)^{\alpha_r - \alpha_l} \right) \approx \frac{(\frac{\sigma_+}{2} - \frac{4\sigma_-}{\sqrt{a\pi}}) \left(\frac{t}{\tau_0} \right)^{\alpha_l}}{\gamma_l \Gamma(\alpha_l + 1)} + \frac{2\sigma_-^2 \left(\frac{t}{\tau_0} \right)^{2\alpha_l}}{\pi \gamma_l^2 \Gamma(2\alpha_l + 1)} \\ &- \frac{4\sigma_-^2 \gamma_{rl} \left(\frac{t}{\tau_0} \right)^{3\alpha_l - \alpha_r}}{\pi \gamma_l^2 \Gamma(3\alpha_l - \alpha_r + 1)} - \frac{8\sigma_- \gamma_{rl}^3 \left(\frac{t}{\tau_0} \right)^{3\alpha_l - 2\alpha_r}}{\pi \sqrt{b}\gamma_r \Gamma(1 + 3\alpha_l - 2\alpha_r)} + \frac{\gamma_{rl} \left(\frac{8-\pi}{2b} - \frac{16+\pi}{2a} + \frac{4}{\sqrt{ab}} \right) \left(\frac{t}{\tau_0} \right)^{2\alpha_l - \alpha_r}}{\pi \gamma_l \Gamma(2\alpha_l - \alpha_r + 1)} + \frac{4 + \pi}{2a\pi}. \quad (41) \end{aligned}$$

Finally, at *long* times for this choice of the model parameters, from Eqs. (39) and (41) we arrive at

$$\langle x(t) \rangle \propto t^{\alpha_l}, \quad (42a)$$

$$\langle x^2(t) \rangle \propto t^{2\alpha_l}, \quad (42b)$$

$$\text{Var}[x(t)] \propto t^{2\alpha_l}, \quad (42c)$$

whereas for $a = b$ (and thus Gaussian JLDs) we get

$$\langle x(t) \rangle = 0, \quad (43a)$$

$$\langle x^2(t) \rangle \propto t^{\alpha_l}, \quad (43b)$$

$$\text{Var}[x(t)] \propto t^{\alpha_l}. \quad (43c)$$

From the scaling relations listed above for the situations (A), (B), and (C), respectively, it follows that for the MSD and for the variance the properties of particle propagation with time depend on the *smaller* of the two exponents of the power-law WTDs. To see this, one should compare the MSD and the variance scaling exponents for the cases (B) and (C) of unequal exponents $\alpha_{r,l}$ examined above.

Note that for *different* AJDLs considered here, *continuous*-space $A_{t,l}$ -CTRWs can (rather naturally) still generate a *nonzero bias* despite having equal total probabilities of jumps to the right and to the left, as quantified by Eq. (8). This feature is in contrast to the behavior of A_t -CTRWs in *discrete* space—with the jumps of a particle being possible only to the *neighboring* sites—where for equal probabilities of jumps right and left *no bias* is found [48]. We also find that when the AMGDS are chosen as the AJDLs, the respective AD

exponents of the MSD and of the variance for such $A_{t,l}$ -CTRWs are *doubled* in value as compared to the situations with $a = b$ and of symmetric JLDs. Therefore, effectively, the AMGDS for the JLDs in *continuous* space have the same effect as nonequal probabilities of jumping right or left on a *discrete* lattice.

E. Comparison of theory with simulations

1. Exponential WTDs

To verify the obtained analytical results, we implemented extensive Monte Carlo simulations of the particles performing such $A_{t,l}$ -CTRWs. First, for the *exponential* WTDs and the AMGDS for the JLDs, some exemplary trajectories are presented in Fig. 2. In Fig. 3 we demonstrate the agreement of our simulation-based results with the analytical solutions for the PDFs (22). Note that the peak of the PDF is exactly at the starting position for the simplest unbiased CTRWs when both the WTDs and the JLDs are left-vs-right symmetric; see the analytical PDF expression (23) and the green triangles in Fig. 3. Once the JLDs are chosen to be *asymmetric*, the PDF peaks start to be off-centered, with the location of this peak for the symmetric WTDs (as in Fig. 3) being equal to the mean position of the particles (*drift* in the first moment).

For the moments with the analytical expressions (D7a) and (D7b) and for the variance with the via-the-moment result (10), we observe a good agreement with the outcomes of computer simulations, as demonstrated in Fig. 4. One can see from Fig. 4 that—for all three types of symbols of three

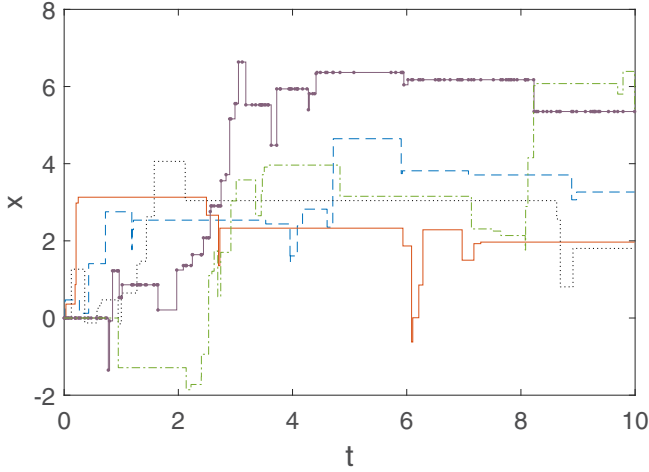


FIG. 2. Simulated trajectories of the particles performing continuous-space $A_{t,l}$ -CTRWs with the *symmetric* Gaussian distribution for the JLDs with $a = b = 1/2$ and with the *asymmetric* exponential WTDs for $\lambda_r = 10$ and $\lambda_l = 1$.

different colors corresponding to three different sets of the model parameters reflecting in this particular situation the asymmetric WTDs and the AMGDs for the JLDs—the moments and the variance are in agreement with the analytics not only in terms of their scaling relations at long times, but also quantitatively both at long and at short times. Clearly, the long-time asymptotics follow the relations (17a) and (17b), while to achieve a quantitative agreement with the short-time results of the simulations we used the exact relations (D7a) and (D7b) valid at all times [see also the approximate short-time results (16a) and (16b) and their respective linear-in-time scaling relations for this situation regarding the values of the model parameters].

The bias of such $A_{t,l}$ -CTRWs stems primarily from the form of the AMGDs used to simulate the AJLDs, even though

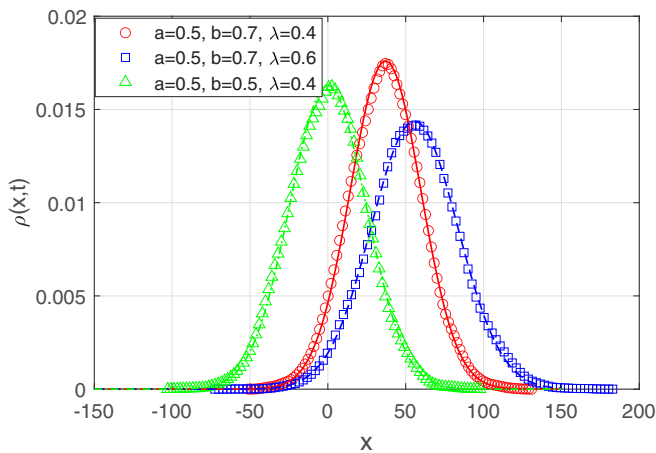


FIG. 3. Results of Monte Carlo simulations (symbols) vs the analytic solutions (curves) of the PDF $\rho(x, t)$ given by Eq. (22) for biased $A_{t,l}$ -CTRWs with the *symmetric* exponential WTDs (12) with (13) and with the AMGD for the JLDs (5), starting with the initial condition (C3) and presented after a diffusion time $t = 1500$. The values of relevant model parameters are provided in the legend.

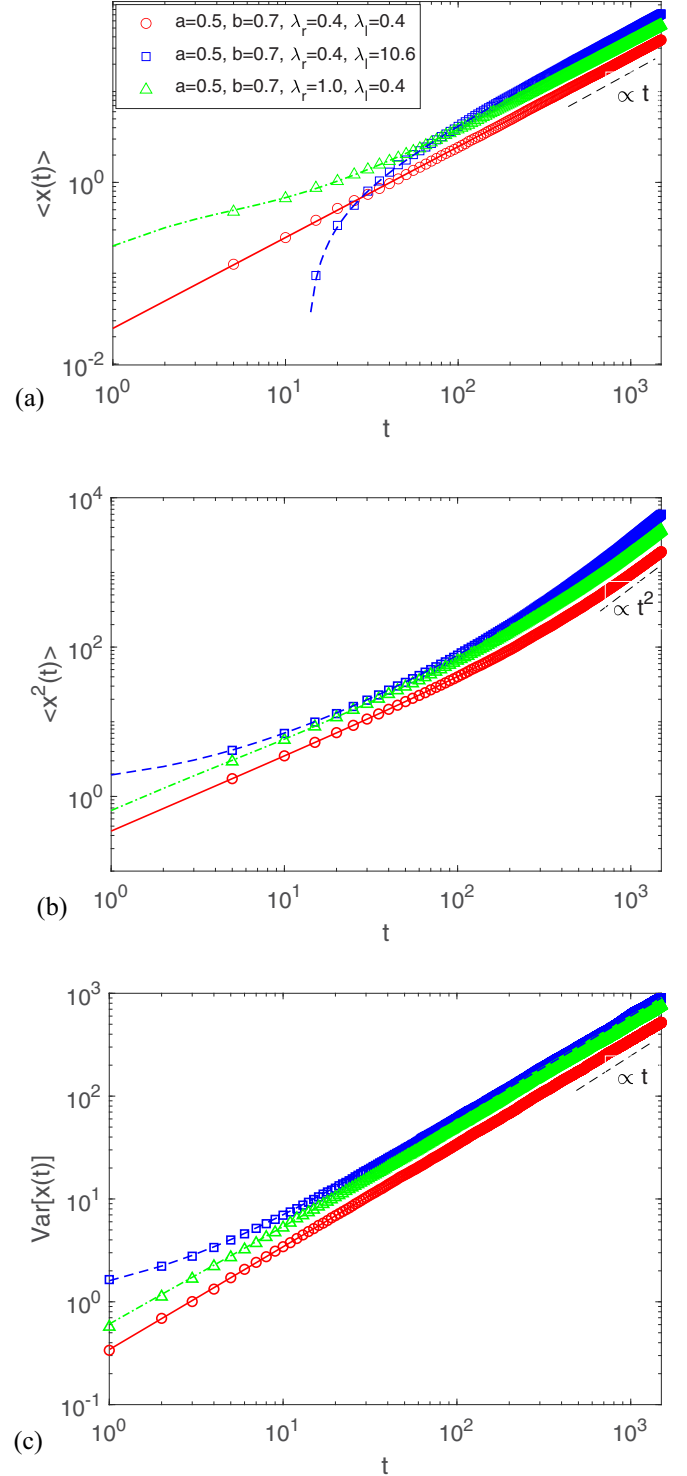


FIG. 4. Results of the simulations (symbols) vs the analytic solutions (curves) for $A_{t,l}$ -CTRWs with the *exponential* WTDs presented in log-log scale for the first and second moments given, respectively, by Eqs. (D7a) and (D7b) as well as for the variance generally given by Eq. (10) and constructed from the two moments listed above. The values of relevant model parameters are given in the legend of panel (a).

particles can have the same jump probability to the right and to the left. This is the major difference compared to the appearance of bias in A_r -CTRWs recently considered

in discrete space [48]. Additional effects of AWTs on the magnitude of bias cannot be ignored, however. The novel results of the current model with the *exponential* WTDs are that (i) with the AMGDs as the JLDs the computed MSD exhibits *ballistic* diffusion regardless of the asymmetry of the WTDs and (ii) the variance has a *linear* long-time scaling in both symmetric and asymmetric forms of the JLDs and of the WTDs.

2. Power-law WTDs

To verify the analytical results for $A_{r,l}$ -CTRWs with the *power-law* AWTs (24), we generate such WTDs in the current Monte Carlo simulations using the Pareto long-time WTD [33] given by (25) with $t > \tau_0$. As, unfortunately, no explicit analytical results for the PDF in real space are generally possible in this case of WTDs, we move directly to examining the behavior of the moments. In Fig. 5 we present the first moments and the variance of such $A_{r,l}$ -CTRWs, which reveal excellent agreement with the analytical predictions at intermediate and, particularly, at long times. Regarding the latter, depending on the choice of the model parameters α_l and α_r , the scaling relations of Eqs. (30a) and (30b) for the case (A) with $\alpha_l = \alpha_r$, Eqs. (36a) and (36b) for the case (B) with $\alpha_r < \alpha_l$, and, finally, Eqs. (42a) and (42b) for the case (C) with $\alpha_r > \alpha_l$ are to be compared with the results of computer simulations.

Note that in Fig. 5, only the intermediate-to-long-time results are shown because of the Pareto-type WTD (25) employed in the simulations, as compared to the Mittag-Leffler form (24) valid at all times used in the theoretical calculations. The theoretical scaling results for the moments are thus valid in the limit of long times. In practice, we do find a certain deviation between the short-time simulated results and the analytical formulas for the moments (simulation data not shown), while slight deviations in the magnitude of the moment at long times are mitigated by using a log-log plot.

III. DISCUSSION AND CONCLUSIONS

In this study, we have introduced a model of $A_{r,l}$ -CTRWs in continuous space, as a natural extension of a somewhat similar model of biased CTRWs we proposed recently in discrete space [48]. Our current model was designed to feature both AWTs and AJLDs. For random waiting times, we implemented both the exponential and the power-law WTDs, as particular examples of functional forms widely used in the theory of random walks. For random jump lengths, we employed the jump-containing AMGD for the JLDs. We derived the generalized Montroll-Weiss equation in Fourier-Laplace space and also the generalized master equation. For the chosen forms of the WTDs and JLDs, we recovered the first two moments in Laplace space, and finally via the inverse Laplace transform we derived these moments and the respective variance in real space. These moments were our main targets, with an incentive to unveil the AD properties of this specific model of biased CTRWs.

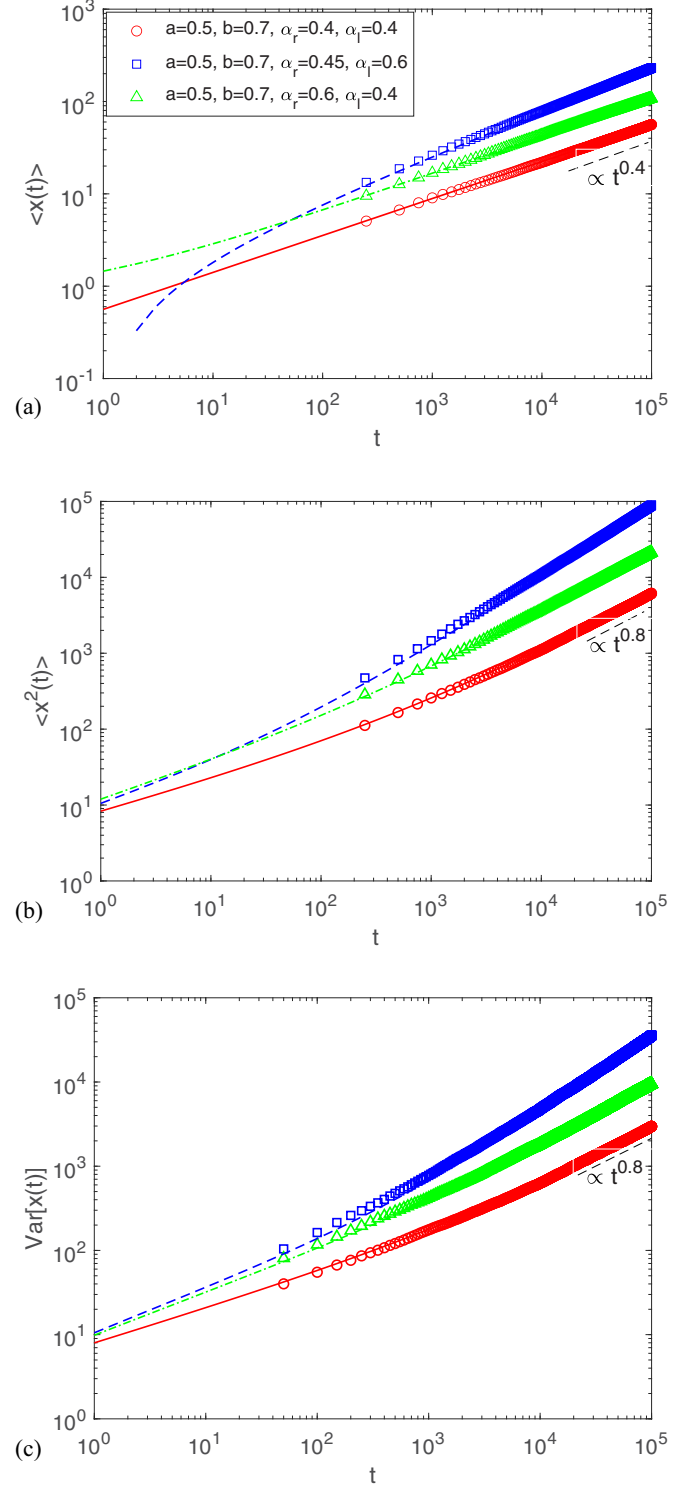


FIG. 5. The same as in Fig. 4, but for biased $A_{r,l}$ -CTRWs with the asymmetric *power-law* WTDs. The first moment is given (for three different situations regarding the relations between the model parameters) by Eqs. (27), (33), and (39), the second moment is described by Eqs. (29), (35), and (41), and, finally, the variance is given via the first two moments by the general Eq. (10). The relevant model parameters are provided in the legend of panel (a).

Our main goal was to understand how the particle-spreading characteristics depend on the properties of the chosen AWTs and AJLDs as well as to categorize the differences of the current continuous versus the previous discrete-lattice [48] approach for examining such biased CTRWs. From the obtained rather cumbersome analytical expressions (including the special three-parameter Mittag-Leffler functions) valid at all times, we derived the asymptotic scaling relations for the first moments and for the variance. The analytical results and scalings in the long-time limit were demonstrated to yield a quantitative agreement with the results of conducted Monte Carlo simulations of such $A_{t,l}$ -CTRWs. We explicitly demonstrated, for instance, that even if the total probability of particles to jump to the left and to the right are equal, for the situations of AJLDs in the model, a sequence of random jumps of the particles—weighted differently for left-vs-right directions as a function of the jump distance—could lead to a nonzero drift in such AJLD-based $A_{t,l}$ -CTRWs.

To predict scaling relations for the moments for *other* functional forms of the AWTs, separate calculations are needed; these can become the subject of our future research. For instance, the next natural step is to consider the Gamma distribution featuring (under proper conditions on its parameters) finite moments of all orders. Other mathematical forms of the JLDs are likewise conceivable and need to be investigated separately to unveil the particle-spreading and drift properties of such $A_{t,l}$ -CTRWs. Our main incentive is to provide this continuous-space model of biased CTRWs with enough tunability with regard to their WTDs and JLDs so that certain currently puzzling experimental data could be rationalized using the new concept of such $A_{t,l}$ -CTRWs.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are available upon reasonable request.

APPENDIX A: FOURIER TRANSFORMS

Here, with the main concepts, functions, and parameters of the model introduced in Sec. II A, we conduct the sin- and cos-Fourier transforms to get the PDF in Fourier-Laplace space [Eq. (A10) used later in Appendix B].

Let $\Psi(t)$ be the probability that no jumps have occurred up to time t . Then this survival probability becomes

$$\Psi(t) = 1 - \int_0^\infty \lambda(\xi) d\xi \int_0^t \psi_r(\tau) d\tau - \int_{-\infty}^0 \lambda(\xi) d\xi \int_0^t \psi_l(\tau) d\tau. \quad (\text{A1})$$

We can obtain the evolution equation for the PDF of the particles as

$$\rho(x, t) = \rho_0(x) \Psi(t) + \int_0^t I(x, t - \tau) \Psi(\tau) d\tau. \quad (\text{A2})$$

Here, the first term on the right-hand side represents the probability of being at the position x times the probability of no jumps up to time t , while the second term takes into account the probability of arriving at position x at time $(t - \tau)$ times the probability of no jumps during the remaining time τ (see Refs. [58,59] for details).

The Fourier-Laplace transform of (A2) multiplied by u with the help of Eq. (A1) becomes

$$u\rho(k, u) = \left[1 - \left(\int_0^\infty \lambda(\xi) d\xi \psi_r(u) + \int_{-\infty}^0 \lambda(\xi) d\xi \psi_l(u) \right) \right] \rho_0(k) + \left[1 - \left(\int_0^\infty \lambda(\xi) d\xi \psi_r(u) + \int_{-\infty}^0 \lambda(\xi) d\xi \psi_l(u) \right) \right] I(k, u). \quad (\text{A3})$$

Here, the quantity $I(x, t)$ obeys the balance equation [59]

$$I(x, t) = \int_{-\infty}^\infty \rho_0(x - \xi) \phi(\xi, t) d\xi + \int_0^t \int_{-\infty}^\infty I(x - \xi, t - \tau) \phi(\xi, \tau) d\xi d\tau. \quad (\text{A4})$$

Combining Eqs. (1), (2), and (3), we get

$$I(x, t) = \int_{-\infty}^\infty \rho_0(x - \xi) [\psi_r(t) \Theta(\xi) + \psi_l(t) \Theta(-\xi)] \lambda(\xi) d\xi + \int_0^t \int_{-\infty}^\infty I(x - \xi, t - \tau) [\psi_r(\tau) \Theta(\xi) + \psi_l(\tau) \Theta(-\xi)] \lambda(\xi) d\xi d\tau. \quad (\text{A5})$$

Note that while the Z-transformation is used to handle A_t -CTRWs in discrete space [48], the sin- and cos-Fourier transforms used here are better suited for $A_{t,l}$ -CTRWs in continuous space (playing the same role). The explicit formulas for these transformations are [60]

$$\mathcal{F}_{\cos}[\lambda(x)] \equiv \int_0^\infty \cos(kx) \lambda(x) dx, \quad \mathcal{F}_{\sin}[\lambda(x)] = \int_0^\infty \sin(kx) \lambda(x) dx, \quad (\text{A6})$$

so that in the complex-number form we get

$$\mathcal{F}[\lambda(x)] = \int_{-\infty}^{\infty} e^{-ikx} \lambda(x) dx = \mathcal{F}_{\cos}[\lambda(x) + \lambda(-x)] - i\mathcal{F}_{\sin}[\lambda(x) - \lambda(-x)]. \quad (\text{A7})$$

Specifically, for the functions (3) the Fourier transforms can be obtained as

$$\mathcal{F}[\lambda^+(x)] = \int_0^{\infty} \cos(kx) \lambda(x) \Theta(x) dx - i \int_0^{\infty} \sin(kx) \lambda(x) \Theta(x) dx = \mathcal{F}_{\cos}[\lambda^+(x)] - i\mathcal{F}_{\sin}[\lambda^+(x)], \quad (\text{A8a})$$

$$\begin{aligned} \mathcal{F}[\lambda^-(x)] &= \int_{-\infty}^0 \cos(kx) \lambda(x) \Theta(-x) dx - i \int_{-\infty}^0 \sin(kx) \lambda(x) \Theta(-x) dx \\ &= \int_0^{\infty} \cos(kx) \lambda(-x) \Theta(x) dx + i \int_0^{\infty} \sin(kx) \lambda(-x) \Theta(x) dx = \mathcal{F}_{\cos}[\lambda^-(x)] + i\mathcal{F}_{\sin}[\lambda^-(x)]. \end{aligned} \quad (\text{A8b})$$

The Fourier-Laplace image for the expression (A5) then turns into

$$I(k, u) = [\rho_0(k) + I(k, u)] \{ \psi_r(u) [\lambda_{\cos}^+(k) - i\lambda_{\sin}^+(k)] + \psi_l(u) [\lambda_{\cos}^-(k) + i\lambda_{\sin}^-(k)] \}. \quad (\text{A9})$$

Here, $\lambda_{\cos}^{\pm}(k)$ and $\lambda_{\sin}^{\pm}(k)$ are the cos- or sin-Fourier transformations of $\lambda^+(\xi)$ and $\lambda^-(\xi)$. Combining Eqs. (A3) and (A9), we obtain the corresponding generalized Montroll-Weiss equation in Fourier-Laplace space as

$$\rho(k, u) = \frac{u^{-1} [1 - (\psi_r(u) \int_0^{\infty} \lambda(\xi) d\xi + \psi_l(u) \int_{-\infty}^0 \lambda(\xi) d\xi)] \rho_0(k)}{1 - [\psi_r(u) (\lambda_{\cos}^+(k) - i\lambda_{\sin}^+(k)) + \psi_l(u) (\lambda_{\cos}^-(k) + i\lambda_{\sin}^-(k))]}. \quad (\text{A10})$$

For a special case of the *same* WTDs in both directions, i.e. with $\psi_r(t) = \psi_l(t) \equiv \psi(t)$, Eq. (A10) reduces to the standard Montroll-Weiss equation

$$\rho(k, u) = \frac{[1 - \psi(u)] \rho_0(k)}{u[1 - \psi(u)\lambda(k)]}, \quad (\text{A11})$$

where according to Eqs. (A8a), (A8b), and (3) we used

$$[\lambda_{\cos}^+(k) - i\lambda_{\sin}^+(k)] + [\lambda_{\cos}^-(k) + i\lambda_{\sin}^-(k)] = \lambda^+(k) + \lambda^-(k) = \lambda(k). \quad (\text{A12})$$

APPENDIX B: MASTER EQUATION

Here, using the Fourier-Laplace result for the PDF given by (A10), we derive the master equation for the real-space PDF used in Sec. II B. Namely, using the Laplace transform (A1), expression (A10) turns into

$$\rho(k, u) = \rho_0(k) \Psi(u) + [\psi_r(u) (\lambda_{\cos}^+(k) - i\lambda_{\sin}^+(k)) + \psi_l(u) (\lambda_{\cos}^-(k) + i\lambda_{\sin}^-(k))] \rho(k, u), \quad (\text{B1})$$

which—after taking the inverse Laplace and inverse Fourier transforms with the help of Eqs. (A8a) and (A8b)—becomes

$$\rho(x, t) = \rho_0(x) \Psi(t) + \int_0^t \int_{-\infty}^{\infty} \rho(x - \xi, t - \tau) [\psi_r(\tau) \Theta(\xi) + \psi_l(\tau) \Theta(-\xi)] \lambda(\xi) d\xi d\tau. \quad (\text{B2})$$

Note here that Eq. (A10) can also be written as

$$u\rho(k, u) - \rho_0(k) = \frac{\psi_r(u)}{\Psi(u)} \left[\lambda_{\cos}^+(k) - i\lambda_{\sin}^+(k) - \int_0^{\infty} \lambda(\xi) d\xi \right] \rho(k, u) + \frac{\psi_l(u)}{\Psi(u)} \left[\lambda_{\cos}^-(k) + i\lambda_{\sin}^-(k) - \int_{-\infty}^0 \lambda(\xi) d\xi \right] \rho(k, u). \quad (\text{B3})$$

Performing the inverse Laplace and Fourier transformations, using Eqs. (A8a) and (A8b), from Eq. (B3) we obtain the following generalized master equation [61]:

$$\begin{aligned} \frac{\partial \rho(x, t)}{\partial t} &= \int_0^t \int_{-\infty}^{\infty} M_r(\tau) \rho(x - \xi, t - \tau) \left[\Theta(\xi) \lambda(\xi) - \delta(\xi) \int_0^{\infty} \lambda(\xi') d\xi' \right] d\xi d\tau \\ &\quad + \int_0^t \int_{-\infty}^{\infty} M_l(\tau) \rho(x - \xi, t - \tau) \left[\Theta(-\xi) \lambda(\xi) - \delta(\xi) \int_{-\infty}^0 \lambda(\xi') d\xi' \right] d\xi d\tau, \end{aligned} \quad (\text{B4})$$

where the memory kernels $M_r(t)$ and $M_l(t)$ have the following Laplace transforms:

$$M_r(u) = \psi_r(u)/\Psi(u), \quad M_l(u) = \psi_l(u)/\Psi(u). \quad (\text{B5})$$

Note that when the JLD satisfies a two-point distribution (4), we find

$$\Theta(\xi)\lambda(\xi) = \gamma\delta(\xi-1), \quad \Theta(-\xi)\lambda(\xi) = (1-\gamma)\delta(\xi+1), \quad \int_0^\infty \lambda(\xi')d\xi' = \gamma, \quad \int_{-\infty}^0 \lambda(\xi')d\xi' = 1-\gamma. \quad (\text{B6})$$

Then Eq. (B4) becomes the master equation with the AWTDS in discrete space [48],

$$\frac{\partial \rho(x, t)}{\partial t} = \gamma \int_0^t M_r(\tau) [\rho(x-1, t-\tau) - \rho(x, t-\tau)] d\tau + (1-\gamma) \int_0^t M_l(\tau) [\rho(x+1, t-\tau) - \rho(x, t-\tau)] d\tau. \quad (\text{B7})$$

If condition (4) is not satisfied, the situation is more complicated, as we demonstrate in Sec. II B where a number of new results for $A_{t,l}$ -CTRWs are presented.

APPENDIX C: PDFs IN THE FOURIER-LAPLACE SPACE

We substitute Eqs. (6) and (8) into Eq. (B3) to obtain (again, up to k^2 -terms only)

$$[u\rho(k, u) - \rho_0(k)]\Psi(u) \approx \psi_r(u) \left(-\frac{k^2}{8a} - \frac{ik}{2\sqrt{a\pi}} \right) \rho(k, u) + \psi_l(u) \left(-\frac{k^2}{8b} + \frac{ik}{2\sqrt{b\pi}} \right) \rho(k, u). \quad (\text{C1})$$

Taking the inverse Laplace-Fourier transform with $u \rightarrow t$ and $k \rightarrow x$, this yields the following diffusion equation:

$$\frac{\partial \rho(x, t)}{\partial t} = \frac{1}{8} \frac{\partial^2}{\partial x^2} \int_0^t \left[\frac{M_r(\tau)}{a} + \frac{M_l(\tau)}{b} \right] \rho(x, t-\tau) d\tau - \frac{1}{2\sqrt{\pi}} \frac{\partial}{\partial x} \int_0^t \left[\frac{M_r(\tau)}{\sqrt{a}} - \frac{M_l(\tau)}{\sqrt{b}} \right] \rho(x, t-\tau) d\tau. \quad (\text{C2})$$

From Eq. (A10), with expressions (6) and (8) as well as the initial condition for all particles being

$$\rho_0(x) = \delta(x), \quad (\text{C3})$$

we obtain

$$\rho(k, u) \approx \frac{u^{-1}[2 - \psi_r(u) - \psi_l(u)]}{2 - \psi_r(u)\left(1 - \frac{k^2}{4a} - \frac{ik}{\sqrt{a\pi}}\right) - \psi_l(u)\left(1 - \frac{k^2}{4b} + \frac{ik}{\sqrt{b\pi}}\right)}. \quad (\text{C4})$$

Using the Laplace transform of Eq. (A1), we get (up to k^2 -order)

$$\rho(k, u) \approx \frac{\Psi(u)}{1 - \frac{\psi_r(u)}{2}\left(1 - \frac{k^2}{4a} - \frac{ik}{\sqrt{a\pi}}\right) - \frac{\psi_l(u)}{2}\left(1 - \frac{k^2}{4b} + \frac{ik}{\sqrt{b\pi}}\right)}, \quad (\text{C5})$$

from where

$$\rho(k, u) = \frac{\Psi(u)}{\left(\frac{\psi_r(u)}{a} + \frac{\psi_l(u)}{b}\right)\frac{k^2}{8} - \frac{ik}{2\sqrt{\pi}}\left(\frac{\psi_l(u)}{\sqrt{b}} - \frac{\psi_r(u)}{\sqrt{a}}\right) - \frac{1}{2}[\psi_r(u) + \psi_l(u)] + 1}. \quad (\text{C6})$$

From this, we finally arrive at the relation

$$\rho(k, u) = 2\Psi(u) \left(\frac{1}{-d_-(u) + ik} + \frac{1}{d_+(u) - ik} \right) / \Omega_{lr}(u), \quad (\text{C7})$$

where

$$d_{\pm}(u) = \left[-\left(\frac{\psi_l(u)}{\sqrt{b}} - \frac{\psi_r(u)}{\sqrt{a}} \right) \pm \sqrt{\pi} \Omega_{lr}(u) \right] / \left[\frac{\sqrt{\pi}}{2} \left(\frac{\psi_r(u)}{a} + \frac{\psi_l(u)}{b} \right) \right] \quad (\text{C8})$$

are the two roots of the quadratic equation

$$\left(\frac{\psi_r(u)}{a} + \frac{\psi_l(u)}{b} \right) \frac{k^2}{8} - \frac{ik}{2\sqrt{\pi}} \left(\frac{\psi_l(u)}{\sqrt{b}} - \frac{\psi_r(u)}{\sqrt{a}} \right) - \frac{1}{2}[\psi_r(u) + \psi_l(u)] + 1 = 0 \quad (\text{C9})$$

in terms of the variable (ik) , with $d_-(u) < 0 < d_+(u)$ and with a newly defined function

$$\Omega_{lr}(u) = \sqrt{\frac{1}{\pi} \left(\frac{\psi_l(u)}{\sqrt{b}} - \frac{\psi_r(u)}{\sqrt{a}} \right)^2 + \left(\frac{\psi_r(u)}{a} + \frac{\psi_l(u)}{b} \right) [2 - \psi_r(u) - \psi_l(u)]}. \quad (\text{C10})$$

Note that expression (C9) stems from the denominator of Eq. (C7). Also note that hereafter all definitions with the subscript $\{lr\}$ *without* a comma imply a dependence on the parameters of the jump-quantities both to the left and to the right, while for the parameters or quantities with the subscript $\{l, r\}$ written *with* the comma some specific dependencies for the respective quantities with the subscript l or r on both sides of a given relation are symbolized.

Taking the inverse Fourier transforms of (C7) gives for $x \geq 0$ the following result:

$$\rho(x, u) = 2\Psi(u)e^{d_-(u)x}/\Omega_{lr}(u), \quad (\text{C11})$$

while for $x < 0$ we get

$$\rho(x, u) = 2\Psi(u)e^{d_+(u)x}/\Omega_{lr}(u). \quad (\text{C12})$$

One can find that expressions (C11) and (C12) satisfy the relation

$$\begin{aligned} \frac{\rho(x, u)}{\rho(-x, u)} &= \exp\{[d_+(u) + d_-(u)]x\} \\ &= \exp\left[\frac{4[b\sqrt{a}\psi_r(u) - a\sqrt{b}\psi_l(u)]x}{\sqrt{\pi}[a\psi_l(u) + b\psi_r(u)]}\right], \end{aligned} \quad (\text{C13})$$

which leads in the simplest case of $\psi_r(u) = \psi_l(u)$ to the so-called fluctuation theorem [42,62] of the form

$$\frac{\rho(x, u)}{\rho(-x, u)} = \exp\left[\frac{4(b\sqrt{a} - a\sqrt{b})x}{\sqrt{\pi}(a + b)}\right]. \quad (\text{C14})$$

Therefore, we call Eq. (C13)—which was not discussed in the discrete case [48]—a generalized fluctuation theorem, proposed for arbitrary WTDs already in Refs. [42,62]. The results of expressions (C11) and (C12) are utilized in Sec. II B of the main text to compute the first two moments (9a) and (9b).

APPENDIX D: FIRST MOMENTS IN REAL SPACE: EXPONENTIAL WTDs

Similarly to the interpretation proposed in Ref. [48], the expressions $\int_0^t \psi_{r,l}(\tau)d\tau$ give the conditional probabilities for a particle to jump starting from time $t = 0$ up to time t in a respective direction. In Laplace space, from expressions (12) we get

$$\psi_{l,r}(u) = \lambda_{l,r}/(\lambda_{l,r} + u), \quad (\text{D1})$$

and, by using Eqs. (A1) and (8), the memory kernels (B5) become

$$M_r(u) = \lambda_r + \frac{\lambda_r(\lambda_l - \lambda_r)}{2u + \lambda_r + \lambda_l}, \quad (\text{D2a})$$

$$M_l(u) = \lambda_l + \frac{\lambda_l(\lambda_r - \lambda_l)}{2u + \lambda_r + \lambda_l}. \quad (\text{D2b})$$

After taking the inverse Laplace transform, we obtain

$$M_r(t) = \lambda_r\delta(t) + \lambda_r(\lambda_l - \lambda_r)e^{-(\lambda_r + \lambda_l)t/2}/2, \quad (\text{D3a})$$

$$M_l(t) = \lambda_l\delta(t) + \lambda_l(\lambda_r - \lambda_l)e^{-(\lambda_r + \lambda_l)t/2}/2. \quad (\text{D3b})$$

We substitute these results into Eq. (C2) in the case of exponential AWTs and of AMGDs to get the generalized

Fokker-Planck-Kolmogorov equation of the form

$$\begin{aligned} \frac{\partial \rho(x, t)}{\partial t} &= \frac{1}{16} \frac{\partial^2}{\partial x^2} \int_0^t \left[2\left(\frac{\lambda_r}{a} + \frac{\lambda_l}{b}\right)\delta(\tau) + \left(\frac{\lambda_r}{a} - \frac{\lambda_l}{b}\right) \right. \\ &\quad \times (\lambda_l - \lambda_r) \exp\left(-\frac{(\lambda_r + \lambda_l)\tau}{2}\right) \left. \right] \rho(x, t - \tau) d\tau \\ &\quad - \frac{1}{4\sqrt{\pi}} \frac{\partial}{\partial x} \int_0^t \left[2\left(\frac{\lambda_r}{\sqrt{a}} - \frac{\lambda_l}{\sqrt{b}}\right)\delta(\tau) \right. \\ &\quad \left. + \left(\frac{\lambda_r}{\sqrt{a}} + \frac{\lambda_l}{\sqrt{b}}\right)(\lambda_l - \lambda_r) \exp\left(-\frac{(\lambda_r + \lambda_l)\tau}{2}\right) \right] \\ &\quad \times \rho(x, t - \tau) d\tau. \end{aligned} \quad (\text{D4})$$

From Eqs. (9a) and (9b), via combining with Eq. (D1), we get in Laplace space

$$\langle x(u) \rangle = -\frac{\sigma_- \lambda_r \lambda_l + (\lambda_r/\sqrt{b} - \lambda_l/\sqrt{a})u}{\sqrt{\pi}u^2(2u + \lambda_r + \lambda_l)}, \quad (\text{D5a})$$

$$\langle x^2(u) \rangle = \frac{2\pi(\lambda_r/a + \lambda_l/b)u^3 + Au^2 + Bu + 4(\sigma_- \lambda_r \lambda_l)^2}{2\pi u^3(2u + \lambda_r + \lambda_l)^2}, \quad (\text{D5b})$$

where we defined the following new parameters:

$$\begin{aligned} A &= 4\left(\frac{\lambda_r}{\sqrt{a}} - \frac{\lambda_l}{\sqrt{b}}\right)^2 + \pi\left(\frac{\lambda_r^2}{a} + \frac{\lambda_l^2}{b} + 3\sigma_+ \lambda_r \lambda_l\right), \\ B &= \lambda_r \lambda_l \left[8\left(\frac{\lambda_r}{a} + \frac{\lambda_l}{b}\right) + (\lambda_r + \lambda_l)\left(\pi\sigma_+ - \frac{8}{\sqrt{ab}}\right) \right]. \end{aligned} \quad (\text{D6})$$

Taking the inverse Laplace transform, we obtain the first two moments for this choice of the exponential AWTs as

$$\begin{aligned} \langle x(t) \rangle &= -\frac{\sigma_- \lambda_r \lambda_l}{\sqrt{\pi}(\lambda_r + \lambda_l)} t \\ &\quad - \frac{(\lambda_r/\sqrt{a} + \lambda_l/\sqrt{b})(\lambda_l - \lambda_r)(1 - e^{-(\lambda_r + \lambda_l)t/2})}{\sqrt{\pi}(\lambda_r + \lambda_l)^2}, \end{aligned} \quad (\text{D7a})$$

$$\langle x^2(t) \rangle = \frac{k_1 t^2 + k_2 t + k_3 + (k_4 t - k_3)e^{-(\lambda_r + \lambda_l)t/2}}{2\pi}, \quad (\text{D7b})$$

where for compactness we introduced the following parameters redefined via expressions (D6) as

$$\begin{aligned} k_1 &= \frac{2(\sigma_- \lambda_r \lambda_l)^2}{(\lambda_r + \lambda_l)^2}, \quad k_2 = \frac{B}{(\lambda_r + \lambda_l)^2} - \frac{16(\sigma_- \lambda_r \lambda_l)^2}{(\lambda_r + \lambda_l)^3}, \\ k_3 &= \frac{48(\sigma_- \lambda_r \lambda_l)^2}{(\lambda_r + \lambda_l)^4} - \frac{4B}{(\lambda_r + \lambda_l)^3} + \frac{A}{(\lambda_r + \lambda_l)^2}, \\ k_4 &= \frac{-8(\sigma_- \lambda_r \lambda_l)^2}{(\lambda_r + \lambda_l)^3} + \frac{B}{(\lambda_r + \lambda_l)^2} - \frac{A}{2(\lambda_r + \lambda_l)} \\ &\quad + \frac{\pi(\lambda_r/a + \lambda_l/b)}{2}. \end{aligned} \quad (\text{D8})$$

The results of expressions (D7a) and (D7b) are used in Sec. II C to compute the first moments in the scenario of exponential WTDs.

APPENDIX E: PROPERTIES OF THE MITTAG-LEFFLER FUNCTIONS

Here, the generalized two- and three-parameter Mittag-Leffler functions are defined as [48,63–66]

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (\text{E1a})$$

$$E_{\alpha,\beta,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_n z^n}{\Gamma(\alpha n + \beta) n!}. \quad (\text{E1b})$$

Here $(\delta)_n = \Gamma(\delta + n)/\Gamma(\delta)$ is the Pochhammer symbol. The respective Laplace transforms of (E1a) and (E1b) for the range of parameters $\{\alpha, \beta, a\} > 0$ are [48]

$$\mathcal{L}[t^{\beta-1} E_{\alpha,\beta}(-at^\alpha)] = \frac{u^{\alpha-\beta}}{u^\alpha + a}, \quad (\text{E2a})$$

$$\mathcal{L}[t^{\beta-1} E_{\alpha,\beta,\delta}(-at^\alpha)] = \frac{u^{\alpha\delta-\beta}}{(u^\alpha + a)^\delta}. \quad (\text{E2b})$$

The asymptotic expansions of the Mittag-Leffler functions in the *long-time* limit are given by

$$E_{\alpha,\beta}(-t^\alpha) \stackrel{t \rightarrow \infty}{\approx} \frac{t^{-\alpha}}{\Gamma(\beta - \alpha)} - \frac{t^{-2\alpha}}{\Gamma(\beta - 2\alpha)}, \quad (\text{E3a})$$

$$E_{\alpha,\beta,\delta}(-t^\alpha) \stackrel{t \rightarrow \infty}{\approx} \frac{t^{-\alpha\delta}}{\Gamma(\beta - \alpha\delta)} - \frac{\delta t^{-\alpha(\delta+1)}}{\Gamma[\beta - \alpha(\delta+1)]}, \quad (\text{E3b})$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \quad (\text{E4})$$

is the Euler Gamma-function. The outlined expansions are used in Appendix F below and in Sec. II D of the main text to compute the first moments for the case of power-law WTDs.

APPENDIX F: FIRST MOMENTS IN REAL SPACE: POWER-LAW WTDs

With the help of relations (E3a) and (E3b), the Laplace transform of the WTD (24) for *long* diffusion times (i.e., at $u \rightarrow 0$) is

$$\psi_{r,l}(u) = \frac{1}{1 + \gamma_{r,l}(\tau_0 u)^{\alpha_{r,l}}} \approx 1 - \gamma_{r,l}(\tau_0 u)^{\alpha_{r,l}}, \quad (\text{F1})$$

where for $0 < \alpha_{r,l} < 1$ we used the definition

$$\gamma_{r,l} = |\Gamma(1 - \alpha_{r,l})|, \quad (\text{F2})$$

for all cases with $\gamma \neq \{0, 1\}$ and we also introduce a parameter [see a note after Eq. (C10) regarding the meaning of the comma in the subscripts in such definitions]

$$\gamma_{rl} = \gamma_r / \gamma_l. \quad (\text{F3})$$

After applying Eqs. (A1), (8), and (F1), from Eq. (B5) the *asymptotic* memory kernels for the power-law AWTd for the case $\alpha_r < \alpha_l$ become

$$M_r(u) \approx \frac{2u(\tau_0 u)^{-\alpha_r} (\gamma_l^{-1} + (\tau_0 u)^{\alpha_l})}{(\tau_0 u)^{\alpha_l - \alpha_r} + \gamma_{rl}}, \quad (\text{F4a})$$

$$M_l(u) \approx \frac{2u(\gamma_l^{-1}(\tau_0 u)^{-\alpha_r} + \gamma_{rl})}{(\tau_0 u)^{\alpha_l - \alpha_r} + \gamma_{rl}}, \quad (\text{F4b})$$

whereas for the situation $\alpha_r > \alpha_l$ they assume the form

$$M_r(u) \approx \frac{2u(\gamma_r^{-1}(\tau_0 u)^{-\alpha_l} + \gamma_{rl}^{-1})}{(\tau_0 u)^{\alpha_r - \alpha_l} + \gamma_{rl}^{-1}}, \quad (\text{F5a})$$

$$M_l(u) \approx \frac{2u(\tau_0 u)^{-\alpha_l} (\gamma_r^{-1} + (\tau_0 u)^{\alpha_r})}{(\tau_0 u)^{\alpha_r - \alpha_l} + \gamma_{rl}^{-1}}, \quad (\text{F5b})$$

see also Appendix G.

From the first moment (9a) with the expansion (F1) we get in Laplace space that

$$\langle x(u) \rangle = -\frac{1}{\sqrt{\pi}u} \frac{\sigma_- + \frac{1}{\sqrt{b}}\gamma_r(\tau_0 u)^{\alpha_r} - \frac{1}{\sqrt{a}}\gamma_l(\tau_0 u)^{\alpha_l}}{2\gamma_r\gamma_l(\tau_0 u)^{\alpha_r+\alpha_l} + \gamma_r(\tau_0 u)^{\alpha_r} + \gamma_l(\tau_0 u)^{\alpha_l}}. \quad (\text{F6})$$

Analogously, for the second moment of such $A_{r,l}$ -CTRWs, starting from expression (9b) and applying (F1) we obtain in Laplace space that

$$\langle x^2(u) \rangle = \frac{1}{u} \left[\frac{\frac{\sigma_-^2}{2} + \frac{1}{2b}\gamma_r(\tau_0 u)^{\alpha_r} + \frac{1}{2a}\gamma_l(\tau_0 u)^{\alpha_l}}{2\gamma_r\gamma_l(\tau_0 u)^{\alpha_r+\alpha_l} + \gamma_r(\tau_0 u)^{\alpha_r} + \gamma_l(\tau_0 u)^{\alpha_l}} + 2 \left(\frac{\frac{\sigma_-}{\sqrt{\pi}} + \frac{1}{\sqrt{b\pi}}\gamma_r(\tau_0 u)^{\alpha_r} - \frac{1}{\sqrt{a\pi}}\gamma_l(\tau_0 u)^{\alpha_l}}{2\gamma_r\gamma_l(\tau_0 u)^{\alpha_r+\alpha_l} + \gamma_r(\tau_0 u)^{\alpha_r} + \gamma_l(\tau_0 u)^{\alpha_l}} \right)^2 \right]. \quad (\text{F7})$$

The Laplace-space results (F6) and (F7) are used in Sec. II D to compute the first- and second-order real-space moments for the case of power-law WTDs.

APPENDIX G: EXACT SOLUTIONS FOR THE MEMORY KERNELS

As referred to in Appendix F, the *exact* (instead of approximate) solutions for the memory kernels can also be obtained in terms of the Mittag-Leffler functions. In Laplace space from Eq. (B5) we get, with the help of geometric progression, that

$$M_r(u) = \frac{2u[1 + (\tau_0 u)^{\alpha_l}]}{(\tau_0 u)^{\alpha_r} + (\tau_0 u)^{\alpha_l} + 2(\tau_0 u)^{\alpha_r+\alpha_l}} = \frac{2u[1 + (\tau_0 u)^{\alpha_l}]}{(\tau_0 u)^{\alpha_r} + (\tau_0 u)^{\alpha_l}} \frac{1}{1 + \frac{2(\tau_0 u)^{\alpha_r+\alpha_l}}{(\tau_0 u)^{\alpha_r} + (\tau_0 u)^{\alpha_l}}} \quad (\text{G1})$$

and then

$$M_r(u) = \frac{2u[1 + (\tau_0 u)^{\alpha_l}]}{(\tau_0 u)^{\alpha_r} + (\tau_0 u)^{\alpha_l}} \sum_{j=0}^{\infty} \left(\frac{-2(\tau_0 u)^{\alpha_r+\alpha_l}}{(\tau_0 u)^{\alpha_r} + (\tau_0 u)^{\alpha_l}} \right)^j. \quad (\text{G2})$$

With further algebraic manipulations, we obtain that

$$M_r(u) = \sum_{j=0}^{\infty} \frac{-(-2)^{j+1} u (\tau_0 u)^{j(\alpha_r+\alpha_l)}}{[(\tau_0 u)^{\alpha_r} + (\tau_0 u)^{\alpha_l}]^{j+1}} + \sum_{j=0}^{\infty} \frac{-(-2)^{j+1} u (\tau_0 u)^{j(\alpha_r+\alpha_l)+\alpha_l}}{[(\tau_0 u)^{\alpha_r} + (\tau_0 u)^{\alpha_l}]^{j+1}} \quad (\text{G3})$$

and thus

$$M_r(u) = \sum_{j=0}^{\infty} \frac{-(-2)^{j+1} u (\tau_0 u)^{j\alpha_l - \alpha_r}}{[1 + (\tau_0 u)^{\alpha_l - \alpha_r}]^{j+1}} + \sum_{j=0}^{\infty} \frac{-(-2)^{j+1} u (\tau_0 u)^{(j+1)\alpha_l - \alpha_r}}{[1 + (\tau_0 u)^{\alpha_l - \alpha_r}]^{j+1}}. \quad (\text{G4})$$

After further simplifications, we get

$$M_r(u) = \sum_{j=0}^{\infty} \frac{-(-2)^{j+1} \tau_0^{j\alpha_r - \alpha_l} u^{j\alpha_l - \alpha_r + 1}}{(\tau_0^{\alpha_r - \alpha_l} + u^{\alpha_l - \alpha_r})^{j+1}} + \sum_{j=0}^{\infty} \frac{-(-2)^{j+1} \tau_0^{j\alpha_r} u^{(j+1)\alpha_l - \alpha_r + 1}}{(\tau_0^{\alpha_r - \alpha_l} + u^{\alpha_l - \alpha_r})^{j+1}} \quad (\text{G5})$$

or

$$M_r(u) = \sum_{j=0}^{\infty} \frac{-(-2)^{j+1} \tau_0^{j\alpha_r - \alpha_l} u^{(j-1)\alpha_r + 1} u^{j(\alpha_l - \alpha_r)}}{(\tau_0^{\alpha_r - \alpha_l} + u^{\alpha_l - \alpha_r})^{j+1}} + \sum_{j=0}^{\infty} \frac{-(-2)^{j+1} \tau_0^{j\alpha_r} u^{\alpha_l + (j-1)\alpha_r + 1} u^{j(\alpha_l - \alpha_r)}}{(\tau_0^{\alpha_r - \alpha_l} + u^{\alpha_l - \alpha_r})^{j+1}}. \quad (\text{G6})$$

These results generalize the results of expressions (F4a) and (F4b) as well as those of expressions (F5a) and (F5b).

Via inverse Laplace transformation, for $\alpha_r < \alpha_l$ we arrive in real-time domain at

$$M_r(t) = \sum_{j=0}^{\infty} -(-2)^{j+1} \tau_0^{(j-1)\alpha_r - 1} {}_0D_t^{(j-1)\alpha_r + 1} \left[\left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r - 1} E_{\alpha_l - \alpha_r, \alpha_l - \alpha_r, j+1} \left(- \left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r} \right) \right] \\ + \sum_{j=0}^{\infty} -(-2)^{j+1} \tau_0^{(j-1)\alpha_r + \alpha_l - 1} {}_0D_t^{\alpha_l + (j-1)\alpha_r + 1} \left[\left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r - 1} E_{\alpha_l - \alpha_r, \alpha_l - \alpha_r, j+1} \left(- \left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r} \right) \right]. \quad (\text{G7})$$

Via taking the inverse Laplace transform of the asymptotic results of Eqs. (F4a) and (F4b) and after substituting them into Eq. (C2) in the case of the AMGD with $\alpha_r < \alpha_l$, we get the generalized Fokker-Planck-Kolmogorov equation of the form

$$\frac{\partial \rho(x, t)}{\partial t} \approx \frac{\sigma_+}{4} \frac{\partial^2}{\partial x^2} {}_0D_t^{1-\alpha_r} \left[\gamma_l^{-1} \tau_0^{-\alpha_r - 1} \left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r - 1} E_{\alpha_l - \alpha_r, \alpha_l - \alpha_r} \left(-\gamma_{rl} \left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r} \right) * \rho(x, t - \tau) \right] \\ + \frac{1}{4a} \frac{\partial^2}{\partial x^2} {}_0D_t^{1+\alpha_l - \alpha_r} \left[t^{\alpha_l - \alpha_r - 1} E_{\alpha_l - \alpha_r, \alpha_l - \alpha_r} \left(-\gamma_{rl} \left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r} \right) * \rho(x, t - \tau) \right] \\ + \frac{1}{4b} \frac{\partial^2}{\partial x^2} {}_0D_t^1 \left[\gamma_{rl} \tau_0^{-1} \left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r - 1} E_{\alpha_l - \alpha_r, \alpha_l - \alpha_r} \left(-\gamma_{rl} \left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r} \right) * \rho(x, t - \tau) \right] \\ + \frac{\sigma_-}{\sqrt{\pi}} \frac{\partial}{\partial x} {}_0D_t^{1-\alpha_r} \left[\gamma_l^{-1} \tau_0^{-\alpha_r - 1} \left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r - 1} E_{\alpha_l - \alpha_r, \alpha_l - \alpha_r} \left(-\gamma_{rl} \left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r} \right) * \rho(x, t - \tau) \right] \\ - \frac{1}{\sqrt{\pi a}} \frac{\partial}{\partial x} {}_0D_t^{1+\alpha_l - \alpha_r} \left[t^{\alpha_l - \alpha_r - 1} E_{\alpha_l - \alpha_r, \alpha_l - \alpha_r} \left(-\gamma_{rl} \left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r} \right) * \rho(x, t - \tau) \right] \\ + \frac{1}{\sqrt{\pi b}} \frac{\partial}{\partial x} {}_0D_t^1 \left[\gamma_{rl} \tau_0^{-1} \left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r - 1} E_{\alpha_l - \alpha_r, \alpha_l - \alpha_r} \left(-\gamma_{rl} \left(\frac{t}{\tau_0} \right)^{\alpha_l - \alpha_r} \right) * \rho(x, t - \tau) \right], \quad (\text{G8})$$

and a similar equation can be obtained for the case $\alpha_r > \alpha_l$. Here, the Riemann-Liouville fractional operator is defined as [67]

$${}_0D_t^{1-\alpha} \rho(x, t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t dt' \frac{\rho(x, t')}{(t - t')^{1-\alpha}} \quad (\text{G9})$$

and the asterisk denotes the convolution, i.e.

$$f(t) * g(t) = \int_0^t f(\tau) g(t - \tau) d\tau, \quad (\text{G10})$$

see, e.g., Eq. (2.237) in Ref. [68].

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