

Criticality in one-dimensional field theories with mesoscopic, infinite-range interactions

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This research investigates a class of one-dimensional theories characterized by a distinctly defined infinite interaction range. We propose that such theories emerge naturally through a mesoscopic feedback mechanism. In this proof-of-concept study, we examine Ising-type models and a model with continuous $O(3)$ symmetry, and demonstrate that the natural emergence of phase transitions, criticality, spontaneous symmetry breaking and previously unidentified universality classes is evident. The framework introduced here holds particular relevance for monolayer spintronics research, where the ultimate goal is to achieve a strong ferromagnetic order at room temperature.

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I. INTRODUCTION

Phase transitions and critical phenomena in low-dimensional systems have long served as a testing ground for statistical field theory. The Hohenberg-Mermin-Wagner theorem [1,2] provides one of the cornerstones of this field, establishing that systems with continuous symmetries and short-range interactions cannot exhibit spontaneous symmetry breaking at finite temperature in one or two dimensions. The theorem applies to models such as the isotropic Heisenberg or XY ferromagnet, where the order parameter transforms continuously under $SU(2)$ or $U(1)$ symmetry. However, the argument does not constrain systems with discrete symmetries—such as the Ising model—or those with infinite- or long-range couplings, where mean-field behavior can dominate even in one dimension.

Dyson [3] demonstrated that phase transitions can occur in one-dimensional (1D) Ising models with algebraically decaying couplings, $J(r) \sim r^{-\alpha}$ for $1 < \alpha < 2$, and later rigorous work by Fröhlich and Spencer [4] established a transition for $1/r^2$ interactions. These studies clarified that the presence or absence of long-range order depends not on dimensionality alone, but crucially on the effective range and structure of interactions. The existence of phase transitions for the 1D ferromagnetic Ising model with long-range two-body interaction has subsequently been an object of much research [5–9].

In this paper we develop 1D systems in which the interaction between elementary degrees of freedom depends on mesoscopic observables such as the magnetization or energy density. This feedback mechanism endows the model

with effectively infinite-range interactions while preserving a clear physical interpretation in terms of collective mesoscopic processes. The resulting theory can be viewed as a generalization of the Curie-Weiss model [10,11], where each spin interacts equally with every other spin. The Curie-Weiss model features an infinite range interaction, but because of the “democratic” interaction of all spins with each other, it does not have a spatial structure to define dimensionality. Our models below, however, share the infinite range interaction with the Curie-Weiss model but inherit dimensionality, geometry, lattice structure, size, and topology of the related model with local interactions. We will find that it retains analytic tractability but allows for nontrivial extensions of criticality.

Our initial focus is on Ising-type models with discrete symmetry. We are interested in robust long-range order and phase transitions. The interest in these topics has seen renewed interest with the discovery of monolayer ferromagnets—atomically thin materials that exhibit ferromagnetism down to a single layer—which are considered a major breakthrough for spintronics [12–14]. Transition temperatures are of particular interest for the quest for room temperature, monolayer ferromagnets [15]. We will demonstrate that a mesoscopic feedback mechanism can produce stable long-range order even in 1D systems. Within our framework, we derive closed-form expressions for the free energy and demonstrate how the nature of the feedback term determines the order of the transition. When the free energy depends quadratically on the order parameter, the transition is continuous (second order); when cubic or higher-order terms dominate, the transition becomes discontinuous (first order). These results reproduce the expected Landau-theoretic structure of phase transitions while revealing new possibilities for feedback-induced universality classes in 1D systems.

The framework naturally extends to continuous symmetries. Motivated by the question of whether the proposed mechanism can induce long-range order in systems with

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continuous degrees of freedom, we examine the 1D $O(3)$ model under analogous mesoscopic coupling. Despite the 1D geometry, we find that the feedback mechanism induces spontaneous symmetry breaking for sufficiently strong coupling. This behavior demonstrates that mesoscopic feedback can effectively act as a nonlocal mediator of interactions, producing ordered phases even where conventional short-range models would remain disordered. The $O(3)$ case thus serves as an important conceptual extension, linking the generalized Landau-theoretic picture developed here to a broader class of continuous-symmetry systems.

Beyond their specific analytic results, these models provide a framework for exploring emergent universality in systems with feedback-induced long-range correlations. Universality—the insensitivity of critical behavior to microscopic details—has long been a central organizing principle of statistical mechanics [16–18]. Its manifestations range from phase transitions in magnets and superfluids to collective behavior in biological and neural systems [19–22]. The density-of-states approach developed here provides a complementary route to universality, in which critical scaling emerges not from proximity in real space, but from a collective coupling across the system.

The remainder of this paper is organized as follows. Section IA introduces the class of models and formulates the mesoscopic feedback mechanism. Section II presents analytical results for the generalized Ising model, including its free energy, order parameters, and critical exponents. Section III studies in detail the criticality of the 1D mesoscopic field theory. Section IV extends the analysis to the $O(3)$ model, demonstrating how feedback-driven coupling can produce symmetry breaking in continuous systems. Section V summarizes our conclusions.

A. Rise of 1D field theory with mesoscopic long-range interactions

In this paper we introduce a class of 1D nonlocal field theories in which the fundamental constituents interact through conventional local interactions, yet the parameters governing these interactions are dynamically dependent on global field configurations.

The standard Ising or Heisenberg model has short-range interactions with no long-range order in one dimension. In the Kac limit [23], the interaction potential is rescaled so that its range becomes large compared to the interparticle distance but remains small compared to the system size. One obtains a mean-field-type description while retaining the system’s underlying geometry. This construction provides a mesoscopic, finite-range analog of infinite-range Curie-Weiss interactions. Our models described below have an infinite range interaction as the Curie-Weiss but retain the system’s underlying geometry and dimensionality as models in the Kac limit.

Our mesoscopic feedback mechanism allows the system’s parameters—such as coupling strengths or temperature-like quantities—to be modulated by the state of the entire field. As a result, these systems naturally exhibit complex collective behavior, including self-organized criticality, emergent phase transitions, and rich dynamical phenomena not present in purely local theories. The interplay between local dynam-

ics and global feedback encapsulates a distinctive paradigm for modeling statistical systems, with potential applications ranging from condensed matter physics to biological and social systems.

In the standard Ising model on the line with open boundary conditions, the degrees of freedom are spins $z_i \in \{-1, +1\}$, $i = 1 \dots L$, where L is the total number of spins. The *local* standard action is given by the standard nearest-neighbor interaction:

$$S \propto \sum_{i=1..L-1} z_i z_{i+1}.$$

In this paper we will explore versions of Ising models, the actions of which are (simple) functions of S , such as

$$S^2 \propto \left(\sum_{i=1..L-1} z_i z_{i+1} \right)^2 = \sum_{i,j=1..L-1} z_i z_{i+1} z_j z_{j+1}.$$

This induces a type of nonlocality where all spins interact with each other. We motivate the study of models with actions of this form below by showing that this type of interaction can occur naturally in many physical systems.

Denote the action of a field theory by $S(\theta; \kappa)$, where $\theta(x)$ encapsulates the degrees of freedom in one dimension spanned by the variable x , and κ encapsulates the model parameters. If S is chosen so that $S(\theta; \kappa)$ has only short-range interactions for the field θ for given parameters κ , nonlocality can be induced by introducing a dependence of the parameters κ on observables (expectation values) of the field theory such as action density or magnetization. Examples include models for social dynamics or opinion models, where individuals modify their openness to influence (noise level) based on the perceived consensus [24].

To illustrate how nonlocality arises, we will explore in more detail the example of feedback-driven effective temperatures in protein thermodynamics. We adopt a rigid polymer model where polymers are described by a set of angles $\theta_1 \dots \theta_L$ specifying the position of rigid links along the polymer chain [see Figs. 1(a) and 1(b)]. A typical local action is given by

$$S = \sum_{a,i=1..L-1} \beta \cos(\theta_{i+1}^{(a)} - \theta_i^{(a)}) + V(\theta),$$

where i labels the position and a the polymer in a multi-polymer ensemble. The parameter β describes stiffness. For low β , the angles are (almost randomly) oriented leading to a low action “crumpled-up” state of the polymers, whereas for large β , the state with $\theta_i = \text{constant}$ is preferred leading to “stretched-out” polymer ensembles. The potential V ensures that the polymers are confined to the encasing [shown as the boxes in Fig. 1(b)]. More interaction terms could be added that ensure self-avoidance.

Let us now introduce mesoscopic feedback by considering polymers in a dense environment. If the polymers are in a low action state and if each of them is mainly confined to a sphere then, with self-avoidance and densely packed spheres, it is more difficult for the polymers to move to the higher action state of a linear configuration. Consequently, the effective stiffness parameter should be lower. Conversely, if the

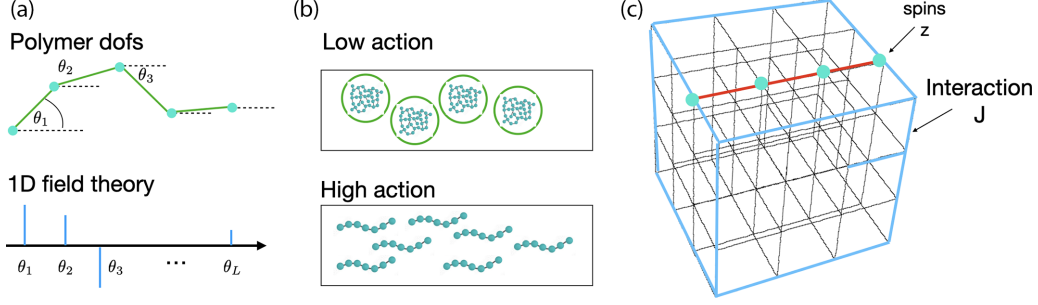


FIG. 1. (a), (b) Effective statistical theory for polymer thermodynamics with mesoscopic feedback on model parameter. (c) Effective 1D field theory emerging at the 1D subdomain of a local 3D statistical field theory.

polymers are in a high action, linear state, it would be more difficult in a dense environment to reach a crumpled state and consequently the effective stiffness parameter should be higher.

Rather than working with a microscopic theory of many polymers, if we are only interested in their geometry we can switch to an effective description of a one-polymer state where the stiffness parameter is now taken as an increasing function of the *action density*

$$s := \frac{1}{L-1} \sum_{i=1..L-1} \cos(\theta_{i+1} - \theta_i),$$

for example, $\beta(s) = \kappa s$. Note that this is an intensive quantity as needed for the stiffness. A simple effective model for the angles $\theta_1 \dots \theta_L$ of a single polymer in a dense environment is then given by

$$S_{\text{eff}} = \sum_{i=1..L-1} \beta(s) \cos(\theta_{i+1} - \theta_i),$$

$$= \frac{\kappa}{L-1} \left(\sum_{i=1..L-1} \cos(\theta_{i+1} - \theta_i) \right)^2.$$

We next discuss a second motivation for a nonlocal 1D field theory, which belongs to the large class of models that emerge from higher dimensional field theory. These emerging theories (QFTs) are central to understanding holography, dimensional reduction, and effective field theories. Examples are rooted in both string theory [25,26] and condensed matter physics [27], where low-dimensional QFTs emerge through compactification, boundary effects, or dualities.

To illustrate how a mesoscopic type of interaction with infinite range emerges from a local three-dimensional (3D) field theory, we consider Ising spins $z_i = \pm 1$ that exist in the 1D subdomain at the surface of a 3D cube with labelled by i [see Fig. 1(c)]. The spins interact with the field J_ℓ associated with the links of the cube. For illustration purposes only, our theory is described by the action

$$S = \sum_{\ell \in (x,y)} z_x J_\ell z_y - \frac{1}{4\kappa} \sum_{\ell} J_\ell^2 - \kappa_2 \sum_{x, (\ell_1 \ell_2 \ell_3) \in x} \left[\frac{(J_{\ell_1} - J_{\ell_2})^2}{J_{\ell_1}^2 + J_{\ell_2}^2} + \frac{(J_{\ell_1} - J_{\ell_3})^2}{J_{\ell_1}^2 + J_{\ell_3}^2} + \frac{(J_{\ell_2} - J_{\ell_3})^2}{J_{\ell_2}^2 + J_{\ell_3}^2} \right].$$

The first sum extends over all links ℓ of the lattice, and x and y are the lattice sites that are located at the boundary of the elementary link ℓ , the second sum extends over all links ℓ , and in the third sum $\ell_1, \dots, \ell_3$ are the three elementary links that “belong” to a particular site x . The theory has only local interactions. It is described by the partition function

$$Z = \sum_{\{z\}} \int_J \exp\{S\}.$$

The 1D effective theory emerges by “integrating out” the interaction field J_ℓ . For the purposes of this illustration we are interested in two limits: first, for $\kappa_2 = 0$, the integral is Gaussian and can be performed by completing the square. By virtue of $(z_x z_y)^2 = 1$, we observe that the emerging action for the spin is constant, i.e., the effective spin theory is that of random spins. Second, in the limit $\kappa_2 \rightarrow \infty$, we find that fluctuations of the interaction field J are not suppressed though its scale:

$$J_1 = J_2 = J_3 = \dots =: J.$$

We find in this limit

$$S = J \sum_{x,y \in \ell} z_x z_y - \frac{L-1}{4\kappa} J^2.$$

Integrating over the scale J , we find that the emerging 1D field theory has infinite range but is of the mild mesoscopic type:

$$S_{\text{eff}} = \frac{\kappa}{L-1} \left(\sum_{x,y \in \ell} z_x z_y \right)^2.$$

II. PHASE TRANSITIONS IN ISING-LIKE MODELS IN ONE DIMENSION

A. Standard Ising model

Recall the standard nearest-neighbor Ising model with action given by

$$S = \sum_{\ell \in (xy)} z_x z_y, \quad (1)$$

where ℓ is an elementary link joining neighboring sites x and y . We start the exploration with a calculation of the density of states for the interaction S from (1):

$$\rho(E) = \sum_{\{z\}} \delta(E, S[z]), \quad (2)$$

where δ is the Kronecker symbol. This can be evaluated in closed form by a shift of variables:

$$z_x, x = 1, \dots, L \Rightarrow z_1, u_x, x = 1, \dots, L-1,$$

where

$$u_x = \begin{cases} 1 & \text{for } z_x = z_{x+1} \\ -1 & \text{for } z_x \neq z_{x+1} \end{cases}.$$

The action only depends on u :

$$S = \sum_{x=1}^{L-1} u_x = n - (L-1-n) = 2n - L + 1,$$

where $n \in [0, L-1]$ is the number u variables that are positive, leaving $L-1-n$ variables negative. The number of possibilities to render n of our $L-1$ variables positive is

$$\binom{L-1}{n}.$$

The density of states is given by

$$\rho(E) = 2 \binom{L-1}{n}, \quad E = 2n - L + 1. \quad (3)$$

Note that the factor 2 rises from summing over z_1 .

With $\rho(E)$ at our fingertips, we can derive the exact partition function. The same approach also works for models with nonlocal interactions, which we demonstrate in the next section. Inserting the “1,”

$$\sum_E \delta(E, S[z]) = 1, \forall \{z\}$$

into the partition function of the standard Ising model,

$$Z(\beta) = \sum_{\{z\}} \exp \{ \beta S[z] \}, \quad (4)$$

we change the order of summations and substitute E for $S[z]$ by virtue of the δ function,

$$\begin{aligned} Z(\beta) &= \sum_E \sum_{\{z\}} \delta(E, S[z]) \exp \{ \beta S[z] \} \\ &= \sum_E \exp \{ \beta E \} \sum_{\{z\}} \delta(E, S[z]) \\ &= \sum_E \exp \{ \beta E \} \rho(E) = \sum_E D^{(1)}(E), \end{aligned}$$

where the density of states for the nonlocal theory can be entirely expressed by the density $\rho(E)$ for the next-to-nearest-neighbors interaction:

$$D^{(1)}(E) = \exp \{ \beta E \} \rho(E). \quad (5)$$

As we illustrate in Sec. II D below, the density $D^{(1)}(E)$ possesses only a single maximum for all finite values of the inverse temperature β . We do not face the situation where two distinct values of E are maxima and equally likely, which signals the presence of a phase transition. We thus recover the familiar result that phase transitions are absent in a 1D statistical field theory with local interactions.

B. Ising model with S^2 nonlocality

We now show how the same approach can be used for the model with the S^2 action:

$$S^2 = \frac{1}{L-1} \sum_{\langle xy \rangle, \langle uv \rangle} z_x z_y z_u z_v. \quad (6)$$

The partition function we consider here is given by

$$Z(\kappa) = \sum_{\{z\}} \exp \left\{ \frac{\kappa}{2(L-1)} S^2[z] \right\}, \quad (7)$$

where S is the nearest-neighbor interaction in (1). We can consider the interaction between those spins depending on a medium and the interaction strength influenced by the *action density* $S/(L-1)$:

$$\beta(S)S, \text{ with } \beta(S) = \frac{\kappa}{2} \frac{S}{L-1}.$$

If more and more spins align, e.g., in a ferromagnetic setting, the effective interaction β between spins even increases further. The present model has a ferromagnetic antiferromagnetic symmetry (apart from the usual spin-flip symmetry of the standard Ising model). The transformation

$$\begin{aligned} z_x &\rightarrow z_x & \text{for } x \text{ even,} \\ z_x &\rightarrow -z_x & \text{for } x \text{ odd,} \end{aligned}$$

maps $S \rightarrow -S$. This is a symmetry since the probabilistic measure only depends on S^2 .

Following the steps of the previous subsection, we express the density of states of the S^2 model in terms of the density $\rho(E)$ of the nearest-neighbor interaction E [28]:

$$D^{(2)}(E) = \rho(E) \exp \left[\frac{\kappa}{2(L-1)} E^2 \right]. \quad (8)$$

Because of the ferro-antiferromagnetic symmetry, it is a symmetric function in E .

Using (3), the distribution E can be easily calculated numerically. The result is shown in Fig. 2 for several values of κ . We observe that if the interaction strength κ exceeds a critical value $\kappa_c \approx 1$, the most likely action is obtained for $\pm E$, $E \neq 0$. This indicates the spontaneous breakdown of the ferromagnetic antiferromagnetic symmetry. This would happen if the transition probability between the two possible ground states, i.e., ferromagnet and antiferromagnet, vanishes in the infinite volume limit $L \rightarrow \infty$. To explore this idea further, we study the marginal distribution $D^{(2)}(E=0)$ over $D^{(2)}_{\max}$ for three values $\kappa > \kappa_c$ as a function of L . We observe an exponential suppression of the probability:

$$D^{(2)}(E=0)/D^{(2)}_{\max} = e^{-\sigma(\kappa)L}. \quad (9)$$

This suppression occurs from faults in the (anti-) ferromagnetic order, and σ is called *interface tension*. If we dare to extrapolate without proof to the limit $L \rightarrow \infty$, we indeed observe the spontaneous breaking of the ferromagnetic antiferromagnetic symmetry. Figure 2(c) shows the behavior of the interface tension as a function of κ . The tension vanishes at the critical coupling κ_c and monotonically increases with κ from there on. The transition is second order.

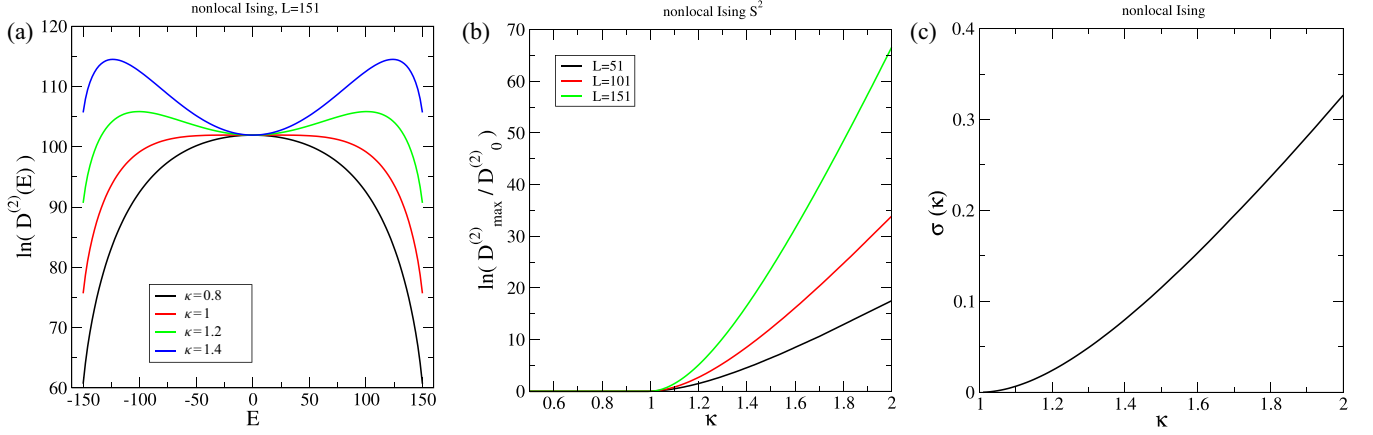


FIG. 2. (a) Marginal distribution for the nearest-neighbor interaction E (nonlocal interaction) for several interactions strengths κ . (b) Maximum marginal probability over that at $E = 0$ for several extensions L . (c) Interface tension σ as a function of the coupling strengths κ .

C. Ising model with S^3 nonlocality

The S^3 model possesses only the spin-flip symmetry, which is familiar from the standard local Ising model,

$$z_x \rightarrow -z_x, \forall x.$$

While the standard 1D Ising model does not have a spontaneous breaking of a symmetry, the interesting question is whether a nonlocal version that shares the same symmetry pattern can break the symmetry spontaneously.

The S^3 model can be understood as an Ising model with effective interaction induced by a medium that is sensitive to the energy stored in the spin system. This time the effective interaction,

$$\beta(S) = \frac{\omega}{3} \left(\frac{S}{L-1} \right)^2,$$

is proportional to S^2 and the partition function is given by

$$\begin{aligned} Z(\omega) &= \sum_{\{z\}} \exp \{ \beta(S[z]) S[z] \} \\ &= \sum_{\{z\}} \exp \left\{ \frac{\omega}{3(L-1)^2} S^3[z] \right\}. \end{aligned} \quad (10)$$

It is ideal to use the nearest-neighbor interaction S as a collective coordinate since it is a good litmus test for symmetry breaking. In the unbroken so-called *paramagnetic phase*, the disorder in a particular spin configuration implies $\langle S \rangle = 0$ already upon the spatial average. The variable S is also insensitive to the spin-flip:

$$z \rightarrow -z \rightarrow S \rightarrow S.$$

In the *ferromagnetic phase* and even for the finite volume ergodic stochastic simulation, we expect $\langle S \rangle \neq 0$. The partition function again can be expressed in closed form as an integral over the nearest-neighbor interaction E :

$$\begin{aligned} Z(\omega) &= \sum_E D^{(3)}(E), \\ D^{(3)}(E) &= \rho(E) \exp \left[\frac{\omega}{3(L-1)^2} E^3 \right], \end{aligned} \quad (11)$$

with $\rho(E)$ already calculated in (3). For small enough ω , the distribution $D^{(3)}(E)$ has the global maximum at $E = 0$ (see Fig. 3). Even at finite volumes, the most likely state is *paramagnetic*. We find that for sufficiently large ω , the distribution $D^{(3)}(E)$ develops a second local maximum at large $E > 0$. The local maximum becomes the global one for $\omega > \omega_c \approx 2.0$. The value for ω_c slightly depends on the volume L but seems to converge to a finite value with increasing L . If we consider the rift between the maximum of the marginal distribution $D^{(3)}(E)$ and its minimum, it increases exponentially with increasing L [see Fig. 3(c)], suggesting that the reflection symmetry is indeed spontaneously broken in the infinite volume limit. Here we observe that the transition is first order.

D. The large volume limit

Let us further explore the phase transitions of the previous subsections in particular the behavior of order under an increase of the volume L . To this aim, we introduce the LLR coefficient $a(E)$ by [29,30]

$$a(E) = - \frac{d}{dE} \ln \rho(E). \quad (12)$$

[We use the notation $a(E; L)$ for this quantity where it is useful to emphasize the dependence on L .] This derived quantity is *intensive*, i.e., there exists a well-defined limiting function $\bar{a} :]-1, 1[\setminus \{0\} \rightarrow \mathbb{R}$ satisfying

$$\lim_{L, E \rightarrow \infty} a(E; L) = \bar{a}(e),$$

where the limit is taken under the restriction $E/(L-1) \rightarrow e$.

Let us now consider the distribution $D^{(2)}(E)$ for the theory with the S^2 action, which can be written in terms of the LLR coefficient as

$$D^{(2)}(E) = D^{(2)}(0) \exp \left\{ - \int_0^E \left[a(E') - \frac{\kappa}{L-1} E' \right] dE' \right\}. \quad (13)$$

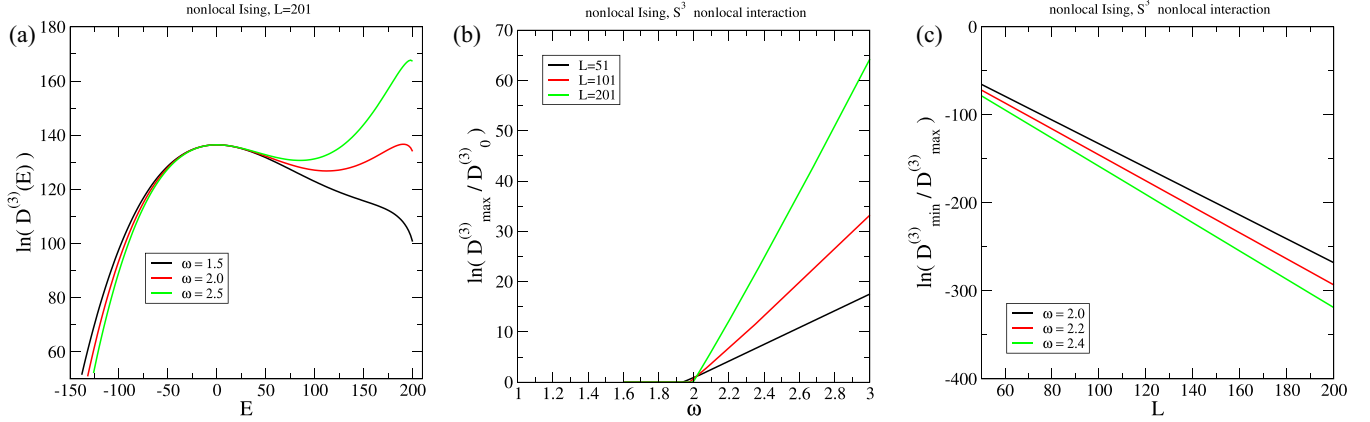


FIG. 3. (a) Distribution marginal distribution for the nearest-neighbor interaction E . (b) Suppression of the symmetric state $E = 0$ in the broken phase. (c) Gap formation in the limit $L \rightarrow \infty$.

Changing the variable from the extensive action E to the interaction density $e := E/(L-1)$, we find

$$D^{(2)}(E) = D^{(2)}(0) \exp \left\{ -(L-1) \int_0^e [\bar{a}(e') - \kappa e'] de' \right\}. \quad (14)$$

We recover the familiar result from thermodynamics that probability densities for aggregate variables sharply peak around their most likely values. Extrema are obtained by

$$\frac{dD^{(2)}(E)}{dE} = 0 \Rightarrow \bar{a}(e) = \kappa e. \quad (15)$$

For the Ising model (3), the energies are discrete, and we define the LLR coefficient by a centered difference:

$$\begin{aligned} a(E) &= -\frac{1}{4} \left[\ln \left(\frac{L-1}{n+1} \right) - \ln \left(\frac{L-1}{n-1} \right) \right] \\ &= -\frac{1}{4} \ln \left(\frac{L}{n+1} - 1 \right) - \frac{1}{4} \ln \left(\frac{L}{n} - 1 \right) \quad (16) \\ E &= 2n - L + 1, n = 1, \dots, L-2. \end{aligned}$$

Asymptotically for large L , we find for the limiting function

$$\bar{a}(e) = \frac{1}{2} \ln \left(\frac{1+e}{1-e} \right). \quad (17)$$

To explore this limit numerically, we define the ratio

$$R = \frac{a(E; L)}{\bar{a}(e)}, e \in]-1, 1[\setminus \{0\}. \quad (18)$$

Even for moderate sizes such as $L = 101$, we find an agreement within 4% for most of the e domain (see Fig. 4). Deviations are most significant at the boundaries of the domain of support. Let us now use these findings to screen for phase transitions in the infinite volume limit using (15).

Standard local Ising model: Here we need to solve $\bar{a}(e) = \beta$. The function $\bar{a}(e)$ is monotonic leaving us with only one intersection point for each β value. This provides the expectation value $\langle S \rangle$, which smoothly moves from negative values ($\beta < 0$) and antiferromagnetic behaviour to ferromagnetic behavior for $\beta > 0$. As in line with the Mermin-Wagner theorem, we do not encounter a phase transition.

Nonlocal Ising model with S^2 action: The crucial equation here becomes $\bar{a}(e) = \kappa e$. Possible intersection points are illustrated in Fig. 4(b). If the slope κ of the straight line is smaller than 1, the only intersection point is at $e = 0$ and the spin system is in the paramagnetic phase. For $\kappa > 1$, we find three intersection points. Those correspond to two maxima and one minimum in the probability distribution $D^{(2)}$. We observe two distinct maxima: a ferromagnetic state and its counterpart, an antiferromagnetic state, have equal probability (symmetry) and are more likely than the paramagnetic state at $e = 0$. If we start increasing κ from low values, the transition from the one to the two maximum scenario occurs smoothly at $\kappa = \kappa_c = 1$. The transition is second order.

Nonlocal Ising model with S^3 action: We now encounter $\bar{a}(e) = \omega e^2$. There is always an intersection point at the paramagnetic point $e = 0$. The question is whether there are others, and then whether those give rise to a more likely state. Again, it is easily observed that if ω is small, the only maximum is at the paramagnetic point. If ω exceeds a critical value, more candidates for the ground state emerge [see Fig. 4(c)]: while $e = 0$ will always be a maximum, the corresponding value for $D^{(3)}$ at the second maximum at large e (ferromagnetic state) will decide whether this state is more likely. The numerical experiments from the previous subsection indicate that this is indeed the case for $\omega > 2$. The transition is abrupt and first order: for ω slightly below the critical value, the spin system is paramagnetic $e = 0$. With ω slightly above ω_c , the system can be already strongly ferromagnetic $e \approx 1$.

III. CRITICALITY OF THE NONLOCAL S^2 ISING MODEL

We now focus on the nonlocal 1D Ising model with partition functions (7) and (10) with respective actions

$$A_2 = \left(\frac{S}{L-1} \right) S, \quad (19)$$

$$A_3 = \left(\frac{S}{L-1} \right)^2 S. \quad (20)$$

As explained in the previous subsections, both theories have different symmetry patterns. In both cases a spontaneous

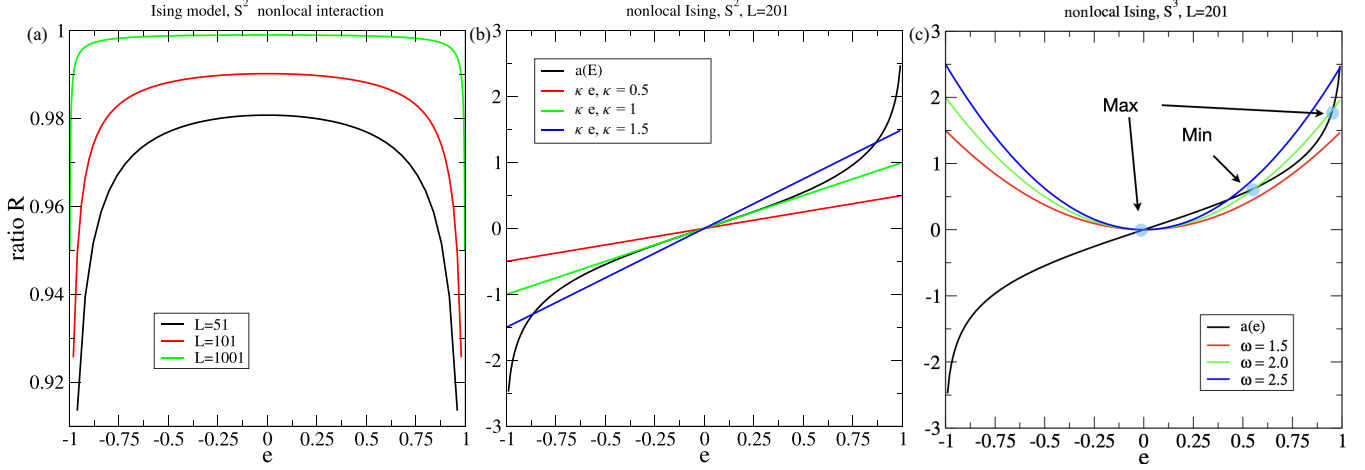


FIG. 4. (a) LLR coefficient at finite volume over its asymptotic value (17). Screening for phase transitions using the LLR coefficient for the S^2 (b) and S^3 (c) case.

breakdown of symmetry occurs, which is second order in the former and first order in the latter case.

A. Internal energy and specific heat

The internal energy density is defined by

$$\langle A_\ell \rangle / (L - 1), \quad \ell = 2, 3. \quad (21)$$

For the S^2 theory, we observe typical second-order behavior: the internal energy density is small in the symmetric phase and then rises smoothly for couplings κ greater than the critical value. For large volumes L , this behavior is more pronounced. The energy density develops a cusp in the infinite volume limit but remains smooth (see Fig. 5). By contrast, the S^3 theory shows a sharp rise at the critical coupling ω_c , the steepness of which increases with increasing volume. Increasing L from a

moderate 101 to 1001 lets us anticipate the first-order jump of the energy density in the infinite volume limit (see Fig. 5).

In the following, we will focus on the interacting case of the second order transition of the nonlocal 1D Ising model with S^2 action.

The specific heat C is a further thermodynamical observable with high phenomenological relevance:

$$C_L(\kappa) = (\langle A^2 \rangle - \langle A \rangle^2) / (L - 1). \quad (22)$$

It can be viewed as an integrated correlation function and is therefore proportional to a power of the correlation length ξ . Close to the second-order transition, the correlation length ξ diverges with a scaling law of the type

$$\xi \propto \left| 1 - \frac{\kappa}{\kappa_c} \right|^{-\nu}, \quad \nu > 0, \quad (23)$$

where ν is the so-called critical exponent. Close to the transition, the singularity is usually described by a *power law* or *logarithmic* behavior:

$$C_L(\kappa) \propto |t|^{-\alpha'}, \quad \text{for } \kappa > \kappa_c, \quad t := \frac{\kappa_c - \kappa}{\kappa_c}, \quad (24)$$

$$C_L(\kappa) \propto -\ln |t|. \quad (25)$$

We can relate the divergence of C to that of the correlation length by noting $|t| \propto \xi^{-1/\nu}$. At finite volume L and for $\kappa = \kappa_c$, the (in the infinite volume divergent) correlation length ξ is limited by L . We then observe

$$C_L(\kappa_c) \propto (|t|^{-\nu})^{\alpha'/\nu} \propto \xi^{\alpha'/\nu} = L^{\alpha'/\nu}, \quad (26)$$

$$C_L(\kappa_c) \propto \ln \xi = \ln L. \quad (27)$$

The later equations are ideal to study finite-size scaling: We calculate the specific heat for several values L as a function of κ . At the (pseudo-) critical coupling $\kappa_{\max} \approx \kappa_c$, the specific heat is maximal. From the function

$$L \rightarrow C_L(\kappa_{\max}),$$

we then can infer the scaling via (26) or (27).

Figure 6(a) shows the specific heat as a function of κ for several lattice sizes L . We observe the signature of a second-

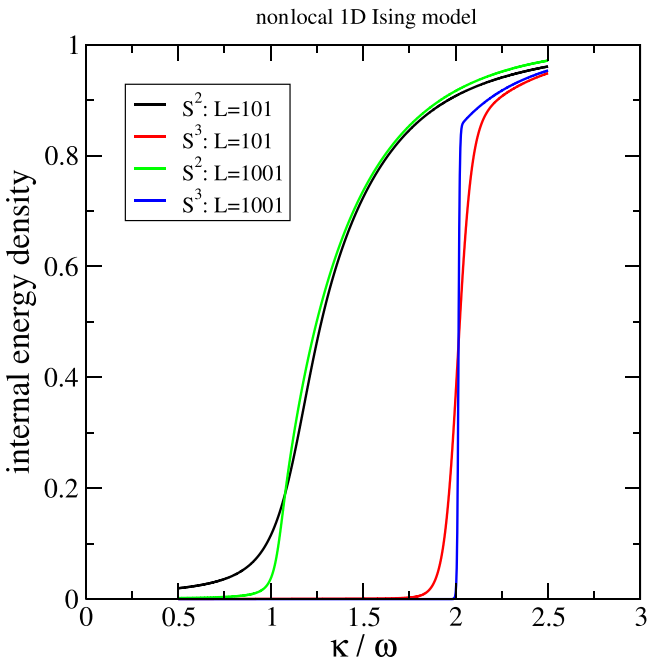


FIG. 5. The internal energy density as a function of the couplings for the S^2 and S^3 theory.

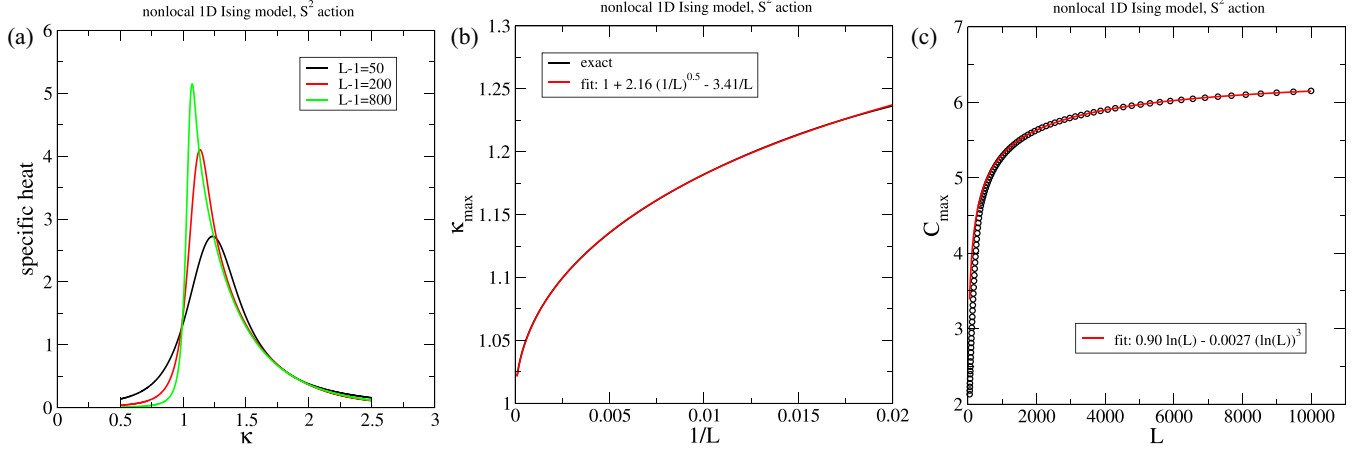


FIG. 6. (a) The specific heat as function of the coupling strength κ for several volumes L . (b) Position of the maximum of the specific heat as a function of $1/L$. (c) Scaling of the maximum of the specific heat with the system size L .

order phase transition: the maximum of the specific heat increases with increasing L , and the position of the maximum, i.e., $\kappa_{\max}$ moves closer to the infinite volume limit $\kappa_c = 1$. Figure 6(b) shows the evolution of $\kappa_{\max}$ with the inverse system size. We model the finite-size corrections to the critical value by the power series

$$\kappa_{\max} = 1 + a_1 \left(\frac{1}{\sqrt{L}} \right) + a_2 \left(\frac{1}{\sqrt{L}} \right)^2 + \dots$$

We find that the first three terms of the expansion excellently fit the curve for L ranging from 50 to 10 000.

Let us now investigate the scaling with the system size L . Figure 6(c) indicates that the scaling with L is logarithmically slow. The data for $L > 1000$ are well represented by the fit function:

$$C_L(\kappa_{\max}) = 0.90 \ln L - 0.0027 \ln^3(L).$$

The scaling is therefore similar to the case of two-dimensional (2D) Ising model with standard next-to-nearest neighbor interaction, which is also proportional to $\ln L$.

B. Scaling of the correlation length ξ

A convenient way to extract the critical coupling and the exponent ν is by means of the Binder cumulant

$$U_L = 1 - \frac{\langle O^4 \rangle}{3 \langle O^2 \rangle^2}, \quad (28)$$

where O is the order parameter of the statistical system. In the broken phase, order parameter expectation values are dominated by a nonvanishing scale O_s :

$$\langle O \rangle = 0 \quad (\text{finite } L), \quad \langle O^4 \rangle \approx O_s^4, \quad \langle O^2 \rangle \approx O_s^2,$$

which suggests that the Binder cumulant approaches $2/3$ for large couplings $\kappa \gg \kappa_c$. In the disordered phase, the probability distribution of the order parameter O is approximately

Gaussian: the moments fulfill the relation

$$\langle O^4 \rangle \approx 3 \langle O^2 \rangle^2$$

and the cumulant vanishes for $\kappa \ll \kappa_c$.

At the critical coupling $\kappa = \kappa_c$, the Binder cumulant U_L becomes independent of the system size (to leading order $1/L$), leading to a crossing point when U_L is plotted as a function of κ for several system sizes L . The critical value U_L for $\kappa = \kappa_c$ depends on the dimensionality of the system and the symmetry of the order parameter, but it is otherwise universal. It therefore can be used to categorize statistical systems in universality classes.

Close to the phase transition, the only relevant ratio of scales is the system size L over the correlation length ξ . The Binder cumulant then obeys a finite-size scaling form:

$$U_L(\kappa) = f((\kappa - \kappa_c) L^{1/\nu}), \quad (29)$$

where f is a universal scaling function.

The critical exponent ν is conveniently estimated from a study of the derivative of the Binder cumulant:

$$\lim_{\kappa \rightarrow \kappa_c^+} \frac{d}{d\kappa} U_L(\kappa) = L^{1/\nu} f'(0^+). \quad (30)$$

With (28), we find

$$\begin{aligned} \frac{d}{d\kappa} \langle O^n \rangle &= \langle O^n A_2 \rangle - \langle O^n \rangle \langle A_2 \rangle, \\ \frac{d}{d\kappa} U_L(\kappa) &= - \frac{\langle O^4 A_2 \rangle - \langle O^4 \rangle \langle A_2 \rangle}{3 \langle O^2 \rangle^2} \\ &\quad + \frac{2 \langle O^4 \rangle [\langle O^2 A_2 \rangle - \langle O^2 \rangle \langle A_2 \rangle]}{3 \langle O^2 \rangle^3}. \end{aligned} \quad (31)$$

Before we can implement this program, we still need to settle the question of the order parameter O . The S^2 theory has a ferromagnetic/antiferromagnetic symmetry. The next-to-nearest neighbor interaction S changes sign under this symmetry transformation, since it is positive in the ferromagnetic phase and negative in the antiferromagnetic one. We therefore choose

$$O = S[z]/(L-1). \quad (32)$$

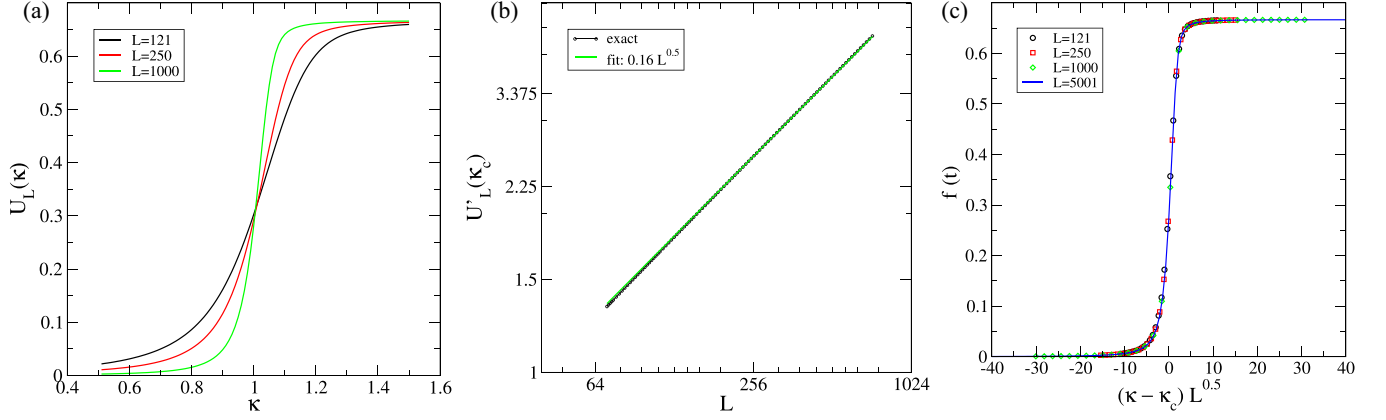


FIG. 7. (a) The Binder cumulant (28) as a function of κ for three system sizes L . (b) The derivative of the Binder cumulant in a log-log plot revealing the scaling (30). (c) Finite-size scaling function f for three values of L .

Again all results can be obtained exactly, and the one for the Binder cumulant is shown in Fig. 7(a). We find that the cumulants for three sizes L intersect close to the critical coupling $\kappa_c = 1$. Asymptotically, the intersection point approaches the critical coupling but the approach is slow due to the “square-root” behavior and thus large finite-size corrections [see Fig. 6(b)]. We have also calculated the derivative of the Binder cumulant (31) at the critical coupling κ_c as a function of L [see Fig. 7(b)]. The result in the log-log plot is perfectly fitted by a straight line bearing witness to the scaling law (30). From the fit, we find the critical exponent is compatible with

$$\nu \approx 2. \quad (33)$$

Figure 7(c) shows the finite-size scaling function f in (29) as a function of the scaling variable

$$t := (\kappa - \kappa_c) L^{1/\nu}, \quad \nu = 2.$$

We perform a scaling test by extracting f for three different lattice sizes. We find excellent scaling over a wide range of t . Extracting the critical Binder value $B_c = f(0)$ from the calculation with the largest system size $L = 5001$, we find

$$B_c \approx 0.2753 \dots \quad (34)$$

For a horizon screening we list the critical exponent and the Binder Cumulant at criticality for several well-studied models in Table I.

TABLE I. Known critical indices and Binder values for selected theories in several dimensions.

Model	Dim	ν	B_c	Comment
Ising	2D	1	0.6107 ...	Exact solution
Ising	3D	0.630..	0.465 ...	Numerical
$O(2)$	2D	BKT	0.636 ...	No true criticality
$O(2)$	3D	0.671..	0.586	Superfluid helium
$O(3)$	3D	0.711..	0.618 ...	Rotational sym break
Three-state Potts	2D	5/6	0.613 ...	Second order
Four-state Potts	2D	2/3	0.66 ...	Log corrections
Ising	$\geq 4D$	1/2	0.2705	Landau theory applies

When it comes to the 1D-Ising model with a long range interaction of the type $J(r) \propto 1/r^{1+\sigma}$, $\sigma < 1$, the model possesses a phase transition for $0 < \sigma \leq 1/2$ with mean field critical exponents. For $1/2 < \sigma < 1$, both ν and B_c , vary with σ [31]. Higher dimensional field theories with local interactions have typically $\nu < 1$.

IV. GENERALIZATION TO CONTINUOUS SYMMETRIES

A. The 1D $O(3)$ model with S^2 action

Continuous symmetries play an important for the inhibition of spontaneous symmetry breaking in low dimensions. The reason is the existence of massless excitations—the Goldstone boson—that creates infrared singularities in low-dimensional momentum integrals. The role of the Goldstone boson can only be explained beyond perturbation theory and is to effectively restore the broken symmetry (see the Coleman theorem [32]). It is therefore important that the above outlined mechanism is still in operation for low-dimensional field theories with continuous symmetry.

We study here a generalized $O(3)$ model with unit vectors $\vec{n} \in \mathbb{R}^3$ as the degrees of freedom. The order parameter is given by

$$S[\vec{n}] = \sum_{l \in \langle xy \rangle} \vec{n}_x \cdot \vec{n}_y. \quad (35)$$

For the field-dependent mesoscopic interaction, we choose one that reinforces the (anti-) ferromagnetic interaction. The interaction strength and partition function are given by

$$\beta(S[\vec{n}]) = \frac{\kappa}{2(L-1)} S[\vec{n}], \quad (36)$$

$$Z(\kappa) = \int \prod_x [d\Omega_x] \exp\{\beta(S[\vec{n}]) S[\vec{n}]\}, \quad (37)$$

where Ω_x is the angular parametrization of the unit vector and $d\Omega_x$ is the related and $O(3)$ invariant Haar measure:

$$\vec{n} = \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix}, \quad d\Omega = d\varphi d\theta \sin \theta.$$

We closely follow the calculation of the partition function for the S^3 -Ising model in Sec. II B and rewrite the partition function in terms of the density of states $\rho(E)$:

$$Z(\kappa) = \int dE D^{(2)}(E),$$

$$D^{(2)}(E) = \rho(E) \exp\left[\frac{\kappa}{2(L-1)} E^2\right], \quad (38)$$

$$\rho(E) = \int \prod_x [d\Omega_x] \delta\left(E - \sum_{l \in \langle xy \rangle} \vec{n}_x \cdot \vec{n}_y\right). \quad (39)$$

We point out the similarity of Ising and $O(3)$ theory in this formulation. Differences only manifest themselves in the density of states. We are going to estimate the density of states and will argue that the qualitative features of ρ are the same so the conclusion of the existence of a phase transition is unchanged.

B. Calculation of the density of states

The big technical advantage of the 1D theory over those in higher dimensions is that the theory can be transformed into a noninteraction theory by a change of degrees of freedom. Consider the unit vector at position $z > 1$, i.e., $\vec{n}_z$, with angle θ_z . We can always perform a $O(3)$ rotation in the z -coordinate system that brings the unit vector $\vec{n}_{z-1}$ to the z axis. This transforms the interaction

$$\cos(\theta_{z-1} - \theta_z) = \vec{n}_{z-1} \cdot \vec{n}_z \rightarrow \vec{e}_z \cdot \vec{n}_z = \cos \theta_z.$$

The Haar measure $d\Omega_z$ is invariant under this transformation. We perform a further variable transformation:

$$d\varphi d\theta \sin \theta = d\varphi d\eta, \eta \in [-1, 1].$$

We point out that the integrands in (39) do not depend on the angles φ and neither on θ_1 . Integrating those variables produces a constant multiplicative factor. Altogether we find

$$\rho(E) = 2(2\pi)^{L-1} \int_{-1}^1 \prod_x d\eta_x \times \delta(E - [\eta_1 + \dots + \eta_{L-1}]). \quad (40)$$

Since $1 \leq \eta \leq 1$, we find that the density of states is only nonvanishing for the allowable energy range:

$$\rho(E) = 0, \text{ for } E > L-1 \text{ or for } E < -(L-1). \quad (41)$$

For studying the density of states in the large particle number limit and moderate energies, the Fourier representation of the Dirac Δ function provides a convenient starting point. We calculate

$$\begin{aligned} \rho(E) &= 2(2\pi)^{L-1} \int dx e^{ixE} \int_{-1}^1 \prod_k d\eta_k \\ &\quad \times \exp\{-ix[\eta_1 + \dots + \eta_{L-1}]\} \\ &= 2(2\pi)^{L-1} \int dx e^{ixE} \left(\int_{-1}^1 d\eta e^{-ix\eta} \right)^{L-1} \\ &= 2(4\pi)^{L-1} \int dx e^{ixE} \left(\frac{\sin x}{x} \right)^{L-1}. \end{aligned} \quad (42)$$

For large L , the $(\sin x/x)^{L-1}$ function sharply peaks at $x = 0$ with other maxima strongly suppressed. We therefore adopt a semiclassical approximation:

$$\begin{aligned} \rho(E) &\approx 2(4\pi)^{L-1} \int dx e^{ixE} \exp\left\{-\frac{(L-1)}{6} x^2\right\} \\ &= 2(4\pi)^{L-1} \sqrt{\frac{6\pi}{L-1}} \exp\left\{-\frac{3}{2(L-1)} E^2\right\}. \end{aligned} \quad (43)$$

The LLR coefficient (12) as a function of $e = E/(L-1)$ is given by

$$\bar{a}(e) \approx 3e, \quad \text{for } e \ll 1. \quad (44)$$

The limit $e \rightarrow 1$ is beyond the semiclassical approximation: we already know that $(e \rightarrow 1) \rightarrow \infty$ since the density of state vanishes at the highest possible energy. The equation for extrema (15) of the $D^{(2)}(E)$ is then given by

$$\bar{a}(e) = \kappa e \Rightarrow 3e \approx \kappa e \quad (e \ll 1). \quad (45)$$

We conclude that for $\kappa < 3$, the only extremum (maximum) is at $E = 0$: the system is in the disordered phase. For $\kappa > 3$, the right-hand side rises more quickly than the left-hand side. Since $a(e \rightarrow 1)$ diverges, there must be another intersection point at $e > 0$ (and its mirror image at $e < 0$) in addition to the extremum at $e = 0$. This implies that $e = 0$ becomes a minimum, and the extrema at $\pm E_0$ correspond to maxima.

We have calculated the critical coupling κ_c in the limit of large L . We now will embark calculating the density of states in analytical form for finite L . For any n with $1 < n \leq L-1$, we define ρ_n and observe the recursion

$$\begin{aligned} \rho_n(E) &= 2(2\pi)^n \int_{-1}^1 \prod_{x=1}^n d\eta_x \delta(E - \eta_1 - \dots - \eta_n) \\ &= 2\pi \int_{-1}^1 d\eta \rho_{n-1}(E - \eta). \end{aligned} \quad (46)$$

The desired density of states is then given by $\rho(E) = \rho_{L-1}(E)$, and the recursion's starting point is

$$\rho_1(E) = \begin{cases} 4\pi & \text{for } E \in [-1, 1] \\ 0 & \text{else.} \end{cases} \quad (47)$$

We can find the density of states by means of $L-1$ integrals. In practice, this involves a representation of ρ in function space and an approximation of the integrals (numerical integration). For this first study, we have represented ρ as equally spaced points on a line between $-(L-1) \dots (L-1)$. We found that an approach with $300 \times (L-1)$ points yields stable results. Details will be presented in future work. Note we have normalized the densities $D^{(2)}(E)$ in (38) by imposing

$$\int dE D^{(2)}(E) = 1.$$

We find that, due to a finite $L = 151$, the critical coupling $\kappa_c = 2.96(1)$ is slightly smaller than the infinite L result of 3. The results are then shown in Fig. 8(a) for $\kappa = 1$ (disordered phase), $\kappa = 2.96$ (at criticality), and $\kappa = 5$ (ordered phase). Note that in the broken phase, the symmetric point $E = 0$ is suppressed by about 20 orders of magnitude.

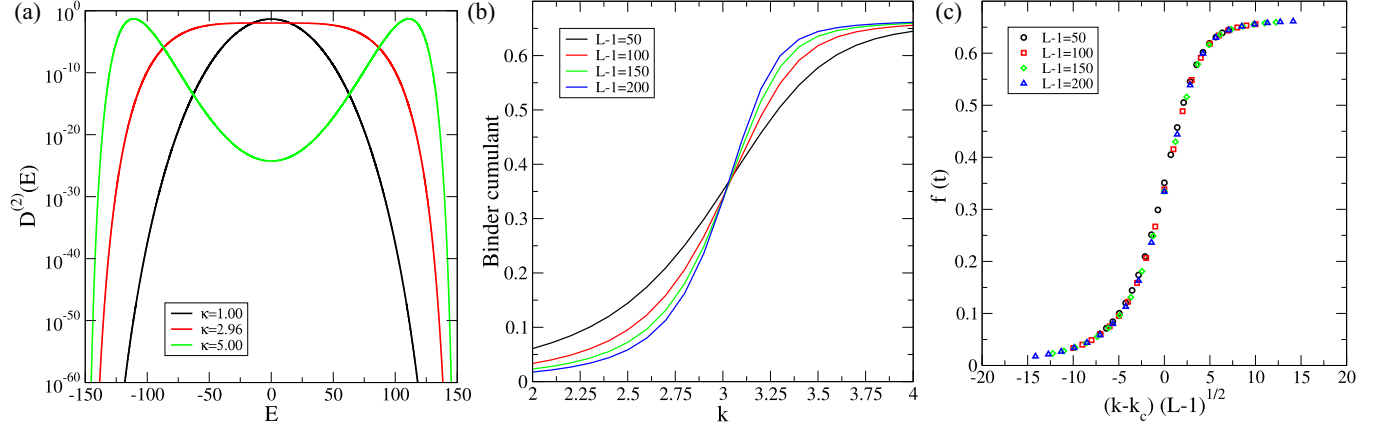


FIG. 8. One-dimensional $O(3)$ model. (a) Marginal distribution $D^{(2)}(E)$ function of the action E for 151 spins. (b) Binder cumulant. (c) Scaling function with a critical index $\nu \approx 2$.

C. Spontaneous symmetry breaking in the S^2 $O(3)$ spin chain

For any finite number of spins L , the $O(3)$ symmetry remains unbroken: the expectation value $\langle S \rangle$ receives equal contributions with opposite sign from the relevant regions around E_0 (ferromagnetic order, i.e., $S > 0$) and $-E_0$ (antiferromagnetic order $S < 0$). For $L \rightarrow \infty$, the tunneling probability from relevant states with $E \approx E_0$ to those with $E \approx -E_0$ vanishes: the system “freezes” with either ferromagnetic or antiferromagnetic order—the $O(3)$ symmetry is spontaneously broken to a residual Z_2 symmetry in either case.

We have then studied the Binder cumulant using the approach from Sec. III B. The unscaled cumulants are shown in Fig. 8(b) as a function of κ for the four system sizes $L = 50, 100, 150, 200$. The intersection of all four curves at the critical coupling $\kappa_c = 3$ is satisfactorily with some small scaling violations for the $L = 50$ case. The universal intersection point is compatible with $1/3$:

$$B(\kappa_c) = 0.33(1). \quad (48)$$

We have then extracted the critical exponent ν using (30) and (31) and find that

$$\nu \approx 2. \quad (49)$$

Equipped with this result and using $\kappa_c = 3$, the finite-size scaling function for the rescaled Binder cumulant is shown in Fig. 8(c), which displays satisfactorily scaling for all system sizes involved.

This model deserves further studies, which are left to future work: the scaling properties of the magnetic susceptibility in relation to the correlation length is an interesting case study to show universality in one dimension. Also the fate of the Goldstone boson, the propagation of which induces infrared singularities at least in perturbation theory (Feynman graphs), is also worthwhile to explore.

V. CONCLUSIONS

We have constructed a family of 1D field theories in which coupling strengths depend explicitly on mesoscopic observables (e.g. magnetization or energy density), thereby generating effective infinite-range interactions that evade the constraints of the Mermin-Wagner theorem. As

a consequence, these models sustain *bona fide* critical phenomena—including spontaneous symmetry breaking and emergent long-range order—despite their intrinsically low dimensionality.

By augmenting the conventional nearest-neighbor Ising action with nonlocal feedback terms proportional to powers of the local action (denoted S^2 and S^3), we identify two 1D universality classes, each characterized by critical behavior irreducible to any known class in one, two, or three dimensions. In particular, we observe the following:

(1) The S^2 model undergoes a continuous, second-order phase transition at $\kappa_c = 1$. The specific heat diverges logarithmically with system size and the correlation-length exponent is anomalously large ($\nu \approx 2$), signaling nontrivial scaling beyond traditional Ising limits.

(2) The S^3 model exhibits a discontinuous, first-order transition at $\omega \approx 2$, marked by an abrupt jump in internal energy and the emergence of a dominant ferromagnetic phase. This dichotomy underscores the decisive role of the feedback functional form in setting the transition order.

We have extended the scope of the investigation to the 1D $O(3)$ model to address the important case of a continuous symmetry. Spontaneous symmetry breaking is accompanied by the emergence of massless Goldstone modes, which could restore the symmetry in low-dimensional field theory. For the case of an S^2 mesoscopic motivated interaction, however, we find a second-order phase transition at the critical coupling $\kappa_c = 3$. The high- κ emerging phase spontaneously breaks the $O(3)$ symmetry, and the emerging phases exhibit either ferromagnetic or antiferromagnetic order, both with a residual Z_2 symmetry.

We have further demonstrated that these nonlocal interactions admit natural physical realizations—ranging from the stiffness-dependent conformations of dense polymer chains to 1D effective theories obtained via dimensional reduction of higher-dimensional field models. Such derivations confirm that mesoscopic feedback and attendant long-range couplings can arise organically in both physical and biological contexts, thereby enhancing the models’ empirical relevance.

Our analysis is supported by exact analytic calculations of partition functions, density of states, and Binder

cumulants, complemented by high-precision numerical studies employing finite-size scaling to extract critical exponents. All results cohere to validate the presence of genuine phase transitions in one dimension and to substantiate the classification of these systems into hitherto unexplored universality classes.

In summary, 1D field theories endowed with mesoscopic feedback-induced infinite-range interactions not only contravene the traditional prohibition of 1D criticality but also inaugurate new avenues for theoretical and applied research. These findings invite further exploration of mesoscopically regulated interactions in complex systems, with potential

implications spanning statistical field theory, soft-matter physics, and beyond.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [33]; embargo periods may apply.

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