

Linear-inverse-modeling approach to estimating Liang-Kleeman information flow in a cyclostationary process under memoryless and persistent noise

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The Liang-Kleeman (LK) information flow provides a powerful framework for quantifying causality among variables, while linear inverse modeling (LIM) offers an effective empirical approach to studying the dynamical evolution of system states from input data. In this study, we unify these two concepts by proposing the LIM-LK framework—a data-driven method for estimating LK information flow from input data using LIM. Beyond capturing causality among state variables, the proposed framework establishes a direct connection between causality and system dynamics, and also enables the quantification of entropy transfer from the ambient environment to the system through both memoryless and persistent stochastic forcing. The effectiveness of this unified approach is demonstrated through an application to the interaction between the Pacific and Indian Oceans, offering both causal and dynamical insight into ocean variability and its seasonal modulation.

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I. INTRODUCTION

Understanding causality is essential for unraveling how variables within a complex system behave and for developing robust strategies to investigate their outcomes [1]. In contrast to mere correlations, causal relationships reveal the underlying mechanisms that drive change, enabling us not just to predict, but to intervene effectively [2]. This makes causality a cornerstone of scientific inquiry, engineering design, and decision making [3–5].

Various approaches have been developed to analyze causality and several studies have recently formalized concepts related to causality using probability theory within a graph-theoretic framework. For instance, LiNGAM is a data-driven method for identifying direct causal relationships among variables in a static system subjected to non-Gaussian noise [6]. Extensions such as DirectLiNGAM, VarLiNGAM, and LayeredLiNGAM have further expanded their utility, especially in

uncovering directional causality and hierarchical structure in multivariate data [7–10]. However, many of these approaches have not yet been extended to dynamical systems where temporal structure and persistent stochasticity are key components.

A notable alternative is an information-theoretic formalism, which quantifies information using Shannon entropy in stochastic systems [11–13]. This formalism has been successfully applied beyond statistical mechanics, including climate science, such as model predictability and ensemble forecast skill [11,14–19]. Building on this foundation, the Liang-Kleeman (LK) information flow, defined via entropy transfer, provides a quantitative measure of causality among variables [12,20,21]. To estimate this empirically, an empirical approach was developed in [2], making use of an approximate autonomous linearized stochastic system to enable data-driven analysis of causality. This approach has since been extended to incorporate factors such as periodicity and Kalman filtering, and has been widely applied in geophysical studies—for example, to analyze causal relationships between geophysical variables (e.g., precipitation and sea surface temperature, snow and surface air temperature, soil moisture and air temperature) [22–24]. However, it remains limited in its capacity to incorporate persistent stochastic influences, which are both prevalent in real-world systems and important in climate modeling [25,26].

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To address this limitation, we adopt linear inverse modeling (LIM), a class of empirical methods that approximates the evolution of input data using linear stochastic differential equations. The most fundamental LIM and its extensions (collectively called LIMs in this article) have seen broad application in climate science, including El Niño-Southern Oscillation (ENSO) predictions, ENSO-type classification, and interbasin teleconnection analysis [27–31]. Although LIM was originally developed without an explicit focus on causality, recent work by [32] demonstrated that the stationary LIMs can be effectively integrated with the LK formulation to estimate causality among state variables. This hybrid method captures the time-independent entropy flow in the presence of persistent noise and offers a physically grounded interpretation of causal influence through both statistical dependencies and dynamical relationships. However, the previous study primarily addressed causality among state variables under the stationary condition [32]. The broader interplay between causality and dynamics, especially under nonstationary regimes, remains underexplored, highlighting key theoretical gaps for further investigation.

This study extends the theoretical foundation and the methodology proposed in [32]. From a theoretical perspective, we show that in a linear stochastic process driven by both white and colored noise, the LK causality can be explicitly formulated in terms of system parameters and statistical quantities. Moreover, through the principle of entropy conservation, a connection between system dynamics and causality is established, yielding a dynamical interpretation of information flow. In practice, LIM is first applied to construct an approximate linear representation of the system based on input data, from which the LK information flow is estimated. This integrated approach, referred to as the unified LIM-LK framework, enables not only the quantification of causal relationships among state variables but also the estimation of entropy transfer from both memoryless and persistent external noise. Furthermore, it offers a more comprehensive analysis by incorporating a dynamical perspective from LIM and a causal perspective from the LK formalism, thereby extending the interpretation beyond causal influence alone.

Instead of presenting the LIM-LK framework for all LIM variants, we particularly use a generalized version, called CS-CW-LIM, which accounts for both the wide-sense cyclostationarity (CS) of the input data and the influence of both colored (C) and white (W) noise in the stochastic forcing. Other LIM variants appeared in [25,30,33–36] and can be viewed as a simplified version of this formulation and hence follow the same workflow. Finally, as an illustrative application, we apply this unified LIM-LK framework to examine causal and dynamical interactions between the Pacific and Indian Oceans. Specifically, we discuss the seasonal variation of LK information flow and the potential pathways through which variability in the Indian Ocean or atmospheric processes influences ENSO. The results enhance the existing understanding by offering insights through an interpretation based on entropy exchange and causal dynamics.

II. METHODOLOGY

In this section, we first introduce the forward formulation of LIM and LK information flow before addressing the associated inverse problems. For clarity, the forward formu-

lation is presented in bold font, with derivations outlined in Appendix A, while inverse problems are written in regular font, with numerical details provided in Appendix B. Throughout, we assume cyclostationarity, although this assumption is not explicitly restated.

A. Forward formulation

1. Linear inverse modeling

The governing equation of the state vector $\mathbf{x}$ in CS-CW-LIM is given by

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}(t)\mathbf{x} + \sqrt{2\mathbf{Q}}\boldsymbol{\eta} + \sqrt{2\mathbf{D}(t)}\boldsymbol{\xi}. \quad (1)$$

In this formulation, $\mathbf{A}$ is the periodic first-order dynamical matrix and $\mathbf{D}$ is the periodic positive definite stochastic matrix characterizing the spatial-temporal patterns of the normalized Gaussian white-noise vector $\boldsymbol{\xi} \in \mathbb{R}^n$ [i.e., $\langle \boldsymbol{\xi}(t) \rangle = 0$ and $\langle \boldsymbol{\xi}(t + \tau)\boldsymbol{\xi}(t)^T \rangle = \delta(\tau)\mathbf{I}$, where the bracket and δ denote the expectation and the Dirac delta function, respectively]. Both $\mathbf{A}$ and $\mathbf{D}$ are assumed to have the same period, normalized to 1 without loss of generality. Meanwhile, $\mathbf{Q}$ is the time-independent positive definite stochastic matrix associated with the normalized colored noise vector $\boldsymbol{\eta} \in \mathbb{R}^n$ characterized by the noise correlation time γ . More precisely, $\boldsymbol{\eta}$ is realized by the steady state of $\frac{d}{dt}\boldsymbol{\eta} = -\frac{1}{\gamma}\boldsymbol{\eta} + \frac{1}{\gamma}\boldsymbol{\xi}$; hence, it has temporal dependence described by $\langle \boldsymbol{\eta}(t + \tau)\boldsymbol{\eta}(t)^T \rangle = \frac{1}{2\gamma}e^{-\frac{|\tau|}{\gamma}}\mathbf{I}$, and formally converges to $\boldsymbol{\xi}$ in the white-noise limit $\gamma \rightarrow 0$ [27,34].

The periodic system parameters $\mathbf{A}$ and $\mathbf{D}$, as well as the statistical quantities of Eq. (1), can be naturally identified as functions on the phase domain, the circle $\mathbb{S}$ of perimeter 1, via the canonical projection map $\omega: \mathbb{R} \rightarrow \mathbb{S}$. For convenience, the phase variable θ and the time variable t are identified via $\omega(t) = \theta$ without explicitly stating if there is no confusion [35].

Unlike the white noise, the colored noise and the state vector are correlated: $\mathbf{C}_{\eta\mathbf{x}}(\theta) = \langle \boldsymbol{\eta}(t)\mathbf{x}(t)^T \rangle \neq \mathbf{0}$ [34,37]. We refer to it as the noise-state coupling matrix, as it represents the extent to which $\boldsymbol{\eta}$ modulates the direction and magnitude of the evolution of $\mathbf{x}$ [25]. Though the colored noise is not part of the system state, the coupling matrix can be written as $\mathbf{C}_{\eta\mathbf{x}} = \sqrt{\frac{\mathbf{Q}}{2}}\mathbf{B}^T$, where $\mathbf{B}$ is a function of $\mathbf{A}$ and γ given by the periodic solution of the following equation:

$$\mathbf{B}' = -\mathbf{B}\gamma^{-1} + \mathbf{A}\mathbf{B} + \gamma^{-1}\mathbf{I}. \quad (2)$$

The correlation function $\mathbf{K}(\tau, \theta) = \langle \mathbf{x}(t + \tau)\mathbf{x}(t)^T \rangle$ of $\mathbf{x}$ and its partial right-derivatives with respect to the lag variable τ satisfy

$$\mathbf{K}_\tau(0, \theta) = \mathbf{A}\mathbf{C} + \sqrt{2\mathbf{Q}}\mathbf{C}_{\eta\mathbf{x}} = \mathbf{A}\mathbf{C} + \mathbf{Q}\mathbf{B}^T \quad (3)$$

and

$$\gamma\mathbf{K}_{\tau\tau}(0, \theta) = \gamma\mathbf{A}'\mathbf{C} + \gamma\mathbf{A}\mathbf{K}_\tau - (\mathbf{K}_\tau - \mathbf{A}\mathbf{C}), \quad (4)$$

where $\mathbf{C}(\theta) = \langle \mathbf{x}(t)\mathbf{x}(t)^T \rangle = \mathbf{K}(0, \theta)$ denotes the covariance of $\mathbf{x}$. The balance equation for the deterministic dynamics and stochastic forcing, called the fluctuation-dissipation relation (FDR), reads

$$\mathbf{C}' = \text{Sym}(\mathbf{A}\mathbf{C} + \mathbf{Q}\mathbf{B}^T + \mathbf{D}), \quad (5)$$

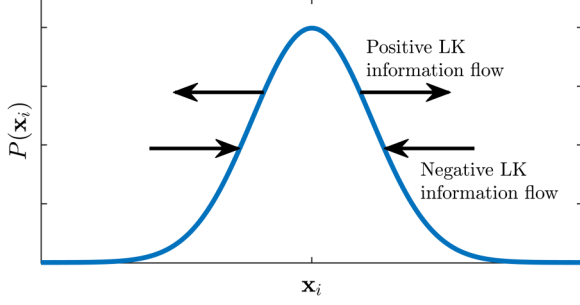


FIG. 1. Schematic illustration of the influence of variables on the probability distribution P of a target variable $\mathbf{x}_i$ through LK information flow. This example is particularly relevant for linear systems of the form given in Eq. (1). A similar conceptual figure is presented in [40].

where $\text{Sym}(\mathbf{X}) = \mathbf{X} + \mathbf{X}^T$ denotes the symmetrization operator (defined without the usual factor of $\frac{1}{2}$). Hence, $\mathbf{K}$ and $\boldsymbol{\gamma}$ characterize the governing system (1).

2. LK information flow

To better understand the interactions both within the system and from external sources, we introduce the concept of Liang-Kleeman information flow, which quantifies the directional causal influence, also referred to as information transfer, between variables [12,21,38,39]. Specifically, consider two components, $\mathbf{a}$ and $\mathbf{b}$, of an n -dimensional process described by a stochastic differential equation. The rate of information flowing from $\mathbf{b}$ to $\mathbf{a}$, denoted by $\mathbf{T}_{\mathbf{b} \rightarrow \mathbf{a}}$ (unit: nats/time), is defined to be the change rate of the marginal (Shannon) entropy of $\mathbf{a}$, minus the same change rate but without the effect from $\mathbf{b}$ to the system [2]. In this context, information flow may also be interpreted as entropy flow. A positive value of $\mathbf{T}_{\mathbf{b} \rightarrow \mathbf{a}} > 0$ indicates that $\mathbf{b}$ excites $\mathbf{a}$ by transferring entropy into $\mathbf{a}$; similarly, a negative value of $\mathbf{T}_{\mathbf{b} \rightarrow \mathbf{a}} < 0$ implies a stabilizing influence from $\mathbf{b}$ to $\mathbf{a}$ through a removal in entropy. In either case, $\mathbf{b}$ is said to be causal to $\mathbf{a}$ [2]. A schematic illustration is provided in Fig. 1.

For a linear system governed by Eq. (1), the time-dependent information flow $\mathbf{T}_{\mathbf{x}_j \rightarrow \mathbf{x}_i}$ from a state variable $\mathbf{x}_j$ to another $\mathbf{x}_i$ can be written as

$$\mathbf{T}_{\mathbf{x}_j \rightarrow \mathbf{x}_i} = \mathbf{A}_{ij} \frac{\mathbf{C}_{ij}}{\mathbf{C}_{ii}}. \quad (6)$$

In addition to the influence determined by the deterministic dynamics, the contribution from persistent colored noise to the state variable can also be captured. The information flow from a colored noise component $\boldsymbol{\eta}_j$ to the state variable $\mathbf{x}_i$ is given by

$$\mathbf{T}_{\boldsymbol{\eta}_j \rightarrow \mathbf{x}_i} = (\sqrt{2\mathbf{Q}})_{ij} \frac{(\mathbf{C}_{\boldsymbol{\eta}\mathbf{x}}^T)_{ij}}{\mathbf{C}_{ii}}, \quad (7)$$

collectively written as $\mathbf{T}_{\boldsymbol{\eta} \rightarrow \mathbf{x}_i} := \sum_j \mathbf{T}_{\boldsymbol{\eta}_j \rightarrow \mathbf{x}_i}$, and from the white-noise vector $\boldsymbol{\xi}$ is given by

$$\mathbf{T}_{\boldsymbol{\xi} \rightarrow \mathbf{x}_i} = \frac{\mathbf{D}_{ii}}{\mathbf{C}_{ii}} > 0. \quad (8)$$

Equations (7) and (8) illustrate the distinct mechanisms of information transfer arising from two types of stochastic forcing. Here, we view the variance $\mathbf{C}_{ii} > 0$ of $\mathbf{x}_i$ as a normalization constant as it appears in Eqs. (6)–(8). In the case of white noise, which is statistically independent of the state vector, the resulting positive information transfer depends solely on the stochastic matrix. In contrast, colored noise interacts statistically with the state vector through the coupling matrix. Consequently, it contributes to information transfer via a combined effect of both the stochastic and coupling matrices. In addition, it is possible for an individual colored-noise component to remove the entropy of certain state variables (i.e., act as a stabilizer), while still contributing a net positive entropy flow to the state vector as a whole.

With the information flow, the balance equation for the variance $\mathbf{C}_{ii}$ (i.e., a diagonal entry of the FDR) can be expressed as

$$\frac{\mathbf{C}'_{ii}}{\mathbf{C}_{ii}} = \frac{d}{d\theta} [\ln(\mathbf{C}_{ii})] = 2 \left(\sum_j \mathbf{T}_{\mathbf{x}_j \rightarrow \mathbf{x}_i} + \mathbf{T}_{\boldsymbol{\eta} \rightarrow \mathbf{x}_i} + \mathbf{T}_{\boldsymbol{\xi} \rightarrow \mathbf{x}_i} \right). \quad (9)$$

This formulation provides a dynamical interpretation of information flow: for a linear stochastic equation of the form (1), the stochastic forcing serves as the source of entropy and the dissipative dynamics removes the entropy from the system. As a result, the information flow quantifies the contribution of each variable to the normalized FDR budget: a positive information flow indicates that a variable increases the variance (or uncertainty) of $\mathbf{x}_i$, whereas a negative value suggests that it has a stabilizing effect on $\mathbf{x}_i$, reducing the variance. This interpretation aligns with the original conceptualization of information flow as the directional transfer of entropy within the system (see, also, Fig. 1) [2,12]. Moreover, summing Eq. (9) over i shows that the total information flow $\sum_{i,j} \mathbf{T}_{\mathbf{x}_j \rightarrow \mathbf{x}_i}$ is associated with the total variance tendency discussed in [36].

The magnitude of an information flow may differ from case to case. It needs to be normalized, just as covariance does, to have its importance assessed. With a slight modification from [39], the normalized information flow is defined as

$$\mathbf{N}_{\mathbf{x}_j \rightarrow \mathbf{x}_i} = \frac{\mathbf{T}_{\mathbf{x}_j \rightarrow \mathbf{x}_i}}{\sum_j |\mathbf{T}_{\mathbf{x}_j \rightarrow \mathbf{x}_i}| + |\mathbf{T}_{\boldsymbol{\eta} \rightarrow \mathbf{x}_i}| + \mathbf{T}_{\boldsymbol{\xi} \rightarrow \mathbf{x}_i}}. \quad (10)$$

Therefore, it can be interpreted from a dynamical perspective as the relative contribution of a variable to the FDR budget.

Finally, the information flow is associated with predictability [2,15,41]. Let $\mathbf{r}_i(\tau, \theta)$ be the correlation coefficient between the target variable $\mathbf{x}_i(t)$ and its τ -lead $\mathbf{x}_i(t + \tau)$. A straightforward computation yields

$$\frac{\partial}{\partial \tau} \mathbf{r}_i(0, \theta) = -\mathbf{T}_{\boldsymbol{\xi} \rightarrow \mathbf{x}_i} = \sum_j \mathbf{T}_{\mathbf{x}_j \rightarrow \mathbf{x}_i} + \mathbf{T}_{\boldsymbol{\eta} \rightarrow \mathbf{x}_i} - \frac{1}{2} \frac{\mathbf{C}'_{ii}}{\mathbf{C}_{ii}}. \quad (11)$$

This expression reveals a key difference in how white and colored noise influence predictability. White noise, being temporally uncorrelated and unpredictable, causes a sharper decline in correlation skill. In contrast, colored noise, due to its temporal structure, tends to sustain correlation and enhance short-term predictability. As for internal entropy transfer, a variable with positive information flow excites the target

variable, reducing stability but transferring dynamics-related information, thus improving short-term prediction. On the contrary, negative information flow has a stabilizing effect on the target variable, but simultaneously damps the dynamically meaningful signal.

These suggest that white noise introduces unstructured randomness and information flow that undermines persistence, while the other components are associated with meaningful variability, either enhancing or suppressing it, depending on the sign of the information flow. This interpretation aligns with the fact that colored noise contributes nontrivially to deterministic forecasting [25].

B. Inverse method

The unified LIM-LK framework follows a two-step procedure: It first applies LIM to construct an approximate system and then computes the LK information flow based on this system.

Given an n -dimensional cyclostationary input dataset $\{x(t)\} \subset \mathbb{R}^n$ with zero mean and uniform sampling interval Δt , the input cyclostationary correlation function K is computed by

$$K(\tau, \theta) = \frac{\sum_{\omega(t)=\theta} x(t + \tau)x(t)^T}{N_s},$$

where N_s denotes the number of valid summation terms that appear in the numerator. As the sampling is divided during the estimation of K , the resulting correlation function may be noisy in the phase direction [35,36]. Therefore, phase averaging must be applied whenever necessary. With a slight abuse of notation, we continue to denote the phase-averaged one as K .

CS-CW-LIM constructs an approximate system of the form (1) using K as follows. For a given noise correlation time γ , it numerically computes the partial right-derivatives K_τ and $K_{\tau\tau}$, and uses them to solve the dynamics A via Eq. (4). The auxiliary function B is obtained as the periodic solution of Eq. (2) determined by γ and A . Then the positive definite stochastic matrix Q associated with the colored noise is obtained as the least square solution of

$$\text{Sym}[K_\tau(0, \theta)] = \text{Sym}(AC + QB^T), \quad (12)$$

determined by all possible θ 's, where $C(\theta) = K(0, \theta)$ is the observed covariance. The stochastic matrix D associated with the white noise is subsequently solved by

$$C' = \text{Sym}(AC + QB^T + D). \quad (13)$$

To determine the noise correlation time, we extend the γ -selection algorithm in [34] in a natural way,

$$\gamma^{\text{CS-CW}} = \arg \min_{\gamma} \|K(\gamma) - K\|_{[0, l]}, \quad (14)$$

where $K(\gamma)$ is the cyclostationary correlation function determined by γ , and $[0, l]$ represents the minimization interval.

Then, with the approximate system, the estimated information flow is simply obtained by using Eqs. (6)–(8). This implies that though the stochastic forcings are latent, their entropy transfer to the state variables can be effectively inferred from the input data.

To determine the confidence interval of the information flow at a given significance level, we apply a bootstrapping procedure based on the correlation function. More precisely, suppose that the input data contains $M + 1$ complete cycles. We divide the data into M (overlapping) segments, where the i th segment spans from $t_i = i$ to $t_f = t_i + 1 + \Delta t$. For each segment, we compute its *correlation function* as

$$K^i(\tau, \theta) = x(t + \tau)x(t)^T,$$

for $t = i, i + \Delta t, \dots, i + 1 - \Delta t$, and $\tau = 0, \Delta t$, and $2\Delta t$. We then randomly sample $M + 1$ such correlation functions K^i , compute their mean, and use the resulting averaged estimate to infer the information flow. Repeating this procedure allows us to construct the empirical distribution of the information flow and thereby estimate its confidence interval.

C. Relationship with previous methods

CS-CW-LIM is closely related to other existing forms of LIMs. When the input data are assumed to be stationary—so that the correlation function is phase independent and the approximate system is autonomous—the resulting models reduce to the classical LIM or the colored LIM, depending on how the random forcing is represented [34,40].

A key distinction from the classical LIM proposed in [40] lies in the estimation of system dynamics: while classical LIM uses the matrix logarithm of the stationary correlation function (assuming exponential decay), Eq. (4) estimates the dynamics directly via the derivative, which more effectively reflects the Jacobian of the underlying system [34].

When the input data are instead assumed to be cyclostationary, but the stochastic forcing involves only one noise type (either white or colored), the resulting models correspond to cyclostationary LIMs, such as l -CS-LIM, e -CS-LIM, or CS-Colored-LIM [25,35]. Moreover, when the noise correlation time is prescribed [i.e., not determined by Eq. (14)], CS-CW-LIM becomes directly comparable to CS-Colored-LIM. Indeed, Eq. (4) depends only on $\mathbf{K}$ and γ instead of the presence of white noise, indicating that the estimated system dynamics remain invariant to whether or not white noise is included. The distinction between the two models lies in the allocation of the residual in the FDR budget: CS-CW-LIM distributes the residual between white and colored noise, whereas CS-Colored-LIM attributes it entirely to colored noise.

The unified LIM-LK framework can also be viewed as an extension of Liang's method [2]. Specifically, it has been shown that for an n -dimensional Markov process (i.e., time-independent and white-noise driven), the information flow $\mathbf{T}_{\mathbf{x}_j \rightarrow \mathbf{x}_i}$ from one component $\mathbf{x}_j$ of the state vector to another $\mathbf{x}_i$ is given by

$$\mathbf{T}_{\mathbf{x}_j \rightarrow \mathbf{x}_i} = \frac{1}{\det \mathbf{C}} \sum_{k=1}^n \Delta_{jk} \mathbf{C}_{k,di} \cdot \frac{\mathbf{C}_{ij}}{\mathbf{C}_{ii}}, \quad (15)$$

where Δ_{jk} denotes the cofactor of $\mathbf{C}$, and $\mathbf{C}_{k,di}$ is the covariance between $\mathbf{x}_k$ and $\frac{d}{dt}\mathbf{x}_i$. Then, given an input dataset, the information flow among state variables can be estimated using a numerical approximation of Eq. (15) [2]. However, as noted in [32], the first term on the right-hand side represents the system dynamics, which can be seen by explicitly expressing

$\mathbf{K}'(0)\mathbf{C}^{-1}$. Consequently, Liang's method is mathematically equivalent to a special case of the proposed framework under the assumptions of stationarity and white-noise forcing [i.e., the combination of derivative-based stationary White-LIM in [27] and Eq. (6)].

Meanwhile, the methodology presented in [32] can be viewed as applying Colored-LIM in [34] and then Eq. (6), except that the dynamical operator is estimated via a minimization technique instead of numerical derivatives. Moreover, a cyclostationary extension of Liang's method, applied in [23], can also be incorporated within this unified LIM-LK approach. It effectively corresponds to applying the e -CS-LIM formulation first, followed by Eq. (15) to estimate periodic information flow between state variables [35].

Consequently, by extending existing LIMs, CS-CW-LIM—when integrated with LK information flow—leads to a two-step approach for analyzing the periodic characteristics of causal influence within a finite time-series dataset. This approach is not limited to interactions among observable state variables, but also captures simultaneous entropy transfer from both white and colored noise. As such, it offers a comprehensive and flexible framework capable of accommodating the complexity of physical systems, while preserving physical interpretability and statistical rigor. This is particularly appropriate for studying processes exhibiting strong periodic dependence while being influenced by stochastic variability, such as large-scale ocean-atmosphere coupled modes.

III. A REAL-WORLD APPLICATION

To illustrate the applicability of our proposed method, we investigate the causal influence between ENSO and the Indian Ocean Dipole (IOD)—two major modes of interannual climate variability. ENSO is a well-known ocean-atmosphere coupled phenomenon centered in the tropical Pacific, affecting global weather patterns and driving extreme events such as droughts and floods [42,43]. It is characterized by anomalous warming and cooling of sea surface temperatures (SSTs) in the tropical Pacific, referred to as El Niño and La Niña events, respectively (collectively termed ENSO events in this study). Monitoring ENSO typically involves analyzing SST anomalies in regions such as Niño 3 (150°W–90°W, 5°N–5°S), Niño 3.4 (170°W–120°W, 5°N–5°S), and Niño 4 (160°E–150°W, 5°N–5°S).

The IOD, by contrast, is characterized by irregular oscillations in SST and atmospheric circulation over the tropical Indian Ocean. Its intensity is often measured using the Dipole Mode Index (DMI), defined as the difference in SST anomaly between the western (50°E–70°E, 10°S–10°N) and southeastern (90°E–110°E, 10°S–0°N) equatorial Indian Ocean [44,45].

The evolution of both ENSO and the IOD is strongly locked to the annual cycle, driven by the seasonally modulated strength of ocean-atmosphere coupling and tropical-subtropical interactions [46–48]. In particular, ENSO typically peaks at the end of the year, whereas the IOD reaches its maximum during boreal autumn. These seasonal variations exert a profound influence on regional and global weather and climate [49–51].

Despite their prominent roles, the nature of ENSO-IOD teleconnections remains debated [52–57]. Some studies

suggest that ENSO predominantly drives the Pacific-Indian interactions, whereas others argue that ENSO may develop in response to preceding IOD events [58–61]. Additional research posits that ENSO and the IOD often synchronize due to shared atmospheric and oceanic teleconnections, while contrasting perspectives emphasize their asymmetric and potentially independent behavior [2,62–64]. Clarifying this relationship is essential for improving seasonal forecasts and mitigating the weather-related impacts on agriculture, water resources, and economies [65]. This scientific uncertainty highlights the need for a rigorous and quantitative causal framework to assess the direction and strength of their interaction, motivating the application of the unified LIM-LK framework.

In the context of physical oceanography, oceanic variability is often conceptualized as the superposition of multiscale oceanic and atmospheric processes [28,30]. To implement CS-CW-LIM and LIM-LK, we construct state vectors using SST over the Pacific and Indian Oceans. Within this framework, the deterministic component captures the intrinsic oceanic dynamics and large-scale remote interactions between the basins, while the colored noise represents persistent monthly atmospheric variability, such as the Madden-Julian Oscillation (MJO) and the westerly wind bursts (WWB), and remote influences from other climate modes external to the tropical Indo-Pacific system [26,66–68]. The remaining high-frequency variability, modeled as memoryless white noise, corresponds to extremely short-term atmospheric perturbations, possibly on the daily or hourly scales [61,69,70].

More precisely, the state vectors, parametrized by latitude and longitude, are constructed using DMI and Pacific monthly SST data over 150°E–70°W and 20°N–20°S at a $2^\circ \times 2^\circ$ resolution, obtained from NOAA Physical Sciences Laboratory [71,72]. The data cover the period 1958 to 2010, consistent with [2]. For preprocessing, the climatological mean over the entire period is first removed, followed by the application of a three-month running average to smooth the data before computing the observed cyclostationary correlation function. To reduce phase uncertainty associated with limited sampling, phase averaging is performed using matlab's built-in lowpass filter. This similar preprocessing approach follows established procedures in the literature and is consistent with previous studies focused on isolating meaningful low-frequency variability [29,33,35,36]. In addition, the minimization window length l , used to determine the noise correlation time, is fixed at six months. The proposed methodology is applied independently to each grid point within the Pacific region.

A. Causal influences on Pacific SSTs

Figure 2 presents the seasonal evolution of information flow across the equatorial Pacific region, based on DMI and SST data at each grid point. Overall, the Pacific SSTs exhibit strong self-causality, predominantly acting to stabilize themselves. A notable exception occurs during boreal autumn (September to November), when the Pacific SSTs shift from stabilizing to exciting themselves over the Niño 3.4 region, one of the most critical areas for monitoring ENSO variability due to the strong coupling between oceanic and atmospheric processes. As the self-stabilizing effect is

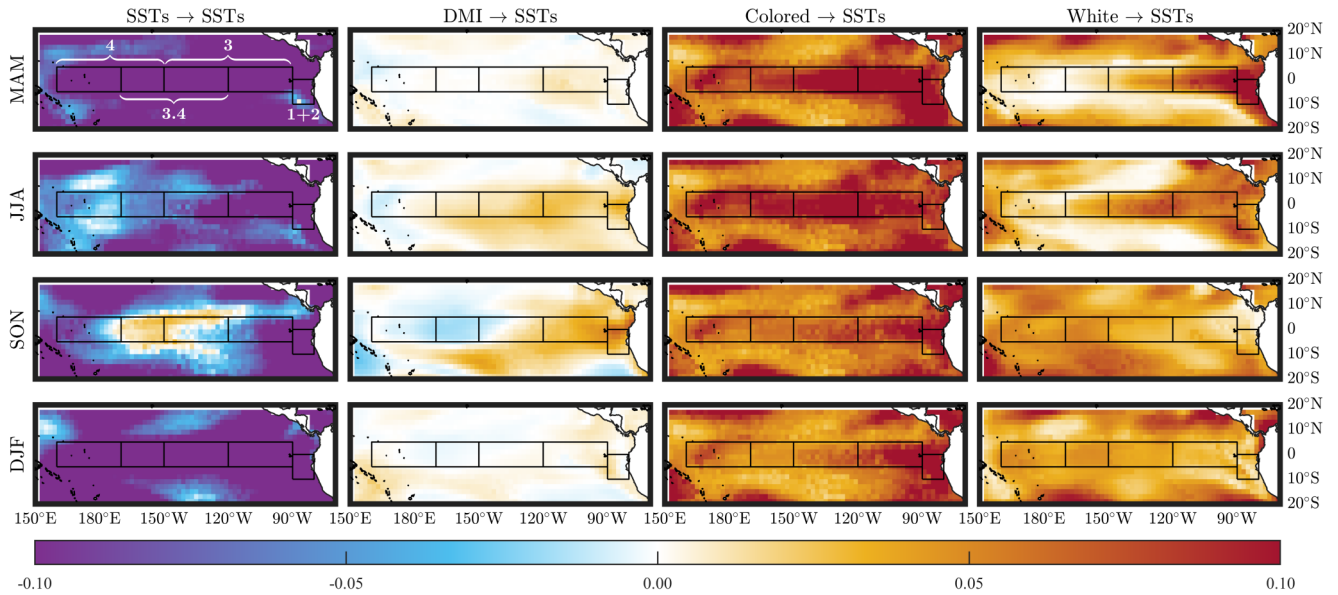


FIG. 2. Spatial-temporal distribution of the estimated LK information flow (nats/month) over the Pacific Ocean across different seasons. Each row corresponds to a season (MAM, JJA, SON, DJF), while the columns show contributions from (left to right): Pacific SSTs, DMI, colored noise, and white noise to the Pacific SSTs. The values are averaged over each respective season. The boxes represent the Niño 1+2, Niño 3, Niño 3.4, and Niño 4 areas.

generally more pronounced than the influence of the Indian-to-Pacific teleconnection, this observation suggests that the role of temporal-spatial self-causality may have been less emphasized in previous applications of the LK information flow framework [2].

Meanwhile, the information flow from the IOD to the Pacific, though relatively weaker in magnitude, demonstrates a clear spatial and temporal structure. Specifically, the IOD tends to excite the eastern Pacific while stabilizing the central tropical Pacific, especially during boreal summer and autumn. This interaction partially counteracts the inherent self-causality over the Pacific Ocean. However, the remote causal influence from the IOD diminishes considerably during boreal spring and winter, in agreement with the seasonality of the IOD, which develops in boreal summer and peaks in autumn [45].

The spatiotemporal pattern of inner- and interbasin causal interactions may be associated with the development of the ENSO Modoki and canonical ENSO, which are distinguished by the location of the extremum SST anomalies in the central and eastern Pacific, respectively [73–75]. ENSO Modoki generally originates within the Pacific basin, initiated at the ocean surface through atmospheric processes such as MJO and the Asian or Australian monsoons [74]. It then evolves through intrinsic Pacific dynamics, such as zonal advection feedback, and is typically less influenced by the IOD [75]. This aligns with the self-exciting causality of the central Pacific during the boreal autumn, the period when ENSO typically begins to intensify. In this context, the stabilizing influence of the IOD appears to play a minimal role in the development of ENSO Modoki.

In contrast, canonical ENSO exhibits a stronger dependence on IOD variability, particularly in the context of extreme or super El Niño events [74]. It has been shown that the eastern Pacific tends to exhibit an inherent self-limiting

behavior, which suppresses SST anomalies and inhibits ENSO amplification, especially when background seasonal cooling prevails. Under such conditions, the IOD can serve as a key modulator, moderately yet persistently enhancing ENSO development when the two phenomena co-occur [76]. This is consistent with our findings: during the development phase of an ENSO event, the IOD causally contributes to the amplification of Pacific SST anomalies by introducing entropy into the system during boreal autumn. As the IOD gradually decays in the boreal winter, the interbasin entropy transfer correspondingly weakens.

Overall, our findings suggest that the IOD and Pacific SSTs exhibit a dipole-like causal interaction, wherein the IOD plays a regulatory role by mitigating excessive excitation or stabilization within the Pacific. This mechanism may contribute to maintaining climate variability within a stable and balanced range.

We now examine the role of stochastic forcing—both memoryless white and persistent colored—in the causal relationship with Pacific SSTs. Overall, both types of stochastic forcing show a strong causal relation, acting to excite the Pacific SSTs across most seasons as expected. Among them, persistent noise demonstrates a stronger and more consistent effect, largely independent of seasonal variations, and is primarily concentrated in the equatorial Pacific and adjacent coastal regions. In contrast, memoryless noise displays a pronounced seasonal dependence: it first emerges in the coastal Niño region (defined as the Niño 1+2 region, 0°N–10°S, 90°W–80°W) during boreal spring, intensifies and expands into the Niño 3 region by summer, and eventually disperses across the entire equatorial Pacific [77].

Stochastic forcing plays a critical role in shaping climate variability. In particular, the strong causality from the white noise to Pacific SSTs over the coastal Niño region indicates that the atmospheric forcing may represent a precursor

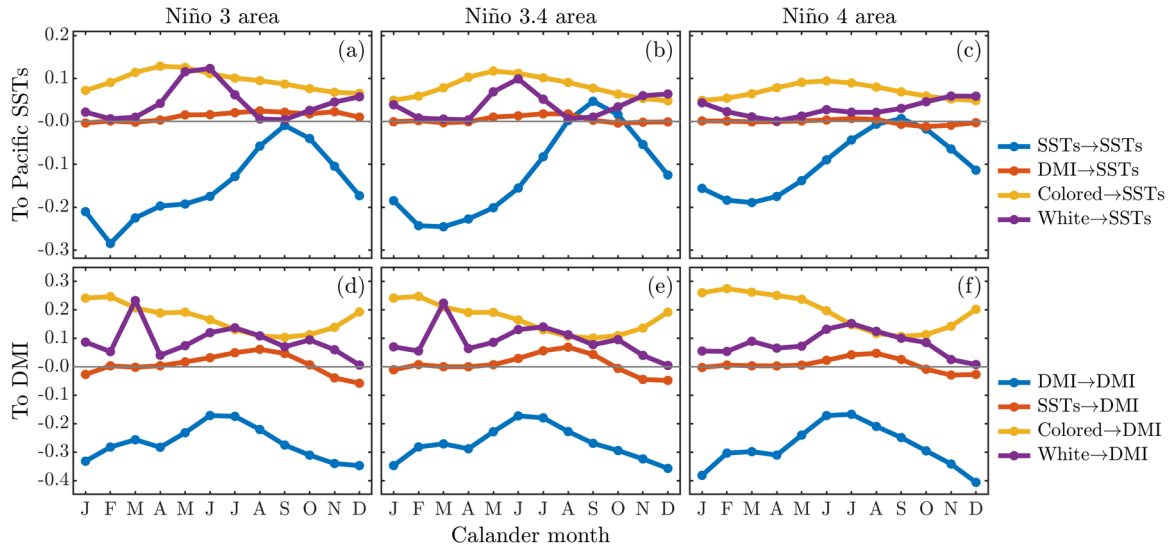


FIG. 3. Monthly variation of the estimated information flow (nats/month) to (a)–(c) Pacific SSTs and (d),(e) DMI averaged over the Niño 3, Niño 3.4, and Niño 4 regions. The curves represent contributions from the target variable itself (blue), the other state variable (i.e., interbasin) (orange), colored noise (yellow), and white noise (purple).

mechanism that triggers the development of coastal Niño events by exciting SSTs in the eastern Pacific during boreal spring [78]. As the season progresses, this causal influence spreads westward into the broader Niño 3 region, supporting the transition from coastal ENSO to basin-scale ENSO under favorable background conditions [79]. This progression may offer insight into the mechanism by which localized SST anomalies evolve into large-scale ENSO events.

Figures 3(a)–3(c) illustrate the seasonal evolution of area-averaged information flow, supporting the observation described above. Over the Niño 3.4 region, the results show the following: (i) strong self-causality, which is generally stabilizing but reverses to an exciting influence during boreal autumn; (ii) strong, seasonally consistent excitation by persistent forcing; (iii) moderate, seasonally modulated excitation by memoryless forcing; and (iv) a comparatively weak but statistically significant causal influence from the IOD to the Pacific, as confirmed by bootstrapping. This implies a real, though minor, interbasin causal effect through excitation from May to August. Over the Niño 3 region, a similar pattern is observed, although self-causality tends to remain negative and the IOD influence consistently positive. The seasonally averaged information flow from the IOD is $+0.0121$ nats/month, consistent with earlier findings such as those reported by [2]. In contrast, the interbasin information flow over the Niño 4 region is limited, with a seasonally averaged value of $+0.0013$ nats/month, supporting the previous discussion that the IOD has minimal influence on ENSO Modoki.

The remainder of this subsection discusses the LK information flow with a focus on the Niño 3.4 region for clarity and conciseness. From a dynamical perspective, our results reveal that the intrinsic dynamics of the Pacific is the primary driver of the seasonal behavior of the Niño 3.4 index. In late boreal winter and early spring, these internal dynamics play a stabilizing role by efficiently dissipating entropy introduced by stochastic forcing, thus reducing system

uncertainty. However, during boreal autumn, this stabilizing effect weakens, where internal dynamics begin to contribute positively to entropy accumulation, leading to increased uncertainty. This seasonal modulation of internal dynamics, as represented within the LIM-LK framework, offers an explanation of why the variance of the Niño 3.4 index, a measure of uncertainty, tends to be locked in boreal winter, consistent with the well-known phenomenon of ENSO phase locking [80].

In contrast, entropy input, arising from both stochastic forcing and interbasin transfer from the IOD, peaks from May to July (MJJ). This indicates that during the developing stage of a potential ENSO event, a substantial amount of entropy is injected into the Pacific SSTs. In particular, while the influence of persistent noise can be predicted to some degree, the inherently unpredictable white noise contributes significantly to entropy accumulation during MJJ. Such an increase hinders the short-range forecast skill of ENSO forecasts initialized over this period, aligning with the well-known challenge referred to as the spring prediction barrier (SPB) [81–83]. On the other hand, the presence of colored noise mitigates the initial drop in correlation skills due to its persistence, thereby improving short-range prediction skills. It is worth noting that this short-range influence is different from the long-range effect of colored noise on prediction skills reported in [25].

Importantly, Figs. 3(a)–3(c) also suggest that SPB can be partially alleviated by interactions with other climate modes. In particular, teleconnections from the Indian Ocean statistically significantly contribute a normalized information flow ranging from $+2.6\%$ to $+14.9\%$ from May to August, computed using Eq. (10). Although moderate in magnitude, the IOD exerts a positive influence on the prediction skill of ENSO, consistent with findings from [26]. This underscores the complex nature of ENSO predictability and highlights the importance of accounting for cross-basin causal interactions in understanding and improving seasonal climate forecasts.

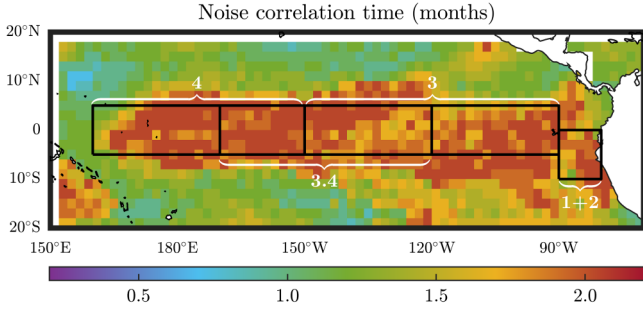


FIG. 4. Spatial distribution of the estimated noise correlation time (months). The boxes represent the Niño 1+2, Niño 3, Niño 3.4, and Niño 4 areas.

Finally, we emphasize that although entropy transfer or removal can be unfavorable for prediction, it plays a crucial role as a driver of climate variability.

B. Niño SSTs to DMI

Figures 3(d)–3(f) present the area-averaged information flow from Pacific SSTs to DMI, based on the Niño regions. In contrast to the seasonally modulated self-causality observed in the ENSO system, DMI exhibits consistent self-stabilizing dynamics throughout the year. Furthermore, the interbasin causal relationship reveals a statistically significant seasonal pattern: Pacific SST anomalies tend to excite DMI during July to September, while exerting a stabilizing influence from November through the subsequent January. The seasonally averaged information flow from ENSO to DMI is estimated as $+0.0080$, $+0.0087$, and $+0.0074$ nats/month from the Niño 3, Niño 3.4, and Niño 4 regions, respectively. These findings indicate that the Pacific and Indian Oceans exhibit mutual causality, with temporal asymmetries at the monthly scale, yet maintain an overall symmetric interaction across the annual cycle.

C. Spatial distribution of noise correlation time

Finally, Fig. 4 presents the spatial distribution of noise correlation time. Overall, the noise memory ranges from approximately 1 to 2 months, suggesting that the persistent noise influences the Pacific SSTs on monthly to subseasonal timescales. Regions exhibiting higher noise memory are primarily concentrated over the Niño areas, aligning with where persistent noise contributes most significantly to the information flow into the Pacific. As a result, the primary pathways through which persistent atmospheric processes such as MJO affect Pacific variability may be largely through the Niño regions. This consolidates the importance of accounting for monthly-scale processes over the Niño regions in inverse climate modeling.

IV. CONCLUDING REMARKS

This study proposes a unified LIM-LK framework, which provides a simple yet effective way to estimate the LK information flow while maintaining the balance between model complexity and physical interpretability via linear inverse modeling. By decomposing complex variability into

deterministic dynamics and stochastic forcing, LIM provides a compact and analytically tractable representation for a cyclostationary process potentially driven by both memoryless and persistent random forcing. Embedding LK information flow within this structure enables the identification and interpretation of causal pathways, even from the external noise to the target state variable, in a physically meaningful manner. In addition, this unified framework connects causality and system dynamics, bridging two distinct fields by interpreting information flow as the individual contribution to the target variable within the FDR budget. The FDR, in turn, leads to the equilibrium of entropy transfer into and out of a dynamical system, offering complementary interpretations of the same underlying phenomenon.

We applied the LIM-LK framework to study the causal relationship between ENSO and the IOD. The results show that the Pacific exhibits strong intrinsic self-causality, primarily acting as a stabilizing mechanism. This stabilizing effect is significantly stronger than the interbasin causal influence from the Indian Ocean, which, although weaker, remains statistically significant and modulates the internal dynamics of the Pacific Ocean seasonally. Notably, this interbasin interaction helps regulate Pacific variability by balancing excitation and stabilization, especially in the Niño 3 and 3.4 regions. In addition, the region of high causal influence from the persistent noise forcing aligns with these regions, suggesting the pathway through which monthly-scale atmospheric processes or other climate modes affect ENSO. In contrast, the transient noise reveals a hot spot at the coastal Niño region, corresponding to atmospheric perturbation as precursor activity [78].

From a dynamical perspective, the temporal evolution of area-averaged entropy flow reflects key characteristics of ENSO phase locking and SPB. Our results indicate that in the LIM-LK framework, phase locking is primarily governed by the seasonal cycle of internal Pacific dynamics, acting to dissipate entropy throughout most of the year. In contrast, SPB is mainly associated with the influence of unpredictable white noise, which peaks during the SPB period. Moreover, interbasin interactions and persistent atmospheric processes contribute to the accumulation of entropy in the Niño areas during the developing stage of ENSO. These external sources of entropy not only facilitate the buildup of Pacific SST anomalies, but also alleviate the impact of the SPB on short-range prediction skill.

Importantly, while this study focuses on CS-CW-LIM, the methodological idea can be naturally extended to other LIM variants. For instance, applying the derivative-based White-LIM in [27] and then Eq. (6) is mathematically equivalent to the original LK formulation proposed by [2]; the formulation used in [23] corresponds to e -CS-LIM in [35], and the framework proposed by [32] aligns with Colored-LIM in [34]. These correspondences illustrate the generality of our approach and offer a unified view of LIM-based causality diagnostics.

We further clarify our choice of the colored noise η with diagonal variance used in Eq. (1). Under an orthogonal change of variables, $\tilde{\eta} = \mathbf{U}\eta$, the component-wise information flows $\mathbf{T}_{\tilde{\eta} \rightarrow \mathbf{x}_i}$, similarly defined by Eq. (7) become basis-dependent, but their summation $\mathbf{T}_{\tilde{\eta} \rightarrow \mathbf{x}_i} := \sum_j \mathbf{T}_{\tilde{\eta}_j \rightarrow \mathbf{x}_i}$ is basis-invariant. However, in the inverse formulation, the transformation $\mathbf{U}$

cannot be identified from data, causing the identification issue discussed in the Concluding Remark of [34]. To avoid this ambiguity, we adopt the normalized colored noise in our inverse formulation.

We also note that the seasonally averaged information flow from Niño 3 to DMI differs from that reported by [2]. This discrepancy arises because the stationary LIM does not accurately capture the mean state of information flow, in contrast to previous studies showing that it can approximate the mean state of the dynamical operator within an acceptable error range [35]. As a heuristic example, consider a two-dimensional linear system of the form (1) where $\mathbf{C}_{12}(\theta) = \mathbf{A}_{12}(\theta) = \sin \theta$ and $\boldsymbol{\gamma} = \mathbf{Q} = \mathbf{0}$ (i.e., a white-noise driven system). In this case, with sufficiently long input data, applying either l -CS-LIM or e -CS-LIM in [35] yields positive information flow almost everywhere, whereas its stationary counterpart results in statistically vanishing information flow. This highlights the importance of correctly accounting for periodicity when analyzing cyclostationary systems, as also emphasized in [36].

Despite its strengths, the proposed framework inherits several limitations from the previous LIMs. Specifically, its performance depends on the characteristics of the dataset, including the sampling length, frequency, and uncertainty [34]. In addition, its linear formulation implicitly approximates the probability distribution of the input dataset with a Gaussian distribution, and hence does not capture non-Gaussian characteristics of the system, such as skewness or higher-order moments of the underlying data [27,84]. Moreover, time series of considerable length often exhibit variability across multiple timescales, requiring not only periodic linear dynamics but also explicit consideration of long-term forcing, particularly in the context of paleoclimate studies [85–87]. Consequently, the effectiveness of the method depends on how well the target system can be approximated by a linear model with additive noise, and future extensions to account for more complex features remain an important direction.

Overall, the unified LIM-LK framework offers a powerful tool for diagnosing entropy transfer in the ocean-atmosphere coupled system and provides insight into interbasin and intrabasin causal structures. In future work, we plan to extend this approach to further investigate the relationship between coastal Niño activity and basin-scale Pacific variability.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [71,72,88]; embargo periods may apply.

APPENDIX A: PROOF OF KEY EQUATIONS

This appendix outlines the derivations of the key equations presented in the manuscript.

A linear stochastic equation of the form (1) can be written in a white-noise-driven augmented form as

$$\frac{d}{dt} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\eta} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \sqrt{2}\mathbf{Q} \\ \mathbf{0} & -\frac{1}{\gamma}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\eta} \end{bmatrix} + \begin{bmatrix} \sqrt{2}\mathbf{D} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\gamma}\mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_x \\ \boldsymbol{\xi}_\eta \end{bmatrix}, \quad (\text{A1})$$

where $\boldsymbol{\xi}_x$ and $\boldsymbol{\xi}_\eta$ denote independent normalized white-noise forcing acting on $\mathbf{x}$ and $\boldsymbol{\eta}$, respectively. By following the derivation presented in [25], we have Eqs. (2)–(5). Also, applying the formula of LK information flow for the white-noise case immediately to Eq. (A1) yields Eqs. (6)–(8) and (10) [2,32,38,39].

To see Eq. (11), we start by expressing the correlation coefficient $\mathbf{r}_i(\tau, \theta)$ for $\mathbf{x}_i$ as

$$\mathbf{r}_i(\tau, \theta) = \frac{[\mathbf{K}(\tau, \theta) \mathbf{G}(\tau, \theta)^T]_{ii}}{\sqrt{\mathbf{C}(\tau + \theta)_{ii}} \sqrt{[\mathbf{G}(\tau, \theta) \mathbf{C}(\theta) \mathbf{G}(\tau, \theta)^T]_{ii}}},$$

where $\mathbf{G}$ satisfies $\mathbf{G}_\tau(0, \theta) = \mathbf{A}(\theta)$ and $\mathbf{G}(0, \theta) = \mathbf{I}$ [25]. A straightforward computation leads to Eq. (11).

APPENDIX B: NUMERICAL DETAILS OF CS-CW-LIM

In this appendix, the details of the numerical implementation for CS-CW-LIM are presented. The procedure follows the algorithm described in the supplementary text of [25], with several modifications.

Given a noise correlation time γ and the input cyclostationary correlation function K^{obs} , CS-CW-LIM computes $K_\tau(0, \theta)$ and $K_{\tau\tau}(0, \theta)$, the first- and second-order partial right-derivatives, using the first-order forward finite-difference schemes. The periodic dynamical matrix A is obtained by solving the discretized form of Eq. (4), where the derivative of A is approximated by the second-order centered difference. To ensure stability of the approximate system, the Floquet exponents associated with A are required to have negative real parts; otherwise, the estimate is regarded as invalid and returned as NaN. With the valid estimated A , the periodic auxiliary function B is computed by numerically integrating Eq. (2) over a sufficiently long time.

The stochastic matrix Q associated with the colored noise is determined by minimizing the residual of Eq. (12) over the space of symmetric matrices,

$$\sum_{\theta} \|\text{Sym}[K_\tau(0, \theta) - A(\theta)C(\theta) - Q(\theta)B(\theta)^T]\|^2,$$

where $C(\theta) = K^{\text{obs}}(0, \theta)$ denotes the observed covariance. If the resulting Q has negative eigenvalues, an eigenvalue adjustment procedure is applied, in which negative eigenvalues are set to zero and the positive ones are rescaled [40].

The stochastic matrix D associated with the white noise is then obtained from the discretization of Eq. (13), where the derivative of C is approximated using the second-order central finite-difference scheme. If $D(\theta)$ exhibits negative eigenvalues at some phase θ , the same adjustment is applied.

The approximate system determined by A , Q , D , and γ may not accurately reproduce C due to eigenvalue adjustments. To address this, the stochastic matrix associated with the colored noise is rescaled by a factor α that minimizes

$$\|C - C(A, \alpha Q, D, \gamma)\|,$$

where $C(A, \alpha Q, D, \gamma)$ denotes the periodic covariance matrix generated by the corresponding approximate system. The updated stochastic matrix, still denoted by Q with an abuse of notation, is set as $Q \leftarrow \alpha Q$.

Rather than using standard numerical optimization techniques such as gradient descent, the search space of α is restricted to $\{0.8, 0.9, 1.0, 1.1, 1.2\}$. A brute-force search with

parallel computation is performed to obtain a computationally efficient suboptimal estimate.

Finally, the well posedness of the γ -selection algorithm (14) is addressed. In the case of white noise (i.e., $\gamma = Q = 0$), the periodic matrix A estimated by Eq. (4) yields stable dynamics by Schur complement provided the sampling interval Δt is sufficiently small. When the approximate system is assumed autonomous, the estimated parameters depend continuously for $\gamma \geq 0$ near 0 [27], and this continuity extends to the periodic setting. Consequently, although the stability of A may fail for some large γ , it is guaranteed for sufficiently small γ . This ensures a well-defined $\gamma^{\text{CS-CW}}$.

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