

# Multicircular static patterns in swarmalator systems: Radii and stability

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We analyze spatial patterns emerging in self-organized systems of swarmalators, focusing on constellations where all entities arrange in a sorted manner along one or more concentric circles. Key properties of these patterns are explored by computing their radii and examining stability criteria based on the number of entities and a coupling parameter. The findings can be applied to swarm robotics and other collective motion systems.

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## I. INTRODUCTION

Many natural systems exhibit the phenomenon of *collective motion*, where coordinated movement and spatial patterns emerge in a self-organized way from local interactions. Examples in biology are insect swarms, bird flocks, fish schools, and microorganisms; those from physics include particles in fluids and magnetic fields. The absence of centralized control enables these systems to scale efficiently, adapt to changing environments, and remain resilient to failures. These features are also desirable in technical systems when mobile entities have to be coordinated to achieve a common goal. One prominent example is swarm robotics, where multiple autonomous entities perform tasks like exploration and monitoring.

Research on collective motion is inherently multidisciplinary, integrating concepts from biology, physics, mathematical modeling, and engineering [1–3]. Models span a wide spectrum of assumptions, abstraction levels, and complexity. They range from basic approaches that mainly align the directions of movement (like [4,5]) over more advanced models that incorporate attraction and repulsion (like [6–8]) to empirical frameworks that derive interaction rules from real-world data (like [9,10]).

A recent approach to collective behavior based on attraction and repulsion is the *swarmalator* model [11]—the name being a neologism for “swarming oscillator.” It describes the spatial coordination of entities in combination with their temporal coordination in terms of synchronization of internal oscillators. Crucially, the model introduces a mutual dependency between the two types of coordination: spatial proximity influences synchronization and oscillator phase similarity affects spatial attraction. As a result, swarmalators evolve into specific space-time patterns whose shape is determined by two parameters:  $J$  controls motion and  $K$  regulates synchronization. Notably, if  $K$  is set to zero, the model reduces to a pure swarming system without synchronization, though the phase values still influence spatial arrangement.

This article explores this special case of the swarmalator model to contribute to the field of collective motion. Specifically, we analyze patterns in which swarmalators form circular constellations and arrange along them in a sorted way according to their phase. This ordered spatial arrangement is a key feature of the swarmalator model.

The formation of circular patterns is relevant for applications in which multiple entities, such as robots or drones, need to arrange around a point of interest [12,13]. For example, in disaster response, a swarm of quadcopters could support emergency teams by providing a 360-degree view of a collapsed building (e.g., as outlined in [14]). Collective motion models facilitate the formation of such arrangements in a self-organized way, eliminating the need to program movement paths and trajectories. This feature is one reason as to why the swarmalator model is used in multirobot research. Figure 1 showcases a real-world instance emerging from our swarmalator implementation on mobile robots [15]. To employ such emergent behavior in practice, however, it is essential to know in advance the properties of the resulting patterns (such as its size) and ensure their stability.

Our goal is to advance the state of the art in this domain. By analyzing systems in which swarmalators converge to a stable state where they eventually lie on a circle and remain static, we find the conditions that cause a single-circle pattern to destabilize and break into multiple circles, for which we compute the resulting radii. More specifically, the contributions are as follows:

**Generalization:** The mathematical analysis of circular patterns has so far been limited to states at which nodes are distributed between two circles, denoted as “ring states” [16], and its extreme case in which all nodes lie on a single circle. We generalize this work by exploring states in which all nodes lie on multiple discrete circles. It is demonstrated, through both mathematical analysis and simulations, that these states can indeed emerge and remain stable, depending on the system parameters and the initial conditions.

**Stability analysis:** We analyze the stability of circular static states using the system’s Jacobi matrix. It is shown that the stability depends on the number of swarmalators and the system parameter  $J$ . If the number of swarmalators exceeds a threshold, a previously stable circular state loses its stability and transitions into a saddle point, while at the same time a state

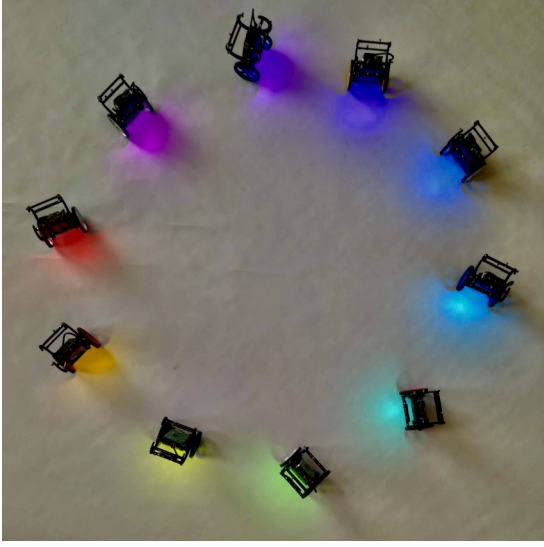


FIG. 1. Mobile robots forming a circular pattern with ordered colors (phases) using the swarmalator model with  $J = 1$  and  $K = 0$  (static phase wave). Photo made by Agata Barciś for the University of Klagenfurt as part of work published in [15].

with more distinct circles becomes stable. These transitions are visualized in a transition graph.

**Radii:** As a necessary by-product of the stability analysis, we compute the radii of the individual circles by identifying joint fixed points of all radii in a numerical fashion. These radii are required in the Jacobi matrices for stability analysis. Knowing the size of emergent patterns in advance is essential for technical applications.

In the terminology of swarmalators, we study the “static phase wave” pattern [11], specifically, its manifestations where all swarmalators lie on one or more discrete circles.

The article is structured as follows: Section II recalls the swarmalator model. Section III investigates static circle patterns in swarmalator systems, covering single-circle, double-circle, and multicircular patterns. Section IV addresses related work. Section V concludes this work. A short video clip on our work can be found in Ref. [17].

## II. SWARMALATOR MODEL

A swarmalator is a node indexed by  $i \in \{1, \dots, N\}$  that is described by its location  $x_i \in \mathbb{R}^2$  and oscillator phase  $\theta_i \in [0, 2\pi]$  at a given time. It changes its location and phase based on the locations and phases of other swarmalators  $j \neq i$ , depending on the distance vectors  $x_j - x_i$  and phase gaps  $\theta_j - \theta_i$ , governed by a first-order system of differential equations [11]:

$$\begin{aligned} \dot{x}_i &= \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \frac{x_j - x_i}{\|x_j - x_i\|} [1 + J \cos(\theta_j - \theta_i)] - \frac{x_j - x_i}{\|x_j - x_i\|^2}, \\ \dot{\theta}_i &= \frac{K}{N} \sum_{\substack{j=1 \\ j \neq i}}^N \frac{\sin(\theta_j - \theta_i)}{\|x_j - x_i\|}. \end{aligned} \quad (1)$$

The parameters  $J$  and  $K$  control the node behavior and determine the type of emergent pattern. The parameter  $J$  controls how a node’s movement is influenced by its phase similarity to other nodes. Positive values make nodes with similar phases attract each other the most; negative values make nodes with opposing phases attract. The parameter  $K$  controls how a node’s synchronization process is influenced by its proximity to other nodes. Depending on  $J$  and  $K$ , five types of space-time patterns can form [11]: static sync, static async, static phase wave, splintered phase wave, and active phase wave.

## III. STATIC PHASE WAVE: CIRCLE PATTERNS

Swarmalator systems often lead to circular patterns, where all nodes arrange within an annulus (“ring”) of certain width, so they are distributed in some way between two concentric circles. If the nodes also sort themselves along this annulus according to their phase and then remain static, the pattern is called “static phase wave” and occurs for the parameters  $J > 0$  and  $K = 0$  [11]. The parameter choice  $K = 0$  means that all nodes keep their initial phase since  $\dot{\theta}_i = 0 \forall i$  in (1). In contrast, for any nonzero value of  $K$ , the phases can change and no static phase wave emerges. A special case of the static phase wave emerges if the number of nodes is small: the width of the annulus vanishes, and all nodes arrange along a discrete circle [16] (as in Fig. 1). Beyond this, simulations indicate that besides single-circle patterns also multicircle patterns can form, where each node is eventually located on one of multiple concentric circles. It is this subclass of the static phase wave that is subject to our investigations. Using methods from nonlinear dynamics and simulations, we analyze aspects like the occurrence, size, and stability of single circles, double circles, and, most general, multiple circles.

The phases  $\theta_i$  are initialized at equidistant intervals over the entire value range (as in [11]) to achieve equidistant placement of the nodes on the circles in the stable state. A random, equally distributed phase selection would lead to virtually identical patterns for a large number of nodes. However, if the number of nodes is low, according to simulations, the circle pattern will have gaps, be deformed, or not form at all. The Euler method is used to numerically solve the differential equation for  $\dot{x}_i$  in an iterative way. For a given location  $x_i(t)$  at time step  $t$ , the location at the next time step  $t + \Delta t$  is calculated by  $x_i(t + \Delta t) = x_i(t) + \Delta t \dot{x}_i(t)$ . A sufficiently small step size  $\Delta t$  must be chosen to ensure convergence. Our simulations employ  $\Delta t = 0.1$ , a value commonly used in this field (see, e.g., [16,18,19]). Since all converged states are *static* patterns, once the system reaches a stable state, the nodes keep their positions  $x_i$  and no longer move. Consequently, we can set  $\dot{x}_i = 0 \forall i$  in (1) in all stable states. To simplify certain calculations without loss of generality, we assume that in the converged pattern, the node with index  $i = 1$  is located on the positive horizontal axis.

### A. Single-circle pattern

If the number of nodes  $N$  is small enough, the system tends to converge to a state in which all nodes are evenly spaced on a circle, with their phases arranged in ascending or descending order [16]. We call this pattern the *monocircle*. Figure 2(a)

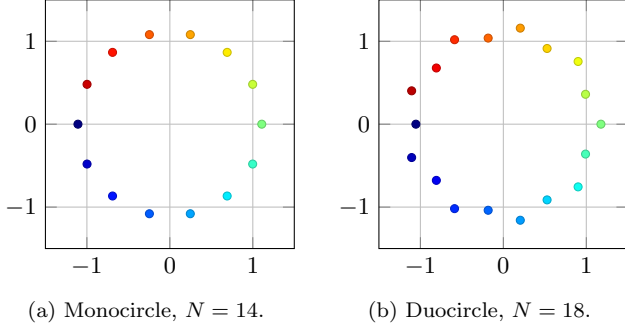


FIG. 2. Swarmalators with  $J = 1$  and  $K = 0$  emerge into a static phase wave pattern with one or two circles, after 10 000 time steps with  $\Delta t = 0.1$ .

illustrates an example. After convergence, the nodes are located at  $x_i = (r \cos \phi_i, r \sin \phi_i)^T$  with circle radius  $r$  and polar angles  $\phi_i = \frac{2\pi i}{N} + \phi_0$ . The phases are given by  $\theta_i = \phi_i + \theta_0$ . The terms  $\phi_0, \theta_0 \in [0, 2\pi]$  are offsets, which we assume to be zero without loss of generality.

The pattern radius depends on  $J$  and  $N$ . It can be derived by substituting  $x_i = (r \cos \phi_i, r \sin \phi_i)^T$  into (1), setting  $\dot{x}_i = 0$ , and solving the resulting equation for  $r$ . This yields

$$r(J, N) = \frac{N - 1}{(2 - J) \cot \frac{\pi}{2N} + J \cot \frac{3\pi}{2N}}. \quad (2)$$

Figure 3 plots the function  $r(J, N)$  for different  $N$ . It is shown how the circle increases with increasing  $J$  and eventually explodes in size. As the denominator of (2) has to be positive, the state is stable if  $J < 1 + 2 \cos \frac{\pi}{N}$ . Two nodes require  $J < 1$ , three nodes need  $J < 2$ , and the limit  $N \rightarrow \infty$  demands for  $J < 3$ . If  $J \geq 3$ , there is no stable monocircle for any  $N$ .

### B. Two-circle pattern

If  $N$  exceeds a certain value, a pattern emerges in which the nodes no longer lie on one circle but distribute to two

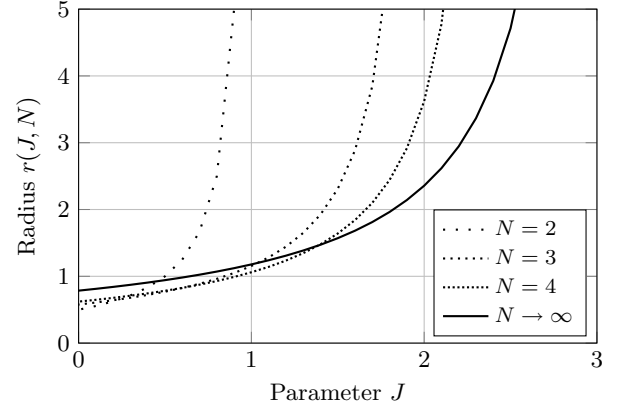


FIG. 3. Monocircle of swarmalators: radius  $r(J, N)$  of (2) over the parameter  $J$  for a given finite number of nodes  $N$  along with the limiting case  $\lim_{N \rightarrow \infty} r(J, N) = \frac{3\pi}{4(3-J)}$ .

concentric circles. We call this pattern the *duocircle*. Typically, half of the nodes lie on the inner circle and the other half on the outer one (for an even  $N$ ), alternating between the two [see Fig. 2(b)]. We now explore two questions: What is the minimum number of nodes required for the transition from a monocircle to a duocircle? What are the radii of the circles?

The threshold for the number of nodes, denoted as  $N_2$ , depends on the parameter  $J$ : if  $J$  is large, the radius of a monocircle is large, providing space for many nodes; as a result,  $N_2$  also increases. For example, eight swarmalators with  $J = 0.2$  arrange on a duocircle, whereas  $J = 0.5$  requires ten nodes to form this pattern.

If  $N$  is even, the node locations are given by

$$x_i = \begin{cases} (r_1 \cos \phi_i, r_1 \sin \phi_i)^T & \text{for } 2 \mid i \\ (r_2 \cos \phi_i, r_2 \sin \phi_i)^T & \text{else,} \end{cases} \quad (3)$$

with polar angles  $\phi_i = \frac{2\pi i}{N} + \phi_0$  and the radii  $r_1$  and  $r_2$ . The phases are  $\theta_i = \phi_i + \theta_0$ . Again, the offsets  $\phi_0$  and  $\theta_0$  can be assumed to be zero.

$$\begin{aligned} \dot{r}_1(r_1, r_2) = & \frac{1}{N} \sum_{j=1}^{N/2-1} \frac{\cos(2j\frac{2\pi}{N}) - 1}{\sqrt{2 - 2\cos(2j\frac{2\pi}{N})}} \left[ 1 + J \cos\left(2j\frac{2\pi}{N}\right) \right] - \frac{\cos(2j\frac{2\pi}{N}) - 1}{2r_1 - 2r_1 \cos(2j\frac{2\pi}{N})} \\ & + \frac{1}{N} \sum_{j=1}^{N/2} \frac{r_2 \cos((2j-1)\frac{2\pi}{N}) - r_1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos((2j-1)\frac{2\pi}{N})}} \left[ 1 + J \cos\left((2j-1)\frac{2\pi}{N}\right) \right] - \frac{r_2 \cos((2j-1)\frac{2\pi}{N}) - r_1}{r_1^2 + r_2^2 - 2r_1 r_2 \cos((2j-1)\frac{2\pi}{N})} \end{aligned} \quad (4)$$

$$\begin{aligned} \dot{r}_2(r_1, r_2) = & \frac{1}{N} \sum_{j=1}^{N/2-1} \frac{\cos(2j\frac{2\pi}{N}) - 1}{\sqrt{2 - 2\cos(2j\frac{2\pi}{N})}} \left[ 1 + J \cos\left(2j\frac{2\pi}{N}\right) \right] - \frac{\cos(2j\frac{2\pi}{N}) - 1}{2r_2 - 2r_2 \cos(2j\frac{2\pi}{N})} \\ & + \frac{1}{N} \sum_{j=1}^{N/2} \frac{r_1 \cos((2j-1)\frac{2\pi}{N}) - r_2}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos((2j-1)\frac{2\pi}{N})}} \left[ 1 + J \cos\left((2j-1)\frac{2\pi}{N}\right) \right] - \frac{r_1 \cos((2j-1)\frac{2\pi}{N}) - r_2}{r_1^2 + r_2^2 - 2r_1 r_2 \cos((2j-1)\frac{2\pi}{N})}. \end{aligned} \quad (5)$$

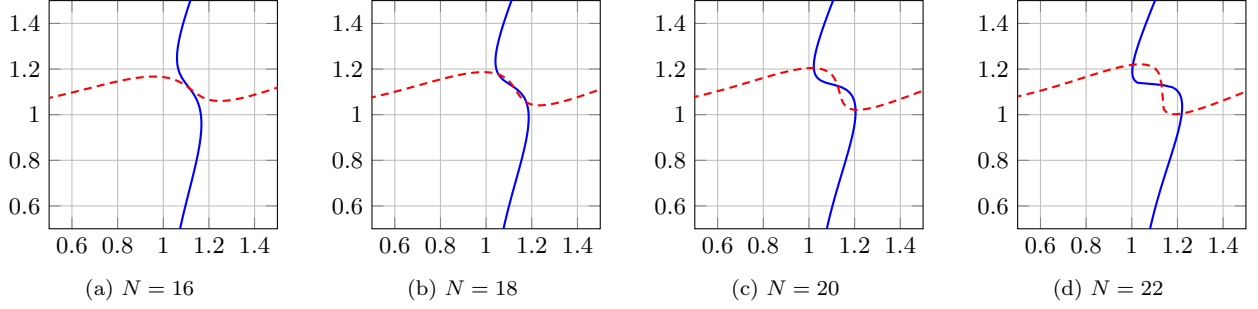


FIG. 4. Analysis of monocircle and duocircle patterns, a contour plot (axes are the radii  $r_1$  and  $r_2$ ) for  $\dot{r}_1 = 0$  (blue solid lines) and  $\dot{r}_2 = 0$  (red dashed lines). The intersections show fixed points of the system. Swarmalator parameters  $J = 1$  and  $K = 0$ .

We now investigate as to which pairs of  $(r_1, r_2)$  exist in stable patterns. Again, we apply (3) in the system equation (1) and set  $\dot{x}_i = 0$  (static nodes). Since pattern stability implies that the radii must not change over time, we require  $\dot{r}_1 = 0$  and  $\dot{r}_2 = 0$ . Figure 4 shows a contour plot with two curves in the  $r_1 r_2$  plane, each curve satisfying one of these two conditions. Each intersection point of the two curves represents a fixed point of the system, revealing potential radii of stable patterns for a given parameter  $J$  (here  $J = 1$ ). For  $N = 16$  [Fig. 4(a)], there is a single fixed point, which satisfies  $r_1 = r_2$ , which implies a single circle.  $N = 18$  nodes [Fig. 4(b)] lead to already three fixed points. Their distances increase with more nodes [like in Figs. 4(c) and 4(d)].

Figure 5 shows the radii that occur in stable patterns. Duocircles form for small values of  $J$ , but as  $J$  increases, the circles move closer to each other and eventually merge to form a monocircle. The more nodes we have, the larger the radii of the duocircle and the later the transition to the monocircle occurs. For any  $N$ , there exists a parameter  $J < 3$  such that a transition to the monocircle occurs. In the limiting case  $N \rightarrow \infty$ , the radius tends to infinity when  $J$  approaches 3 [as in Fig. 3].

To determine the type of fixed points, we investigate the dynamics around them by means of stream plots, as shown in Fig. 6. For  $N = 16$  nodes, the single fixed point at  $r_1 = r_2$  is attractive as the derivatives tend toward the fixed point from

all directions. This confirms that this pattern is a stable monocircle. For  $N = 18$ , the three fixed points show the following behavior: the outer two points are attractive; the middle point is attractive from the  $r_1 = r_2$  direction but repelling in other directions, classifying it as a saddle point. The two stable fixed points have their radii just swapped, making them manifestations of the same duocircle state. For  $N = 24$ , the fixed point properties are analogous, thus again resulting in a duocircle with a unique pair of possible radii.

These insights guide us to calculate the threshold  $N_2$  as a function of  $J$ . We assume all nodes to be placed on duocircle as given by (3). Without loss of generality, one node ( $i = 1$ ) is placed on the positive  $x$  axis, so that its  $x$  coordinate corresponds to the radius of the circle it is placed on. Using (3) in (1), we calculate the derivatives of the radii, yielding (4) and (5). Solving the system  $\dot{r}_1(r_1, r_2) = 0$  and  $\dot{r}_2(r_1, r_2) = 0$  yields the fixed points, which depend on  $N$ . If  $N < N_2$ , one fixed point occurs; otherwise, there are three, with  $N_2$  depending on  $J$ .

To formally assess the stability of a fixed point, we calculate the Jacobi matrix of the corresponding vector  $(\dot{r}_1, \dot{r}_2)$ , defined as

$$J_2 = \begin{pmatrix} \frac{\partial}{\partial r_1} \dot{r}_1(r_1, r_2) & \frac{\partial}{\partial r_2} \dot{r}_1(r_1, r_2) \\ \frac{\partial}{\partial r_1} \dot{r}_2(r_1, r_2) & \frac{\partial}{\partial r_2} \dot{r}_2(r_1, r_2) \end{pmatrix}, \quad (6)$$

and evaluate it at each fixed point. The stability of a fixed point is found via the eigenvalues of the Jacobi matrix. The point is an attractor if all eigenvalues are negative and a saddle point if both positive and negative eigenvalues exist. All eigenvalues are real valued, since no spiral-shaped trajectories occur due to the lack of angular forces on swarmalators. Equivalently, one can use the determinant of the matrix: the condition  $\det(J_2) < 0$  means that a fixed point is a saddle, whereas stability requires  $\det(J_2) > 0$  and  $\text{trace}(J_2) < 0$ .

These rules can be used to determine the boundary conditions between a monocircle and a duocircle of swarmalators. For given  $J$ , the threshold  $N_2$  equals the largest  $N$  value for which both  $\det(J_2) > 0$  and  $\text{trace}(J_2) < 0$  hold. Or, if  $N$  is given, we solve  $\det(J_2) = 0$  for  $J$  to obtain the threshold value for  $J$ , given that  $\text{trace}(J_2) < 0$ . In all our numerical analyses, the condition  $\det(J_2) > 0$  implies  $\text{trace}(J_2) < 0$ . Thus, it suffices to verify the sign of determinant. The first approach is less precise, since  $N$  is an even number. The resulting  $(J, N)$

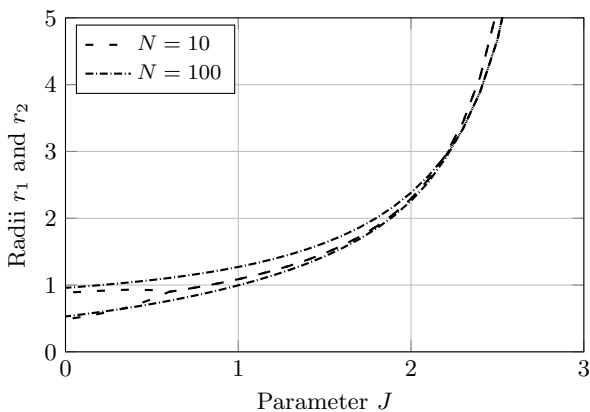


FIG. 5. Duocircle of swarmalators: radii  $r_1, r_2$  of stable states over the parameter  $J$  for a given finite number of nodes  $N$ .

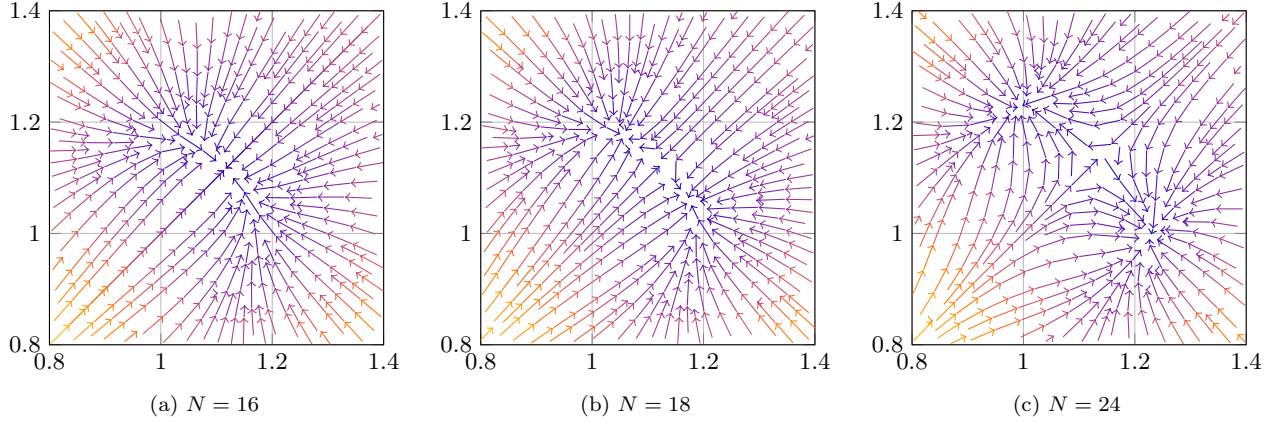


FIG. 6. Analysis of monocircle and duocircle patterns via a stream plot (axes  $r_1, r_2$ ) for  $\dot{r}_1$  and  $\dot{r}_2$ . The colors encode the magnitude of the derivatives. The black (red) dots indicate stable (unstable) fixed points. Parameters  $J = 1$  and  $K = 0$ .

pairs are plotted in Fig. 7, illustrating the stability regions of monocircle and duocircle states.

The existence of the saddle point gives rise to the following swarming dynamics, as also evidenced in simulations. If nodes are initialized on a circle, they first stay on a common circle but increase or decrease their radius until the saddle point is reached. In the stream plot, this corresponds to a trajectory on the diagonal  $r_1 = r_2$  to the saddle point in the middle. Once the saddle point is reached, the swarmalators stay at their positions for some time. However, after many iterations, numerical inaccuracies accumulate and the system moves slightly off the fixed point, leading to marginally different radii of the swarmalators. In the stream plot, this corresponds to the system moving slightly away from the fixed point in the direction of the second diagonal toward one of the black dots. Once this happens, the system converges to one of the two stable fixed points, which represent the duocircle state. For an odd number of nodes  $N$ , the general behavior is the same as for even  $N$ . However, the outer circle has one node more than the inner circle, which leads to slightly deformed (not perfectly circular) patterns.

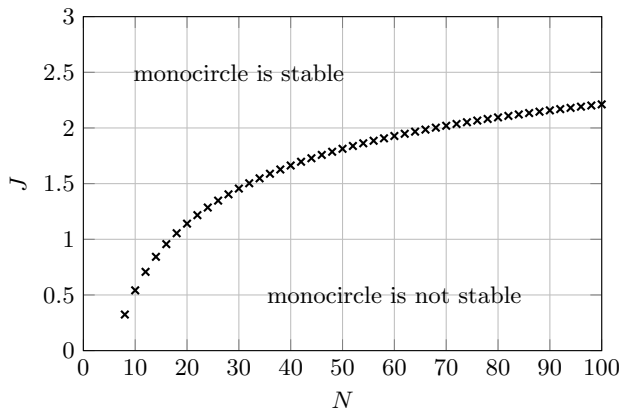


FIG. 7. Stability region of the monocircle state of swarmalators ( $K = 0$ ). Minimum value of the parameter  $J$  for a given number of nodes  $N$  to obtain a stable monocircle state.

### C. Multicircle patterns

Up to this point, we have seen that if the number of swarmalators exceeds a threshold, the monocircle becomes unstable and a duocircle emerges. This raises the question of whether even more circles will form if we further increase the number of nodes. Simulations reveal that this progression indeed takes place. The goal of this section is to analyze the resulting states, called *multicircle* states or *s-circle* states for a specific number  $s$ .

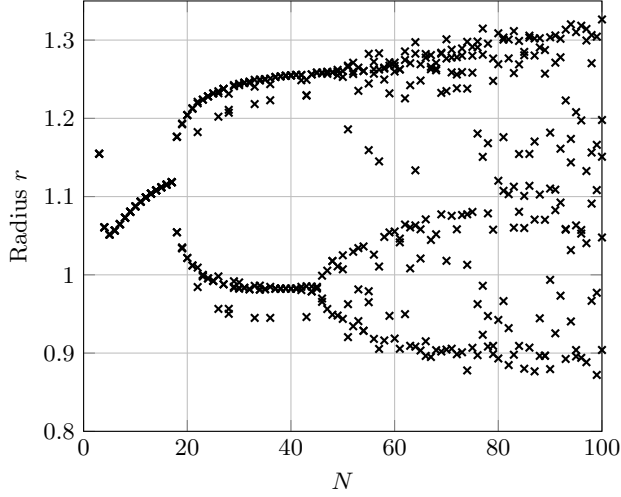
#### 1. Existence

To start with, Fig. 8 shows simulation results demonstrating that swarmalators can indeed form multicircle states for sufficiently large  $N$ .

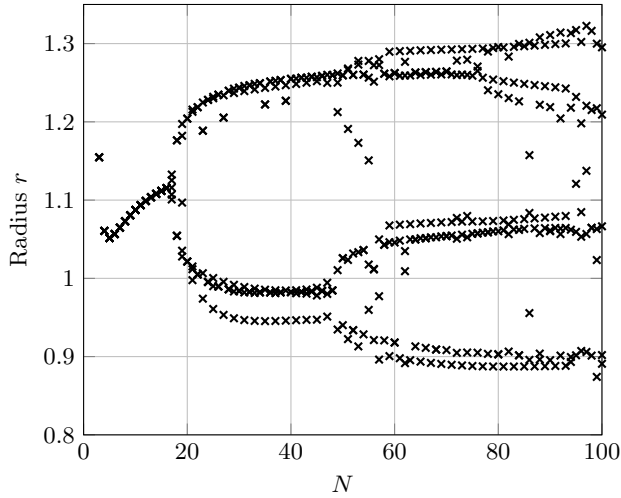
#### 2. Number and radii of circles

Striving for a mathematical analysis of multicircle states, we assume that  $s \mid N$  and divide the nodes into  $s$  groups, each having radius  $r_k$  with  $k = 1, \dots, s$ , in an interleaved fashion. Thus, node  $i$  is located at distance  $\|x_i\| = r_k$  with  $k = 1 + (i \bmod s)$  from the origin. Based on this, we compute expressions for the derivatives  $\dot{r}_k$ , set  $\dot{r}_k = 0$  for all  $k$ , and solve for the radii  $r_k$  numerically. Depending on  $J$  and  $N$ , some of these radii may be identical, meaning that an  $s$ -circle pattern does not form but rather one with fewer circles. The value of  $s$  is basically the number of potentially different radii.

The plots in Fig. 9 show the radii of patterns that emerge for  $J = 1$  for a given  $s$  as a function of  $N$ . The following observations can be made: In all cases, independent of  $s$ , a monocircle forms if the number of nodes is below 18. A bifurcation to a duocircle occurs for even values of  $s$  at  $N = 18$  [Fig. 9(a)] and for odd values at higher  $N$ . The subsequent bifurcations to states with more circles, occurring as we increase  $N$ , also depend on the value of  $s$ . If we set  $s = 3$  [Fig. 9(b)], the bifurcation from one to two circles occurs at  $N = 21$ ; the transition to three occurs at  $N = 30$ , where the outer circle splits into two. If we set  $s = 4$  [Fig. 9(c)], the inner circle is the first to split at  $N = 48$ , while the outer circle splits later at  $N = 80$ . The qualitative difference between these two cases can be explained as follows: For  $s = 3$ , there are eventually



(a) Initial placement of the nodes on a circle



(b) Initial placement of the nodes on two circles

FIG. 8. A set of  $N$  swarmalators converges to a circular pattern with several distinct circles. Radii from simulations. The initial positions of the  $N$  nodes are on one or two circles; this favors the formation of multicircle states if  $N$  is sufficiently large compared to uniform random placement. Swarmalator parameters:  $J = 1$  and  $K = 0$ .

three circles with  $N/3$  nodes each. The corresponding duo-circle has  $1/3$  of the nodes on the inner circle and  $2/3$  on the outer circle (where the nodes on the outer circle are not equidistantly placed but occur in pairs). The bifurcation to the three-circle state (i.e., the split of the outer circle) occurs when there are  $2N/3 = 32$  nodes on the outer circle and  $N/3 = 16$  on the inner one. For  $s = 4$ , the bifurcation to the four-circle state occurs when there are  $2N/4 = 40$  nodes on the outer circle and  $2N/4 = 20$  nodes on each of the inner circles.

If we set  $s = 6$  [Fig. 9(d)], the bifurcations occur at  $N = 21, 30$ , and  $87$ ; and for  $s = 8$  [Fig. 9(e)], they occur at  $N = 24, 48, 80, 144, 280$ , and  $320$ .

Furthermore, for a given  $N$ , the number of circles may vary depending on  $s$  in Fig. 9. For example, with 40 nodes, we have three circles for  $s = 3$  but only two for  $s = 4$ ; and with

100 nodes, there are still three circles for  $s = 3$  but now four for  $s = 4$ . The reason for this is twofold: First, the value of  $s$  limits the maximum number of distinct circles that can form. Second, as each radius is associated with a fixed number of nodes that depends on  $s$ , the thresholds for bifurcations vary with  $s$ .

### 3. Stability analysis

To calculate the relationship between  $N$  and  $J$  at the bifurcation points, we again calculate the eigenvalues of the Jacobi matrix of the derivatives  $\dot{r}_k(r_1, \dots, r_s)$ . The general Jacobi matrix for  $s$  different radii is given by

$$J_s = \begin{pmatrix} \frac{\partial}{\partial r_1} \dot{r}_1(r_1, \dots, r_s) & \cdots & \frac{\partial}{\partial r_s} \dot{r}_1(r_1, \dots, r_s) \\ \vdots & & \vdots \\ \frac{\partial}{\partial r_1} \dot{r}_s(r_1, \dots, r_s) & \cdots & \frac{\partial}{\partial r_s} \dot{r}_s(r_1, \dots, r_s) \end{pmatrix}. \quad (7)$$

When substituted with the fixed points,  $J_s$  generally possesses  $s$  real eigenvalues. For small enough  $N$ , all eigenvalues are negative, implying stability. At the bifurcation point, a fixed point splits into three fixed points (pitchfork bifurcation), and one of the eigenvalues becomes positive at the middle fixed point. This middle fixed point is now a saddle point; the newly occurring fixed points are attractors. The points of bifurcation can be calculated by determining the values  $N$  and  $J$  at which this eigenvalue vanishes.

### 4. State transition graph

The graph in Fig. 10 illustrates some of the transitions between the states. Each vertex represents a state, where the number inside the vertex indicates the number of circles and the numbers below are the fractions of nodes on each of them (sorted from the innermost to the outermost circle). The edges represent the possible transitions, annotated with the threshold values of  $N$  at which the transitions take place.

The following observations can be made: First, a higher number of nodes typically leads to more circles. The transitions occur in discrete steps. Second, transitions take place in a way that one or several circles split into two circles each. Hence, for a low number of circles, most transitions double the number of circles. Exceptions occur for uneven node distributions, where the circle that contains more nodes splits up first, or when the smaller circles split up earlier than the larger ones in a given state, since the distances between nodes are smaller at inner circles due to their smaller radii (such as in the transition from  $2 \rightarrow 3 \rightarrow 4$ ). As an example, for an uneven split, a single circle could split into two, where one-third of the nodes are located on the inner and one-third on the outer circle. This split occurs when  $N > 18$ . For  $N > 27$ , the outer circle also splits into two, where each of them obtains half of the nodes. Overall, this yields a three-circle state in which each circle contains one-third of the points. Note, for example, that the split from one to two circles occurs at different thresholds for different node distributions on the circles. Third, some transitions can be “masked” by others, which means that they do not necessarily occur in simulations (in which all nodes have freedom of movement). For example, the transition from one to two circles with uneven node distribution as well as the transition from one to three circles occurs for a higher number

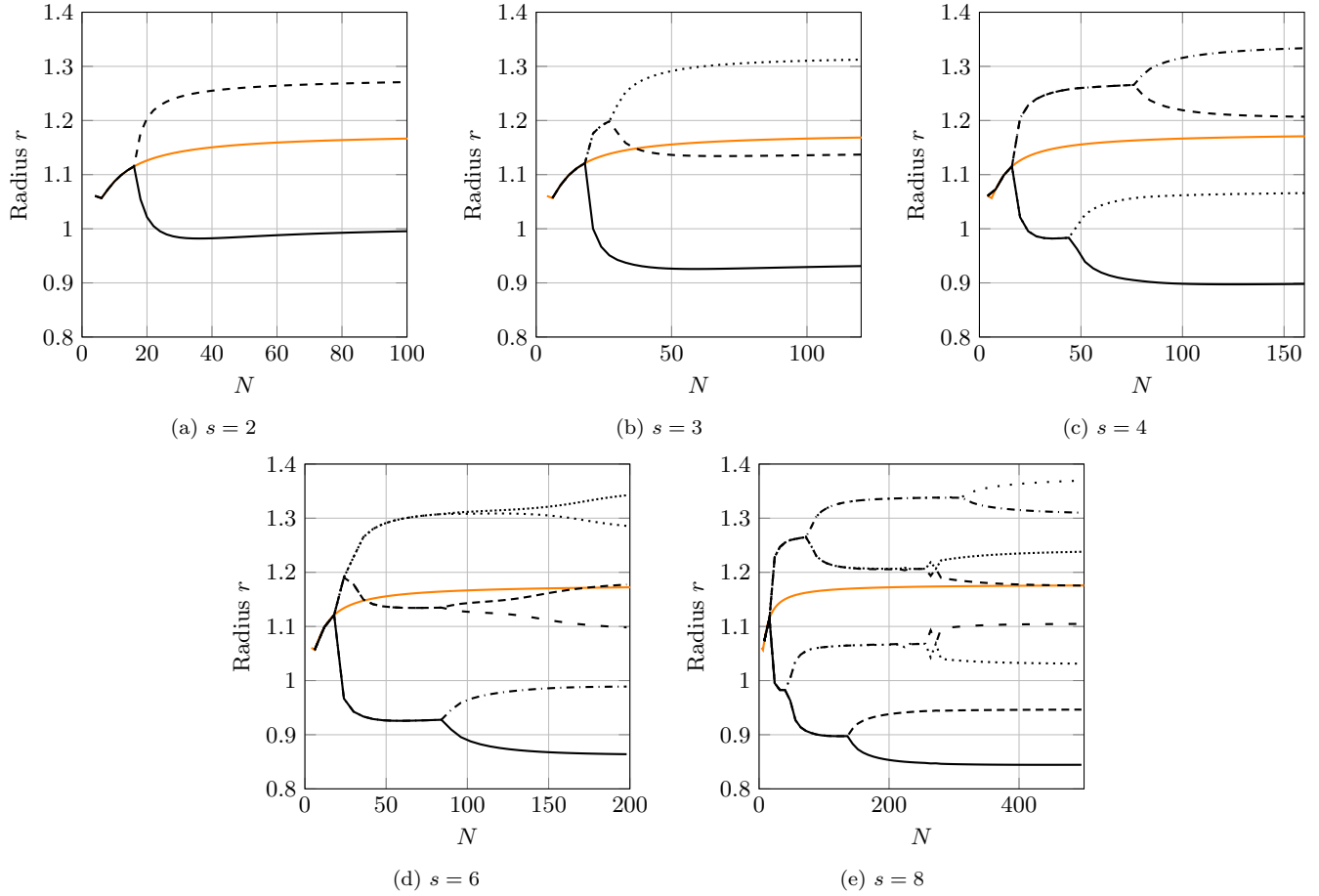


FIG. 9. Radii of multicircle states or varying  $N$ , calculated numerically. The orange lines indicate the single circle radius for comparison. Swarmalator parameters:  $J = 1$  and  $K = 0$ .

of nodes than the transition from one to two circles with even node distribution. Thus, the latter would occur instead of the other two, and once this split happens, the other two

states are no longer reachable. If we, however, allow only three radii, each allocated to the same number of nodes, an even split to two circles becomes impossible and the uneven

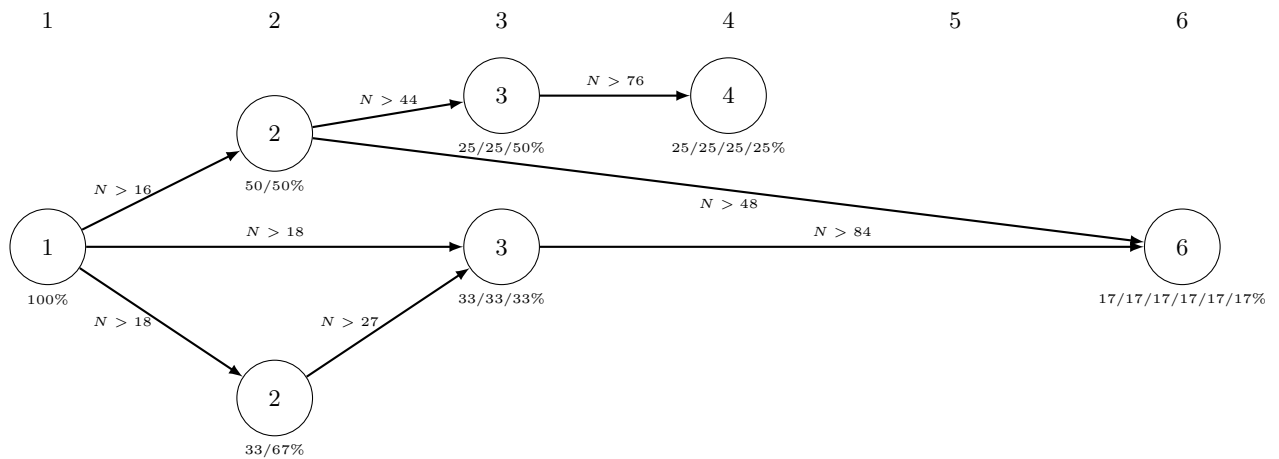


FIG. 10. Graph of transitions between static circular states for swarmalators with parameters  $J = 1$  and  $K = 0$ . The vertices show the number of circles and the percentage of nodes distributed to each of them, rounded to integer values. The edges show the state transitions, annotated with the number of nodes required for a transition to take place. The notation  $N > n$  means that the transition takes place somewhere in the range  $n < N \leq n + s$ , where  $s$  is the number of circles. This graph is incomplete and presents only a small fraction of all possible multicircle states and their transitions.

split to two circles or an even split to three circles can be observed.

Also note that the transition graph is derived from a stability point of view using the Jacobi matrices (7). Hence, at each transition point, the previous state changes from being a stable attractor to a saddle point and a pitchfork bifurcation takes place. The succeeding state becomes stable and attractive after the bifurcation and possesses a certain basin of attraction. However, it still is only reached if the system starts within this basin of attraction or enters it for some other reason; otherwise it converges to some other attractor.

#### IV. RELATED WORK

Related research can be found in publications on “ring states” in swarmalator systems, the analysis of circular patterns in other models of collective motion, and algorithms to achieve circular patterns in multirobot systems.

In swarmalator systems, O’Keeffe *et al.* describe the presence of “ring states” and analyze their radius and stability [16], including the case with all nodes on a monocircle. Schilcher *et al.* derive the inner and outer radii for all static patterns, including the static phase wave [19].

A series of papers by O’Keeffe and different coauthors studies the behavior of swarmalators whose movements are confined to a “1D ring” [20], where the location  $x_i$  of each node  $i$  is now a one-dimensional variable, for reasons of mathematical tractability. Among other findings, the emergence of a one-dimensional counterpart to the static phase wave is observed. Other publications address the effects of node heterogeneity [21], noise [22], node-individual parameters  $J_i$  and  $K_i$  [18], and external forces [23]. Moreover, Lizárraga and de Aguiar study the impact of advanced phase dynamics [24]. These studies relying on one-dimensional models enhance our general understanding of swarmalators. However, their modeling approach differs fundamentally from the article at hand. Here, swarmalators move in a two-dimensional space and eventually emerge into a circular pattern rather than being confined to locations on a circle from the very beginning.

Most recently, Lu *et al.* modify the swarmalator model by incorporating the concept of chirality [25], where each node continuously rotates and moves in the direction of its current phase  $\theta_i(t)$ , leading to dynamic patterns with nodes moving along multiple concentric circles.

Beyond swarmalators, circular patterns are also highly relevant in the broader literature on collective motion. Research in this field is motivated by empirical findings showing that several animal species exhibit collective behavior known as “milling,” in which they move together in circles around a common center [26]. However, these models typically do not perform spatial *sorting* of nodes according to their phases or any other state variable or identifier (as done here), nor do they link the movements with an adjustment or alignment of such variable (as done in swarmalators with  $K \neq 0$ ).

Finally, in the field of technical sciences, researchers design and test distributed algorithms for circle formation and control in multirobot and vehicle systems [12]. Various real-world constraints of mobile entities are taken into account, like limited interaction range [27], collision

avoidance [28], node heterogeneity [29], and the lack of a common coordinate system [30]. The body of research in this area covers not only collective motion but also aspects of control theory and mechanics of individual entities. Applications include cooperative surveillance and monitoring as well as target enclosing [31].

In summary, even when focusing solely on constellations without phase dynamics, the study of swarmalators contributes to the modeling of collective behavior, especially in self-organized formation of circular patterns with spatially ordered entities. Moreover, this research direction may become of practical use when swarmalators gain traction in applied fields. In the field of swarm robotics, implementations of the swarmalator model have already been demonstrated on real-world mobile robots [15] and quadcopters [32] and simulated for microscale robots [33,34].

#### V. SUMMARY AND APPLICATIONS

This article analyzed circular patterns that emerge as stable states in self-organized systems of swarmalators. Specifically, we explored the formation of stable patterns where all nodes arrange themselves in a sorted manner on one or more circles and then remain static. These patterns can be regarded as instances of the static phase wave. Using mathematical analysis and simulation, we calculated the radii of the circles and analyzed how the pattern stability depends on the number of nodes and the swarmalator parameter  $J$ .

The methods used and the results obtained can be applied to the dimensioning of swarmalator systems in the real world. For example, consider a scenario in which multiple entities—like robots or drones—should build a circular formation around a point of interest. Our results enable us to dimension such a system with respect to the number of entities needed to achieve a stable pattern with a desired number of circles. The pattern radii and stability can be controlled by adjusting the parameter  $J$  according to the needs of the application. In the context of next-generation semi-autonomous systems, circlelike swarm formations of unmanned entities are envisioned to operate around human-operated vehicles and aircraft. Finally, beyond the scope of technology, the approach and methods may be helpful to analyze swarmalatorlike behavior in nature. For example, there are suggestions that swarmalators can describe the collective behavior of certain animals, like microswimmers [35] and frogs [36].

While the work at hand studied swarmalators that keep their phase fixed ( $K = 0$ ) as a baseline case, future work should try to extend this analysis to circular patterns that emerge from systems with coupled phases ( $K \neq 0$ ). Furthermore, the analysis could be generalized from two to three-dimensional patterns (see [11,37]). The ultimate goal is to identify all stable fixed points of swarmalator systems and categorize them into groups of similar type. Another contribution would be determining the basin of attraction for the emergent patterns, enabling us to select suitable initial conditions to achieve a desired pattern (or predict the pattern resulting from given initial conditions) without relying on simulations.

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## DATA AVAILABILITY

The data that support the findings of this article are openly available [39].

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