

Extended thermodynamics of optimal networks

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We introduce a thermodynamic formalism for nonequilibrium transport networks based on generalized transport potentials (GTPs). When endowed with Legendre duality and the associated contact structure, this framework quantifies the distance from nonequilibrium operating points in analogy with thermodynamic availability, and clarifies how different optimality principles emerge under distinct constraints. Classical extremizations of dissipation and entropy production are recovered as special cases, requiring power-law resistances and, for entropy production, isothermal conditions. Stability analysis shows that changes between multiple operating points can be interpreted as nonequilibrium phase transitions governed by generalized equations of state. Finally, cost-based optimization of transport properties reveals new connections to generalized dissipation, particularly for power-law costs and linear resistance laws, though typically associated with unstable states that give rise to branched optimal transport.

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I. INTRODUCTION

Regardless of their epistemological objectivity [1,2], the intrinsic sense of design and purpose of variational approaches has long shaped research in thermodynamics and transport (see, e.g., [3–5]). This influence is evident in the enduring tradition of optimization principles for macroscopic nonequilibrium systems, beginning with Kelvin and Maxwell (minimum heat theorem in linear electrical circuits), continued by Helmholtz, Dirichlet, and Rayleigh (principle of least dissipation), and later advanced by Onsager, Prigogine, Biot, Gyarmati, Ziegler, and Paltridge, among many others [2,6,7].

In nonequilibrium thermodynamics, debate persists over maximization principles, in part due to the existence of dual

variational formulations that arise from a min–max structure and Legendre-transform duality [3]. While such a duality is similar to equilibrium thermodynamics [8], where entropy is maximized in isolated systems and different free energies are minimized in open systems depending on the environmental constraints, in nonequilibrium the conflation of entropy-production and dissipation extremizations is justified only under isothermal conditions, where entropy production is directly proportional to dissipation [see Eq. (6)]. For power-law systems, the generalized transport potentials (GTPs; see Sec. III) are themselves proportional to dissipation (and hence also to entropy production in isothermal cases), which has reinforced the idea that entropy production and dissipation serve as the universal governing quantities for extremization, even in nonlinear systems. This is misleading: variational principles based on entropy production often fail, as their extremization proves problematic even in reproducing Fourier’s law of heat conduction, leading to a T^{-2} dependence of thermal conductance rather than the constant value more consistent experimentally [9–11].

The pioneering works of Millar and Cherry [12,13] in the context of networks, and later those of Edelen, Verhas, and Presnov in nonequilibrium thermodynamics [14], extended optimization principles to transport problems with nonlinear constitutive laws. These principles rely on generalized

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transport potentials (GTPs; see Sec. III), constructed through Legendre transforms, with the product of generalized currents and driving forces as generators. As we will show, the resulting Lagrangians provide a unified framework that clarifies several earlier optimization principles, including maximum or minimum entropy production (MEP) and dissipation principles, which emerge naturally under suitable constraints.

Alongside the optimization of currents and their associated driving forces, another important line of research has examined the optimization of network configurations by adjusting resistances or conductances. Building on Murray's classical approach [15], which accounts not only for dissipation but also for the maintenance costs of transport systems, numerous subsequent studies [16–20] have shown that the long-term coupling between flow and optimal conductance induces competition among branches. In some cases, this competition generates unstable transport laws and branch pruning, ultimately favoring tree-like configurations. Such mechanisms underlie the theory of optimal-channel networks (OCNs) [4,21], continuous models of fractal landscape evolution [22,23], and the broader class of branched optimal transport models [5,24].

In this work, we develop a unified thermodynamic framework for optimal transport problems on discrete networks, formulated by extremizing Lagrangians constructed with GTPs. Our approach is mainly theoretical and based on a synthesis of previous theories; however, it is important to stress that several of the transport laws on which our work is founded are empirical in nature, as they result from observations rather than first-principle derivations. These not only involve the classic Fourier, Ohm, and fluid transport laws, but also their nonlinear extensions, as well as cases which undergo a transition to more dissipative conditions, such as turbulence [7,25], crystal polymorphism [25,26], Rayleigh–Bénard convection [27], and biological organization [28,29].

In Secs. II–V, we introduce the formalism and show how it provides a consistent treatment of several previously disconnected results, yielding transport laws that link generalized fluxes and forces in both linear and nonlinear networks. These laws exhibit a Legendre duality analogous to that of equilibrium thermodynamics. The associated contact structure [30] offers a natural measure of the distance from a nonequilibrium steady state (NESS) and introduces gradient flows that drive the network toward its operating point. We emphasize that our scalar-potential formulation is restricted to systems with gradient-based flux–force relations; many nonequilibrium settings (e.g., vortex-mediated transport or coupled reaction networks) exhibit intrinsically nonpotential couplings, for which no global scalar potential is expected, and the present approach may not directly apply.

The thermodynamic analogies are further explored in Sec. VII, where nonlinearities give rise to multiple solutions of the constitutive laws. Generalized dissipation potentials then allow us to discriminate their stability, drawing parallels with nonequilibrium phase transitions. In Sec. VIII, we extend the analysis to the optimization of transport properties (i.e., resistance laws). Owing to the thermodynamic parallel between transport characteristics and material properties—captured by the second derivatives of the Gibbs free energy (see Sec. IV)—this optimization can be viewed

as a macroscopic nonequilibrium counterpart of modern material science. The inclusion of maintenance costs leads to apparent transport laws that are typically unstable, promoting competition among branches and resulting in optimal tree-like structures related to branched optimal transport and evolutionary game theory [18,31]. In the special case of power laws, the optimization reduces to the minimization of generalized dissipation under current-conservation constraints.

II. TRANSPORT LAWS

We study transport phenomena on directed networks of resistive elements, a prototypical class of dissipative nonequilibrium systems. Currents arise only from boundary conditions; no loop circulations driven by internal sources are allowed [32]. The N branches of such a network may represent electrical resistors, fluid pipes, or one-dimensional thermal conductors. We focus on conditions under which the amount of the transported quantity remains unchanged within each link, so that no capacitance effects have to be taken into account. This either implies a steady state or an ‘incompressibility’ assumption, and it allows us to have a single, well-defined current for each link. As a result, the balance equation for the quantity being transported in a branch reduces to

$$I = O = J, \quad (1)$$

where I and O are the inflow and outflow, and J is the current (flux) through the link. The dimensions of the current depend on the transported quantity, such as charge, mass, or energy.

The constitutive relation, in the form of a generalized force or momentum balance, defines the operating point,

$$X = R(J)J \quad \text{or} \quad J = C(X)X, \quad (2)$$

where X is the generalized force, $R(J)$ is the resistance, and $C(X) = 1/R(J(X))$ is the conductance. In the linear case (constant R , C), this recovers Ohm's, Fourier's, Fick's, or Poiseuille's laws. Nonlinear generalizations describe, for example, turbulent flows or state-dependent electrical resistances. A common model is the power-law resistance,

$$R = \Re J^{\alpha-1}, \quad C = \Re^{-1/\alpha} X^{(1-\alpha)/\alpha}, \quad (3)$$

with parameter $\Re$ set by geometry or material properties. The case $\alpha = 1$ corresponds to laminar flow in pipes, while $\alpha = 2$ describes rough turbulent flow. For laminar flow in circular pipes, for instance, $\Re \propto r^{-4}$, while for electrical resistors, $\Re \propto r^{-2}$, where r denotes the pipe radius.

Multiplying the constitutive law by J yields a generalized power balance at the operating point,

$$XJ = R(J)J^2 = C(X)X^2, \quad (4)$$

where the left-hand side is the generalized power input and the right-hand side the generalized dissipation. For the power-law case, this becomes

$$XJ = \Re J^{\alpha+1} = \Re^{-1/\alpha} X^{(1+\alpha)/\alpha}. \quad (5)$$

Some clarifications are important regarding these transport laws:

(1) *NESS*. Our optimal solutions correspond to nonequilibrium steady states (NESS). Away from steady state, the

evolution must be described by relaxation dynamics that include inertial (or, more generally, storage/capacitance) effects, which are not explicitly represented in our variational formulation. For sufficiently slow driving, these effects can be treated in a quasisteady approximation: the same transport laws apply instantaneously while time t enters as an external parameter, in analogy with quasistatic thermodynamic transformations. In this interpretation, virtual perturbations about the steady-state optimum can be associated with the neglected inertial or capacitance contributions. When the dynamics are purely dissipative, the approach to (or departure from) a stable (or unstable) steady state may be expected to be monotonic in an appropriate time or time-like parameter, as assumed in some parts of the paper; for non-purely-dissipative systems, such monotonicity need not hold.

(2) *Systems that are not purely dissipative.* For systems that are not purely dissipative, such as viscoelastic fluids, transport laws of the form $X = R(J)J$ typically remain valid in nonequilibrium steady states. For instance, identifying $X \equiv \Delta p$ and $J \equiv Q$, a Maxwell fluid in a straight, fully developed conduit satisfies the steady Poiseuille relation $\Delta p = (8\eta L/\pi R^4)Q$, while viscoelasticity enters only in transients through the first-order dynamic extension $\Delta p + \lambda \dot{\Delta p} = (8\eta L/\pi R^4)Q$. In such cases, the lack of purely dissipative dynamics implies that the approach to, or departure from, extrema need not be monotonic in time.

(3) *Single energy mode.* We restrict attention to the transport of a single conserved quantity [33], neglecting cross-coupled fluxes as in nonequilibrium thermodynamics, where

$$J_i = \sum_k C_{ik} X_k,$$

with Onsager reciprocity $C_{ik} = C_{ki}$ [32,34]. Extension to simultaneous transport of different quantities will be explored in future contributions.

(4) *Generalized dissipation vs. entropy production.* Our focus is on generalized dissipation, not on entropy production. Only when Eq. (4) refers to actual energy rates (e.g., heat conduction at uniform temperature T) does it coincide with the entropy production rate,

$$\sigma = \frac{1}{T} R(J)J^2, \quad (6)$$

provided that T is uniform along the branch. In all other cases, extremizing entropy production can be misleading [11,35], since it does not, in general, correspond to extremizing the Lagrangian built on generalized dissipation rates (defined in the next section). In the present formulation, the objective functional involves a generalized dissipation XJ based on the phenomenological force-flux pairing adopted for each transport mode; in general (and especially in non-isothermal settings) XJ should not be identified *a priori* with either the mechanical energy dissipation rate or the entropy production rate. This is particularly clear for Fourier heat conduction, where one may naturally take $J \equiv q$ (heat flux) and a temperature-type driving X (e.g., ∇T): in that case, XJ carries “mixed” units (temperature times heat flux, or temperature-gradient times heat flux) and therefore has no direct interpretation as either power dissipation or an entropy-production rate. For heat conduction, the entropy-production

density is canonically written as $\sigma = q \cdot \nabla(1/T)$, corresponding to pairing the heat flux with the thermodynamic force $\nabla(1/T)$; if instead one adopts a temperature-type driving such as ∇T , the resulting product XJ is generally different from σ (the two are related by $\nabla(1/T) = -(1/T^2)\nabla T$) [9–11]. The use of generalized dissipation functionals of this type traces back to Biot [36,37].

III. VARIATIONAL APPROACHES BASED ON GENERALIZED TRANSPORT POTENTIALS

Generalized transport potentials (GTPs) provide a unifying criterion for the optimization of both linear and nonlinear transport laws. Originally introduced by Millar and Cherry for electrical networks [12,13] and later extended by Edelen, Verhas, and Presnov in nonequilibrium thermodynamics [14], GTPs have remained relatively unexplored and not fully connected to the classical thermodynamic formalism.

To illustrate their role, we first consider a single branch. Virtual variations are allowed either in the flux J or in the force X , while the other variable is held fixed at the operating point. In this way, perturbations need not satisfy the constitutive law (2) (physically, one may think of these representing inertial terms related to transient conditions); the extremization of suitable Lagrangians then identifies the stable operating point. This construction parallels the Onsager–Machlup variational approach for linear laws in the large-system limit [38]. Depending on whether J or X is varied, one obtains dual principles connected by Legendre transformations. Importantly, dissipation or entropy production alone cannot serve as functionals with an extremum at the operating point.

In the *flux representation*, the variational principle maximizes

$$\mathcal{L}_J(J) = XJ - \Phi(J), \quad (7)$$

where the GTP (or content) is defined as [12]

$$\Phi(J) = \int_0^J X(J') dJ'. \quad (8)$$

This case corresponds to the Rayleigh–Onsager principle [14] (Rayleigh’s original sign convention instead yielded a minimum dissipation principle [39,40]). Because X is externally imposed, this is often referred to as the force-driven case. The stable operating point (2) is the maximizer of $\mathcal{L}_J$. As shown later in Sec. VII, when the extremum of $\mathcal{L}_J$ is a minimum, the operating point is dynamically unstable.

The *dual force representation*, also called the Biot principle [14], maximizes

$$\mathcal{L}_X(X) = XJ - \Psi(X), \quad (9)$$

with the co-content

$$\Psi(X) = \int_0^X J(X') dX'. \quad (10)$$

At the extremum of (7), $\mathcal{L}_J = \Psi$; at the extremum of (9), $\mathcal{L}_X = \Phi$. Thus, the generalized power input XJ acts as the generator of the Legendre transformation

$$\Psi(X) = XJ - \Phi(J), \quad (11)$$

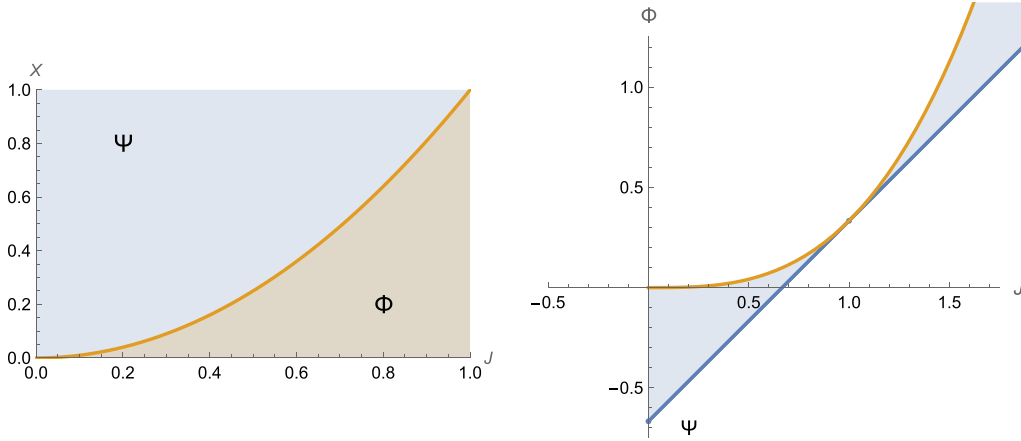


FIG. 1. Generalized transport potentials for a nonlinear transport law with $\alpha = 2$ and $\mathfrak{R} = 1$ (left). Right: Legendre-transform structure with shaded area representing the Lagrangian $\mathcal{L}_J$ in Eq. (7).

endowing nonequilibrium steady states (NESS) with a gradient structure [41]. This terminology has roots in elasticity, where the duality between energy and co-energy—already implicit in Castigliano’s theorem (1873)—corresponds to internal energy-Helmholtz free energy Legendre transform pair [42,43].

In the linear case, Φ and Ψ are quadratic in J and X , equal to half the generalized power input and the generalized dissipation rate. Under isothermal conditions, they are also proportional to the entropy production rate (6). For nonlinear power-law laws (3), Φ and Ψ remain proportional to dissipation, but they are distinct from each other. Figure 1 illustrates this difference for $\alpha = 2$, where content, co-content, and $\mathcal{L}_J$ are compared. For general nonlinear laws, neither Φ nor Ψ is proportional to dissipation.

Gyarmati extended these ideas by combining the two Lagrangians, allowing variations in both X and J [14,44]. The resulting principle extremizes

$$\mathcal{L}(X, J) = XJ - \Psi - \Phi, \quad (12)$$

whose value at the operating point is always zero. This follows because Φ and Ψ are Legendre transforms of each other, cf. Eq. (11). The constitutive law (2) is typically monotonic and single valued. More complex behaviors connect to catastrophe theory [43] and hysteresis phenomena [45]—see Sec. VII. As shown in Fig. 2, for power laws (3), the Lagrangian landscape $\mathcal{L}_J$ exhibits a saddle point structure: for $\alpha > 0$, the extremum is a maximum, while for $\alpha < 0$, it becomes a minimum, corresponding to instability. For $\alpha \leq -1$, the GTPs fail to converge.

Ziegler’s maximum dissipation principle can also be recast in the GTP framework. His Lagrangian [34,46] imposes the steady-state energy balance (4) via a Lagrange multiplier,

$$\mathcal{L}_{J\lambda}(J) = R(J)J^2 + \mu(XJ - R(J)J^2), \quad (13)$$

or equivalently, with $\lambda = \mu - 1$,

$$\mathcal{L}_{J\lambda}(J) = XJ + \lambda(XJ - R(J)J^2). \quad (14)$$

Stationarity with respect to λ enforces the constitutive law $X = R(J)J$. Eliminating λ yields a reduced Lagrangian de-

pending only on J , which shares the same extremum as $\mathcal{L}_J$. For power-law resistances (3), one finds $\lambda = 1/\alpha$, and $\mathcal{L}_J$ and $\mathcal{L}_{J\lambda}$ are proportional. For more general nonlinear laws, they differ in form but still possess the same extremum, as illustrated in Fig. 3. Thus, while their physical interpretation may differ, both recover identical NESS operating points. This situation resembles ensemble equivalence in statistical mechanics, where fluctuations differ under distinct constraints but macroscopic averages remain the same [47]. Finally, for the dual force-driven case, repeating Ziegler’s construction yields $\lambda = 2$ for the linear law ($\alpha = 1$). This value frequently appears in the literature [48], though often interpreted in terms of entropy production. As stressed earlier, only in isothermal

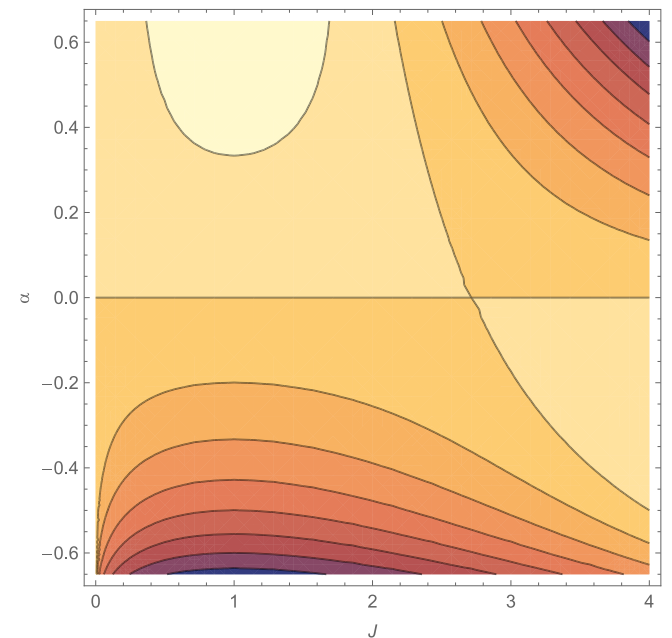


FIG. 2. Contour plot of $\mathcal{L}_J$ [7] for the power-law resistance (3) as a function of J and α . Lighter colors correspond to higher values. The extremum switches from a maximum ($\alpha > 0$) to a minimum ($\alpha < 0$), revealing a saddle point.

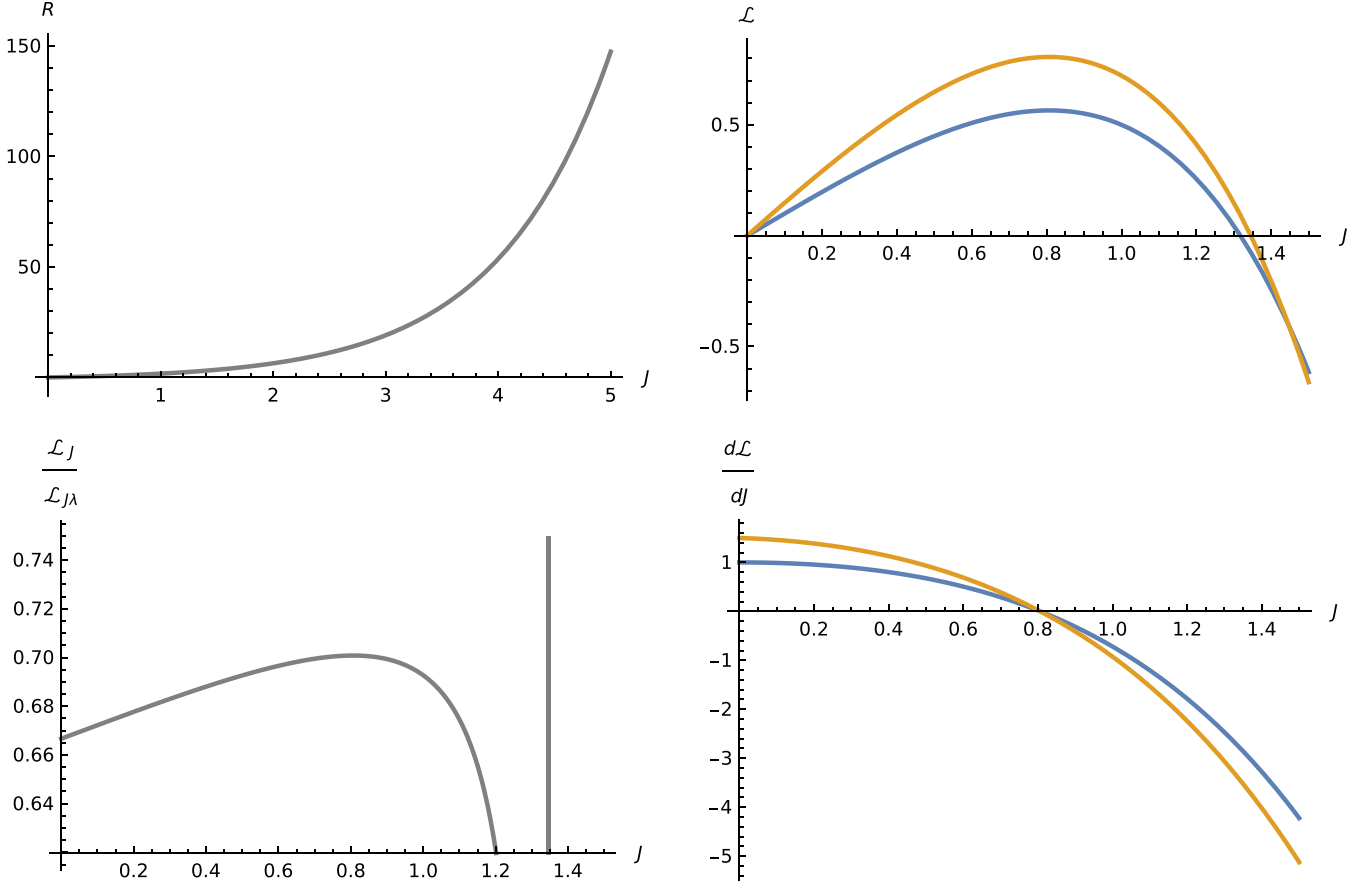


FIG. 3. Comparison of $\mathcal{L}_J$ (blue curve) and $\mathcal{L}_{J\lambda}$ (orange curve) for an exponential resistance law $R(J) = e^J - 1$. Although the two Lagrangians differ, their maxima coincide at the same flux value.

systems does entropy production extremization coincide with dissipation extremization [see Eq. (6)].

IV. THERMODYNAMIC FORMALISM FOR NESS

From a thermodynamic perspective, the operating point of a network corresponds to a nonequilibrium steady state (NESS) described by the transport law (2). In this section, we highlight an analogy between the formalism of generalized transport potentials (Sec. III) and classical equilibrium thermodynamics [8], which is built on the Gibbs fundamental relation. For a one-dimensional thermal system, the fundamental equation reads $U = U(S)$, where U is internal energy and S is entropy. In the NESS framework, the analogous role is played by $\Psi(X)$, leading to a correspondence between Legendre transforms,

$$F(T) = U(S(T)) - TS, \quad (15)$$

and

$$\Phi(J) = -\Psi(X(J)) + JX. \quad (16)$$

The only difference lies in the sign convention adopted in thermodynamics. In this analogy, the entropy representation corresponds to the flux representation in J .

First derivatives of the potentials define the nonequilibrium analogues of equations of state,

$$\frac{\partial \Psi}{\partial X} = J, \quad (17)$$

$$\frac{\partial \Phi}{\partial J} = X, \quad (18)$$

recovering the constitutive laws of Eq. (2).

Similarly, the second derivatives define the transport properties of the nonequilibrium system—generalizations of material properties in equilibrium thermodynamics,

$$\frac{\partial^2 \Psi}{\partial X^2} = C(X), \quad \frac{\partial^2 \Phi}{\partial J^2} = R(J). \quad (19)$$

These properties are inherently nonequilibrium in character. In the linear multivariate case, they satisfy the Onsager reciprocal relations [34], while in the thermodynamic analogy, they play a role akin to Maxwell-type relations [41,49].

Gyarmati's Lagrangian (12) has a natural analog in equilibrium thermodynamics: the *availability* A between two equilibrium states. Availability represents the maximum reversible work extractable from a system (or, equivalently, the minimum reversible work required to realize a given change). For a thermal system at constant mass and volume (canonical ensemble), combining the first and second laws yields the

availability when moving from a state $\hat{U}, \hat{T}, \hat{S}$ to U, T, S :

$$A(T, \hat{S}) = \hat{U} - U - T(\hat{S} - S) = F(T) - T\hat{S} - \hat{U}. \quad (20)$$

Formally, this is the difference between a Legendre transform and its definition when the arguments are treated as independent. It vanishes only when T matches the slope of the fundamental equation at S , but in general defines a nonzero surface endowed with a *contact structure* [30]. By the Stodola–Guoy theorem, the availability (or entropy production) is zero at equilibrium, i.e., when $T\hat{S} - U$ is exactly the Legendre transform of $U(S)$.

The nonequilibrium analog is obtained by treating X and J as independent. Moving from $(\hat{\Psi}, \hat{J}, \hat{X})$ to (Ψ, J, X) gives

$$-\mathcal{L}(J, \hat{X}) = \Phi(J) - J\hat{X} + \hat{\Psi}(\hat{X}). \quad (21)$$

Thus, the negative Lagrangian $-\mathcal{L}$ plays the role of a nonequilibrium availability, measuring the potential dissipation that would occur as the system relaxes dynamically to the transport law. Equivalently, it quantifies the excess dissipation compared to the operating point—an energetic measure of the distance from the constitutive law attained at the maximum of $\mathcal{L}$. In this way, $\mathcal{L}$ can serve as the basis for defining a nonequilibrium metric that measures distance from NESS, including the distance between equilibrium and NESS configurations [50–54]. This possibility will be explored in future work.

V. REDUCED LAGRANGIANS FOR NETWORKS

The variational principles based on the Lagrangians (7) and (9) extend naturally to (nonlinear) networks by summing over all branches. For the force-driven case, for example, the network Lagrangian is

$$\mathcal{L}_J = \sum_{i=1}^N \mathcal{L}_{J_i} = \sum_{i=1}^N X_i J_i - \sum_{i=1}^N \Phi(J_i), \quad (22)$$

where N is the number of branches. This formulation carries over the same variational properties and thermodynamic analogies introduced in Sec. III, now in the multivariate space of currents J_i .

In practice, however, this formulation is often not convenient, since the branch forces X_i are typically unknown. An alternative arises by exploiting Kirchhoff’s current law (KCL). Specifically, the sum of power inputs across the N branches reduces to a constant G for prescribed forces X_i , leaving only $M < N$ independent currents,

$$\sum_{i=1}^N X_i J_i = G, \quad (23)$$

where G is the generalized power absorbed from the external environment.

Substituting Eq. (23) into (22) eliminates the X_i , but also removes explicit information about the network topology. This information must then be recovered through KCL at the nodes, which further reduces the number of independent currents,

$$\mathcal{L}_J = G - \sum_{i=1}^N \Phi(J_i). \quad (24)$$

Coupled with KCL, this yields a reduced Lagrangian in terms of the M independent currents,

$$\mathcal{L}_J^*(J_i), \quad i = 1, \dots, M. \quad (25)$$

Although $\mathcal{L}_J^*$ still has an extremum at the operating point, it lacks the full contact structure associated with the Legendre transform. This structure can be restored by reconstructing the corresponding forces for the M independent currents and returning to the definition (7),

$$\mathcal{L}_J^{**} = \sum_{i=1}^M X_i J_i - \sum_{i=1}^M \Phi(J_i), \quad (26)$$

where the M forces are treated as fixed. In Callen’s terminology [8,55], this formulation may be viewed as *microcanonical* with respect to the nonindependent currents (which remain fixed) and *canonical* with respect to the independent ones [56].

Equations (24) and (26) imply the existence of a potential landscape in the space of independent currents, such that the approach to the operating point follows a gradient flow. Thus, in a practical dynamic optimization, one can imagine following a dynamical system of the type

$$\tau \frac{dJ_i}{dt} = -\frac{\partial \mathcal{L}_J^{**}}{\partial J_i}, \quad (27)$$

for $i = 1, \dots, M$, with t time and τ a suitable heuristic timescale of evolution. Convergence to an absolute maximum or to local minima depends on the convexity or concavity of the branch Lagrangians (cf. Fig. 2).

A simple but illustrative case is that of two parallel pipes. When the pipes follow different nonlinear (power-law) transport laws, the generalized dissipation formalism yields consistent solutions, as shown in Fig. 4. This goes beyond the limitations of entropy-based principles (e.g., maximum entropy production, MEP), which may fail in such cases [57]. The left panels show the full Lagrangian (22), while the right panels display the reduced form $\mathcal{L}_J^*$ obtained by substituting $J_2 = J - J_1$ and $X_1 = X_2 = X$ with $J = 2$ and $X = 1$.

The unstable case with negative exponent $\alpha = -2/3$ is also instructive. Here, the two flows “compete” for the total current, in a process reminiscent of evolutionary games [18,31], leading to a single-pipe configuration that carries all the flow. This type of instability, with multiple local extrema, reappears in Sec. VIII, where subadditive cost functions drive the emergence of optimal tree-like networks.

VI. LIMITS OF ENTROPY-BASED EXTREMUM PRINCIPLES

For the specific case of power-law resistances, the previous Lagrangians acquire a special form related to generalized dissipation. In this section, we clarify the validity of entropy-based extremum principles by showing that both maximum entropy production and minimum dissipation emerge as special cases of the general Lagrangian, Eq. (22), under different constraints. For power laws (3), the total content is propor-

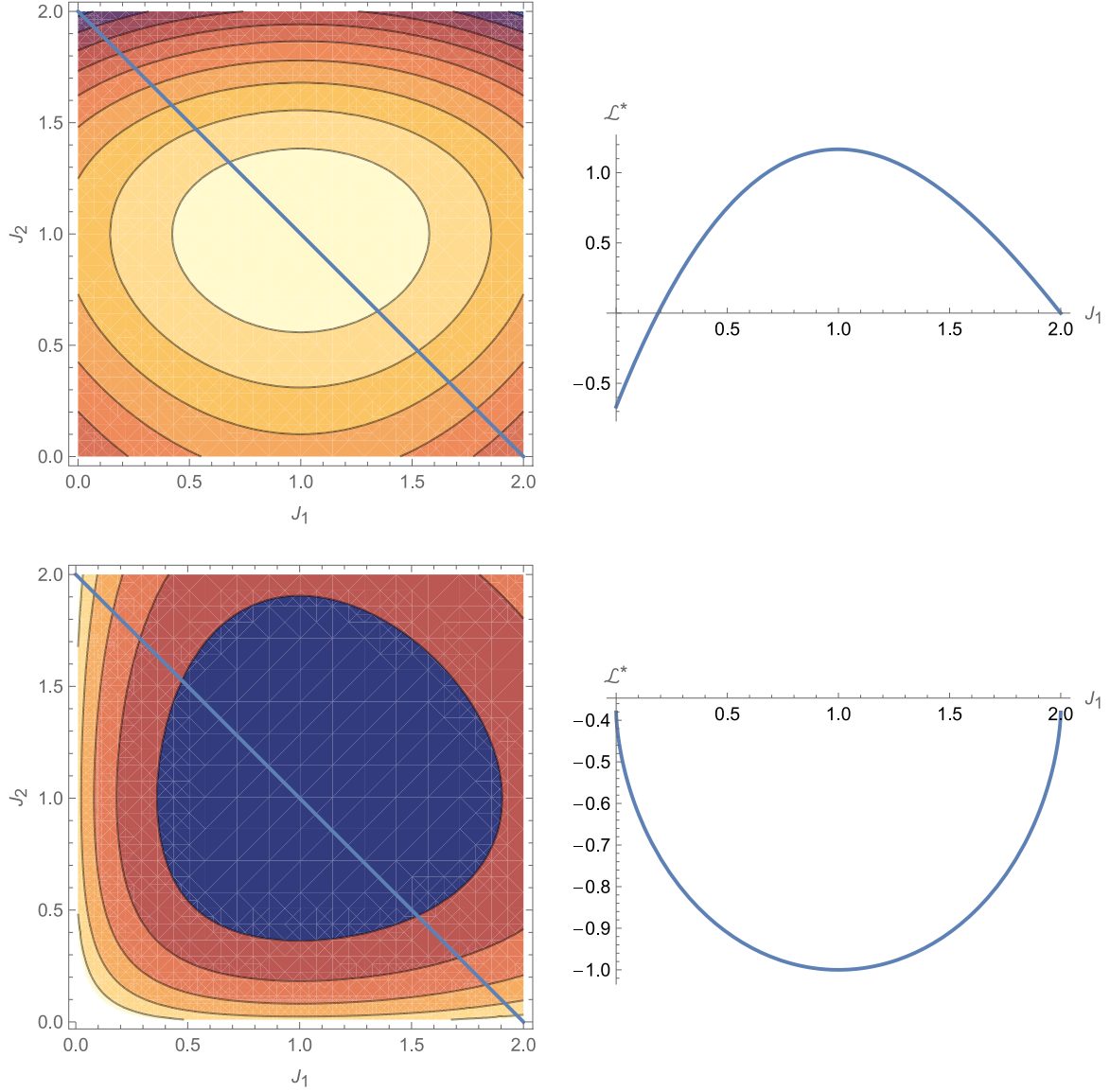


FIG. 4. Parallel pipes: contour plots of the full Lagrangian $\mathcal{L}_J$ (left) and the reduced Lagrangian $\mathcal{L}_J^*$ (right). Lighter/darker colors denote higher/lower values of the Lagrangian. Top panels: stable case with different power laws ($\alpha_1 = 1$, $\alpha_2 = 2$). Bottom panels: unstable case ($\alpha = -1/3$ for both pipes).

tional to the total virtual dissipation $\mathcal{D}$, i.e.,

$$\sum_{i=1}^N \Phi(J_i) = \frac{1}{\alpha + 1} \sum_{i=1}^N \mathfrak{R}_i J_i^{\alpha+1} = \frac{\mathcal{D}}{\alpha + 1}. \quad (28)$$

Then, with the use of KCL, the reduced Lagrangian becomes

$$\mathcal{L}_J^* = G - \frac{\mathcal{D}^*}{\alpha + 1}, \quad (29)$$

where $\mathcal{D}^*$ is the reduced virtual dissipation, obtained by varying only the $M < N$ remaining independent currents. As a result, at the operating point, when such currents satisfy KVL, the reduced virtual dissipation is minimized, while the reduced Lagrangian above is maximized. In isothermal conditions, these principles directly translate into entropy-production extremizations: the maximization of $\mathcal{L}_J^*$ is Onsager's and Ziegler's principle of maximum entropy pro-

duction [34], while the dissipation minimization corresponds to Prigogine's minimum entropy production principle [10] (and therefore also the minimum heat theorem of Maxwell and the least energy dissipation of Rayleigh).

It should be clear at this point that similar results can also be obtained starting with the formulation (14) with the Lagrange multiplier. In fact, for the power-law case, adding the KCL as a constraint ($\lambda = 1/\alpha$), one obtains

$$\mathcal{L}_{J\lambda}^* = G + \frac{1}{\alpha} G - \frac{1}{\alpha} \mathcal{D}^* = \frac{\alpha + 1}{\alpha} \left(G - \frac{\mathcal{D}^*}{\alpha + 1} \right), \quad (30)$$

which has the same variational properties as (29), being proportional to it. It is important to stress that for non-power-law resistances, the GTPs are no longer proportional to generalized dissipation; as a result, dissipation or entropy production principles no longer hold.

VII. STABILITY AND DYNAMIC PHASE TRANSITIONS

Transitions between dynamic regimes as the driving force increases are often interpreted *ad hoc* using optimality theories based on dissipation or entropy production [25,57–59]. Such cases typically involve a crossover from linear response at weak forcing to nonlinear behavior at strong forcing, with examples including turbulence onset in pipes [7,25], crystal polymorphism [25,26], Rayleigh–Bénard convection [27], and even the transition from abiotic to biological organization [28,29].

Here, we show that the stability of a constitutive law follows directly from the Lagrangians (7) and (9) introduced in Sec. III, depending on whether an extremum is a maximum (stable) or a minimum (unstable). These stability switches may be regarded as *dynamic phase transitions*, in analogy with equilibrium phase transitions governed by thermodynamic potentials such as the Gibbs free energy [8,60].

For a flux-driven 1D system, the dynamic relaxation near an operating point may be expressed as

$$\tau \frac{dJ}{dt} = -\frac{\partial \mathcal{L}_J(J)}{\partial J} = -X + \frac{\partial \Phi(J)}{\partial J} = -X + R(J)J, \quad (31)$$

assuming—as in (27)—that because of the dissipative nature of the transport problems considered, the evolution is first order in t ; it should be clear, however, that the generalized dissipation potentials do not encode any information regarding the actual timescales of approach to NESS (much like the thermodynamic potential does not contain information on the timescale to approach equilibrium). In cases with multiple extrema, the physical solution is determined by their stability type, as shown in the nonlinear models of Kawazura, Yoshida, and co-workers [59,61–63]. These examples demonstrate that neither dissipation nor entropy production provides a universal optimality criterion to choose a system’s state, contrary to other proposals such as the Glansdorff–Prigogine criterion [64] or the maximum entropy production principle (MEP) [34]. Instead, the correct selection of NESS follows from the appropriate Lagrangians and their second variations.

Accordingly, nonequilibrium transitions [60] are recast here as changes in stability between solutions of a “dynamic equation of state” [e.g., (17), (18)] with multiple roots. Integration of these equations yields the corresponding potentials, and their second variations determine stability. Importantly, this integration—see (8) and (10)—implies that the stability criteria are generally *nonlocal*, depending on integrals over the entire domain of current or flux. This agrees with Landauer’s critique [65] of the locality assumed in the Glansdorff–Prigogine criterion, and parallels nonlocal features found in statistical mechanics of NESS, even when local equilibrium holds [66–68].

In the following subsections, we analyze the nonlinear model of Kawazura and co-workers [59,62] under constant-force and constant-current conditions. We note that when generalized transport potentials (GTPs) are not convex, Legendre duality breaks down. In such macroscopic systems, a Maxwell construction is typically not present for lack of macroscopic fluctuations, in contrast to microscopic systems [60].

A. Self-organized plasma: Force-driven case

We first examine the nonlinear resistance induced by a self-organized boundary layer in a plasma flow [59,62]. The situation is analogous to turbulence transition in pipe flow [7,25], where the resistance law shifts from linear to nonlinear as the regime changes from laminar to turbulent. The system consists of a constant resistor R_0 in series with a nonlinear contribution,

$$R_1 = aJ \left(\frac{T_0}{T_0 + R_0 J} - \frac{T_0}{X + T_0} \right),$$

where a is a parameter, T_0 is a reference temperature, and the driving force is $X = T - T_0$. The nonequilibrium “equation of state” is then expressed in terms of an equivalent resistance,

$$X = \frac{\partial \Phi}{\partial J} = \left[R_0 + a \left(\frac{T_0}{T_0 + R_0 J} - \frac{T_0}{X + T_0} \right) J \right] J. \quad (32)$$

This relation can be rearranged as

$$(X - R_0 J) \left(\frac{a T_0 J^2}{R_0 (X + T_0) J + X + T_0} - 1 \right) = 0, \quad (33)$$

yielding two possible current branches,

$$J = \begin{cases} X/R_0, \\ \frac{R_0(X + T_0) + \sqrt{(R_0(X + T_0))^2 + 4aT_0^2(X + T_0)}}{2aT_0}. \end{cases} \quad (34)$$

These branches intersect at the critical force

$$X_c = \frac{T_0 \sqrt{aT_0}}{\sqrt{aT_0} - R_0} - T_0. \quad (35)$$

The stable solution is identified through the potential $\mathcal{L}_J$, obtained from Eq. (32) as

$$\begin{aligned} \mathcal{L}_J &= \int^J \left(X - \frac{\partial \Phi}{\partial J'} \right) dJ' = XJ - \Phi(J), \\ &= \frac{1}{6} J \left(\frac{2aJ^2 T_0}{X + T_0} - \frac{3J(aT_0 + R_0^2)}{R_0} + 6 \left(\frac{aT_0^2}{R_0^2} + X \right) \right) \\ &\quad - \frac{aT_0^3 \ln(JR_0 + T_0^2)}{R_0^3}. \end{aligned} \quad (36)$$

With dissipation defined as $\Upsilon = XJ$, the observed branch corresponds to the maximum of $\mathcal{L}_J = \Upsilon - \Phi(J)$. Thus, the linear solution is selected for $X < X_c$, while the nonlinear branch is realized for $X > X_c$. Figures 5(b) and 5(c) illustrate how the stability changes as X crosses the critical value. Interestingly, if only dissipation ($\Upsilon = XJ$) or entropy production [25] are considered, the selected branch appears to correspond to a *minimum* of Υ [Fig. 5(a)], opposite to the flux-driven case discussed below.

B. Self-organized plasma: Flux-driven case

For the flux-driven setup, the two steady-state branches are

$$X = \begin{cases} R_0 J, \\ \frac{aJ^2 T_0}{JR_0 + T_0} - T_0, \end{cases} \quad (37)$$

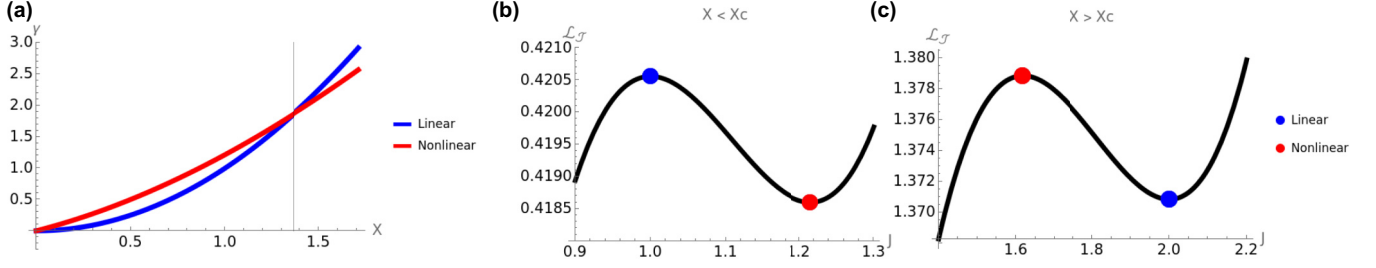


FIG. 5. Self-organized boundary layer in a force-driven plasma flow. (a) Dissipation for the linear (blue) and nonlinear (red) branches, with a vertical line marking the critical force X_c . (b),(c) $\mathcal{L}_J(J)$ for $X < X_c$ [panel (b)] and $X > X_c$ [panel (c)]. Parameters: $R_0 = 1$, $a = 3$, $T_0 = 1$, giving $X_c = 1.366$. Panel (b): $X = 1 < X_c$; panel (c): $X = 2 > X_c$.

which cross at the critical current

$$J_c = \frac{T_0}{\sqrt{aT_0} - R_0}. \quad (38)$$

In this case, the selected solution corresponds to the branch with *maximum* dissipation [Fig. 6(a)], opposite to the force-driven case. The generalized potential again provides the correct stability criterion. Rewriting Eq. (32) in dual form,

$$J = \frac{\partial \Psi}{\partial X} = \frac{X}{R_0} - \frac{a}{R_0} \left(\frac{T_0}{T_0 + R_0 J} - \frac{T_0}{X + T_0} \right) J^2, \quad (39)$$

one obtains the potential

$$\begin{aligned} \mathcal{L}_X &= \int^X \left(J - \frac{\partial \Psi}{\partial X'} \right) dX' = XJ - \Psi(X) \\ &= \frac{X \left(\frac{2J(aT_0 + R_0^2) + R_0 T_0}{JR_0 + T_0} - X \right) - 2aJ^2 T_0 \ln(T_0 + X)}{2R_0}. \end{aligned} \quad (40)$$

The stable operating point corresponds to the maximum of $\mathcal{L}_X$, just as for $\mathcal{L}_J$ in the force-driven case. The dynamics follow

$$\tau \frac{dX}{dt} = -\frac{\partial \mathcal{L}_X}{\partial X}, \quad (41)$$

with stability switching from linear to nonlinear branch as J crosses J_c [see Figs. 6(b) and 6(c)].

VIII. OPTIMAL TRANSPORT PROPERTIES

Another form of optimality can be sought by varying the transport properties themselves, i.e., the resistances in (2).

Physically, this type of optimization is quite different from those discussed earlier, but it exhibits a formal similarity with previous variational principles, particularly in relation to dissipation minimization (in the case of power-law cost functions) and the emergence of unstable apparent transport laws that favor tree-like network structures. In the thermodynamic analogy of Sec. IV, transport properties play the same role for NESS as material properties do in equilibrium thermodynamics: they are associated with the curvatures of the fundamental equation and its Legendre transforms (e.g., Gibbs free energy), cf. (19). Adjusting transport properties thus mirrors the design of materials in modern material science, where properties are engineered to achieve specific objectives [69]. These ideas also connect to the adaptation and evolution of natural systems (including biological ones, often framed in terms of fitness maximization [70–72]), the design of engineered networks, and classical optimal transport problems in the Monge–Kantorovich framework [5,24,73].

A simple example is the design optimization of a pipe cross section for a given flow or driving potential, a classical integral variational problem (cf. Dido’s isoperimetric problem [21,74]). Bejan used such arguments in support of his constructal theory [17], showing that a circular cross section is optimal in conductance by maximizing flow (and dissipation) for a given driving force X , or equivalently by minimizing X for a given current J . This illustrates the duality (min–max) character of the optimization [3].

Here, we focus on Murray’s classical biological optimization [15], which incorporates a maintenance cost alongside energy dissipation. In this formulation, optimality arises even for a single pipe, without the need to impose a global constraint on cross sectional area. For a given branch, the cost

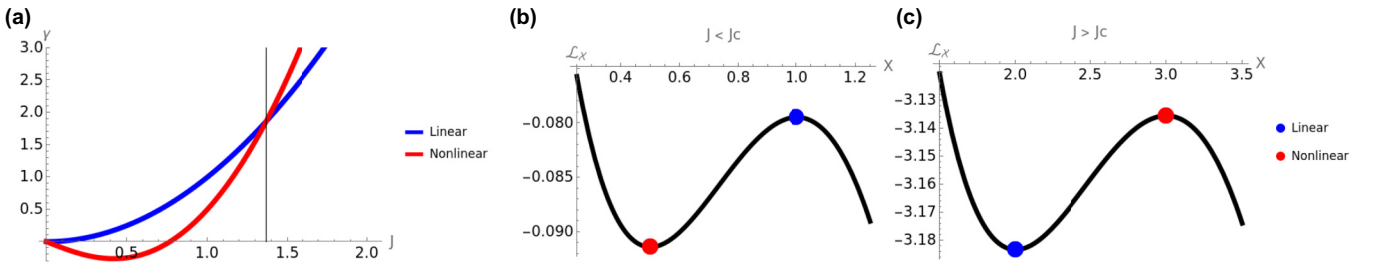


FIG. 6. Self-organized boundary layer in a flux-driven plasma flow. (a) Dissipation for the linear (blue) and nonlinear (red) branches, with a vertical line marking J_c . (b),(c) $\mathcal{L}_X(X)$ for $J < J_c$ [panel (b)] and $J > J_c$ [panel (c)]. Parameters: $R_0 = 1$, $a = 3$, $T_0 = 1$, giving $J_c = 1.366$. Panel (b): $J = 1 < J_c$; panel (c): $J = 2 > J_c$.

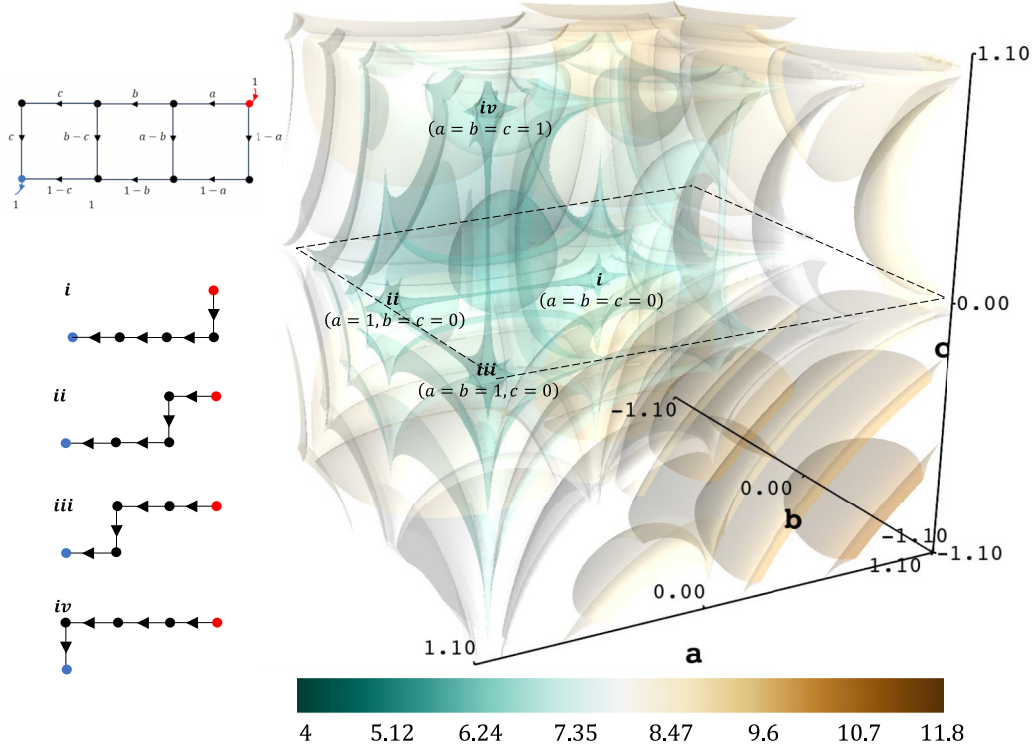


FIG. 7. Selection of optimal network configurations for the triple-loop case, with unit inflow at the top left and outflow at the bottom right. The dissipation is shown for $\check{\alpha} = 0.5$. Panels (i)–(iv) display the different configurations corresponding to the local optima.

function is defined as the sum of dissipation and maintenance cost,

$$\mathcal{M}_i = R_i(J_i, \mathfrak{R}_i)J_i^2 + f(\mathfrak{R}_i), \quad (42)$$

where $\mathfrak{R}_i$ is a structural parameter (e.g., radius) as in (3). Since property optimization is a slow process, the flow can be assumed to have already reached the NESS defined by (2) at each stage of the property optimization. Thus, the optimization here concerns only the resistances, not the X – J relationship as done before.

For power-law resistances and costs,

$$\mathcal{M}_i = \mathfrak{R}_i J_i^{\alpha+1} + \frac{k}{\mathfrak{R}_i^\beta}, \quad (43)$$

minimization with respect to $\mathfrak{R}_i$ at fixed J_i yields

$$J_i = (k\beta)^{\frac{1}{\alpha+1}} \mathfrak{R}_i^{-\frac{1+\beta}{1+\alpha}}, \quad \mathfrak{R}_i = (k\beta)^{\frac{1}{\beta+1}} J_i^{-\frac{1+\alpha}{1+\beta}}. \quad (44)$$

The resulting *apparent* resistance law has an effective exponent,

$$\check{\alpha} = \frac{\beta - \alpha}{1 + \beta}. \quad (45)$$

For Murray's case ($\alpha = 1$, $\beta = 1/2$), $\check{\alpha} = -1/3$, corresponding to an unstable law (cf. Fig. 2). An apparent linear law, $\check{\alpha} = 1$, cannot arise for finite β with $\alpha = 1$; it requires $\alpha = -1$.

For networks, summing (43) over all branches and using (44) yields

$$\mathcal{M}^* = \sum_i (\beta k)^{\frac{1}{\beta+1}} [1 + (\beta k)^\beta J_i^{\frac{\beta(\alpha+1)}{\beta+1}}]. \quad (46)$$

For $\alpha = 1$, this reduces to

$$\mathcal{M}^* = \sum_i \frac{1+\beta}{\beta} \mathcal{D}_i \propto \sum_i J_i^{\check{\alpha}+1}, \quad (47)$$

so that the optimization again involves extremization of dissipation, but now with apparent resistance laws derived from the cost. In this setting, the system tends toward maximum current uniformity: loops persist for $\check{\alpha} > 1$, whereas tree-like structures emerge for $\check{\alpha} < 1$ as redundant branches are pruned.

This framework has been widely applied to optimal channel networks [4,16,20,75]. The functional $\mathcal{M}^*$ can also be interpreted as a potential governing the slow evolution of transport properties. Again, assuming an optimization evolving as a dissipative gradient flow [76], one can assume a dynamical system of the type

$$\tau \frac{dR_i}{dt} = -\frac{\partial \mathcal{M}^*}{\partial R_i}, \quad (48)$$

a formulation closely related to adaptive models of conductance evolution [18,77]. Depending on β , the resulting Lagrangian landscape ranges from convex (congested flow, $\check{\alpha} > 1$, yielding looped networks) to rugged with multiple local minima (branched transport, $\check{\alpha} < 1$, yielding tree networks). In the latter case, the system evolves by pruning redundant loops until a minimum-dissipation tree structure is achieved, analogous to a discrete Monge–Kantorovich transport problem [5].

As shown in [75], $\mathcal{M}^*$ is convex with a unique optimum for $\check{\alpha} > 1$, while for $\check{\alpha} < 1$ it develops multiple singular minima associated with tree configurations. Figure 7 illustrates this behavior with multiple local optima for the case $\check{\alpha} = 1/2$ in

a triple-loop network. The 3D figure shows the values of $\mathcal{M}^*$ as a function of the values of the three independent currents (a , b , and c) in the three loops. Already for this relatively simple case, the rugged landscape yields several quasioptimal tree structures. As the number of independent currents grows and the complexity of the network increases, the landscape acquires fractal-like properties and the branched networks become self-similar [4,20,23,75].

IX. CONCLUSIONS

We have expounded a thermodynamic formalism for nonlinear resistive transport networks based on generalized thermodynamic potentials (GTPs), first introduced by Millar [12] and later extended through Legendre-transform symmetries [14]. The resulting Lagrangians apply broadly to complex networks with nonlinear transport laws, and their variational structure provides natural gradient flows leading to quasistatic evolution toward nonequilibrium steady states (NESS).

Our analysis clarified the *ad hoc* nature of optimality principles based on dissipation and entropy production, which hold only in special cases (e.g., power-law resistances, so that content and co-content scale with generalized dissipation). By explicitly considering the role of constraints, we showed how different extremum principles (least dissipation, maximum entropy production, and related criteria) emerge as special cases of the same Lagrangian framework. The same potentials also determine the stability of systems with multiple solutions [10], recasting nonequilibrium phase transitions [60] in terms of equations of state directly analogous to equilibrium thermodynamics.

We also linked the optimization of transport properties (i.e., resistances), pioneered by Murray [15] and later developed in many contexts [4,19,75,76], to dissipation minimization under power-law costs. In this setting, whether networks organize into tree-like or looped structures depends on the convexity or concavity of the cost functional [78], with direct connections to optimal transport theory [5].

The optimal points in our theory always correspond to NESS. Transient evolution, thus, may be seen as perturbations around these extrema of the GTP-based Lagrangians. There is an extensive literature on networks, including capacitors and inductances, which include the possibility of temporal fluctuations (see e.g., [79,80]), and it would be interesting to analyze whether these approaches have a connection with the present work and a similar thermodynamic analogy. Similarly, it is currently unclear whether extensions to cases with cross-currents linked to multi-energy modes and extensions of Onsager reciprocity relations would be amenable to a formulation like the one here for single energy modes.

Implementing the present framework at large-scale networks raises natural questions of computational tractability, since the variational potentials may be extremized in effectively high-dimensional spaces. Along with the reduced Lagrangian formulations of Sec. V, for network optimization, we anticipate that quasiuniversal scaling behaviors reported in the branched-transport literature can be leveraged to construct reduced or approximate schemes and to benchmark numerics, and that problem-specific constraints can frequently prune physically irrelevant regions and excluded topologies.

Future directions may also include the development of an ensemble theory for networks with thermodynamic fluctuations around operating points (see, e.g., [56]), and the extension of this formalism to families of self-similar tree networks (such as optimal channel networks [4,20]). Such efforts may also shed light on analogies with glassy states in disordered systems [73,81], providing a bridge between ground-state optimality and metastable, suboptimal configurations [82].

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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