

Motion of an ellipsoidal particle in shear flow: Analyzing memory effect influence through exact solutions investigation

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This study introduces an approach to extend the Keller and Skalak (KS) theory by integrating the modified Riemann-Liouville fractional derivative. Our focus is on investigating the transition of red blood cells from flipping to stationary motion within shear flows. Expanding upon the predictions outlined by KS regarding flipping periods and alignment with flow, our research aims to delve deeper into this transition process, which remains largely unexplored experimentally. The rationale for employing fractional derivatives stems from their capacity to capture memory effects and nonlocal interactions, thus providing a more nuanced modeling framework for complex particle dynamics.

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I. INTRODUCTION

In the field of fluid dynamics, understanding the motion of particles immersed in shear flows is crucial for various applications ranging from biomedical to industrial processes. The behavior of particles, such as red blood cells, in fluid environments plays a significant role in determining overall system dynamics and performance.

Keller and Skalak [1] introduced the KS theory, which assumes shape preservation (meaning the particle maintains an ellipsoidal geometry throughout its motion). Their investigation under shear flow revealed two primary modes of motion: tank treading (TT), in which the particle, such as a red blood cell, adopts an inclined ellipsoidal shape and maintains a steady orientation while its membrane undergoes continuous rotational motion; and tumbling (TB), or flipping, where the entire particle rotates periodically, resembling the motion of a rigid body.

The presence of these two modes and their transition are governed by crucial control parameters that play a key role in determining particle behavior under shear flow conditions. These include the excess area relative to a sphere (Δ), which represents the difference in surface area between the particle and a sphere of the same volume, computed as $\Delta = (A - 4\pi r_0^2)/r_0^2$, where A denotes the particle's surface area and r_0 is its effective radius; the viscosity ratio (λ), defined as the ratio of the internal to external fluid viscosities, denoted

η_{int} and η_{ext} , respectively; and the capillary number (Ca), given by $\eta_{\text{ext}}\gamma r_0^3/\kappa$, where γ is the shear rate and κ is the bending rigidity of the particle's membrane.

Furthermore, Keller and Skalak predicted that the flipping period would progressively increase, ultimately leading to a stationary orientation. While this behavior was introduced as a theoretical insight rather than a rigorously derived result, it has not, to the best of our knowledge, been analytically confirmed. Although several studies have since addressed the transition between tank treading (TT) and tumbling (TB) [2–4], the predicted increase in the flipping period remains theoretically unexplored.

On the other hand, subsequent research efforts have largely focused on deformable particles, particularly vesicles. A notable breakthrough was made by Misbah in Ref. [5], who discovered a novel dynamical regime known as the vacillating-breathing (VB) mode. This intermediate behavior between TT and TB is characterized by asymmetric oscillations of the vesicle's orientation, coupled with periodic, breathinglike deformations of its shape. Later studies, such as Ref. [6], demonstrated that the transition between TB and VB depends sensitively on the system's initial configuration. These findings opened a rich avenue of exploration in the dynamics of shape-changing systems, and naturally shifted attention away from the case of rigid, fixed-shape particles.

While previous models have significantly contributed to our understanding of vesicle and rigid particle dynamics, they often focus on instantaneous mechanical responses, overlooking the system's dependence on past configurations. In the context of vesicles and rigid particles suspended in viscous flows (especially in the low Reynolds number regime where inertial effects are negligible) memory effects become non-trivial and physically relevant. The membrane of a vesicle is not a simple interface: It possesses shear and bending elasticity, as well as surface viscosity, all of which contribute to memory effects. For instance, during successive deformations

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or shape transitions, the time a vesicle spends in a given state can affect its subsequent response [7–9]. This is particularly important near transition thresholds (such as between tumbling and tank treading) where hysteresis and path-dependent behavior clearly indicate memory effects. Ignoring them can lead to oversimplified models and inaccurate predictions. Accounting for memory enables a more realistic description of shape evolution and flow interaction, often aligning better with experimental observations. For example, Kantsler *et al.* observed wrinkling and budding in vesicles under time-dependent elongation flows, linked to deformation history [10]. Likewise, Kumar *et al.* reported double-mode relaxation in deformed vesicles, reflecting memory-influenced dynamics [11]. Including such effects provides deeper insight into the role of membrane rheology in cellular dynamics. Thus, incorporating memory effects not only bridges the gap between theoretical predictions and experimental observations but also provides deeper insights into the role of membrane rheology in cellular dynamics.

Building on these observations, it becomes essential to adopt modeling strategies that naturally incorporate memory. In this context, fractional calculus provides a powerful framework, as shown by several authors, who demonstrated that fractional derivatives can effectively capture memory effects in viscoelastic materials while requiring fewer parameters and avoiding noncausal responses compared to classical integer-order models [12–14]. We employ fractional calculus not as a mathematical abstraction but rather because its integral operators (see Sec. II) inherently embody the continuous temporal memory observed in experiments. For vesicles, this approach has already revealed the damped TT regime [15,16], an intermediate state that conventional models cannot capture. Extending this framework to rigid particles reveals that similar memory-dependent phenomena may either explain previously unobserved behaviors or validate Keller and Skalak's predictions.

To introduce this memory effect into the Keller and Skalak framework, we employ the modified Riemann-Liouville fractional derivative [17,18], whose definition preserves certain properties of classical derivatives (such as the vanishing derivative of a constant), making it particularly well suited for

physical modeling. This operator enables a natural incorporation of historical dependence without diverging significantly from the classical differential equation structure. Building on this, we explore how fractional calculus (specifically through the modified Riemann-Liouville derivative) can deepen our understanding of the transition from flipping to stationary motion in red blood cells under varying viscosity ratios in shear flow. Classical approaches that neglect memory may fail to capture not only subtle changes in the motion period but also other complex dynamical behaviors emerging near transition thresholds. Thus, adopting a fractional framework not only extends the original Keller and Skalak theory [1], but also offers a more realistic depiction of red blood cell dynamics, especially in regimes where the effects of deformation history are non-negligible. Structured into three main sections, this article systematically explores the dynamics of ellipsoid particles under shear flow. section II introduces the mathematical framework, notably fractional calculus, essential for analyzing the system's dynamics. Subsequently, Sec. III presents rigorous mathematical formulations and analytical solutions derived from the model, offering insights into vesicle behavior under shear flow conditions and delves into the implications of these solutions, elucidating how ellipsoid particles respond within the shear plane. Lastly, Sec. IV summarizes the findings and discusses their implications, offering avenues for future research in the field.

II. FUNDAMENTAL MATHEMATICAL TECHNIQUES

Before delving into the intricacies of our primary findings, let us first shed light on the fundamental mathematical tools that underlie our exploration. Within this section, we provide a comprehensive overview of key mathematical constructs essential to our investigation.

Definition 1. Let $f \in C([a, b])$ and $\alpha \in \mathbb{R}_+$, the Riemann-Liouville integral of f is defined as follows:

$$(I_a^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds. \quad (1)$$

Definition 2. The Jumarie derivative of order α for the function f is expressed as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} [f(s) - f(0)] ds, \quad 0 < \alpha < 1, \\ D_t^\alpha f(t) = (f^{(\alpha-n)})^{(n)}, \quad n \leq \alpha < n+1, \quad n \geq 1, \quad (2)$$

where the Γ function is defined as follows,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad z \in \mathbb{C}, \quad (3)$$

and we have

$$D_t^{\alpha_1} t^{\alpha_2} = \frac{\Gamma(\alpha_2 + 1)}{\Gamma(\alpha_2 + 1 - \alpha_1)} t^{\alpha_2 - \alpha_1}, \quad \alpha_2 > 0, \alpha_2 \geq \alpha_1, \quad D_t^\alpha C = 0, \quad (4)$$

$$D_t^\alpha [f(t)g(t)] = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \quad (5)$$

$$D_t^\alpha f[g(t)] = \left(\frac{d}{dg} f[g(t)] \right) D_t^\alpha g(t) = D_t^\alpha f[g(t)] \left(\frac{d}{dt} g(t) \right)^\alpha. \quad (6)$$

Jumarie's modified Riemann-Liouville derivative definition not only refines the Riemann-Liouville definition but also establishes a fractional calculus framework analogous to classical calculus.

III. EXACT SOLUTIONS AND PARTICLE MOTION

The following section outlines the specific contributions of this study, detailing the objectives, methodology, results, and implications. This investigation aims to assess the extent to which memory effects influence the mobility of an ellipsoidal particle.

Building upon the foundational two-dimensional framework introduced by Misbah [5], which effectively captures the three principal vesicle motion regimes (TT, VB, and TB) we incorporate memory effects using fractional calculus. Specifically, we generalize the original system by employing Jumarie fractional derivatives, so that the dynamics of a vesicle under shear flow are governed by the following equations,

$$\begin{aligned} D_t^\alpha \mathcal{R} &= h \left[1 - 4 \frac{\mathcal{R}^2}{\Delta} \right] \sin(2\psi), \\ D_t^\alpha \psi &= -\frac{1}{2} + \frac{h}{2\mathcal{R}} \cos(2\psi), \end{aligned} \quad (7)$$

where ψ denotes the vesicle's orientation angle, with $-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$, and $\mathcal{R}$ represents the amplitude of the vesicle's deformation (the measure of how much the vesicle deviates from a reference spherical shape). The first equation describes the evolution of the vesicle's shape, while the second governs its orientation in the shear flow. The parameter $h = \frac{60\sqrt{2\pi/15}}{32+23\lambda}$, and D_t^α denotes the Jumarie fractional derivative of order $0 < \alpha \leq 1$.

As outlined in the introduction, our objective in this study is to focus specifically on the dynamics of ellipsoidal particles, which corresponds to the case where $\mathcal{R} \equiv \frac{\sqrt{\Delta}}{2}$ (shape-preserving theory). In this case, the system (7) reduces to the generalization of the Jeffery equation [19],

$$D_t^\alpha \psi = -\frac{1}{2} + \frac{h}{\sqrt{\Delta}} \cos(2\psi), \quad (8)$$

where the particle dynamics are exclusively characterized by the variation of the angle ψ , simplifying the dynamics. Keller and Skalak [1] thoroughly explored equation (8) in the classical case (i.e., $\alpha = 1$). In this study, we employ Jumarie's modified Riemann-Liouville derivative in which it can reveal a spectrum of new behaviors that have not yet been detected in the literature, or even behaviors that align with the predictions of Keller and Skalak. The purpose of this section is to derive exact solutions for the fractional equation (8) using the properties of the modified Riemann-Liouville fractional derivative, as described in Sec. II. This approach allows us to quantify the influence not only of the parameters h and Δ , but also of the fractional derivative order α , on the mobility of the ellipsoidal particle. By utilizing this framework, we are able to investigate the deeper effects of fractional-order derivatives on the dynamics and associated memory effects. To further explore the implications of these findings, we will analyze these aspects in the following sections.

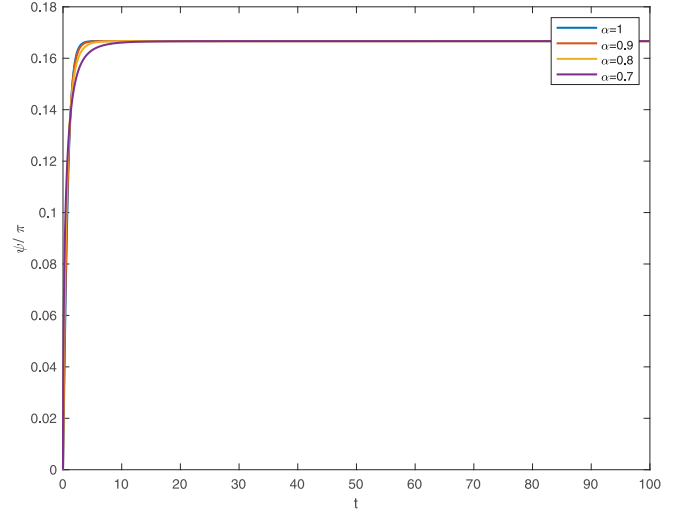


FIG. 1. The orientation angle ψ of the ellipsoidal particle for four different fractional order values when the parameter $\chi = 2$ ($\chi > 1$).

A. TT mode

In this section, we investigate the tank-treading mode (TT mode), a distinct behavior observed in specific fluid dynamics scenarios. This behavior is analyzed using Jumarie's modified Riemann-Liouville fractional derivative and its properties to determine the expression of ψ . The resulting expression reveals the manifestation of the tank-treading mode (see the Appendix for details),

$$\psi(t) = \arctan \left[-\frac{\chi - 1}{\sqrt{\chi^2 - 1}} \frac{1 + K \exp \left(\frac{\sqrt{\chi^2 - 1}}{\Gamma(\alpha + 1)} (t - t_0)^\alpha \right)}{1 - K \exp \left(\frac{\sqrt{\chi^2 - 1}}{\Gamma(\alpha + 1)} (t - t_0)^\alpha \right)} \right], \quad (9)$$

where $\chi = \frac{2h}{\sqrt{\Delta}}$, $K = \frac{\tan(\psi_0) + \frac{\chi-1}{\sqrt{\chi^2-1}}}{\tan(\psi_0) - \frac{\chi-1}{\sqrt{\chi^2-1}}}$, and $\psi_0 = \psi(t_0)$, where t_0 is the initial value.

Figure 1 illustrates the orientation angle ψ for different values of α . The solutions show that ψ remains constant for different values of α , indicating that the solution tends toward a stable equilibrium regardless of the order of the derivative. This observation confirms that the particle exhibits tank-treading (TT) motion. Additionally, the orientation lies in the shear plane, corresponding to the TT motion for the vesicle.

From the above figure, it is evident that the order of the fractional derivative has no impact on the stability of the equilibrium. Therefore, increasing the value of α enables the solutions to converge to the steady state more rapidly. The strength of memory effects, governed by α , significantly influences the dynamics of rigid particles. Consequently, the dynamics of the ellipsoidal particle are influenced by both the current time and the time history, which can be effectively captured using fractional calculus concepts.

B. TB mode

Embarking on the pivotal segment of our research, we now pivot our attention to the oscillatory solution where $\chi < 1$.

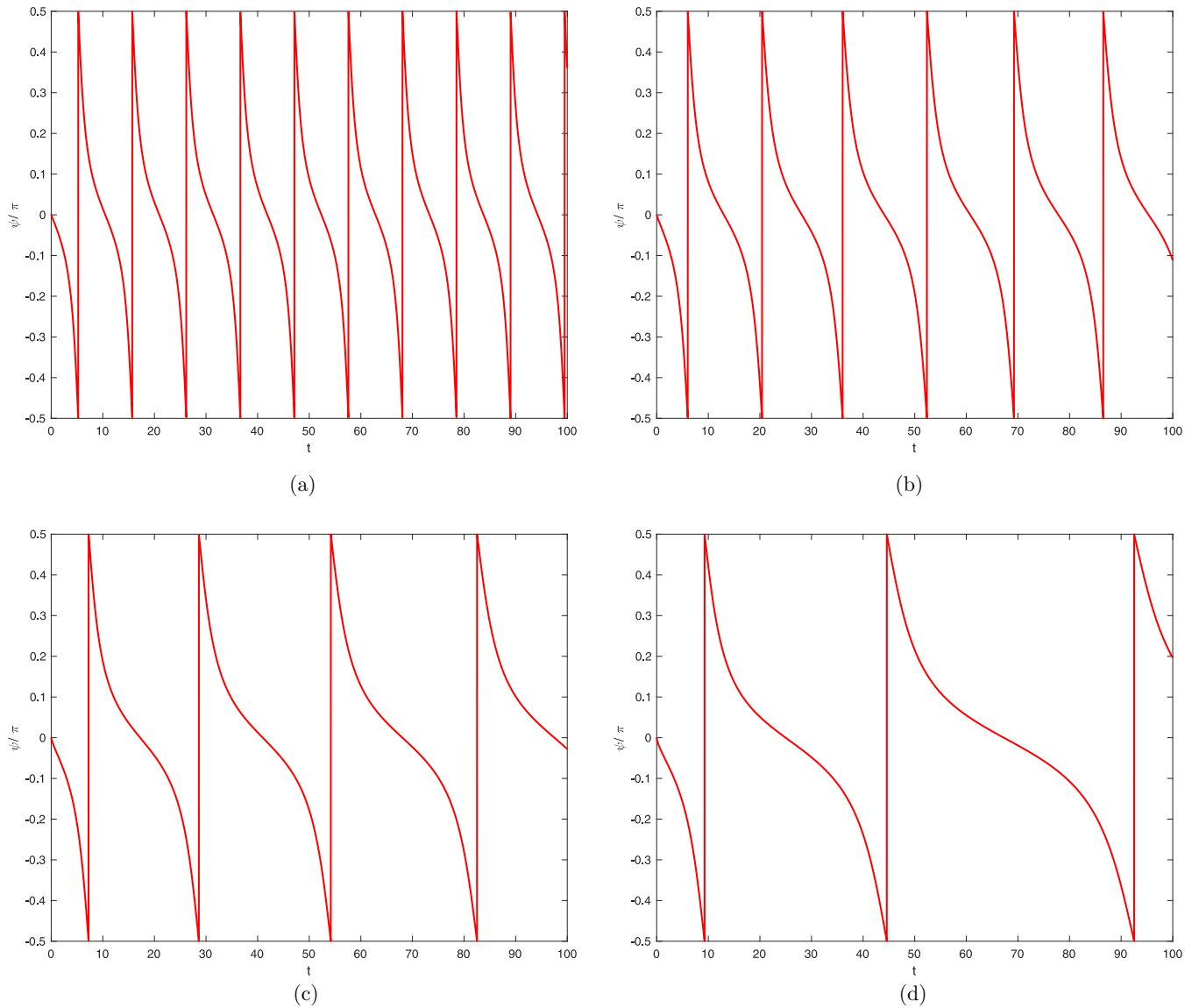


FIG. 2. The particle's orientation angle ψ for four different fractional order values when the parameter $\chi = 0.8$ ($\chi < 1$). (a) The orientation angle ψ when $\alpha = 1$, (b) the orientation angle ψ when $\alpha = 0.9$, (c) the orientation angle ψ when $\alpha = 0.8$, and (d) the orientation angle ψ when $\alpha = 0.7$.

Motivated by prior results involving fractional derivatives in vesicle dynamics [15,16], we hypothesize that neglecting memory effects may have limited the theoretical observation of additional motion modes for ellipsoidal particles. As in the case $\chi > 1$, our objective is to solve (8) by applying the properties of the Jumarie derivative introduced in Sec. II. Through careful derivation (see the Appendix for details), we obtain the following exact expressions:

$$\psi(t) = \arctan \left\{ -\frac{1-\chi}{\sqrt{1-\chi^2}} \tan \left[\frac{\sqrt{1-\chi^2}}{2\Gamma(\alpha+1)} (t-t_0)^\alpha \right. \right. \\ \left. \left. + \arctan \left(\frac{-\sqrt{1-\chi^2}}{1-\chi} \tan(\psi_0) \right) \right] \right\}, \quad (10)$$

where $\psi_0 = \psi(t_0)$, where t_0 is the initial value.

Figure 2 illustrates the orientation angle ψ for four different values of α when $\chi < 1$. In the classical case ($\alpha = 1$), we observe that the particle flips increasingly. However, for $0 < \alpha < 1$, the tumbling period increases, meaning that the vesicle rotates more slowly, and the amplitude of oscillations gradually dampens over longer times. This behavior is due to the introduction of the modified Riemann-Liouville fractional derivatives, which extend the period of the ψ angle and reveal previously unexplored dynamics. These findings highlight the significant influence of viscosity on red blood cell dynamics in shear flow, and show that the memory effect induced by the fractional operator plays a key role in modulating the flipping behavior.

The memory effect is crucial for how particles react and orient over time. Fractional-order derivatives allow particles to respond based on recent history, influencing current orientation and dynamically adapting to environmental changes. The

increase in motion period as the fractional derivative order decreases suggests that lower-order derivatives introduce more pronounced memory effects into the system, where motion is influenced by a wider range of historical information or past states, resulting in slower motion evolution. The relationship between fractional derivative order of the modified Riemann-Liouville fractional derivative and motion period underscores the complexity of system dynamics.

Our utilization of the Jumarie derivative unveils a spectrum of behaviors consistent with the predictions of KS, including an increase in period duration that was not evident in the KS theory when employing classical derivative.

The flipping period significantly affects particle dynamics, with variations impacting motion types. Changes in the flipping period due to alterations in the viscosity ratio between inner and outer liquids can shift the particle's behavior between steady stationary orientation and unsteady flipping motion. Moreover, the flipping period can indicate transitions between different motion modes exhibited by the ellipsoidal particle, particularly as the viscosity ratio approaches critical threshold values. By studying the flipping period and its correlation with viscosity ratio, researchers gain insights into ellipsoidal particle dynamics in shear flows, emphasizing the parameter's importance in understanding particle motion in complex fluid environments.

This can be explained by the fact that, when considering the memory effect (the entire time history), the period of the flipping motion and the time spent aligned with the flow will increase, eventually leading the particle to assume a stationary orientation. In summary, the increase in the ψ angle period with the introduction of fractional-order derivatives signifies more complex and adaptive system dynamics, with notable memory effects and emergent properties. Understanding and interpreting these changes are vital for accurately modeling and predicting system behavior. By incorporating Jumarie's modified Riemann-Liouville derivative, we present a more realistic and comprehensive model.

IV. CONCLUSION

The integration of fractional derivatives in the study of ellipsoidal particle motion in shear flow unveils profound insights into the influence of memory effects on the dynamics of blood cells and particles. By capturing nonlocal interactions and memory phenomena, fractional derivatives introduce a higher level of complexity and adaptability to the system dynamics. This innovative approach not only enhances our understanding of particle transitions but also emphasizes the crucial role of memory effects in modeling intricate phenomena. The transition from flipping to rigidlike motion remains a subject of theoretical and experimental interest, further highlighting the complexity of particle dynamics.

Extending the Keller and Skalak theory through modified Riemann-Liouville derivatives offers valuable insights into the transition of red blood cells in shear flow and provides a novel perspective for accurately predicting particle behavior in diverse fluid environments. By addressing the gap in experimental investigations and providing a more detailed understanding of the flipping to stationary motion transition, this study contributes to the advancement of particle dynam-

ics research. These findings underscore the significance of fractional calculus in modeling real-world complexities, with promising implications for biomedical research and fluid mechanics.

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DATA AVAILABILITY

Data are available from the authors upon reasonable request.

APPENDIX: DETAILS ON THE DERIVATION OF ANALYTICAL SOLUTIONS

We provide here a detailed outline of the steps followed to derive the analytical solutions presented in the main text. Specifically, to solve Eq. (8), it is equivalent to consider the fractional differential equation,

$$D_t^\alpha \psi = -\frac{1}{2} + \frac{h}{\sqrt{\Delta}} \frac{1 - \tan^2(\psi)}{1 + \tan^2(\psi)}, \quad (\text{A1})$$

which can be rewritten using trigonometric identities as

$$\begin{aligned} & \left(-\frac{1}{2} + \frac{h}{\sqrt{\Delta}}\right)(1 + \tan^2 \psi) D_t^\alpha \psi \\ &= \left(-\frac{1}{2} + \frac{h}{\sqrt{\Delta}}\right)^2 + \left[\left(\frac{-1}{2}\right)^2 - \frac{h^2}{\Delta}\right] \tan^2(\psi). \end{aligned} \quad (\text{A2})$$

Introducing the dimensionless parameter $\chi = \frac{2h}{\sqrt{\Delta}}$, we distinguish two cases based on the value of χ :

Case 1: $\chi < 1$. Using the properties of fractional calculus described in Sec. II, and appropriate transformations, the equation reduces to

$$D_t^\alpha \left[\arctan \left(\frac{-\sqrt{1-\chi^2}}{1-\chi} \tan \psi \right) \right] = \frac{1}{2} \sqrt{1-\chi^2}. \quad (\text{A3})$$

Applying the Riemann-Liouville fractional integral, and after meticulous calculations involving the properties of fractional operators and trigonometric identities, we derive the explicit expression for $\psi(t)$,

$$\begin{aligned} \psi(t) = \arctan & \left\{ -\frac{1-\chi}{\sqrt{1-\chi^2}} \tan \left[\sqrt{1-\chi^2} 2\Gamma(\alpha+1)(t-t_0)^\alpha \right. \right. \\ & \left. \left. + \arctan \left(\frac{-\sqrt{1-\chi^2}}{1-\chi} \tan(\psi_0) \right) \right] \right\}, \end{aligned} \quad (\text{A4})$$

where $\psi_0 = \psi(t_0)$, with t_0 denoting the initial time.

Case 2: $\chi > 1$. In this case, the original equation transforms into

$$D_t^\alpha \psi (1 + \tan^2 \psi) = -\frac{\chi^2 - 1}{2(\chi - 1)} \left[\tan^2 \psi - \left(\frac{\chi - 1}{\sqrt{\chi^2 - 1}} \right)^2 \right]. \quad (\text{A5})$$

Using the identities and properties from Sec. II, we obtain

$$D_t^\alpha \ln \left(\frac{\tan \psi + \frac{\chi-1}{\sqrt{\chi^2-1}}}{\tan \psi - \frac{\chi-1}{\sqrt{\chi^2-1}}} \right) = \sqrt{\chi^2 - 1}. \quad (\text{A6})$$

Applying the Riemann-Liouville fractional integral and solving the resulting equation yields

$$\psi(t) = \arctan \left[-\frac{\chi - 1}{\sqrt{\chi^2 - 1}} \frac{1 - K \exp \left(\frac{\sqrt{\chi^2 - 1}}{\Gamma(\alpha+1)} (t - t_0)^\alpha \right)}{1 + K \exp \left(\frac{\sqrt{\chi^2 - 1}}{\Gamma(\alpha+1)} (t - t_0)^\alpha \right)} \right], \quad (\text{A7})$$

where the constants are defined as

$$K = \frac{\tan(\psi_0) + \frac{\chi-1}{\sqrt{\chi^2-1}}}{\tan(\psi_0) - \frac{\chi-1}{\sqrt{\chi^2-1}}}, \quad \chi = \frac{2h}{\sqrt{\Delta}}, \quad \psi_0 = \psi(t_0),$$

with t_0 denoting the initial time.

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