

Strong-damping limit of quantum Brownian motion in a disordered environment

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We consider a microscopic model of an inhomogeneous environment where an arbitrary quantum system is locally coupled to a harmonic bath via a finite-range interaction. We show that in the overdamped regime the position distribution obeys a classical Kramers-Moyal equation that involves an infinite number of higher-order derivatives, implying that the finite bath correlation length leads to non-Gaussian Markovian noise. We analytically solve the equation for a harmonically bound particle and analyze its non-Gaussian diffusion as well as its steady-state properties.

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I. INTRODUCTION

The description of quantum systems coupled to an external environment is of central importance in many different fields, including condensed matter physics [1], quantum optics [2], and quantum information science [3], as well as the theory of open quantum systems [4], quantum dissipation [5], and environment-induced decoherence [6]. Starting from a microscopic model of the environment, usually a bath of harmonic oscillators [1–6], one aims at describing the dynamics of the system of interest in terms of approximate master equations for its reduced density operator. In the high-temperature limit, such master equations reduce to the Klein-Kramers equation for phase-space distributions known from the theory of classical Brownian motion [7,8]. On the other hand, in the strong system-bath coupling domain, the almost classical dynamics can be described with the help of a Fokker-Planck equation whose noise properties are strongly impacted by quantum fluctuations [9–18]. This so-called quantum Smoluchowski regime has recently been investigated experimentally using superconducting tunnel junctions at ultralow temperatures [19].

The standard theory of quantum Brownian motion assumes that the system is coupled to the bath via a global, infinite-range interaction, where the system-bath coupling is homogeneous in the position of the system [1–6]. There is, however, a large class of microscopic objects where the bath coupling is local, with a finite interaction range given by a typical length scale of the surroundings. Concrete examples include Brownian motion in a scattering environment made of a gas of molecules (when the coupling is limited by the wavelength of the environment) [20–25] and diffusion in a disordered medium (when the coupling is limited by the spatial correlation length of the disorder potential) [26–33]. The restricted range of the system-bath interaction will in

general strongly affect the dynamics of the system, including its dephasing and diffusion properties [20–33]. These studies all consider weak coupling between system and reservoir.

In this paper, we derive a generalized Smoluchowski equation for an arbitrary quantum system strongly coupled to an (Ohmic) bath of harmonic oscillators via a finite-range interaction, which can be regarded as a model of a dynamically disordered environment (Sec. II) [28–31]. We obtain a classical Kramers-Moyal equation that involves an infinite number of higher-order derivatives, and thus includes both Gaussian and non-Gaussian white noise contributions (Sec. III) [34–36]. The usual Smoluchowski equation with purely Gaussian noise, which corresponds to the first two terms of the expansion, is recovered in the limit of an infinite interaction range. Finally, we analytically determine the solution of the generalized Smoluchowski equation for the case of a Brownian particle trapped in a harmonic potential. We analyze its non-Gaussian properties as well as its steady-state distribution (Sec. IV), thus providing useful insight into the effect of a finite bath correlation length.

II. MODEL FOR LOCAL BATH COUPLING

We consider a general one-dimensional quantum system, with position x and momentum p , that is locally coupled to a heat bath, initially in a thermal state at temperature T , made of infinitely many harmonic oscillators, with canonical coordinates (q_n, p_n) , mass m_n , and frequency ω_n (Fig. 1). The total Hamiltonian reads

$$\mathcal{H} = \mathcal{H}_S + \mathcal{H}_I + \mathcal{H}_B, \quad (1)$$

where the respective system, interaction, and bath Hamiltonians, $\mathcal{H}_S$, $\mathcal{H}_I$, and $\mathcal{H}_B$, are given by [28–31]

$$\mathcal{H}_S = \frac{p^2}{2} + V(x), \quad (2)$$

$$\mathcal{H}_I = - \sum_n c_n q_n u(x - x_n), \quad (3)$$

$$\mathcal{H}_B = \sum_n \frac{p_n^2}{2m_n} + \frac{1}{2} m_n \omega_n^2 q_n^2. \quad (4)$$

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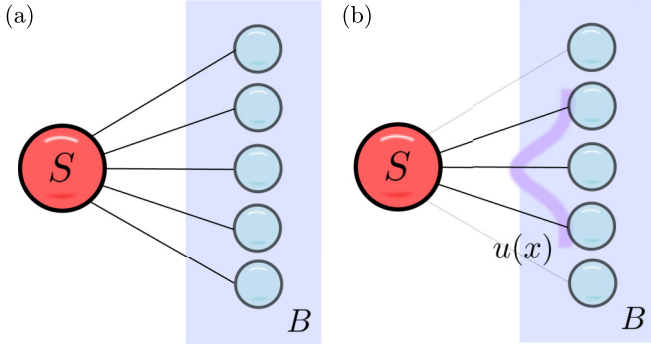


FIG. 1. Schematic representation of the system-bath interaction. (a) A quantum system S is globally coupled to a bath B of harmonic oscillators via an infinite-range interaction. (b) A local system-bath coupling with a finite spatial correlation length is characterized by a finite-range interaction $u(x)$.

The system, with mass $m = 1$, moves in a confining potential $V(x)$. The scattering potential $u(x - x_n)$ further specifies the local interaction between system and bath; it depends solely on the distance between the system and the fixed location x_n of the n th bath oscillator. The disordered environment is characterized by the spatial autocorrelation function $w(r) = \int_{-\infty}^{\infty} dx u(r - x)u(x)$, with a finite correlation length l . For concreteness, we will use the Gaussian form, $w(r) = l^2 \exp[-r^2/2l^2]$ [28–31]. The bath spectral density, $J(\omega) = (\pi/2) \sum_n (c_n^2/m_n \omega_n) \delta(\omega - \omega_n) \sim \omega$, is additionally assumed to be linear at low frequencies, corresponding to an Ohmic, that is, Markovian bath (c_n are the bath coupling constants) [1–6]. This can be realized microscopically by choosing a (continuous) bath frequency distribution of the form $g(\omega) \sim 2m(\omega)\omega^2/\pi c^2(\omega)$ (up to some high-frequency cut-off), with $J(\omega) = (\pi/2) \int_0^\infty d\omega' g(\omega') [c(\omega')^2/m(\omega')\omega'] \delta(\omega - \omega')$ [1–6]. Hamiltonian (1) is often referred to as a “diffusion, localization and dissipation” model [28,29] or as a “mattress” model [30].

In the high-temperature limit, the reduced Wigner function $W(x, p, t)$ of the system satisfies the Markovian phase-space equation [31]

$$\partial_t W = -p \partial_x W + V'(x) \partial_p W + \gamma \partial_p [G_F * (pW)] + \nu (G_N * W), \quad (5)$$

where γ and ν denote the respective friction and diffusion coefficients. The two functions $G_F(p)$ and $G_N(p)$ are the corresponding friction and noise kernels, whose expressions are given in terms of Fourier transforms (denoted by $\mathcal{FT}$) of the correlation function $w(r)$ as [31]

$$G_F(p) = \mathcal{FT}_{r \rightarrow p/\hbar} \left\{ -\frac{w'(r)}{r} \right\} = \frac{1}{\sqrt{2\pi}(\hbar/l)} e^{-\frac{(p-p')^2}{2(\hbar/l)^2}} \quad (6)$$

and

$$G_N(p) = \frac{1}{\hbar^2} \mathcal{FT}_{r \rightarrow p/\hbar} \{ [w(r) - w(0)] \} = \frac{1}{(\hbar/l)^2} \left[\frac{1}{\sqrt{2\pi}(\hbar/l)} e^{-\frac{(p-p')^2}{2(\hbar/l)^2}} - \delta(p - p') \right]. \quad (7)$$

Both expressions, which are nonlocal in p , capture the effects of a finite bath correlation length l . The symbol $*$ represents the convolution operation.

In the limit of a global system-bath interaction, $l \rightarrow \infty$ (no disorder), $w(r) \rightarrow -r^2/2$. As a result, $G_F(p)$ tends to $\delta(p - p')$ and the operator $G_N(p) * (\cdot)$ can be replaced by ∂_p^2 [31]. In this case, Eq. (5) reduces to the familiar Klein-Kramers equation for classical Brownian motion [7,8]

$$\partial_t W = -p \partial_x W + V'(x) \partial_p W + \gamma \partial_p [pW] + \nu \partial_p^2 W. \quad (8)$$

It should be noted that the high-temperature limit of model (1) does not coincide with the classical limit $\hbar \rightarrow 0$ [28,29]. The classical model with white noise indeed leads to standard Gaussian diffusion irrespective of the value of the spatial length scale l . By contrast, even though noise is white in the high-temperature regime, diffusion in the quantum model is non-Gaussian in general (and depends explicitly on the correlation length l). The reduced dynamics of the system only agree in the two cases in the absence of coupling to the bath [28,29]. In that sense, the high-temperature limit may be seen as a semiclassical approximation, in which the dynamics of the system is classical but that of the bath is quantum.

III. STRONG-DAMPING LIMIT

We next consider the strong-friction limit of Eq. (5). For large damping, the momentum is expected to quickly relax to its stationary limit, so that (for infinite friction) the phase-space distribution can be factorized as $W(x, p, t) = \varrho_s(p) \rho(x, t)$, where $\varrho_s(p)$ denotes the stationary momentum distribution [7,8]. In the following, we shall derive a generalized Smoluchowski equation by eliminating the fast momentum variable [37,38].

We begin by rewriting the phase-space equation (5) in the form of an infinite sum that will be more convenient to analyze. To that end, we note that the friction and noise kernels can be expressed as (see Appendix)

$$G_F(p) * [p \cdot] \rightarrow \exp \left(\frac{(\hbar/l)^2 \partial_p^2}{2} \right) [p \cdot], \quad (9)$$

$$G_N(p) * [\cdot] \rightarrow \frac{1}{(\hbar/l)^2} \left(\exp \left(\frac{(\hbar/l)^2 \partial_p^2}{2} \right) - 1 \right) [\cdot]. \quad (10)$$

By expanding the two exponentials in Eqs. (9) and (10) in a power series, we obtain

$$\begin{aligned} \partial_t W = & -p \partial_x W + V'(x) \partial_p W + \gamma \partial_p [pW] + \nu \partial_p^2 W \\ & + \gamma \sum_{n=1}^{\infty} \gamma_n \partial_p^{2n+1} [pW] + \frac{\nu}{(\hbar/l)^2} \sum_{n=2}^{\infty} \gamma_n \partial_p^{2n} W, \end{aligned} \quad (11)$$

where $\gamma_n = (\hbar/l)^{2n}/(2^n n!)$. The first line corresponds to the standard Klein-Kramers equation ($l \rightarrow \infty$), while the two infinite sums on the second line, which involve all higher derivatives, describe the influence of a finite length l . In order to simplify the notation, we will henceforth set $\hbar = 1$ (the $\hbar$ dependence can be easily reintroduced by replacing the spatial length scale l by $l/\hbar$).

Following Refs. [37,38] (see also Ref. [8]), we rewrite Eq. (11) in the form

$$\begin{aligned} \partial_t W = & \frac{1}{\gamma} \partial_x [V'(x)W] + \nu \partial_p^2 W \\ & + \gamma \left(\partial_p - \frac{1}{\gamma} \partial_x \right) \left(pW + \frac{V'(x)}{\gamma} W \right) \\ & + \gamma \sum_{n=1}^{\infty} \gamma_n \partial_p^{2n+1} [pW] + \nu l^2 \sum_{n=2}^{\infty} \gamma_n \partial_p^{2n} W, \end{aligned} \quad (12)$$

and proceed by integrating both sides of the equation along the straight phase-space line $x + \gamma^{-1}p = x'$ with line element ds [8]. In the strong damping limit, the marginal $\rho(x, t)$ follows from $W(x, p, t)$ as

$$\int_{x+\gamma^{-1}p} ds W \xrightarrow{\gamma \rightarrow \infty} \int dp W = \rho(x', t), \quad (13)$$

where we have used the factorization $W(x, p, t) = \varrho_s(p)\rho(x, t)$. In addition, we have $\partial_x^n = \partial_{x'}^n$ and $\partial_p^n = (-\gamma)^{-n} \partial_{x'}^n$ along the integration path. As a consequence, $\partial_p - \gamma^{-1} \partial_{x'}$ is a null operator along this path, which eliminates the contribution coming from the first two terms in Eq. (12). Moreover, since $\int ds \partial_p^{2n+1} [pW] \xrightarrow{\gamma \rightarrow \infty} \gamma^{-2n} \partial_{x'}^{2n} \rho$, the fourth term in Eq. (12) reduces to

$$\gamma \sum_{n=1}^{\infty} \gamma_n \int ds \partial_p^{2n+1} [pW] \xrightarrow{\gamma \rightarrow \infty} \gamma \sum_{n=1}^{\infty} \frac{(1/\gamma l)^{2n}}{2^n n!} \partial_{x'}^{2n} \rho. \quad (14)$$

A similar calculation applied to the fifth term yields

$$\nu l^2 \sum_{n=2}^{\infty} \gamma_n \int ds \partial_p^{2n} W \xrightarrow{\gamma \rightarrow \infty} \nu l^2 \sum_{n=2}^{\infty} \frac{\gamma_n}{\gamma^{2n}} \partial_{x'}^{2n} \rho. \quad (15)$$

Combining all the terms, we obtain the generalized overdamped Smoluchowski equation

$$\partial_t \rho = \frac{1}{\gamma} \partial_x [V'(x)\rho] + \sum_{n=1}^{\infty} \frac{a_{2n}}{(2n)!} \partial_x^{2n} \rho, \quad (16)$$

where $a_{2n} = (\gamma + \nu l^2)(2n)!/[2^n(\gamma l)^{2n}n!]$; without loss of generality, we have replaced $x' \rightarrow x$. Equation (16) has the form of a classical Kramers-Moyal equation that includes both Gaussian and non-Gaussian white noise contributions [34,35]: The first two terms describe deterministic drift plus continuous Gaussian diffusion, whereas the higher-order terms ($n \geq 2$) correspond to discrete jumps and to non-Gaussian white noise [34,35]. In the limit of a global system-bath interaction, $l \rightarrow \infty$, the standard overdamped Smoluchowski equation is recovered:

$$\partial_t \rho = \frac{1}{\gamma} \partial_x [V'(x)\rho] + \frac{\nu}{2\gamma^2} \partial_x^2 \rho, \quad (17)$$

with only Gaussian noise. Note that in this case the Einstein relation, $\nu/2\gamma = kT$ [34,35], where k is the Boltzmann constant, implies the coefficients

$$a_{2n} = \frac{\gamma[1 + 2kTl^2]}{2^n n! (\gamma l)^{2n}}. \quad (18)$$

We may thus conclude that the effect of a finite correlation length l is the occurrence of discrete non-Gaussian Markovian

jump events in the overdamped dynamics. It is worthwhile mentioning that this strongly differs from the coupling to a finite number of bath modes, which usually leads to non-Markovian dynamics [39–41]. In the present case, the system is still coupled to all the bath modes, but with a modulated coupling strength.

The above results can in principle be generalized to Brownian motion in arbitrary dimension d . Equations (5)–(7) have been extended to d dimensions in Ref. [31]. In that case, the Fourier transforms in Eqs. (6) and (7) are replaced by their d -dimensional counterparts. The corresponding Kramers-Moyal coefficients may then be defined and computed analogously to the one-dimensional case, leading to drift vectors and diffusion tensors [7].

IV. HARMONICALLY BOUND PARTICLE

Let us now investigate the dynamics of a harmonically bound particle with potential $V(x) = \kappa x^2/2$, where κ is the spring constant. For this purpose, we solve Eq. (16) in Fourier space, and introduce the characteristic function $\varphi(z, t) = \mathcal{FT}_{x \rightarrow z}\{\rho(x, t)\}$ which is defined via [7,8]

$$\rho(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz e^{izx} \varphi(z, t). \quad (19)$$

The master equation for $\varphi(z, t)$ accordingly reads

$$\frac{\partial \varphi}{\partial t} = -\frac{\kappa}{\gamma} z \frac{\partial \varphi}{\partial z} + \frac{kT}{\gamma} \sum_{n=1}^{\infty} D_{2n} (iz)^{2n} \varphi, \quad (20)$$

with the coefficients $D_{2n} = \gamma a_{2n}/kT(2n)!$. Equation (20) can be solved with the method of characteristics [7,8]. With the initial condition $\rho(x, 0) = \delta(x)$, we find

$$\begin{aligned} \varphi(z, t) = & \exp \left[\varepsilon \frac{D_2}{2} (iz)^2 (1 - e^{-2\kappa t/\gamma}) \right] \\ & \times \exp \left[\varepsilon \sum_{n=2}^{\infty} \frac{D_{2n}}{2n} (iz)^{2n} (1 - e^{-2\kappa t/\gamma}) \right] \\ = & \varphi^G(z, t) \varphi^{\text{NG}}(z, t), \end{aligned} \quad (21)$$

where $\varepsilon = kT/\kappa$. We observe that $\varphi(z, t)$ can be written as a product of two, Gaussian and non-Gaussian, components, $\varphi^G(z, t)$ and $\varphi^{\text{NG}}(z, t)$, where the latter term encodes the contribution of the finite bath correlation length l . The asymptotic steady state $\varphi_s(z)$ is, moreover, approached exponentially in time.

Additional insight into the time evolution of the system and the effect of the inhomogeneous bath coupling may be gained by evaluating the moments of the position distribution $\rho(x, t)$, $M_m(t) = i^m \partial_z^m \varphi(z, t)|_{z=0}$ [7,8]. Variance and kurtosis are explicitly given by

$$\text{Var}(t) = M_2(t) = \frac{1 + 2kTl^2}{2\kappa l^2} (1 - e^{-2\kappa t/\gamma}), \quad (22)$$

$$\text{Kurt}(t) = \frac{M_4(t)}{\text{Var}(t)^2} = 3 + \frac{3\kappa \coth(t\kappa/\gamma)}{\gamma^2(1 + 2kTl^2)}. \quad (23)$$

We note that all the odd moments ($m \geq 3$) of $\rho(x, t)$ vanish, since the Kramers-Moyal expansion (16) only contains even higher derivatives ($n \geq 2$). We additionally mention that the

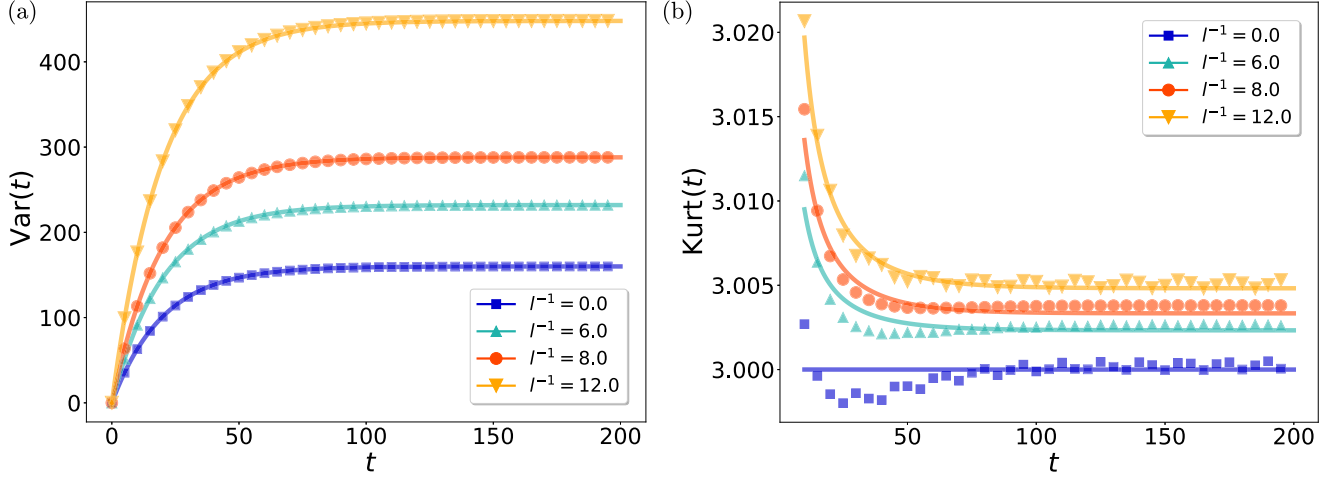


FIG. 2. (a) Time evolution of the position variance $\text{Var}(t)$ for a harmonically bound particle for five values of the inverse bath correlation length, $l^{-1} = 0, 6.0, 8.0, 12.0$. Symbols correspond to numerical solutions of the generalized Smoluchowski equation, (16), up to fifth order, while solid lines correspond to the analytical expression (22). A finite value of the correlation length l enhances the diffusion. (b) Corresponding kurtosis $\text{Kurt}(t)$, where the solid lines correspond to the analytical expression (23). A finite value of l increases the non-Gaussianity of the process. Parameters are $kT = 40.0$, $\gamma = 10.0$, and $\kappa = 0.5$, yielding $\varepsilon = 80$.

exponential relaxation of the mean, $M_1(t) = x_0 e^{-\kappa t/\gamma}$, from an initial position $x_0 \neq 0$ is not affected by the value of l . For long times, we further have

$$\text{Var} = \lim_{t \rightarrow \infty} \text{Var}(t) = \frac{1 + 2kTl^2}{2\kappa l^2}, \quad (24)$$

$$\text{Kurt} = \lim_{t \rightarrow \infty} \text{Kurt}(t) = 3 + \frac{3\kappa}{\gamma^2(1 + 2kTl^2)}. \quad (25)$$

A finite value of the bath correlation length l therefore enhances the value of the variance above its Gaussian limit [Fig. 2(a)], indicating that the system diffuses faster owing to the additional non-Gaussian fluctuations. At the same time, it increases the kurtosis, making the position distribution non-Gaussian [Fig. 2(b)].

The case of free diffusion can be easily recovered by taking the limit $\kappa \rightarrow 0$. The corresponding variance reads

$$\text{Var}_0(t) = \left(\frac{1 + 2kTl^2}{\gamma l^2} \right) t, \quad (26)$$

showing that it is linear in time, as for normal Brownian motion [7,8], albeit with an enhanced diffusion constant, $D = (1 + 2kTl^2)/(2\gamma l^2)$, which can be regarded as a generalized Einstein relation [7,8]. We hence have the remarkable result that the local bath coupling does not modify the linear time dependence of the variance. The diffusion constant of standard Brownian motion, $D_0 = kT/\gamma$, is again obtained for an infinite correlation length.

We proceed by determining the steady-state distribution $\rho_s(x)$, which is given by the inverse Fourier transform

$$\rho_s(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz e^{\varepsilon \Gamma(z)}, \quad (27)$$

where the exponent reads

$$\Gamma(z) = \frac{iz}{\varepsilon} x + \varphi_s(z) = \frac{iz}{\varepsilon} x + \sum_{n=1}^{\infty} \frac{D_{2n}}{2n} (iz)^{2n}. \quad (28)$$

The integral (27) cannot be evaluated exactly. However, since $\varepsilon = kT/\kappa \gg 1$ a saddle-point method is applicable [42]. In the limit $\varepsilon \gg 1$, the integral is indeed dominated by the maximum, $z = q$, of the function $\Gamma(z)$, which satisfies $\partial_z \Gamma = 0$. To leading order, one has [42]

$$\rho_s(x) = \exp[\varepsilon \Gamma(q)] / \sqrt{2\pi \varepsilon |\partial_z^2 \Gamma(q)|}. \quad (29)$$

Setting the derivative of Eq. (28) to zero, we find

$$\partial_q \varphi(q) = \sum_{n=1}^{\infty} (-1)^n D_{2n} (q)^{2n-1} = -\frac{ix}{\varepsilon}, \quad (30)$$

whose solution determines $q = q(x)$. We further have

$$|\partial_q^2 \Gamma| = \left| \sum_{n=1}^{\infty} (2n-1) D_{2n} (iq)^{2n-2} \right|. \quad (31)$$

In order to solve Eq. (30), we note that since ε is large, we may, to leading order, use the approximation $q = q_0 + \mathcal{O}(1/\varepsilon)$. We hence obtain

$$iq_0 = -\frac{x}{\varepsilon D_2} = -\frac{2\kappa x}{\gamma a_2}. \quad (32)$$

Combining everything, we eventually arrive at the steady-state position distribution

$$\begin{aligned} \rho_s(x) &= \frac{\sqrt{2\pi\gamma/\kappa}}{\left| \sum_{n=1}^{\infty} \frac{2n-1}{(2n)!} a_{2n} \left(\frac{2\kappa x}{\gamma a_2} \right)^{2n-2} \right|^{1/2}} \\ &\times \exp \left[-\frac{\kappa x^2}{\gamma a_2} \right] \exp \left[\frac{\gamma}{\kappa} \sum_{n=2}^{\infty} \frac{a_{2n}}{2n(2n)!} \left(\frac{2\kappa x}{\gamma a_2} \right)^{2n} \right] \\ &= \rho_s^G(x) \rho_s^{\text{NG}}(x). \end{aligned} \quad (33)$$

As before, the stationary distribution can be expressed as a product of Gaussian and non-Gaussian contributions, $\rho_s^G(x)$ and $\rho_s^{\text{NG}}(x)$. Since $a_2 \rightarrow 2kT/\gamma$ and $a_{2n} \rightarrow 0$ (for $n > 2$),

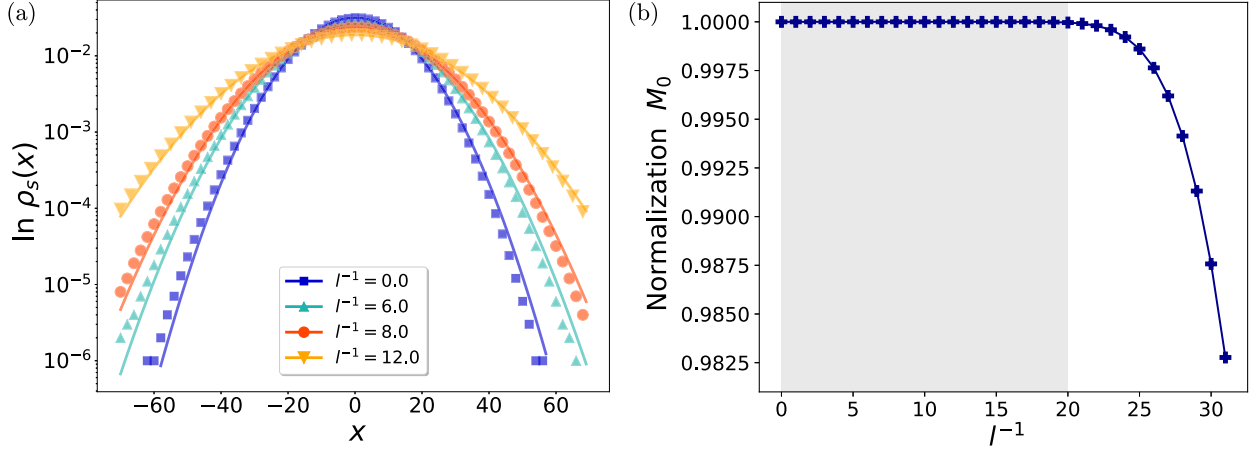


FIG. 3. (a) Steady-state position distribution $\rho_s(x)$ for a harmonically bound particle for five values of the inverse bath correlation length, $l^{-1} = 0, 6.0, 8.0, 12.0$. Symbols correspond to numerical solutions of the generalized Smoluchowski equation (16), up to fifth order, while solid lines correspond to the analytical saddle-point approximation (33). Non-Gaussian tails appear as l^{-1} departs from the Gaussian limit $l^{-1} = 0$. (b) The steady-state distribution is only properly normalized, with unit zeroth moment M_0 , for small values of l^{-1} . The shaded gray area represents the analytical estimate $l^{-1} < l_{\min}^{-1}$, where $l_{\min}^{-1} = 2\gamma$ is determined by the damping coefficient γ . Parameters are $kT = 40.0$, $\gamma = 10.0$, and $\kappa = 0.5$, yielding $\varepsilon = 80$.

when $l \rightarrow \infty$, the exact Gaussian position distribution is recovered in the limit of large l [7,8]

$$\rho_s^G(x) = \sqrt{\frac{\kappa}{2\pi kT}} \exp\left[-\frac{\kappa x^2}{2kT}\right]. \quad (34)$$

Figure 3(a) displays the stationary position distribution $\rho_s(x)$ (1) determined using the Kramers-Moyal equation (16) by keeping the first five terms (symbols) and (2) given by the saddle-point approximation (33) (solid lines), for different values of the inverse correlation length l^{-1} . We observe that the distribution becomes broader, and more and more non-Gaussian, as l decreases, as expected from the previous analysis of the moments (24) and (25). We moreover have excellent agreement between exact and approximated steady-state solutions.

It is important to realize that due to the infinite sum appearing in the exponent of the non-Gaussian part, the steady-state solution (33) is not always properly normalized, that is, the zeroth moment of the distribution, M_0 , is not always equal to one. This is the case for small values of the correlation length l , when the distribution becomes too broad to be normalized. We estimate the minimal value of the parameter l that leads to a properly normalizable stationary position distribution, by analyzing the convergence of the infinite sum that appears in the Kramers-Moyal equation (16). A necessary condition for convergence is that $a_{2(n+1)}/(2(n+1))! < a_{2n}/(2n)!$ [43]. In particular, the condition $a_4/4 < a_2/2$ implies that $l > l_{\min} = 1/2\gamma$. The minimal value of the correlation length is therefore determined by the damping coefficient γ . Figure 3(b) shows the normalization M_0 of the stationary position distribution (33) as a function of l^{-1} for $\gamma = 10$. The zeroth moment is equal to one until the value $l^{-1} = 20$ is reached, after which it sharply drops. This threshold agrees with the estimate $l_{\min}^{-1} = 2\gamma = 20$ (gray shaded area). We further note that the condition $a_{2n} \ll a_2$ ensures a positive distribution $\rho(x, t)$ when the

Kramers-Moyal equation is truncated at a finite order ($n > 2$) [7,8].

V. CONCLUSIONS

The microscopic origin of non-Gaussian noise in systems coupled to a heat bath is still unclear [44–47]. Arguments based on the central-limit theorem, which require a large number of independent random events, indeed always predict Gaussian noise [35,36]. In the present study, we have found that non-Gaussian noise naturally occurs in the overdamped limit of a quantum particle locally coupled to an inhomogeneous high-temperature environment that can be modeled as an ensemble of linear harmonic oscillators. In such a disordered environment, strong correlations between components of the bath may lead to a violation of the independence condition. Contrary to the usual case of infinite-range bath coupling, a finite spatial correlation length leads to a position distribution that evolves according to a Kramers-Moyal equation involving all higher derivatives. Such Markovian equation describes Gaussian diffusion plus non-Gaussian discrete jump events. As a consequence, diffusion is enhanced by the presence of non-Gaussian fluctuations, and the steady-state solution of a damped harmonic oscillator exhibits non-Gaussian tails.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: EXPANSION OF NOISE AND FRICTION KERNELS

In this Appendix, we demonstrate the identities given in Eqs. (9) and (10) for the friction and noise operators, $G_F(p) * (\cdot)$ and $G_N(p) * (\cdot)$. Applying the Fourier transform (FT) and its inverse (IFT) to the kernels (6) and (7), we have

$$G_F(p) * [pW] = \text{IFT}_{k \rightarrow x} \{ \tilde{G}_F(k) \text{FT}_{x \rightarrow k} \{ pW \} \} = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{l^{-2n}}{2^n n!} [\text{IFT}_{k \rightarrow x} \{ (iz)^{2n} \} * pW] \quad (\text{A1})$$

$$= \sum_{n=0}^{\infty} \frac{l^{-2n}}{2^n n!} [\delta^{(2n)} * pW] = \sum_{n=0}^{\infty} \frac{l^{-2n}}{2^n n!} \frac{\partial^{2n}}{\partial p^{2n}} [pW] = \exp\left(\frac{l^{-2} \partial_p^2}{2}\right) [pW], \quad (\text{A2})$$

and

$$G_N(p) * W = \text{IFT}_{k \rightarrow x} \{ G_N(k) \varphi(k) \} = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{l^{-2n}}{2^{n+1} (n+1)!} [\text{IFT}_{k \rightarrow x} \{ (iz)^{2n+2} \} * W] \quad (\text{A3})$$

$$= \sum_{n=0}^{\infty} \frac{l^{-2n}}{2^{n+1} (n+1)!} [\delta^{(2n+2)} * W] = \sum_{n=0}^{\infty} \frac{l^{-2n}}{2^{n+1} (n+1)!} \frac{\partial^{2n+2}}{\partial p^{2n+2}} W = \frac{1}{l^{-2}} \left(\exp\left(\frac{l^{-2} \partial_p^2}{2}\right) - 1 \right) [W], \quad (\text{A4})$$

where we have used $\text{IFT}_{k \rightarrow x} \{ (iz)^n \} = \sqrt{2\pi} \delta^{(n)}$ and $(\delta^{(n)} * F)(p) = F^{(n)}(p)$.

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