

Temperature-Induced Quantum Migration and local heat capacity in confined Fermi gases

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Temperature-Induced Quantum Migration (TIQM) is a macroscopic quantum phenomenon that fundamentally reshapes the local thermodynamics of confined systems. While previously explored in Maxwell-Boltzmann and Bose gases, we here extend the TIQM framework to noninteracting Fermi gases confined in anisometric rectangular domains. These domains are characterized by D_I , representing the number of dimensions that remain unconfined. We derive the local heat capacity and demonstrate that the standard temperature derivative of local energy yields unphysical negative regions, a discrepancy resolved by the TIQM correction. We show that quantum degeneracy in Fermi gases generally suppresses the TIQM contribution to global heat capacity compared to other statistics, yet it drives a distinct redistribution of local heat capacity. Notably, for the most constrained case ($D_I = 0$), a Schottky-type anomaly emerges at low densities, which is progressively “washed out” by increasing dimensionality ($D_I = 1, 2$). Based on an idealized noninteracting model, our calculations suggest that a measurable (22%) enhancement in the local-to-global electronic heat capacity ratio could emerge at approximately 84 nm from the longitudinal boundaries of a $12 \times 12 \times 5600$ nm³ bismuth nanowire at 30 K, offering a potential experimental signature for TIQM. These results complete a systematic trilogy of studies across all fundamental quantum statistics, establishing TIQM as a necessary correction for the local thermodynamics of nanoscale devices.

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I. INTRODUCTION

In statistical thermodynamics, global properties have traditionally received primary attention, whereas local quantities are often overlooked. This tendency is justified in the macroscopic (classical) limit, since local properties reduce to their global counterparts per particle, multiplied by the average density of the system. In macroscopic systems, changes in occupation numbers do not alter the density profile, which remains homogeneous; therefore local and global descriptions coincide. In recent decades, however, growing interest has focused on the thermodynamics of systems confined to finite domains, particularly at the nanoscale, where such classical assumptions often fail.

For nanoscale systems, it is crucial to systematically derive local thermodynamic properties by incorporating the wave nature of particles. One of the most fundamental differences results from recently introduced quantum migration process of particles [1]. In more general terms, any process changing occupation numbers causes a change in density profile, and particles migrate from one region into another within the system. Temperature change is one of the reasons inducing

quantum migration. Although the system is globally closed, it can locally behave as an open system due to quantum migration of particles. Therefore, deriving local quantities requires accounting for local particle exchange, which leads to heat and work interactions between the local volume and its surroundings within the confinement domain.

The local heat capacity of a confined gas in a nanoscale domain provides a subtle example of how local thermodynamic balance equations are influenced by temperature-induced quantum migration (TIQM). Very recently, TIQM was introduced and the local heat capacity of a noninteracting confined Maxwell-Boltzmann (MB) gas was derived and analyzed [1]. It was shown that, without the quantum-migration correction, the bare temperature derivative of the local energy at constant confinement size, referred to as the apparent local heat capacity, yields nonphysical results, featuring adjacent regions of negative and positive values within the system. The correct formulation of the local heat capacity was then established and examined in detail, demonstrating that TIQM is the sole mechanism responsible for both the thermal part of the confinement energy and the global excess heat capacity of a finite MB gas, which was a long-standing research topic since the 1960s [2–7].

Local heat exchange was shown to consist of two components: one arising from conductive heat transfer across the local boundaries, and another originating from quantum migration under the entropy-per-particle difference between the local and global volumes. The domain integral (total) of the conductive contribution was found to remain less than or equal to the classical limit of 0.5 per degree of freedom, whereas quantum migration provides 12.5% over heat capacity, resulting in a maximum of 0.0625 at a specific confinement

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or temperature. Moreover, the local heat capacity exhibits a pronounced maximum; more than twice the classical value of 0.5 [1].

TIQM has been introduced as a macroscopic quantum phenomenon that guarantees the non-negativity of the local heat capacity and constitutes the thermal part of the confinement energy, which underlies many of the unusual behaviors observed in confined systems.

As an extension of the problem, TIQM and the local heat capacity of a noninteracting finite Bose gas were investigated to reveal how quantum degeneracy influences TIQM and its contribution to the local heat capacity [8]. Recent optical techniques enable the confinement of Bose gases in boxlike potentials [9,10], revealing a range of unconventional behaviors [11–16]. Accordingly, a finite Bose gas was examined in three distinct rectangular confinement domains, which induce dimensional crossover during temperature variation and allow exploration of both degeneracy and dimensional effects on TIQM and the local heat capacity in Ref. [8].

It was found that quantum degeneracy enhances TIQM and yields a global heat-capacity maximum that surpasses the value of the unconfined system, provided the domain remains confined in only one direction. Under higher degeneracy conditions, the local heat capacity exceeds twice the classical value of $3/2$. When the confinement is too strong in one or two directions, however, this enhancement effect diminishes with increasing quantum degeneracy [8].

In addition to the Bose gas, the experimental verification of TIQM and the resulting excess local heat capacity can be most practically realized in an electron gas confined at the nanoscale. Therefore, this study extends the local heat capacity problem to the Fermi gas in order to investigate the effects of quantum degeneracy on TIQM and local heat capacity. Building on previous results for Maxwell and Bose gases, this study completes a systematic investigation across all fundamental quantum statistics and provides a comprehensive foundation for experimental verification of TIQM and its effects on local heat capacity.

The global heat capacity of finite and degenerate Fermi gases has been extensively studied in the literature. For noninteracting Fermi gases confined in infinite domains of various integer dimensions, the dependence of global heat capacity on dimensionality, along with other thermodynamic properties, has been examined in detail [17,18]. Similar analyses have also been performed for finite domains to investigate quantum size effects on global heat capacity and other thermodynamic quantities of noninteracting gases [19–28]. In addition, it has been shown that size-invariant shape transformations can modify—and even induce oscillations, in the global heat capacity of a finite Fermi gas [29]. These findings fit into a broader effort to understand how heat capacity encodes changes in the underlying energy-level structure, which also includes the study of the relationship between energy fluctuations and global heat capacity in a finite and noninteracting electron gas [30].

The influence of magnetic fields on the heat capacity of Fermi (and in particular electron) gases has also been a major area of research. It has long been known that strong magnetic fields induce oscillations in the global heat capacity, a phenomenon that has attracted considerable interest [31–36].

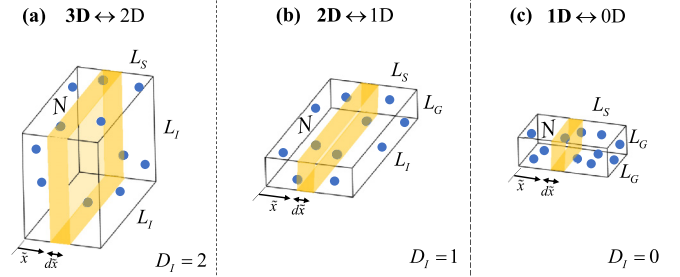


FIG. 1. Schematic representation of the finite confinement domains. The configurations are classified by their dimensionality D_I , representing the number of unconfined (free) directions: (a) $D_I = 2$, (b) $D_I = 1$, and (c) $D_I = 0$. The bold 3D, 2D, and 1D labels in (a), (b), and (c) correspond to the high-temperature or high-density limit behavior of the systems. Subscripts I , S , and G denote the characteristic size scales where integral, summation, and ground-state approaches are used for the calculation of summations over energy components in the corresponding directions. The yellow regions illustrate the local volume elements $d\tilde{x}$ used for local property calculations, with the dimensionless coordinate defined as $\tilde{x} = x/L_S$. (Reprinted from Ref. [8])

These oscillations have been explored not only theoretically but also experimentally, forming a long-standing research on the heat capacity of electron gases [37–40].

Beyond magnetic-field effects, many other physical mechanisms that modify the energy spectrum, such as interactions, confining potentials, noise, or pressure, have also been investigated for their impact on the global heat capacity of Fermi gases [41–44]. Even in strongly confined hadronic systems, variations in heat capacity have been analyzed through the Schottky anomaly, where the finite number of accessible energy levels produces distinct heat-capacity peaks [45].

Despite the extensive body of research on global heat capacity, the local heat capacity of Fermi gases has received no attention and remained unexplored. In this study, using the recently proposed TIQM correction, which guarantees a positive local heat-capacity profile, we derive and analyze the local heat capacity of a noninteracting confined (finite) Fermi gas.

Particle migration leads local work and heat exchanges and makes the calculation of local heat capacity different from the global one. The results obtained here uncover how the TIQM process change the *local* heat capacity and contribute to the *global* one in a noninteracting but confined (finite) Fermi gas. To investigate the role of effective dimension of confinement and to obtain results that are directly comparable, we use the same confinement domains as in our recently published Bose-gas study [8]. Therefore, the three anisometric rectangular boxes shown in Fig. 1 are taken as the confinement domains. These geometries yield effectively three-dimensional (3D), two-dimensional (2D), or one-dimensional (1D) behavior at sufficiently high temperature or density, whereas at low temperature and low density they cross over to 2D, 1D, or zero-dimensional (0D) behavior, producing temperature- and density-driven dimensional crossovers.

Considering a noninteracting Fermi gas confined in these domains, we first analyze in Sec. II the temperature

dependence of the global heat capacity for different densities, which serves as a benchmark for the domain integral of the local heat capacity. We also examine how the maximum excess heat capacity and its corresponding temperature depend on density. Furthermore, we demonstrate that the commonly used temperature derivative of the local energy, here referred to as the apparent local heat capacity, can yield unphysical negative values in each confinement domain.

In Sec. III, we derive the local heat capacity and its conductive and quantum-migration components, and analyze their variations with temperature and position for the selected domains. We also investigate how the maxima of the local heat-capacity profile and of each component depend on density. The contributions of the conductive and migration parts to the global heat-capacity behavior are then evaluated. Finally, we show that the thermal component of the confinement energy is predominantly stored near the boundaries, while a smaller portion is distributed throughout the interior in an oscillatory manner.

II. GLOBAL HEAT CAPACITY IN FINITE-SIZE NANO DOMAINS

A. Confinement domains, model parameters, and geometric constraints

In a Fermi gas, the Pauli exclusion principle forces particles to occupy progressively higher single-particle energy levels as density increases, resulting in substantial occupation of high-energy states even in the zero-temperature limit ($T \rightarrow 0$). At zero temperature, the highest occupied energy level defines the Fermi energy ϵ_F . The corresponding Fermi temperature, $T_F = \epsilon_F/k_B$, increases monotonically with the particle density as $n^{2/D}$ in a D -dimensional system. While T quantifies thermal excitations, T_F quantifies the role of degeneracy in populating higher energy levels, the statistical forcing driven by the Pauli exclusion principle. For clarity, we introduce $T_* = \max(T, T_F)$ as the effective temperature, which unifies the thermal and Fermi scales into a single quantity, indicating the dominant mechanism governing the occupation of higher-energy single-particle states. Similarly, the thermal and Fermi-de Broglie wavelengths, $\lambda_{th}(T)$ and $\lambda_F = \lambda_{th}(T_F)$, can be combined into a single characteristic particle wavelength at the effective temperature: $\lambda_* = \lambda_{th}(T_*)$.

In the confinement domains illustrated in Fig. 1, the ground-state energy component in direction i , $\epsilon_{0,i}$, increases as the confinement length L_i decreases. When confinement is sufficiently strong such that the corresponding ground-state temperature $T_{0,i} = \epsilon_{0,i}/k_B$ greatly exceeds T_* ($T_* \ll T_{0,i}$), or equivalently $L_i \ll \lambda_*$, particles occupy only the ground subband of direction- i . Consequently, the effective momentum-space dimensionality along that direction reduces to zero (0D). In the opposite limit ($L_i \gg \lambda_*$ or $T_* \gg T_{0,i}$), confinement becomes negligible, multiple excited subbands are populated, and the effective dimensionality approaches a full dimension (1D). Between these limits, the accessible momentum space acquires a noninteger dimensionality, giving rise to the fractional-dimensional behavior of the confined Fermi gas.

In Fig. 1, the subscripts I , S , and G denote the evaluation methods for the energy summations along the corresponding directions: integral approximation (I), discrete summation (S), and ground-state approximation (G). The domain sizes are chosen such that the hierarchy $L_G \ll \lambda_{th}(T_*^{\max}) \lesssim L_S \lesssim \lambda_{th}(T_*^{\min}) \ll L_I$ is maintained over the considered temperature and density ranges. This ensures that the integral and ground-state approximations remain valid across the entire parameter space; notably, the discrete summation requires no such constraints as it constitutes an exact calculation method.

Along the L_I direction, an effectively infinite number of momentum states are accessible; thus, each unconfined direction contributes 1D to the total momentum-space dimensionality. In contrast, along the L_G direction, only the ground subband is occupied, and strongly confined directions contribute 0D. The intermediate length L_S corresponds to partial confinement, resulting in a fractional contribution to the total dimensionality [46] that approaches 1D in the weak-confinement limit and 0D in the strong-confinement limit. For a fixed-size domain, the degree of confinement is governed by $T_*(T, n)$.

For $T_F(n) < T$, the effective temperature reduces to $T_* = T$, and the strength of confinement is governed by thermal excitations. Since λ_{th} scales as $T^{-1/2}$, the system in Fig. 1(a) becomes strongly confined along L_S at low temperatures, exhibiting 2D behavior whenever $L_S \ll \lambda_{th}(T)$. Conversely, at high temperatures ($L_S \gg \lambda_{th}(T)$), the system is effectively unconfined and exhibits three-dimensional (3D) characteristics. Thus, the system undergoes a temperature-driven dimensional crossover. Similar behavior is observed for the lower-dimensional domains in Figs. 1(b) and 1(c), enabling a systematic study of the global heat capacity as the effective dimensionality changes. In the degenerate regime where $T_F(n) > T$, we have $T_* = T_F$. If $T_F \gg T_{0,S}$, the Fermi gas in Fig. 1(a) remains effectively unconfined because a substantial number of excited states are populated via Pauli exclusion, causing the gas to behave as a 3D system even at low temperatures ($T \ll T_{0,S}$). On the contrary, if $T \ll T_F \ll T_{0,S}$, the gas in Fig. 1(a) is kinematically frozen into the ground subband of direction- S by both low temperature and restricted density, thereby behaving as a strictly 2D system. By extension, this mechanism of quantum-statistical confinement applies to the lower-dimensional domains in Figs. 1(b) and 1(c). In these cases, the effective temperature T_* determines the transition from 2D to 1D and from 1D to 0D, respectively, allowing for a unified description of dimensionality-driven effects across all considered geometries.

For particles confined in a rectangular box, the energy eigenvalues along each spatial direction i are determined by the Schrödinger equation: $\epsilon_i = \pi^2 \hbar^2 k_i^2 / 2mL_i^2$, where $k_i = 1, 2, 3, \dots$, m is the particle mass, and L_i is the domain size. To facilitate the analysis, we introduce the dimensionless energies $\tilde{\epsilon}_i = \epsilon_i/k_B T = (\alpha_i k_i)^2$. Here, $\alpha_i = \sqrt{T_{0,i}/T}$ is the confinement parameter, where $T_{0,i} = \epsilon_{0,i}/k_B$ corresponds to the single-particle ground-state energy $\epsilon_{0,i} = \pi^2 \hbar^2 / 2mL_i^2$.

We adopt $T_0 = T_{0,S}$ as the reference temperature to define a dimensionless temperature, $\tilde{T} = T/T_0$, which yields

$\alpha_S = 1/\sqrt{\tilde{T}}$. By utilizing the aspect ratios $r_I = L_I/L_S$ and $r_G = L_S/L_G$, the confinement parameters along the remaining directions can be expressed as $\alpha_I = \alpha_S/r_I$ and $\alpha_G = \alpha_S r_G$.

Similarly, we define a temperature-independent reference quantum density at T_0 as $n_q(T_0) = g_s/\lambda_{th}^{D_I}(T_0)$, where $\lambda_{th}(T_0) = 2L_S/\sqrt{\pi}$ is the thermal de Broglie wavelength at the reference temperature and $g_s = 2$ is the spin degeneracy. This allows the introduction of the dimensionless density $\tilde{n}_0 = \bar{n}/n_q(T_0)$, where $\bar{n} = N/V_f$ represents the global density relative to the “free” volume $V_f = L_I^{D_I}$ associated with the unconfined directions. The total particle number N is then related to $\tilde{n}_0$ via $N = g_s \tilde{n}_0 (\sqrt{\pi} r_I/2)^{D_I}$. For cases where $D_I \neq 0$ [Figs. 1(a) and 1(b)], the condition $r_I \gg 1/\tilde{n}_0^{1/D_I}$ ensures a macroscopically large particle number ($N \gg 1$). For the fully confined case where $D_I = 0$ [Fig. 1(c)], the large- N limit is ensured when $\tilde{n}_0 \gg 1$.

To justify the validity of the integral and ground-state approximations, the confinement parameters must satisfy $\alpha_I \ll 1$ and $\alpha_G \gg 1$ when evaluated at the effective temperature $T_* = \max(T, T_F)$. In terms of characteristic temperatures, these conditions require $T_{0,I} \ll T_*^{\min}$ and $T_{0,G} \gg T_*^{\max}$. Consequently, the aspect ratios must satisfy:

$$r_I^2 \gg \frac{1}{\tilde{T}_{\min}^2} \quad \text{and} \quad r_G^2 \gg \tilde{T}_{\max}^2. \quad (1)$$

These requirements maintain the hierarchy $L_G \ll \lambda_{th}(T_*^{\max}) \lesssim L_S \lesssim \lambda_{th}(T_*^{\min}) \ll L_I$, which is necessary for the consistent application of the approximations across the entire considered parameter space.

B. Chemical potential

Using the Fermi-Dirac distribution together with the relation $N = g_s \tilde{n}_0 * (\sqrt{\pi} r_I/2)^{D_I}$, and applying the integral and ground-state approximations along the L_I and L_G directions, respectively, we obtain

$$\tilde{n}_0(D_I) = -\tilde{T}^{D_I/2} \sum_{k_S=1}^{\infty} Li_{D_I/2} \left[-\exp \left(\Lambda_C - \frac{k_S^2}{\tilde{T}} \right) \right]. \quad (2)$$

Here Li denotes the Polylogarithm function, and $\Lambda_C = \mu_C/k_B T$ is the *effective dimensionless chemical potential*, which incorporates the contribution of the ground-state energy associated with the strongly confined directions. It is expressed as $\Lambda_C = \Lambda - (2 - D_I)r_G^2/\tilde{T}$, and takes values in the range $-\infty < \Lambda_C < \infty$. For the domain shown in Fig. 1(a), where $D_I = 2$, the effective chemical potential coincides with the conventional one, $\Lambda_C = \Lambda$. For the domains in Figs. 1(b) and 1(c), with $D_I = 1$ and $D_I = 0$, respectively, the chemical potential is shifted by the dimensionless ground-state energy, $\alpha_G^2 = r_G^2/\tilde{T}$ along each strongly confined direction.

For given values of $\tilde{n}_0$, D_I , and $\tilde{T}$, the inverse numerical solution of Eq. (2) gives the value of Λ_C . The derivation of Eq. (2) is presented in the Appendix. For the high and low asymptotic regimes of effective excitation temperature, $\tilde{T}_* \gg 1$ and $\tilde{T}_* \ll 1$, the summation in Eq. (2) can be calculated by continuum (integral) and ground-state approaches, respectively. These regimes are discussed in the following subsections.

1. High temperature regime ($\tilde{T}_* \gg 1$): continuum approach

In the high-temperature regime ($\tilde{T} \gg 1$), the factor of $1/\tilde{T}$ in Eq. (2) is too small allowing the discrete sum to be approximated by an integral. In the low-temperature limit, $\tilde{T} \ll 1$, the state space becomes strongly discrete since $1/\tilde{T} \gg 1$. Unlike the Bose gas [8], however, the sum cannot be represented by considering only the first term if $\Lambda_C \gg 1/\tilde{T} \gg 1$. Because, due to the Pauli exclusion principle, the number of occupied states is still large enough to consist numerous states and still allow us to use integral approach. Considering that $\Lambda_C \rightarrow \tilde{T}_F/\tilde{T}$ at low temperature limit as $\tilde{T} \rightarrow 0$, the temperature condition for $\Lambda_C \gg 1$ requires $\tilde{T}_F \gg 1$.

Thus, as long as $\tilde{T}_* \gg 1$ (high-temperature or high-density limit), a continuum (integral) approach can be used to calculate the summation in Eq. (2) and we obtain the standard continuous form

$$\tilde{T}_* \gg 1 \Rightarrow \tilde{n}_0 = -\tilde{T}^{(D_I+1)/2} \frac{\sqrt{\pi}}{2} Li_{(D_I+1)/2} [-\exp(\Lambda_C)]. \quad (3)$$

In this regime, the thermal wavelength satisfies $\lambda_* \ll L_S$ meaning that the confinement along the L_S direction becomes effectively negligible. As a result, the geometry in Fig. 1(a) behaves as a 3D system, instead of 2D, while the configurations in Figs. 1(b) and 1(c) correspond to effective 2D and 1D systems, respectively. This dimensional crossover is clearly seen in the order of Li functions and the powers of temperature appearing in Eqs. (2) and (3), $D_I \rightarrow D_I + 1$.

For strong degeneracy ($\Lambda_C \gg 1$), Li_x approaches its high- Λ_C asymptotic expansion:

$$Li_x[-\exp(\Lambda_C)] \rightarrow -\frac{\Lambda_C^x}{\Gamma(x+1)} \left[1 + \frac{\pi^2}{6} \frac{x(x-1)}{\Lambda_C^2} \right], \quad (4)$$

where the second term in brackets represents the second-order contribution. Neglecting this second-order term, Eq. (3) simplifies to

$$\tilde{n}_0 \rightarrow \frac{\sqrt{\pi} \tilde{T}_F^{(D_I+1)/2}}{2\Gamma(\frac{D_I+3}{2})}. \quad (5)$$

Here $\tilde{T}_F = T_F/T_0 = \tilde{T} \Lambda_C$ is the dimensionless Fermi temperature defined for the limit of $\Lambda_C \rightarrow \infty$. This limit is approached either when the density becomes very large ($n_0 \rightarrow \infty$) or when the temperature approaches zero ($T \rightarrow 0$). From Eq. (5), the dimensionless Fermi temperature follows as $\tilde{T}_F = [(2\tilde{n}_0/\sqrt{\pi})\Gamma[(D_I+3)/2]]^{2/(D_I+1)}$. Finally, note that the high-temperature strong-degeneracy conditions at requires the hierarchy $1 \ll \tilde{T} \ll \tilde{T}_F$.

When the degeneracy becomes negligible, $\Lambda_C \ll -1$ (high-temperature and low-density regime), Li_x function reduces to its low- Λ_C asymptotic limit, $Li_x \rightarrow -\exp(\Lambda_C)$. Therefore, Eq. (3) recovers the classical MB limit, $\tilde{n}_0 \rightarrow [\sqrt{\pi} \tilde{T}^{(D_I+1)/2}/2] \exp(\Lambda_C)$. The condition for high-temperature and nondegeneracy regime is $\tilde{T}_F \ll 1 \ll \tilde{T}$.

2. Low temperature regime ($\tilde{T}_* \ll 1$): ground-state approach

In the low temperature and nondegeneracy regime $\tilde{T}_* \ll 1$, the discreteness of energy states dominates for the energy component in L_S direction and the summation in Eq. (2) can

be represented by considering only the first term (ground-state approach). Thus,

$$\tilde{T}_* \ll 1 \Rightarrow \tilde{n}_0 = -\tilde{T}^{D_I/2} Li_{D_I/2} \left[-\exp \left(\Lambda_C - \frac{1}{\tilde{T}} \right) \right]. \quad (6)$$

Under such a condition, as $\lambda_* \gg L_S$, confinement in the direction of L_S becomes too strong and the systems in Figs. 1(a)–1(c) reduce to 2D, 1D and, 0D systems, respectively. The low-temperature and nondegeneracy conditions allow us to use the low- Λ_C asymptote of the Polylogarithm function, $Li_{D_I/2}[-\exp(\Lambda_C - \frac{1}{\tilde{T}})] \rightarrow -\exp(\Lambda_C - \frac{1}{\tilde{T}})$, and Eq. (6) reduces to

$$\tilde{n}_0 = \tilde{T}^{D_I/2} \exp \left(\Lambda_C - \frac{1}{\tilde{T}} \right). \quad (7)$$

The condition for this low-temperature and nondegeneracy regime is $\tilde{T}_F \ll \tilde{T} \ll 1$.

C. Global heat capacity

A general expression for dimensionless global heat capacity of a noninteracting Fermi gas confined in rectangular domains with constant sizes illustrated in Figs. 1(a)–1(c) is obtained as

$$\begin{aligned} \tilde{C} = \frac{C_L}{Nk_B} = & \left(\frac{{}_2A_{-1}}{{}_0A_0} - \frac{{}_1A_{-1}^2}{{}_0A_0 {}_0A_{-1}} \right) \\ & + \frac{D_I}{2} \left[\frac{{}_0A_1}{{}_0A_0} + 2 \left(\frac{{}_1A_0}{{}_0A_0} - \frac{{}_1A_{-1}}{{}_0A_{-1}} \right) \right] \\ & - \left(\frac{D_I}{2} \right)^2 \left(\frac{{}_0A_0}{{}_0A_{-1}} - \frac{{}_0A_1}{{}_0A_0} \right), \end{aligned} \quad (8)$$

where ${}_xA_y = {}_xA_y(\Lambda_C, \tilde{T})$ is the energy moment function defined by

$${}_xA_y = - \sum_{k_S=1}^{\infty} \left(\frac{k_S^2}{\tilde{T}} \right)^x Li_{\frac{D_I}{2}+y} \left[-\exp \left(\Lambda_C - \frac{k_S^2}{\tilde{T}} \right) \right]. \quad (9)$$

Note that the subscript of L in $\tilde{C}$ is dropped for simplicity in later notations. The derivation Eq. (8) can be found in the Appendix. The inverse solution of Eq. (2) gives the value of $\Lambda_C(\tilde{n}_0, \tilde{T})$ to be used for A functions in Eq. (8) on which the global heat capacity results in this study are based. Note that Eq. (8) represents the generalized form of the global heat capacity for arbitrary ratios of λ_*/L_S whereas previous analytical results in Ref. [21] were restricted to the $\lambda_*/L_S \ll 1$ regime via a first-order approximation.

In the continuum limit, $\tilde{T}_* \rightarrow \infty$, the integral approximation can be used to calculate the summation in Eq. (9) and the ${}_xA_y$ functions reduce to following ${}_xB_y = {}_xB_y(\Lambda_C, \tilde{T})$ functions:

$${}_xB_y = - \left(\frac{\sqrt{\tilde{T}}}{2} \right) \Gamma \left(x + \frac{1}{2} \right) Li_{\frac{D_I+1}{2}+x+y} [-\exp(\Lambda_C)]. \quad (10)$$

Therefore, in continuum regime, the total dimension of momentum space is $D = D_I + 1$, thereby implying that D can only take integer values. Using the B functions, Eq. (10) in Eq. (8) leads to the following generalized global heat

capacity expression in the literature [17,28] for any integer dimension D :

$$\tilde{C}_I = \left(\frac{D}{2} \right) \left[\left(\frac{D+2}{2} \right) \frac{Li_{(D+2)/2}}{Li_{D/2}} - \frac{D}{2} \frac{Li_{D/2}}{Li_{(D-2)/2}} \right], \quad (11)$$

where the subscript I in $\tilde{C}$ is used to emphasize that it applies solely to integer dimensions, Li is used for the compact representation, instead of $Li[-\exp(\Lambda_C)]$.

For low-density and high-temperature regime, the classical expression, $D/2$, is recovered by considering the low- Λ_C asymptotic expansion of Li_x functions in Eq. (11). For the degenerate regime (high-density and low temperature), using the high- Λ_C asymptotes of Li functions in Eq. (4) reduces Eq. (11) to the following well-established analytical expression:

$$\tilde{T}/\tilde{T}_F \ll 1 \Rightarrow \tilde{C}_I = \left(\frac{\tilde{T}}{\tilde{T}_F} \right) \left(\frac{\pi^2}{6} \right) D, \quad (12)$$

where $D = 1 + D_I$ due to continuum regime. Equations (11) and (12) represent the thermodynamic limit of the model in Ref. [21], obtained in the limit of infinite system size where finite-size effects vanish.

In finite systems, confinement leads to an effective momentum space with noninteger (fractional) dimensionality, which necessitates a description of the global heat capacity in terms of Eq. (8). By contrast, infinite systems are characterized by integer dimensionality, and their heat capacity is described by the well-established expressions given in Eq. (11) and, in the degenerate regime, Eq. (12).

For systems with integer dimensionality, the global heat capacity is described by Eq. (11) and is bounded above by the classical value $D/2$; and therefore, it exhibits no maximum at finite temperature or density. In contrast, finite systems acquire a noninteger effective dimensionality and develop a pronounced maximum in the heat capacity captured by Eq. (8).

For a noninteracting Fermi gas confined within the domains shown in Fig. 1, the resulting global heat capacities are displayed in Figs. 2(a)–2(c) for different densities. Each curve approaches its classical limit $(1 + D_I)/2$ in the high-temperature regime and vanishes as $\tilde{T} \rightarrow 0$, except for the nondegenerate case ($\tilde{n}_0 = 0.01$), which instead approaches $D_I/2$ because $\tilde{T}_F \ll \tilde{T}^{\min}$ over the considered temperature interval. These high- and low-temperature limits correspond to effectively integer-dimensional regimes with dimensionalities $D_I + 1$ and D_I , respectively. Although these limiting behaviors are straightforward, the heat capacity exhibits markedly different behavior at intermediate temperatures, reflecting the underlying dimensional crossover.

It is observed that, for $D_I = 2$, the dimensionless heat capacity $\tilde{C}$ remains strictly below the MB maximum $\tilde{C}_{\max}^{\text{MB}}$ over the entire parameter range. For $D_I = 1$, $\tilde{C}$ slightly exceeds this limit only marginally (by less than 3%) in the vicinity of $\tilde{T} \approx 1$ and $\tilde{n}_0 \approx 1$. In contrast, when $D_I = 0$, $\tilde{C}$ exhibits a pronounced peak that significantly exceeds $\tilde{C}_{\max}^{\text{MB}}$, around $\tilde{T} \approx 0.5 - 1$ at $\tilde{n}_0 = 1$, arising from a Schottky-type anomaly associated with the discrete and finite number of accessible energy states. In a finite system, increasing temperature makes more energy states available to the particles, causing the anomaly to decay. As the density increases, this peak shifts to

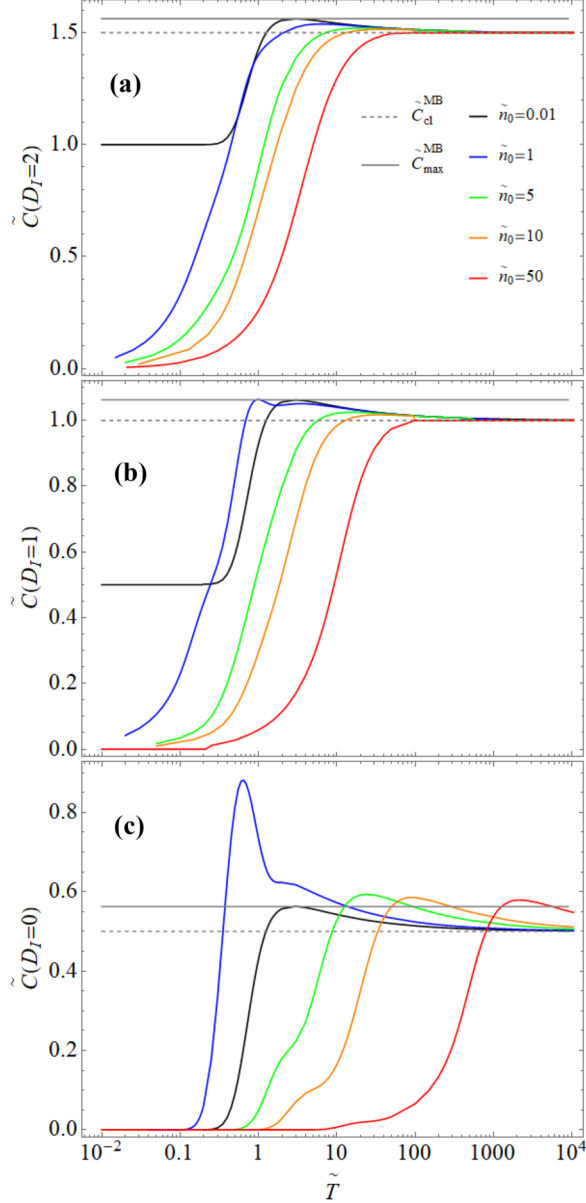


FIG. 2. Temperature dependence of the global heat capacity for a confined Fermi gas at various densities $\tilde{n}_0$, corresponding to the domains illustrated in Fig. 1. Solid colored curves represent finite-sized systems with different densities, while dashed gray lines denote the classical MB (MB) limit for an infinite domain ($D = 3, 2$, and 1 in (a), (b), and (c), respectively). The solid gray lines indicate the maximum global heat capacity achievable for a finite MB gas. Note that for $D_I = 0$, where the total particle number is $N = g_s \tilde{n}_0$, the $\tilde{n}_0 = 0.01$ case is included primarily to represent the mathematical limit of the MB case, rather than a physically realizable system.

higher temperature and is progressively suppressed, whereas in the low-density limit ($\tilde{n}_0 = 0.01$), it approaches $\tilde{C}_{max}^{MB}$.

Figure 3 provides further insight into the dependence of the excess heat capacity and its corresponding temperature on the density for different values of D_I . Since the total particle number is related to the density via $N = g_s \tilde{n}_0 * (\sqrt{\pi} r_I / 2)^{D_I}$, with spin degeneracy $g_s = 2$, the density $\tilde{n}_0$ is quantized in integer multiples of $1/2$ for $D_I = 0$, whereas it varies

continuously for $D_I = 1$ and $D_I = 2$. Consequently, only the discrete values $\tilde{n}_0 = 0.5, 1, 1.5, \dots$ correspond to physically realizable systems in the $D_I = 0$ case. These physically allowed cases, considered during the calculations, are indicated by red markers in Fig. 3. These red dots for the $D_I = 0$ case will not be shown again in the other figures.

Figure 3(a) demonstrates that $\tilde{C}_{max}$ exceeds the $\tilde{C}_{max}^{MB}$ level only when $D_I = 0$, provided the negligible overshoot for $D_I = 1$ is ignored. While $\tilde{C}_{max}$ decays monotonically toward zero with increasing density for $D_I = 2$, it exhibits a local maximum before vanishing in the $D_I = 1$ case. In contrast, the $D_I = 0$ curve approaches a nonzero value, namely the maximum classical excess heat capacity, in the high-density limit, whereas all cases converge to the same expected value in the low-density limit.

Figure 3(b) illustrates the density dependence of the temperature $\tilde{T}$ at which the heat capacity reaches its maximum, denoted as $\tilde{T}_{max}$. In the low-density limit, all curves converge to $\tilde{T}_{max} = 2.86$, recovering the MB gas behavior [1]. As density increases, $\tilde{T}_{max}$ rises monotonically for $D_I = 2$. In contrast, for the $D_I = 1$ and $D_I = 0$ cases, $\tilde{T}_{max}$ initially decreases and exhibits discontinuities near $\tilde{n}_0 \approx 1$, after which it rises and approaches a linear profile. This linearity, observed on a logarithmic scale in the high-density limit, is consistent with the power-law dependence of the Fermi temperature on density. Furthermore, while the Fermi temperature perfectly matches $\tilde{T}_{max}$ for $D_I = 0$ in the high-density limit, there is a near-constant shift for $D_I = 1$ and $D_I = 2$ that increases with the dimensionality of the momentum space due to the finiteness of the system. The discontinuities at intermediate densities are explained by the temperature evolution of $\tilde{C}$ shown in Fig. 2: for the $D_I = 1$ and $D_I = 0$ cases, the heat capacity is bimodal, featuring two distinct maxima near $\tilde{T} \approx 1$. At lower densities, the lower-temperature peak is dominant. However, as density increases, this peak is suppressed more rapidly than its higher-temperature counterpart. Consequently, the global maximum shifts abruptly from the lower to the higher temperature peak, resulting in the observed jump in $\tilde{T}_{max}$ as a function of density.

III. LOCAL HEAT CAPACITY AT NANO SCALE

One of the fundamental requirements for thermodynamic stability is the non-negativity of the local heat capacity. If this condition is violated, any infinitesimal energy fluctuation, such as a transfer from a region of positive heat capacity to one of negative heat capacity, would irreversibly disrupt the temperature homogeneity of the system. Such a process triggers a thermal runaway: the temperature in the negative-capacity region would diverge toward infinity or absolute zero depending on the direction of transfer, ultimately leading to a collapse of the equilibrium state. Therefore, regions of opposing local heat capacity signs cannot coexist in a stable system. As demonstrated in this work, simply taking the temperature derivative of the local energy for a Fermi gas yields nonphysical negative values. This discrepancy highlights the necessity of employing the rigorous definition of local heat capacity introduced in Ref. [1]. Originally formulated for MB gases and subsequently applied to Bose systems [8], this formulation

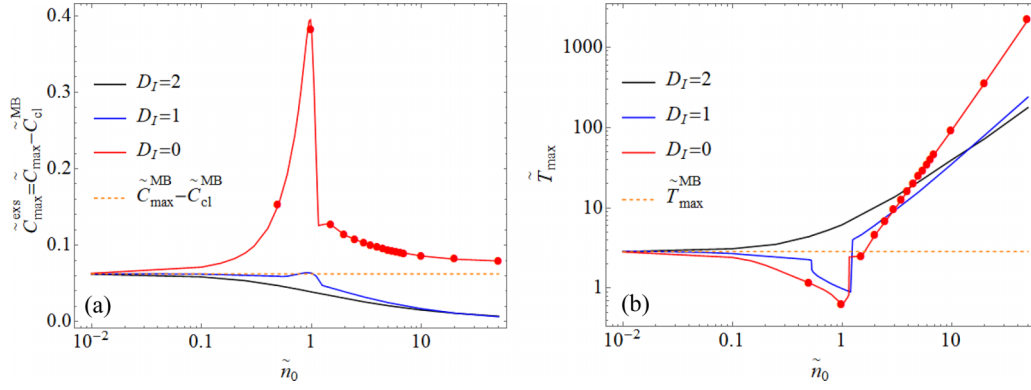


FIG. 3. (a) Dependence of the maximum excess global heat capacity, $\tilde{C}_{\max}^{\text{ex}} = \tilde{C}_{\max}^{\text{MB}} - \tilde{C}_{\text{cl}}^{\text{MB}}$, on particle density $\tilde{n}_0$ for the different confinement domains shown in Fig. 1. (b) Density dependence of the dimensionless temperature $\tilde{T}_{\max}^{\text{MB}}$ at which the heat capacity attains its maximum value for the same domains. Red markers denote the configurations for the $D_I = 0$ case, corresponding to an integer number of particles ($N = g_s \tilde{n}_0$) considered during the calculations.

ensures physical consistency by accounting for the essential quantum migration contributions.

The theoretical framework for local heat capacity employed in this work follows the general formulation developed in Ref. [1]. By adapting this model to Fermi-Dirac statistics, we provide the final component of a systematic study across all fundamental quantum statistics. This allows for a detailed analysis of the local heat capacity in Fermi gases, specifically examining the consequences of quantum degeneracy and Pauli exclusion. This approach decomposes the thermal response into distinct contributions from heat conduction and quantum migration. Within this framework, the dimensionless local heat capacity is defined as the sum of the conduction and quantum migration components, $\tilde{c} = \tilde{c}_{\text{cnd}} + \tilde{c}_{\text{qmh}}$, where $\tilde{c}_{\text{cnd}} = c_{\text{cnd}}/\tilde{n}k_B$, and $\tilde{c}_{\text{qmh}} = c_{\text{qmh}}/\tilde{n}k_B$. In the energy representation, these contributions are expressed as

$$\tilde{c}_{\text{cnd}} = \tilde{c}_0 + \hat{\omega} \tilde{\gamma}_L, \quad (13a)$$

$$\tilde{c}_{\text{qmh}} = (\hat{\Omega} - \hat{\omega}) \tilde{\gamma}_L. \quad (13b)$$

$$\tilde{c} = \tilde{c}_0 + \hat{\Omega} \tilde{\gamma}_L = \tilde{c}_0 + \tilde{c}_{\text{QC}}. \quad (13c)$$

In these expressions, $\tilde{c}_0 = (1/\tilde{n}k_B)(\partial u/\partial T)_L = \tilde{n}(\partial \hat{u}/\partial \tilde{T})_L - \hat{u} \tilde{\gamma}_L$ represents the dimensionless apparent local heat capacity, which is determined by the temperature derivative of the local energy u at a fixed location. Also, $\tilde{\gamma}_L = -T(\partial \tilde{n}/\partial T)_L$ denotes the isometric local thermal expansion coefficient of the dimensionless density, and the term $\tilde{c}_{\text{QC}}$ represents the quantum correction to the apparent local heat capacity $\tilde{c}_0$.

We define the dimensionless local energy as $\tilde{u} = u/(\tilde{n}k_B T)$, normalized by the position-independent global density $\tilde{n}$. In contrast, the dimensionless energy per particle, $\hat{u} = u/(nk_B T)$, is defined relative to the local density n . These two quantities are related via the dimensionless density $\tilde{n} = n/\tilde{n}$ such that $\tilde{u} = \hat{u} \tilde{n}$. Following a similar convention, the state functions per particle in Eq. (13) are expressed as $\hat{\omega} = \omega/(nk_B T)$ and $\hat{\Omega} = \Omega/(Nk_B T)$. These functions are derived from the local (φ) and global (Φ) grand potentials and their corresponding energies (u and U) through the relations $\omega = u - \varphi$ and $\Omega = U - \Phi$.

While the linear relationship between pressure and the grand potential remains valid at the local scale $p = -\varphi$, it is known to break down globally $P \neq -\Phi/V$ in finite systems,

as discussed in Ref. [1]. Consequently, ω can be interpreted as the local enthalpy, $h = u + p$, which is a positive-definite quantity, implying $\hat{\omega} = \hat{h}$.

The application of thermodynamics to an open local control volume leads the components of the local heat capacity given in Eq. (13). In this framework, the spatial number of particles is not a conserved quantity due to the inherent wave nature of the constituents. Consequently, the conductive heat exchange across local boundaries deviates from the standard Tds form; instead, it is expressed as $\delta q_{\text{cnd}} = nTd(s/n) = Tds - T(s/n)dn$, as detailed in Refs. [1,8].

Furthermore, the migration of particles between local and global volumes involves a change in the entropy per particle. This process drives a quantum-migration-induced local heat exchange, defined by $\delta q_{\text{qmh}} = T[(s/n) - (S/N)]dn$. In the energy representation, these heat exchange components can be reformulated as $\delta q_{\text{cnd}} = du - (\omega/n)dn$ and $\delta q_{\text{qmh}} = [(\omega/n) - (\Omega/N)]dn$. These expressions provide the physical basis for the components defined in Eqs. (13a) and (13b), respectively.

The quantities appearing in Eq. (13) are evaluated using their fundamental definitions as discrete summations over the quantum state variable k . To obtain analytical expressions, we apply an integral and ground-state approximations for the states associated with the dimensions L_I and L_G , respectively, while maintaining the explicit summation over the k_S states. Consequently, these local quantities are formulated using the local A functions, LA , defined as follows.

The probability density of a single particle in a state k_S for a local position (indicated in yellow in Fig. 1) is given by $|\psi_{k_S}(\tilde{x})|^2 = (2/L_S) \sin^2 \pi k_S \tilde{x}$, where $\psi_{k_S}(\tilde{x})$ represents the eigenfunction of the Schrödinger equation for the state k_S and $\tilde{x} = x/L_S$. By substituting this probability density into the A -function definition in Eq. (9), we arrive at the local A functions:

$${}_x LA_y = -2 \sum_{k_S=1}^{\infty} \left(\frac{k_S^2}{\tilde{T}} \right)^x Li_{\frac{D_I}{2}+y} \left[-\exp \left(\Lambda_C - \frac{k_S^2}{\tilde{T}} \right) \right] \times \sin^2(\pi k_S \tilde{x}). \quad (14)$$

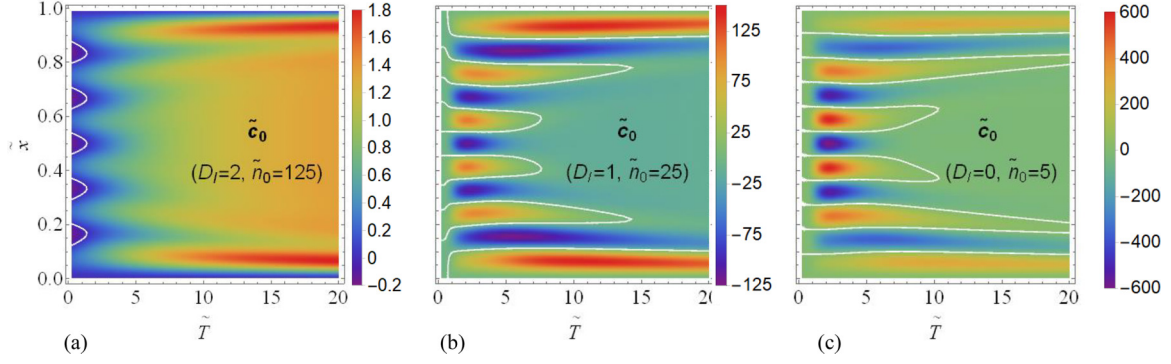


FIG. 4. Position and temperature dependence of the apparent local heat capacity $\tilde{c}_0$ for different D_I values and $\tilde{n}_0 = 5^{1+D_I}$. The white curves denote the zero-crossing zones.

Based on the definition of dimensionless density, it is given in its general form and in terms of LA functions as

$$\tilde{n} = \frac{n}{\tilde{n}} = \frac{\sum_k f_k |\psi_k|^2}{\frac{1}{V} \sum_k f_k} = \frac{{}_0LA_0}{{}_0A_0}. \quad (15)$$

Here, f_k and ψ_k denote the Fermi-Dirac distribution and the translational eigenfunctions for state k , respectively. The index k serves as a collective symbol for the momentum components $\{k_S, k_G, k_I\}$ relative to the confinement geometry shown in Fig. 1, where $V = L_I L_S L_G$ represents the total geometric volume.

Dimensionless local energy $\tilde{u} = u/\tilde{n}k_B T$ is determined in terms of its directional components, $\tilde{u} = \tilde{u}_S + \tilde{u}_G + \tilde{u}_I$, by LA functions as

$$\tilde{u} = \frac{\sum_k \tilde{\epsilon}_k f_k |\psi_k|^2}{\frac{1}{V} \sum_k f_k} = \frac{{}_1LA_0}{{}_0A_0} + \frac{(2-D_I)r_G^2}{{\tilde{T}}} \frac{{}_0LA_0}{{}_0A_0} + \frac{D_I}{2} \frac{{}_0LA_1}{{}_0A_0}. \quad (16)$$

Figure 4 illustrates the spatial and temperature dependence of the apparent local heat capacity $\tilde{c}_0$ for the considered D_I values. The densities for each case are strategically selected to maintain a consistent degree of quantum degeneracy across different dimensions. Given that the Fermi temperature scales as $\tilde{T}_F \sim \tilde{n}_0^{2/(D_I+1)}$, selecting $\tilde{n}_0 = 5^{1+D_I}$ (yielding $\tilde{n}_0 = 5, 25$, and 125 for $D_I = 0, 1$, and 2 , respectively) ensures that the Fermi temperature remains approximately at the same level for all D_I values ($\tilde{T}_F \approx 25, 28$, and 33). This choice provides a robust basis for comparing properties under the almost same quantum degeneracy level across systems of different dimensionality; otherwise, using a fixed density for all cases would lead to an unequal comparison where higher-dimensional systems appear significantly less degenerate. Following this same principle, the density values in Figs. 6 and 10 are also determined by the scaling rule $\tilde{n}_0 = 5^{1+D_I}$ to ensure that observed differences are due to dimensionality rather than varying degrees of degeneracy.

Notably, $\tilde{c}_0$ in Fig. 4 exhibits entirely unphysical behavior, with the white curves marking transitions into strongly negative regimes. This discrepancy primarily arises from the substantial confinement energy, $(2-D_I)r_G^2/\tilde{T}$, carried by migrating particles; here, we set $r_G = 100$ to bound these divergent values while ensuring the validity of the ground state approximation ($\tilde{T} \ll r_G^2$ and $\tilde{n}_0 \ll r_G^{1+D_I}$). As a distinct

signature of Friedel-like oscillations in a degenerate and confined Fermi gas, $\tilde{c}_0$ displays multiple oscillatory regions of alternating sign and fluctuating amplitudes, a behavior that differs significantly from that observed in MB and Bose gases.

These negative predictions, consistent with the results reported in Refs. [1,8], confirm that $\tilde{c}_0$ is merely an apparent quantity. Its failure stems from: (i) treating energy transferred via quantum migration as boundary heat exchange, and (ii) neglecting the associated heating/cooling effects and local expansion/compression work induced by quantum migration. Consequently, the quantum migration correction defined in Eq. (13c) is essential. Analyzing these corrected formulations for a finite Fermi gas provides critical insights into how quantum degeneracy governs migration processes and dictates the local heat capacity of confined Fermi gases.

The local thermal expansion coefficient per particle $\hat{\gamma}_L = \tilde{\gamma}_L/\tilde{n}$ was previously derived for Bose gases in Ref. [8] in terms of the occupancy variance. Its extension to Fermi statistics, presented below, maintains a consistent and clear physical interpretation:

$$\begin{aligned} \hat{\gamma}_L &= -\frac{\tilde{T}}{\tilde{n}} \left(\frac{\partial \tilde{n}}{\partial \tilde{T}} \right)_L \\ &= \frac{\sum_k \sigma_k |\psi_k|^2}{\sum_k f_k |\psi_k|^2} \left[\frac{\sum_k \tilde{\epsilon}_k \sigma_k}{\sum_k \sigma_k} - \frac{\sum_k \tilde{\epsilon}_k \sigma_k |\psi_k|^2}{\sum_k \sigma_k |\psi_k|^2} \right] \\ &= \hat{\sigma}_I (\hat{U}_\sigma - \hat{u}_\sigma), \end{aligned} \quad (17)$$

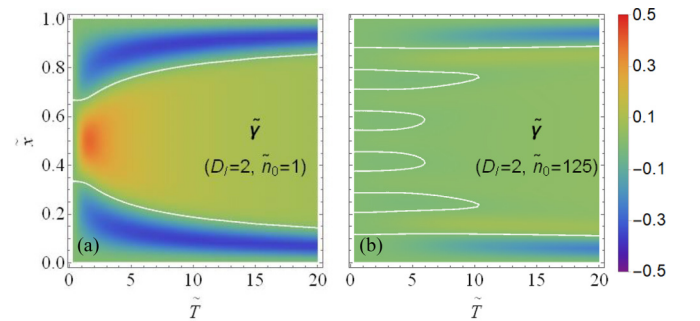


FIG. 5. (a) The thermal expansion coefficient as a function of position and temperature for $D_I = 2$ at a density of $\tilde{n} = 1$, and (b) for $\tilde{n} = 125$. The white curves indicate the zero-crossing contours.

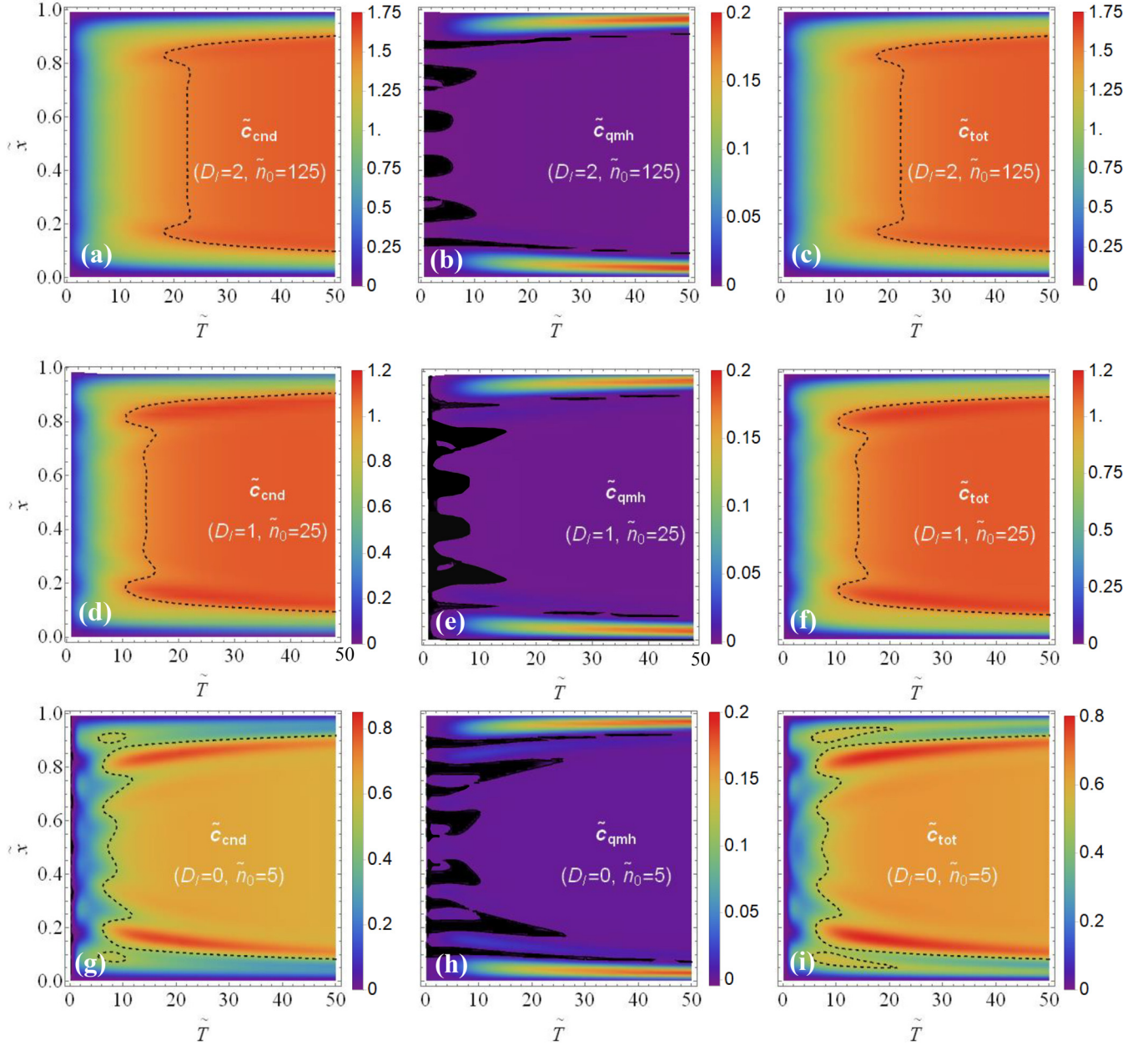


FIG. 6. Position and temperature dependences of the local heat capacity and its constituent components for $\tilde{n}_0 = 5^{1+D_I}$. (a), (d), and (g) display the conductive contribution ($\tilde{c}_{\text{cnd}}$), while (b), (e), and (h) show the quantum migration component ($\tilde{c}_{\text{qmh}}$). The total local heat capacity ($\tilde{c}_{\text{tot}}$) is presented in (c), (f), and (i). Each row corresponds to a distinct confinement domain as illustrated in Fig. 1: $D_I = 2$ (top), $D_I = 1$ (middle), and $D_I = 0$ (bottom). Black regions highlight zones of negligibly small negative values, and dashed black curves indicate the classical heat capacity values, defined as $(1 + D_I)/2$.

where $\sigma_k = \partial f_k / \partial (\mu / k_B T) = f_k(1 - f_k)$ represents the variance of the particle occupancy for a state k . Here, $\hat{\sigma}_l$, $\hat{U}_\sigma$ and $\hat{u}_\sigma$ denote the local occupancy variance per particle, the dimensionless global variance-weighted energy average, and the corresponding local variance-weighted energy average, respectively. For the yellow local regions illustrated in Fig. 1, $\hat{\gamma}_L$ can be expressed explicitly using the LA functions as

$$\hat{\gamma}_L = \frac{{}_0LA_{-1}}{{}_0LA_0} \left[\frac{{}_1A_{-1}}{{}_0A_{-1}} + \frac{D_I}{2} \frac{{}_0A_0}}{{}_0A_{-1}} \right] - \left[\frac{{}_1LA_{-1}}{{}_0LA_0} + \frac{D_I}{2} \right]. \quad (18)$$

In the MB regime ($\tilde{T}_F \ll 1$), quantum degeneracy becomes negligible as the occupancy numbers follow $f_k \ll 1$. In this limit, the variance reduces to the occupancy itself ($\sigma_k \rightarrow f_k$), causing the variance-weighted energy averages to converge with the standard thermal averages ($\hat{U}_\sigma \rightarrow \hat{U}$ and $\hat{u}_\sigma \rightarrow \hat{u}$). Consequently, with the local occupancy variance per particle approaching unity ($\hat{\sigma}_l \rightarrow 1$), Eq. (17) simplifies to $\hat{\gamma}_L \rightarrow \hat{U} - \hat{u}$. This result recovers the classical interpretation established in Ref. [1], where the local thermal expansion is governed purely by the deviation of the global energy per particle from its local value. Note that this simple interpretation does not

hold for the general forms [Eqs. (17) and (18)], as they involve products of local and global averages, involving the terms with and without the $|\psi_k|^2$ weighting, respectively.

Furthermore, in the limit of strong confinement with negligible degeneracy ($\tilde{T}_F \ll \tilde{T} \ll 1$), particles predominantly occupy the ground state; thus, the summations in Eq. (17) collapse, and we obtain $\hat{\gamma} \rightarrow 0$. Conversely, in the highly degenerate limit where degeneracy dominates confinement effects ($\tilde{T} \ll 1 \ll \tilde{T}_F$), the occupancy f_k approaches either 1 or 0 depending on the state energy relative to the Fermi level. In both scenarios, the variance vanishes ($\sigma_k \rightarrow 0$), again resulting in $\hat{\gamma} \rightarrow 0$. These limiting behaviors are clearly reflected in the numerical results presented in Fig. 4.

$\tilde{\gamma}_L$ plays a central role in Eqs. (13); therefore, characterizing its behavior is fundamental to understanding the quantum migration process. Figures 5(a) and 5(b) illustrate the spatial and temperature dependence of $\tilde{\gamma}_L$ for $D_I = 2$ at $\tilde{n} = 1$ and $\tilde{n} = 125$, respectively. The white contours represent the boundaries between regions of positive (expansion) and negative (contraction) local thermal expansion. In the low-density regime, the global maximum occurs at the center while the minima emerge near the boundaries, a behavior consistent with recent findings for the MB gas [1]. At high densities, the overall magnitudes decrease; while the minima remain near the boundaries, the maximum shifts from the center toward the vicinity of the negative-expansion regions.

Dimensionless local ($\hat{\phi}$) and global ($\hat{\Phi}$) grand potentials are also defined and calculated, respectively, as

$$\hat{\phi} = \frac{\varphi}{nk_B T} = \frac{\sum_k \ln(1 - f_k) |\psi_k|^2}{\sum_k f_k |\psi_k|^2} = -\frac{0LA_1}{0LA_0} \leq 0, \quad (19a)$$

$$\hat{\Phi} = \frac{\Phi}{Nk_B T} = \frac{\sum_k \ln(1 - f_k)}{\sum_k f_k} = -\frac{0A_1}{0A_0} \leq 0. \quad (19b)$$

In the low temperature regime (strongly confined), $\tilde{T}_* \ll 1$, $\tilde{c}_{\text{qmh}} \rightarrow 0$, since $\hat{U} \rightarrow \hat{u}$, $\hat{\phi} \rightarrow \hat{\Phi}$, and $\hat{U}_\sigma \rightarrow \hat{u}_\sigma$, leading to $\tilde{\gamma}_L \rightarrow 0$. In the MB regime ($\tilde{T}_F \ll 1$), however, as $f_k \ll 1$ and $\ln(1 - f_k) \rightarrow -f_k$, both $\hat{\phi} \rightarrow -1$ and $\hat{\Phi} \rightarrow -1$. Therefore, $(\hat{\Omega} - \hat{\omega}) \rightarrow (\hat{U} - \hat{u})$ and $\tilde{\gamma}_L \rightarrow (\hat{U} - \hat{u})$, yielding $\tilde{c}_{\text{qmh}} \rightarrow (\hat{U} - \hat{u})^2 \tilde{n} \geq 0$.

Figure 6 displays the local heat capacity calculated using the quantum migration correction [Eq. (13c)], alongside its conductive [Eq. (13a)] and migration [Eq. (13b)] components, for the confinement configurations in Fig. 1. The density for each D_I is selected such that the Fermi temperature remains approximately constant across configurations. While the quantum migration component takes slightly negative values (on the order of -10^{-3} , indicated by the black regions), the total local heat capacity remains strictly non-negative ($\tilde{c}_{\text{tot}} \geq 0$), as shown in Figs. 6(c), 6(f), and 6(i). This ensures the fulfillment of the thermodynamic stability criterion. It is important to emphasize that the localized negativity of the migration component does not imply instability; rather, stability is dictated by the total local heat capacity. Furthermore, since $\tilde{\gamma}_L$ enters both the conductive and migration formulations, these components respond coupled to the migration process. Consequently, this refined framework successfully resolves the negativity problem associated with the uncorrected $\tilde{c}_0$ (see Fig. 4), guaranteeing the positivity of the total heat capacity and the stability of the thermodynamic equilibrium. As

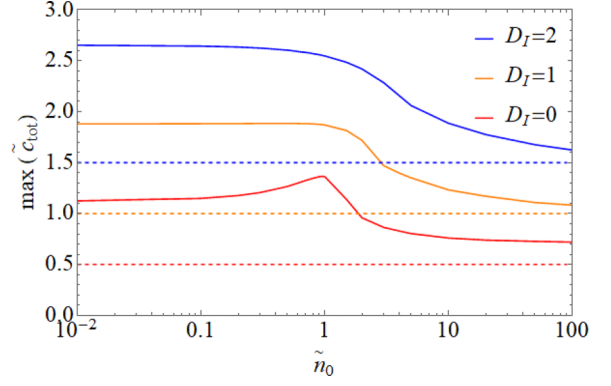


FIG. 7. Variation of the maximum local heat capacity, $\max(\tilde{c}_{\text{tot}})$, with density $\tilde{n}_0$ for the different confinement domains illustrated in Fig. 1. The dashed lines represent the corresponding classical limits $(1 + D_I)/2$ for each dimensionality.

seen in Fig. 6, the extrema for both components emerge near the boundaries. Furthermore, the conductive contribution is sufficiently dominant across the entire domain to offset any negative contributions from the migration term, ensuring that their sum remains positive at every position.

By substituting $\tilde{c}_0 = \tilde{u} - \tilde{\gamma}\hat{u} + \tilde{n}\tilde{T}(\partial\hat{u}/\partial\tilde{T})_L$ and $\hat{\omega} = \hat{u} + \hat{p}$ into Eq. (13a), the conductive component can also be expressed as $\tilde{c}_{\text{cnd}} = \tilde{u} + \tilde{n}\tilde{T}(\partial\hat{u}/\partial\tilde{T})_L + \hat{p}\tilde{\gamma}_L$. While the first two terms are inherently positive, the sign of the third term ($\hat{p}\tilde{\gamma}_L$) depends on the local thermal expansion coefficient $\tilde{\gamma}_L$. However, the magnitude of the positive terms consistently dominates, ensuring that $\tilde{c}_{\text{cnd}}$ remains positive across all configurations, as shown in Figs. 6(a), 6(d), and 6(g).

In contrast, the migration component $\tilde{c}_{\text{qmh}}$ exhibits localized negative values for all D_I configurations [Figs. 6(b), 6(e), and 6(h)]. This negativity arises from a spatial mismatch between the zero-crossing points of $\tilde{\gamma}_L$ and the term $(\hat{\Omega} - \hat{\omega})$ in Eq. (13b). This phase difference leads to small negative values of $\tilde{c}_{\text{qmh}}$, particularly in the low-temperature regime. As discussed following Eq. (19b), in the MB regime (high-temperature or low-density), $\hat{\Omega} - \hat{\omega} \rightarrow \hat{\gamma}$, rendering $\tilde{c}_{\text{qmh}}$ positive-definite and eliminating these negative regions. Conversely, in the limit of low effective temperature, both $(\hat{\Omega} - \hat{\omega})$ and $\tilde{\gamma}_L$ vanish, causing the negative contributions to vanish. These behaviors are clearly reflected in the numerical results shown in Fig. 6.

Our results indicate that the conductive component ($\tilde{c}_{\text{cnd}}$) dominates the central regions of the domain, whereas the quantum migration component ($\tilde{c}_{\text{qmh}}$) becomes more prominent near the boundaries.

A significant enhancement of the total local heat capacity is observed near the boundaries; its density dependence is illustrated in Fig. 7 for various D_I values. The peak value, $\max(\tilde{c}_{\text{tot}})$, increases monotonically with density only for the fully confined case ($D_I = 0$), eventually almost approaching three times the classical value of $1/2$. In contrast, for $D_I = 1$ and $D_I = 2$, $\max(\tilde{c}_{\text{tot}})$ decreases continuously as density rises. For these higher-dimensional cases, the global maxima are reached in the low-density (nondegenerate) limit, where they approach double the corresponding classical values of 1 and $3/2$, respectively.

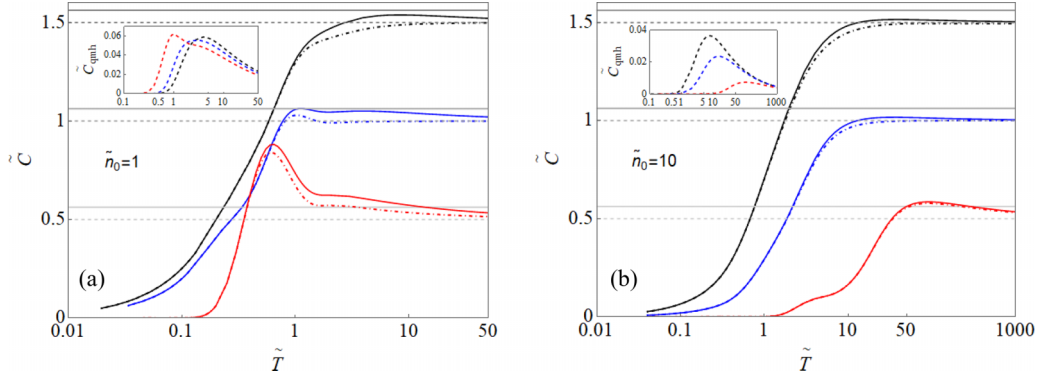


FIG. 8. Temperature dependence of the global heat capacity ($\tilde{C}_{\text{tot}}$) and its constituent conductive ($\tilde{C}_{\text{cnd}}$) and migration ($\tilde{C}_{\text{qmh}}$) components for (a) $\tilde{n}_0 = 1$ and (b) $\tilde{n}_0 = 10$. The red, blue, and black curves correspond to the confinement configurations $D_I = 0, 1$, and 2 , respectively. Solid curves denote the total heat capacity, while dashed-dotted curves represent the conductive contribution. The inset displays the isolated quantum migration contribution, $\tilde{C}_{\text{qmh}}$. Solid and dotted gray horizontal lines indicate the maximum heat capacity of a finite MB gas and its corresponding infinite (classical) limit, respectively.

The global heat capacity ($\tilde{C}_{\text{tot}}$), alongside its conductive ($\tilde{C}_{\text{cnd}}$) and migration ($\tilde{C}_{\text{qmh}}$) components, are obtained by integrating their respective local counterparts over the spatial domain. Figure 8 illustrates the temperature dependence of these global quantities for the configurations defined in Fig. 1 at two distinct densities, $\tilde{n}_0 = 1$ and $\tilde{n}_0 = 10$. The results reveal that quantum migration provides a measurable enhancement in the global heat capacity for the lower-dimensional cases ($D_I = 0$ and $D_I = 1$). In the $D_I = 2$ case, quantum migration acts as the sole driver of the observed excess heat capacity, as the conductive component approaches the classical limit from below, $\tilde{C}_{\text{cnd}} \rightarrow \tilde{C}_{\text{cl}}^{\text{MB}}$. At the higher density, Fig. 8(b), both the conductive and migration contributions to this excess capacity decay, with the total quantity asymptotically approaching the classical limit. Furthermore, the insets in Fig. 8 reveal a dimensionality-dependent shift in the migration peaks: at low density ($\tilde{n}_0 = 1$), the maximum of $\tilde{C}_{\text{qmh}}$ shifts to higher temperatures as dimensionality increases. Conversely, at high density ($\tilde{n}_0 = 10$), this trend is reversed, with the peak occurring at lower temperatures for higher dimensionality.

Another notable feature in Fig. 8(a) is that $\tilde{C}_{\text{cnd}}$ reaches a local maximum at a finite temperature for $D_I = 0$ (red) and $D_I = 1$ (blue). In contrast, for $D_I = 2$ (black), the conductive component lacks a finite-temperature peak and instead approaches the classical limit asymptotically. This suggests that while lower-dimensional confinement forces a maximum in the conductive response at specific temperature, the higher-dimensional configuration ($D_I = 2$) requires infinite temperature to fully recover the classical heat capacity.

Figure 9 provides a comparative analysis of the peak contributions to the global heat capacity as a function of density. Figure 9(a) illustrates the maximum excess conductive contribution, defined as $\max(\tilde{C}_{\text{cnd}}^{\text{exs}}) = \tilde{C}_{\text{cnd}}^{\text{max}} - \tilde{C}_{\text{cl}}^{\text{MB}}$. For the $D_I = 2$ case, this excess is zero because $\tilde{C}_{\text{cnd}}$ approaches the MB limit from below. In contrast, for $D_I = 1$ and $D_I = 0$, $\max(\tilde{C}_{\text{cnd}}^{\text{exs}})$ exhibits a nonmonotonic dependence on density, peaking near $\tilde{n}_0 = 1$. As the system becomes more dense, the excess contribution for the $D_I = 0$ case decays toward zero, while it reaches a stable plateau for the $D_I = 1$ case.

The peak quantum migration contribution, $\tilde{C}_{\text{qmh}}^{\text{max}}$, is shown in Fig. 9(b). This component is most prominent in the low-density, classical limit and undergoes a nearly monotonic decay as density increases across all configurations. Notably, the decay rate is sensitive to the dimensionality of the system: the migration contribution decays most rapidly in the lowest-dimensional ($D_I = 0$) configuration as density takes hold. It suggests that in the case of $D_I = 0$, the “migration” mechanism is quickly suppressed as the states become occupied, whereas in higher dimensions, the available degrees of freedom allow these effects to persist over a broader density range.

It is also instructive to analyze the thermal component of the confinement energy, $\hat{u}_c^{\text{th}}$, defined as

$$\hat{u}_c^{\text{th}} = \hat{u}^{\text{th}} - \hat{u}_{\text{uc}}^{\text{th}}, \quad (20)$$

where $\hat{u}^{\text{th}}$ is the per-particle local thermal energy in the confined system and $\hat{u}_{\text{uc}}^{\text{th}}$ is the corresponding quantity in the absence of confinement along L_S . The thermal energy is extracted by subtracting the zero-temperature limit (the non-thermal contribution induced by the Pauli exclusion principle) from the total dimensionless energy $\hat{u} = u/(nk_B T)$. Thus, $\hat{u}^{\text{th}}$ and $\hat{u}_{\text{uc}}^{\text{th}}$ are determined via:

$$\hat{u}^{\text{th}} = \hat{u} - \frac{1}{\tilde{T}} \lim_{\tilde{T} \rightarrow 0} \tilde{T} \hat{u}, \quad (21)$$

$$\hat{u}_{\text{uc}}^{\text{th}} = \hat{u}_{\text{uc}} - \frac{1}{\tilde{T}} \lim_{\tilde{T} \rightarrow 0} \tilde{T} \hat{u}_{\text{uc}}. \quad (22)$$

The unconfined energy $\hat{u}_{\text{uc}}$ is obtained by applying the integral approximation to the summations in Eqs. (15) and (16):

$$\hat{u}_{\text{uc}} = \frac{1 + D_I}{2} \frac{\text{Li}_{(3+D_I)/2}[-\exp(\Lambda_C)]}{\text{Li}_{(1+D_I)/2}[-\exp(\Lambda_C)]} + \frac{2 - D_I}{\tilde{T}} r_g^2. \quad (23)$$

Under this integral approach, the eigenfunctions transform as $(2/L_S) \sin^2(\pi k_S \tilde{x}) \rightarrow 1/L_S$, necessitating the replacement of the LA functions with their classical counterparts, ${}_x LA_y \rightarrow {}_x B_y$. Consequently, the difference $\hat{u}^{\text{th}} - \hat{u}_{\text{uc}}^{\text{th}}$ represents the local thermal energy purely induced by confinement.

Figure 10 illustrates the spatial distribution of this confinement-induced thermal energy, normalized by the total

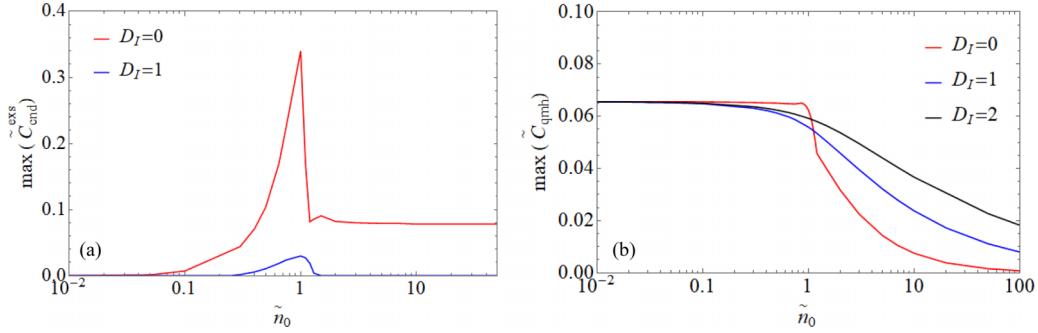


FIG. 9. Density dependences of (a) the maximum excess conductive contribution, $\max(\tilde{C}_{\text{qmh}}^{\text{exc}})$, and (b) the maximum quantum migration contribution, $\tilde{C}_{\text{qmh}}$, for different confinement conditions. The red, blue, and black curves correspond to $D_I = 0, 1$, and 2 , respectively. In (a), the excess is defined relative to the MB limit.

local energy per particle (excluding the ground state energy in the L_G direction), $\hat{u}_{\text{exc}} = \hat{u} - \hat{u}_G$. Note that $\hat{u}_{\text{exc}}$ represents the relevant local energy per particle including confinement effects along L_S but excluding the constant contribution from L_G . Finally, note that $\hat{u}_c^{\text{th}}/\hat{u}_{\text{exc}} = \tilde{u}_c^{\text{th}}/\tilde{u}_{\text{exc}}$, as the normalization factors cancel within the ratio; consequently, the choice between hat ($\hat{u}$) or tilde ($\tilde{u}$) notation does not affect the dimensionless results presented here.

Thermal confinement energy serves as the primary driver for the anomalous behavior observed in the thermal state functions of confined systems. Generally, the thermal components of state functions vanish as the temperature approaches absolute zero ($T \rightarrow 0$). In contrast, nonthermal components, such as zero-point energy ($N\epsilon_0$) and zero-point pressure, retain finite values even in the ground state. As illustrated in Fig. 10, the thermal confinement energy significantly accumulates near the system boundaries, a phenomenon driven by quantum migration. In this context, quantum migration acts as an energy-filtering transport process that redistributes higher-energy particles from the central regions toward the boundaries as temperature increases. This accumulation of thermal confinement energy becomes increasingly pronounced as the dimensionality D_I is reduced.

Furthermore, the spatial profiles exhibit periodic positive and negative regions. The central regions of the domain

accumulate relatively lower positive and even negative thermal confinement energy compared to the boundary regions. In Figs. 10(b) and 10(c), the central region is almost entirely negative, except for the boundary regions and the areas within the white curves at low temperatures. On the other hand, Fig. 10(a) shows that multiple negative and positive regions appear for the case of $D_I = 0$ and a completely negative central region can only emerge at much higher temperatures. For a given density, the number of positive and negative regions is characterized by $1 + (\tilde{n}_0)^{1/(1+D_I)}$ and $(\tilde{n}_0)^{1/(1+D_I)}$, respectively. These spatial oscillations mimic Friedel oscillations, reflecting the underlying quantum interference. The emergence of high-contribution regions near the boundaries is consistent with the patterns observed in Figs. 6(b), 6(e), and 6(h). Collectively, these results suggest that the buildup of confinement-induced thermal energy is mediated by quantum migration, which disrupts the local energy homogeneity of the system and leads to the observed localized energy accumulation.

IV. DISCUSSIONS AND CONCLUSION

The analyses in Figs. 2 and 3 demonstrate that for $D_I = 0$ and $D_I = 1$, the global heat capacity of a noninteracting Fermi gas in a finite domain exceeds that of a MB gas. Interestingly,

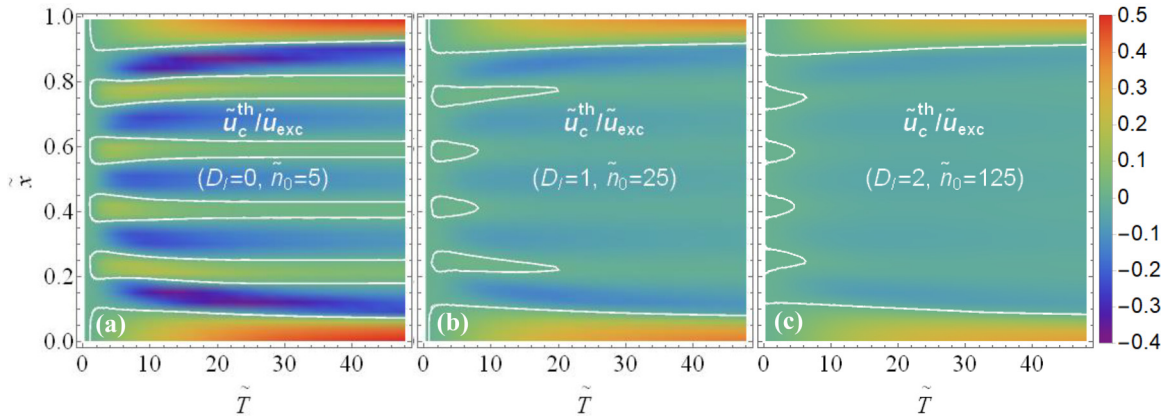


FIG. 10. Spatial distribution of the confinement-induced thermal energy ratio, $\tilde{u}_c^{\text{th}}/\tilde{u}_{\text{exc}}$, for (a) $D_I = 0, \tilde{n}_0 = 5$, (b) $D_I = 1, \tilde{n}_0 = 25$, and (c) $D_I = 2, \tilde{n}_0 = 125$. The color bar indicates the local magnitude of the thermal energy, highlighting the accumulation near the boundaries and the oscillatory behavior in the central domain. The white curves indicate the zero-crossing points. Density values are chosen such that $(\tilde{n}_0)^{1/(1+D_I)}$ remains constant across all D_I values.

this trend reverses for $D_I = 2$, where $\tilde{C}_{\max} \leq \tilde{C}_{\max}^{\text{MB}}$. This global behavior is completely opposed to that observed in Bose gases [8]. A similar contrast appears in the behavior of $\tilde{T}_{\max}$: while it initially decreases and then increases with rising density for $D_I = 0$ and $D_I = 1$, it increases monotonically for $D_I = 2$.

A distinct behavioral difference also emerges in the local heat capacity $\tilde{c}_0$. Unlike the confined Bose gas, where $\tilde{c}_0$ is negative at the center and positive near boundaries, the degenerate finite Fermi gas exhibits a pronounced oscillatory nature (Fig. 4). These oscillations stem from Friedel-like density fluctuations caused by the breaking of translational symmetry at the boundaries. These density oscillations propagate into its temperature derivative $\tilde{\gamma}$, subsequently inducing oscillatory behavior in $\tilde{c}_0$, $\tilde{c}_{\text{qmh}}$, $\tilde{c}_{\text{cnd}}$, and $\tilde{u}_c^{\text{th}}$.

The influence of quantum degeneracy further distinguishes the two statistics. While increased degeneracy enhances the local heat capacity of a Bose gas for $D_I = 2$, Fig. 7 shows a different trend for Fermi gases: $\tilde{c}_{\text{tot}}$ decreases monotonically for $D_I = 2$. In contrast, for $D_I = 1$ and $D_I = 0$, $\tilde{c}_{\text{tot}}$ exhibits an initial peak around $\tilde{n}_0 \approx 1$ (which is nearly negligible for the $D_I = 1$ case) before ultimately decaying toward the classical limit.

This peak behavior in Figs. 3 and 7 can be understood through the dimensionality of the density of states. For $D_I = 0$, as density increases from near-zero toward $\tilde{n}_0 = 1$, the population of low-lying states near the ground state begins. Note that since the number of particles for $D_I = 0$ is $N = 2\tilde{n}_0$, the system contains a nonzero, integer number of particles only for $\tilde{n}_0 \geq 1/2$. In this regime, the sharp peak occurs as the system transitions from a sparse state to one where the first quantum energy gap strongly influences the heat capacity. For $D_I = 1$, however, the extra dimensions provide a smoother density of states compared to the sharp one for the $D_I = 0$ case. This effectively washes out the low-density peak, leaving only a tiny signature. For $D_I = 2$, the inclusion of further dimensions provides even more available states, completely eliminating the peak behavior and leading to a monotonic decay. This explains the behavior observed in Figs. 3 and 7.

Furthermore, as shown in Fig. 8, the migration contribution is the sole mechanism inducing global overhear capacity for $D_I = 2$, whereas it merely supplements the existing conductive overhear capacity for $D_I = 1$ and $D_I = 0$.

We find that very high densities ($\tilde{n}_0 > 100$) force the local heat capacity toward the classical limit, masking the excess local signatures (Fig. 7). Thus, unlike Bose gases, strong degeneracy is unfavorable for observing excess local capacity. Nevertheless, at $\tilde{n}_0 = 50$ and $D_I = 0$, the excess global heat capacity reaches its maximum ($\tilde{C}_{\max} = 0.579$) at $\tilde{T}_{\max} = 2175$. At this condition, the local heat capacity reaches $\tilde{c} = 0.710$ at $\tilde{x} = 0.015$, representing a 22% enhancement over the global value, a definitive signature of quantum migration.

For an experimental verification of this local enhancement, we propose a bismuth quantum wire with a cross-section of $12 \text{ nm} \times 12 \text{ nm}$ and a length of 5600 nm . This geometry allows for the observation of TIQM effects along the longitudinal axis, where the redistribution of electronic local heat capacity is prominent. bismuth's low effective electron mass ($m^* \approx 0.01m_e$) allows quantum effects to manifest in relatively large domains. We identify $T = 30 \text{ K}$ as the

optimal regime; here, the lattice heat capacity is significantly suppressed (as $T \ll \Theta_D \approx 119 \text{ K}$), and the sizes facilitate high crystalline integrity. In this configuration, the 22% enhancement occurs approximately 84 nm from the boundary, providing an accessible target for scanning thermal microscopy.

To realize $\tilde{n}_0 = 50$ ($N = 100$), we propose a field-effect geometry using an electrostatic back gate on a dielectric substrate. The occupancy can be precisely “locked” by counting exactly 100 conductance peaks via Coulomb Blockade spectroscopy [47]. This ensures that electronic heat capacity measurements are performed at the density yielding the maximum spatial contrast between the global ($\tilde{C}_{\text{el}} = 0.58$) and the local ($\tilde{c}_{\text{el}} = 0.71$). Given bismuth's low carrier density, the gate-tunability of the electronic local heat capacity provides a clear experimental path to distinguish this quantum migration effect from the residual phonon background.

This specific configuration, $D_I = 0$ and $\tilde{n}_0 = 50$, serves as a representative example of a strong local deviation; however, the model allows for the investigation of diverse confinement geometries and densities, each potentially yielding distinct experimental signatures of quantum migration.

While our model assumes a noninteracting Fermi gas within impenetrable boundaries, this approximation is well-justified by Landau-Fermi Liquid Theory and remains robust under realistic experimental conditions. In bismuth, long-range Coulomb interactions are heavily screened, and many-body effects are effectively captured by the use of the crystalline effective mass ($m^* \approx 0.01m_e$). These electron-electron interactions may provide additional “stiffness” to the gas, potentially enhancing the suppression of the global heat capacity. Furthermore, the 12 nm cross section ensures “quasi-1D” behavior where Fermi Liquid theory remains valid, even as the system approaches the quantum limit.

Under realistic conditions, the presence of finite potential barriers and image-charge effects at the nanowire interface tends to relax the strict spatial constraints on the wave functions. This “softening” allows wave penetration into the barrier, effectively increasing the local density of states and extending the region of high local heat capacity into the transitional tail. Consequently, these boundary effects may cause a slight spatial shift or an enhancement of the local heat capacity peaks ($\tilde{c}$) beyond the values calculated in the idealized infinite-well limit. Nonetheless, as long as the thermal energy remains comparable to the subband spacing, the fundamental redistribution of local heat capacity toward the boundaries, the hallmark of quantum migration, persists as a dominant and measurable feature.

In this study, which concludes our trilogy on the subject, we have incorporated the TIQM correction to rigorously define and analyze the local heat capacity of a noninteracting finite Fermi gas. This framework allowed us to examine the contribution of quantum migration to both local and global heat capacity, revealing how quantum degeneracy dictates the TIQM response. Specifically, we demonstrated that while the apparent local heat capacity distribution for a Fermi gas exhibits multiple unphysical negative regions, the TIQM correction eliminates these anomalies, ensuring positive and physically consistent results throughout the entire domain.

Our results indicate that for $D_I = 2$, the TIQM correction is the sole source of the excess global heat capacity, though quantum degeneracy generally acts to suppress this contribution. The observed accumulation of thermal confinement energy near system boundaries is fundamentally a result of the energy transfer mechanism driven by quantum migration. Ultimately, these findings establish that TIQM plays a vital role in determining all temperature-dependent local quantities in quantum-confined systems, providing a necessary correction for the thermodynamics of nanoscale devices.

DATA AVAILABILITY

The data that support the findings of this article are openly available [48], embargo periods may apply.

APPENDIX: GENERALIZED DERIVATION OF GLOBAL HEAT CAPACITY OF A FERMI GAS CONFINED IN THE DOMAINS OF FIG. 1

Based on the dimensionless energy per particle, $\tilde{\epsilon}_i = \epsilon_i/k_B T = (\alpha_i k_i)^2$, and the defined confinement parameter

relations $\alpha_S^2 = 1/\tilde{T}$, $\alpha_I = \alpha_S/r_I$, $\alpha_G = \alpha_S r_G$, the following equation can be written to find the chemical potential Λ :

$$N = \sum_{k_S=1}^{\infty} \int_0^{\infty} \frac{dk_{I1} \dots dk_{ID_I}}{\exp \left[-\Lambda + \frac{k_S^2}{\tilde{T}} + \frac{(2-D_I)r_G^2}{\tilde{T}} + \frac{(k_{I1}^2 + \dots + k_{ID_I}^2)}{r_I^2 \tilde{T}} \right] + 1}. \quad (\text{A1})$$

Notice that integral and ground state approaches are used to calculate the contributions of summations over k_{Ii} and k_{Gi} , respectively. As the total dimension $D = 3$ and there is only one α_S , while the numbers of α_I and α_G are D_I and $2 - D_I$. By considering the change of variable $u_i = k_{Ii}/(r_I \sqrt{\tilde{T}})$ and transforming the cartesian coordinates to spherical ones for $dk_{I1} \dots dk_{ID_I}$ as $dk_{I1} \dots dk_{ID_I} = (2\pi^{D_I/2} s^{D_I-1} ds)/(2^{D_I} \Gamma(D_I/2))$, Eq. (A1) is rewritten as

$$N = \left(\frac{r_I \sqrt{\tilde{T}}}{2} \right)^{D_I} \frac{2\pi^{D_I/2}}{\Gamma(D_I/2)} \sum_{k_S=1}^{\infty} \int_0^{\infty} \frac{s^{D_I-1} ds}{\exp \left[-\Lambda + \frac{k_S^2}{\tilde{T}} + \frac{(2-D_I)r_G^2}{\tilde{T}} + s^2 \right] + 1}. \quad (\text{A2})$$

Calculating the integral in Eq. (A2), calling the equality $N = (\sqrt{\pi} r_I/2)^{D_I} \tilde{n}_0$ and using the modified dimensionless chemical potential $\Lambda_C = \Lambda - (2 - D_I)r_G^2/\tilde{T}$, the following normalization equation is obtained for chemical potential, which is the one given in Eq. (2):

$$\tilde{n}_0 = -\tilde{T}^{D_I/2} \sum_{k_S=1}^{\infty} \text{Li}_{D_I/2} \left[-\exp \left(\Lambda_C - \frac{k_S^2}{\tilde{T}} \right) \right]. \quad (\text{A3})$$

Using the same variables, the total energy is expressed as

$$U = k_B T \sum_{k_S=1}^{\infty} \int_0^{\infty} \frac{\left[\frac{k_S^2}{\tilde{T}} + \frac{(2-D_I)r_G^2}{\tilde{T}} + \frac{(k_{I1}^2 + \dots + k_{ID_I}^2)}{r_I^2 \tilde{T}} \right] dk_{I1} \dots dk_{ID_I}}{\exp \left[-\Lambda + \frac{k_S^2}{\tilde{T}} + \frac{(2-D_I)r_G^2}{\tilde{T}} + \frac{(k_{I1}^2 + \dots + k_{ID_I}^2)}{r_I^2 \tilde{T}} \right] + 1}. \quad (\text{A4})$$

By following the same change of variables and transformation of integral over cartesian coordinates into spherical one, Eq. (A4) becomes

$$U = -k_B T \left(\frac{\sqrt{\pi} r_I}{2} \right)^{D_I} \tilde{T}^{D_I/2} \left[\sum_{k_S=1}^{\infty} \frac{k_S^2}{\tilde{T}} \text{Li}_{D_I/2} \left[-\exp \left(\Lambda_C - \frac{k_S^2}{\tilde{T}} \right) \right] - (2 - D_I) \frac{r_G^2}{\tilde{T}} \sum_{k_S=1}^{\infty} \text{Li}_{D_I/2} \left[-\exp \left(\Lambda_C - \frac{k_S^2}{\tilde{T}} \right) \right] - \frac{D_I}{2} \sum_{k_S=1}^{\infty} \text{Li}_{(2+D_I)/2} \left[-\exp \left(\Lambda_C - \frac{k_S^2}{\tilde{T}} \right) \right] \right]. \quad (\text{A5})$$

Each term, from the first to last, represents contribution from confined (S), strongly confined (G) and unconfined (I) directions, respectively. Considering $N = (\sqrt{\pi} r_I/2)^{D_I} \tilde{n}_0$ and Eq. (A3), the dimensionless energy per particle can be expressed in terms of A functions defined in Eq. (9) by

$$\hat{U} = \frac{U}{N k_B T} = \frac{1A_0}{0A_0} + \frac{(2 - D_I)r_G^2}{\tilde{T}} + \frac{D_I}{2} \frac{0A_1}{0A_0}. \quad (\text{A6})$$

In terms of $\hat{U}(\tilde{T}, \Lambda_C)$, the dimensionless global heat capacity per particle is expressed as

$$\hat{C}_L = \frac{C_L}{N k_B} = \left(\frac{\partial \hat{U}}{\partial \tilde{T}} \right)_{L, \Lambda_C} = \hat{U} + \tilde{T} \left(\frac{\partial \hat{U}}{\partial \tilde{T}} \right)_{L, \Lambda_C} + \tilde{T} \left(\frac{\partial \hat{U}}{\partial \Lambda_C} \right)_{L, \tilde{T}} \left(\frac{\partial \Lambda_C}{\partial \tilde{T}} \right)_L. \quad (\text{A7})$$

Due to $N = (\sqrt{\pi}r_l/2)^{D_l}\tilde{n}_0$, a constant number of particles N requires

$$\left(\frac{d\tilde{n}_0}{d\tilde{T}}\right)_L = \left(\frac{\partial\tilde{n}_0}{\partial\tilde{T}}\right)_{L,\Lambda_C} + \left(\frac{\partial\tilde{n}_0}{\partial\Lambda_C}\right)_{L,\tilde{T}} \left(\frac{\partial\Lambda_C}{\partial\tilde{T}}\right)_L = 0, \quad (\text{A8})$$

which gives the following equation in terms of A functions,

$$\left(\frac{\partial\Lambda_C}{\partial\tilde{T}}\right)_L = -\frac{\left(\frac{\partial\tilde{n}_0}{\partial\tilde{T}}\right)_{L,\Lambda_C}}{\left(\frac{\partial\tilde{n}_0}{\partial\Lambda_C}\right)_{L,\tilde{T}}} = -\frac{1}{\tilde{T}} \left[\frac{D_l}{2} \frac{{}_0A_0}{{}_0A_{-1}} + \frac{{}_1A_{-1}}{{}_0A_{-1}} \right]. \quad (\text{A9})$$

For the temperature and chemical potential derivatives of Eq. (A3) in Eq. (A9), the subsequent derivative equalities of A functions are used,

$$\frac{\partial({}_xA_y)}{\partial\tilde{T}} = \frac{1}{\tilde{T}} ({}_{x+1}A_{y-1} - {}_xA_y), \quad (\text{A10})$$

$$\frac{\partial({}_xA_y)}{\partial\Lambda_C} = {}_xA_{y-1}. \quad (\text{A11})$$

Employment of Eqs. (A6), (A9), (A10), and (A11) in Eq. (A7) leads to the global heat capacity given by Eq. (8).

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