

Wall-enhanced dendritic growth: An exact Wiener-Hopf-Hankel solution for oblique reflection

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A solidifying dendrite advancing toward an oblique impermeable wall senses the wall through reflection of its own diffusive depletion field. The strength of this reflection depends on both wall obliquity and the unsteady kinematics of the closing gap, and cannot be captured by any quasistatic frozen offset construction. We solve the moving boundary diffusion problem exactly in the laboratory frame (wall fixed, dendrite advancing at constant speed) by recasting it as a Wiener-Hopf-Hankel system on a pair of spectral branches. The resulting upper triangular operator is inverted sequentially: first as a Fredholm integral equation on the lower branch, then as a scalar Wiener-Hopf problem on the upper branch, reaching a closed form solution via Cagniard-de Hoop. A Padé-Abrahams rational reduction of the spectral radical reduces the entire scheme to a low cost algebraic system at each Laplace node. The exact unsteady solution inverts the classical starvation picture. Measured against an isolated dendrite at the same elapsed time, the wall correction is weakly suppressive while the tip is more than about two diffusion lengths from the wall, where the reflected depletion returns with little retardation and acts as ordinary blockage, but the suppression never exceeds one percent. As the gap closes the correction changes sign and the feeding rises to 31 percent above the isolated reference at grazing incidence, because the reflected field carries a memory of wider past gaps and delivers more solute than the instantaneous geometry would allow. The enhancement is governed by the oblique Péclet number $Pe_\theta = vd_0 \sin \theta / D$ and is strongest at grazing incidence, where it exceeds the preceding suppression by a factor of 30.

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I. INTRODUCTION

Dendritic solidification governs the as-cast microstructure of essentially every metallic alloy of industrial interest, from aluminum-copper castings and nickel-base single-crystal turbine blades to the rapidly solidified microstructures of laser powder-bed fusion. The morphology, the spacing, and the orientation of the primary dendritic branches set the local segregation pattern, the residual stress distribution, and ultimately the mechanical performance of the solidified part [1]. Yet most of the analytical theory through which solidification has been understood concerns a single dendrite advancing into an unbounded undercooled melt. The conditions encountered in actual castings, where dendrites grow in close proximity to mould walls, to one another, and to the boundaries of melt pools whose extent is often comparable with the diffusion length itself, lie outside the scope of that theory. This work concerns one specific instance of this gap: the exact unsteady reflection of a single oblique dendrite by a flat impermeable wall, treated in the laboratory frame and without recourse to the quasistatic approximation that is otherwise universal in the analytical literature.

The classical framework for the free dendrite begins with the parabolic similarity Ivantsov solution of the diffusion

equation in a comoving frame [2]. This solution provides a one-parameter family of needle crystals indexed by the Péclet number but does not by itself fix either the tip velocity or the tip radius. The morphological instability of a planar interface advancing under diffusion controlled growth, identified shortly thereafter [3,4], supplied the physical mechanism by which a planar front breaks up into cells and dendrites and motivated the search for a selection principle. The original marginal stability hypothesis [5] was eventually superseded by the microscopic solvability theory [6,7], in which a discrete spectrum of admissible velocities is selected by exponentially small terms generated by anisotropic surface tension. The mature picture, summarized in Ref. [8] and more recently, [1], treats the dendrite as the matched composition of an outer Ivantsov-like envelope and an inner process zone in which capillarity acts. Within this picture, the dendrite is most naturally idealized as a half line [9]: an infinitely thin, semi-infinite needle whose tip carries a singular flux, and whose interior structure is encoded in a small number of matching variables inherited from the inner zone solution. The half-line idealization is the formal limit in which the tip radius is sent to zero relative to the diffusion length whilst the product $\rho^2 v$ is held fixed by solvability. It preserves the solute or thermal plume of the dendrite while collapsing its geometry to that of a moving line source, and it is the abstraction underlying both the classical solvability constructions and the modern dendritic needle network approach to interacting dendritic arrays [9,10].

The interaction of a dendrite with a nearby wall has been treated almost exclusively under the assumption that the wall

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offset is fixed in the comoving frame, an assumption we shall refer to as the frozen offset approximation. Under this approximation, the diffusive plume relaxes to a steady form in which the wall acts as a stationary mirror, and the modification of the inner driving force can be read off from a quasistatic image construction. The physical content of the frozen offset picture is that the impermeable wall, by blocking diffusive escape on one side, depresses the available undercooling and slows the tip. The defining quantitative result in this direction is the $v \propto \Delta T^5$ velocity scaling for a needle crystal growing in steady state along an adiabatic wall [11], which has shaped the intuition of the field ever since: walls are understood as starvers of the tip. The $v \propto \Delta T^5$ scaling holds for a needle in steady state along a wall at fixed comoving distance, so the approach itself is not represented. Still, the same intuition is reinforced by the channel solvability analysis, in which a free dendrite under a steady front is gradually confined between two parallel adiabatic boundaries until its tip selection condition is replaced by a array selection condition controlled by the channel half-width [6,12]. The morphology diagram of two-dimensional diffusional growth assembled from these and related steady-state analyses partitions the parameter plane into dendritic, doublon, seaweed and fractal regions [13], and provides the standard reference for what is currently known about dendrite-like growth in confined geometries. Current array spacing theories, such as the Warren–Langer model [14] and the Hunt–Lu model and its descendants [12], inherit the same symmetric free space assumptions, and the interaction between a dendrite and a confining boundary is either neglected entirely or treated through a frozen offset approximation within them. These array spacing models return a criterion at negligible cost, but approximate the diffusion field instead of solving the transport equation with dendrite and wall imposed as internal boundaries. Slender-body theory [15] poses the correct outer problem and admits several dendrites, though it is solved numerically and is not matched to the capillary tip.

This entire body of analytical work is steady state. The price paid for analytical tractability is that the wall is held at a fixed distance from the tip in the comoving frame, which is to say that the time-dependent approach of a free dendrite toward a fixed laboratory boundary is not represented at all. Time-dependent dendrite-wall interactions enter the modern literature only through phase field and dendritic needle network simulations of equiaxed and columnar arrays [9,10], and through the corresponding *in situ* synchrotron radiography experiments on aluminium-copper and related alloys, which have directly resolved the slowing and arrest of dendrite tips as their solute boundary layers begin to overlap [16]. The dendrite network model supplies the flux intensity factor for arrays but computes the outer field numerically, so the tip-wall interaction is never available in closed form. Phase field simulations [17–19] can handle complex morphologies and do not require the kinematic simplifications adopted here, but they are computationally expensive, suffer from grid anisotropy at the singular tip scale unless adaptive refinement is used, and yield numerical output from which scaling laws must be extracted empirically rather than derived, yielding no analytical transport law for the interaction. Both the simulations and the experiments confirm the depression picture as a description

of the late stage in which impingement is essentially complete. Neither addresses the question of whether the picture is correct during the approach itself, when the wall offset is decreasing on a timescale comparable with that of the diffusive relaxation. The transient regime is also the regime that is technologically most relevant: the boundary layer near the mould wall in directional casting of single-crystal superalloys, where dendrite arm spacing and orientation are anisotropically perturbed within roughly the first 0.2 mm of the cast [20], is precisely a region in which the time-dependence of the wall offset cannot be neglected. Indeed, the normal separation between the dendrite tip and the wall enters the transport history, and replacing it by its instantaneous value discards the retardation carried by the field reflected by the wall. An exact treatment of a dendrite approaching a fixed wall at arbitrary obliquity must therefore retain the moving (decreasing) gap, the semi-infinite dendrite and the impermeable wall together, and still deliver a quantity that can be matched to the inner tip selection problem.

The aim of the present work is to remove the frozen offset approximation entirely, and to obtain an exact analytical solution for the unsteady diffusion field generated by an oblique semi-infinite dendrite advancing in the laboratory frame past a flat impermeable wall. The geometry is that of a wedge, with the dendrite occupying a half line inclined at angle θ to the wall and translating uniformly along its own axis at the dendrite velocity v . The advection-diffusion equation in the half plane bounded by the wall, with a moving line source on the inclined ray and a no-flux condition on the wall, reduces under Laplace transformation to a Wiener-Hopf functional equation in the wedge geometry. The technique used here to factorize this equation has been developed in the context of elastodynamic interface problems [21–23], where wedge and half-plane Wiener-Hopf factorizations of similar analytical structure arise in the treatment of transient frictional contact and bimaterial cracks, with the difference that reflections at the wall will lead to a Hankel structure that prevents the factorization of the matricial Wiener-Hopf problem that results. Despite this, here we will show how the resulting Wiener-Hopf-Hankel problem can be decomposed into a Frenholm equation and a scalar Wiener-Hopf problem, which is solved in closed form; its noncommutative matrix counterpart is treated by the Padé approximant procedure introduced by Abrahams [24,25], which yields a rapidly convergent rational reduction of the kernel that can be inverted analytically.

To the best of our knowledge, no Wiener-Hopf treatment of any solidification problem has previously appeared in the literature; the closest analog is the use of the technique in the wave scattering and crack mechanics literature surveyed in Refs. [26,27]. The analytical solution provides three things that a phase field simulation of the same geometry does not: infinite resolution at the tip, where the $1/\sqrt{s}$ singularity of the line density is captured exactly rather than smeared over a diffuse interface width; closed form scaling laws, such as the oblique Péclet number $Pe_\theta = vd_0 \sin \theta / D$ that controls the departure from the frozen offset prediction, which emerge from the spectral structure of the operator; and an exact benchmark against which any approximate or numerical treatment of the same geometry can be validated without ambiguity.

The Wiener-Hopf treatment of dendrite-wall interactions is not obvious. To tackle it, we have adopted a wedge Wiener-Hopf-Hankel transform approach (e.g. Refs. [28,29]), which reduces the moving gap problem exactly in the laboratory frame, exposing a rank one reflected kernel and an upper triangular branch matrix that inverts sequentially. This enables us to reconstruct the wall-corrected line density and concentration field without freezing the offset; we validate the approach against an independent boundary integral calculation. The outer solution we thus achieve in this work is then converted into a process zone feeding functional that can be supplied to an inner mobility or solvability law. This enables us to study the physics of wall-dendrite interaction with analytic detail, showing that the wall correction changes sign along the approach: a weak suppression relative to the isolated dendrite while the wall is distant, overcome as the gap closes and followed by a substantial enhancement, strongest near grazing incidence, which no frozen offset treatment reproduces. The calculation concerns transport along a prescribed constant velocity trajectory, so the feeding functional is an outer input to a coupled tip selection calculation rather than a substitute for one.

The analytical solution that emerges inverts the frozen offset starvation picture in a precise sense. The process-zone matching variable, which measures the inner driving force seen by the tip, is depressed by at most one percent while the wall is distant, in accordance with the quasistatic expectation, and is then enhanced by up to 30 percent as the dendrite closes on the wall. Its time evolution is correspondingly nonmonotonic: the natural decay of the plume is overtaken in the late stage of the approach by the wall feedback, so that the inner driving force first decreases and then recovers. A critical wall offset is identified, in the range one to two diffusion lengths and depending on the angle of incidence, beyond which the wall feedback is insufficient to push the tip above its inner threshold; below this critical offset the tip is enhanced and continues to advance. In the isotropic limit the solution admits a geometric continuation that describes the post-contact behavior of the tip, which turns to follow the wall tangent at an angle equal to the original angle of incidence. None of these features is visible to a quasistatic treatment, and each is in direct contrast with the long-standing depression intuition built up from the steady-state analyses summarized above. The single tip plus wall problem solved here is not the final answer for a dense array, but it is the mathematically necessary foundational building block: it establishes the baseline interaction kernel between one tip and one wall, with the full unsteady history resolved exactly, and it is this kernel that must be integrated into any future multitip model that claims to treat wall proximity correctly. The present work and phase field modeling are therefore complementary rather than competing: the analytical solution provides the scaling structure and the benchmark, and the phase field provides the morphological complexity and the nonlinear kinematic feedback that the analytical model does not attempt.

The remainder of the paper is organized as follows. Section II states the canonical reflection problem in the laboratory frame and fixes notation. Section III reduces the problem to a moving endpoint boundary integral equation via the half-space Neumann kernel. Section IV derives

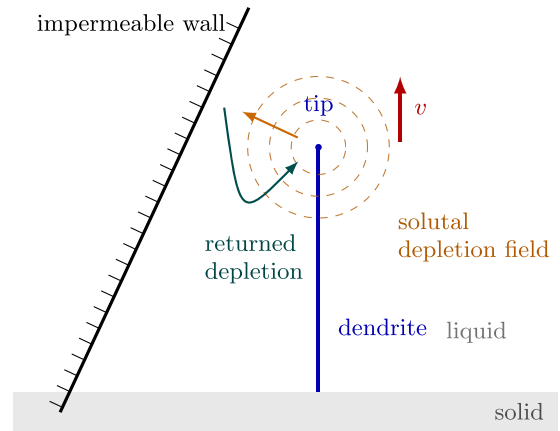


FIG. 1. Physical setting. A dendrite advances at v toward an impermeable wall set obliquely to the growth direction, so the gap closes as it grows. The wall returns the tip's own depletion field after a diffusive delay, by which time the gap has narrowed. Figure 2 defines the variables.

the Wiener-Hopf-Hankel structure of the integral equation and its rank one reflected kernel. Section V introduces the wedge transform and assembles the upper triangular branch matrix problem. Section VI carries out the exact sequential inversion of the branch matrix by Fredholm and scalar Wiener-Hopf methods. Appendix B introduces the Padé-Abrahams rational reduction and quantifies its convergence. Section VIII constructs the process zone functional, identifies the oblique Péclet number, and derives the critical wall offset. Section VII presents the field-level results, the wall-corrected line density and concentration field, and the postcontact tip turning. Section IX draws the implications for the tip drive, the inversion of the frozen offset picture, and the critical wall offset. Finally, Sec. X summarizes the main findings and identifies the natural extensions of this work.

II. CANONICAL REFLECTION PROBLEM

We consider a dendrite advancing at prescribed speed v into an undisturbed supersaturated liquid, with a straight impermeable wall at small oblique distance acting as a specular reflector for the diffusive depletion field, as depicted in Fig. 1. This is the canonical local model for directional solidification near a container boundary, a grain boundary, or a neighboring precipitate, and the question of interest is how the wall modifies the solute feeding available to the tip.

We work in the laboratory frame throughout. Riding with the tip converts the problem into a drift diffusion equation with a gauge transformed Robin wall condition whose reflection coefficient depends on θ , and turns the wall distance into a parameter that must be specified by hand. Freezing it to any single value solves a different boundary value problem from the one posed by a genuine fixed obstacle, and the two differ at leading order for oblique walls. The laboratory formulation avoids all of this: the wall stays fixed, the Neumann condition stays Neumann, and the time dependence of the tip to wall offset is a kinematic consequence rather than an input.

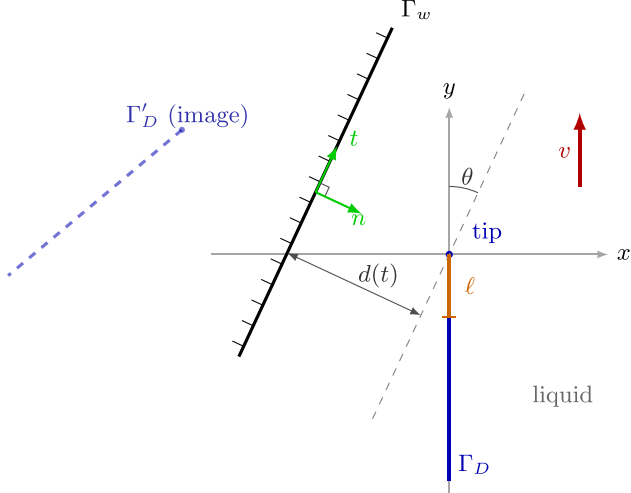


FIG. 2. Geometry of the laboratory wall reflection problem. The wall Γ_w (hatched) is fixed and the tip advances at speed v along $+Y$; the instantaneous offset is $d(t) = d_0 - vt \sin \theta$, built into the analysis through $u = s - vt$. The orange segment is the process zone of length ℓ on which Ψ_{ex} is evaluated (see Sec. VIII), and the dashed blue line Γ'_D is the image dendrite summed by the Neumann term in Eq. (16). At $\theta = 0$ the offset is constant; for $\theta \neq 0$ the wall approaches the tip kinematically.

In that frame, depicted in Fig. 2, the concentration $c(X, Y, t)$ satisfies the pure diffusion equation

$$\partial_t c = D(c_{XX} + c_{YY}), \quad (1)$$

with $D > 0$ the isotropic solute diffusivity. No drift term appears: the liquid is at rest in the laboratory, and the role ordinarily played by advection is played here by the kinematic motion of the absorbing boundary. The initial state is quiescent,

$$c(X, Y, 0) = 0, \quad (2)$$

so every concentration variation is generated either by the absorbing dendrite or by the reflection of its depletion field from the wall.

The wall (see Fig. 2) is the straight line at perpendicular distance $d_0 > 0$ from the initial tip position, inclined at angle $\theta \in [0, \pi/2)$ to the positive Y axis. We orient it by the orthonormal frame

$$t_w = (\sin \theta, \cos \theta), \quad n = (\cos \theta, -\sin \theta), \quad (3)$$

with t_w the unit tangent in the direction of increasing Y and n the outward normal into the liquid. The wall is the line

$$\Gamma_w = \{(X, Y) : d_0 + X \cos \theta - Y \sin \theta = 0\}, \quad (4)$$

the liquid occupies the open half plane

$$d_0 + X \cos \theta - Y \sin \theta > 0, \quad (5)$$

and impermeability is the homogeneous Neumann condition

$$\partial_n c = 0 \quad \text{on } \Gamma_w. \quad (6)$$

Because the medium is at rest, Eq. (6) is a genuine no flux condition and the reflection coefficient at the wall is

identically unity. The image construction underlying the half-space Green function works without modification, and no θ -dependent Robin factor appears.

The dendrite is an absorbing half line whose tip advances along the positive Y axis at constant speed $v > 0$. At time t its dendrite is

$$\Gamma_D(t) = \{(0, Y) : Y < vt\}, \quad (7)$$

parametrized by the arc length s behind the tip via

$$R(s, t) = (0, vt - s), \quad s > 0, \quad (8)$$

and carrying the local equilibrium Dirichlet element

$$c(0, vt - s, t) = c_{\text{LC}}, \quad s > 0, t > 0, \quad (9)$$

the standard internal Dirichlet representation of the absorbing dendrite in the sharp interface limit [9,10]. The condition is thus imposed on a moving set of points, and this time dependence of the Dirichlet support is the source of the analytic difficulty in what follows: a stationary tip would decouple the Dirichlet and Neumann traces, whereas the advancing one couples them nontrivially in time.

We note that the constant speed assumption is not an uncontrolled simplification: it is the self-consistent steady state of the DNN model [9], in which each dendrite advances at constant v set by the Ivantsov-solvability selection at its operating Péclet number. In the low Péclet regime characteristic of directional solidification, this steady state is quantitatively accurate and the kinematic relation $d(t) = d_0 - vt \sin \theta$ is exact for the geometry it describes. The question the present analysis answers is whether the wall-corrected transport field is consistent with sustained advance at the selected v or whether the reflected depletion perturbs it beyond the selection threshold; the observable that enables this comparison is constructed in Sec. VIII.

A direct geometric consequence of Eqs. (4) and (8) is that the perpendicular distance from the tip to the wall is

$$d(t) = d_0 - vt \sin \theta, \quad (10)$$

an exact kinematic corollary rather than an assumed evolution law. For $\theta = 0$ the offset is constant and the geometry reduces to the parallel wall limit. For $0 < \theta < \pi/2$ the wall approaches the tip at rate $\dot{d} = -v \sin \theta$, so the reflected depletion arriving at time t has been built on a geometry closer than the one in place at earlier times, with an amplitude and phase that cannot be recovered from any single fixed offset. Freezing d at d_0 understates the returning depletion; freezing it at $d(t)$ misses the history during which the offset was larger; only the full laboratory formulation treats the history exactly.

Evaluating Eq. (5) on a dendrite point $(0, Y)$ with $Y < vt$ gives

$$d_0 - Y \sin \theta > d_0 - vt \sin \theta = d(t), \quad (11)$$

strictly positive so long as $d(t) > 0$, that is

$$t < t_* := \frac{d_0}{v \sin \theta}. \quad (12)$$

The dendrite therefore lies entirely on the liquid side throughout $0 < t < t_*$, and we restrict the analysis to that interval. The reflected depletion matures on the diffusive scale d_0^2/D , and for any Péclet number $\text{Pe}_\theta = vd_0 \sin \theta/D$ not extremely small

the competition between diffusive maturation and kinematic shrinkage plays out well inside $[0, t_*)$.

With Eqs. (1), (2), (6), and (9) on the dendrite Eq. (7), the problem is fully posed. The obstruction to a direct analysis is that the absorbing support moves, so no ordinary one-sided Laplace transform in s preserves the structure of the equation. The change of variable that resolves this, and drives the whole analysis from Sec. III onward, is the passage from the laboratory arc length s to the fixed line coordinate $u = s - vt$. In u the dendrite becomes a time independent oblique line whose physically active portion is the half line $u > -vt$, and the problem acquires the wedge structure on which a Wiener-Hopf treatment can operate.

III. EXACT HALF-SPACE REDUCTION AND THE MOVING ENDPOINT BOUNDARY INTEGRAL EQUATION

The problem posed at the end of the previous section is a mixed Dirichlet-Neumann initial boundary value problem for the heat equation in a half plane, cut internally by a moving absorbing line. Its awkward feature is that the absorbing support itself moves: the position of the tip and the set of dendrite points on which the Dirichlet element is enforced both depend on time, so the spatial domain is not invariant under time translations and no direct separation of variables applies. The remedy is standard for mixed problems of this kind: eliminate the Neumann boundary through an appropriate Green function and convert the Dirichlet condition on the moving line into an integral equation for a single-layer density. The resulting equation is still posed on a moving domain, but the motion is confined to a single endpoint once the right coordinate is chosen, and the associated spacetime wedge is the natural object on which the transform methods of Sec. V will act.

The reduction proceeds in three steps. We pass to coordinates aligned with the fixed wall, in which the outer domain is a plain half space and the exact Green function is the half-space heat kernel obtained by the method of images. We then substitute the tip kinematics and observe that the image of the dendrite depends on s and t only through $u = s - vt$, so that in wall fixed coordinates the dendrite becomes a static oblique line with a moving endpoint at $u = -vt$. Finally, we represent the field as a single-layer potential supported on the moving dendrite, enforce the Dirichlet element on its trace, and obtain the exact Volterra integral equation of the laboratory problem, which splits into a direct and a reflected part.

A. Wall-fixed coordinates and the half-space Neumann kernel

We rotate and shift to coordinates adapted to the obstacle. Let

$$\sigma = X \sin \theta + Y \cos \theta, \quad \zeta = d_0 + X \cos \theta - Y \sin \theta, \quad (13)$$

so that σ is the signed arc length along the wall in the direction of increasing Y and ζ is the signed perpendicular distance from the wall, positive in the liquid. The wall is the locus $\zeta = 0$, the liquid occupies the half space $\zeta > 0$, and the change of variables is orthogonal with unit Jacobian. The transport equation Eq. (1) retains its isotropic form,

$$\partial_t c = D(c_{\sigma\sigma} + c_{\zeta\zeta}), \quad \zeta > 0, \quad (14)$$

and acquires no drift term. This is the analytic payoff of the laboratory frame: the frozen offset formulation would at this stage leave an effective drift $(v \cos \theta, -v \sin \theta)$ which a gauge transformation would then have to remove by converting the Neumann condition into a Robin condition. None of that is needed here, and Eq. (6) translates without modification into

$$\partial_\zeta c(\sigma, 0, t) = 0 \quad (15)$$

on the line $\zeta = 0$.

The Green function we need is the fundamental solution of the heat equation on the half space with this Neumann condition. It is constructed by the method of images: to the free space heat kernel from a unit source at (σ', ζ') we add an identical kernel from an image source at $(\sigma', -\zeta')$, so that the normal derivative of the sum vanishes on $\zeta = 0$. Explicitly,

$$\begin{aligned} G_N(\sigma, \zeta, t; \sigma', \zeta', \tau) &= \frac{H(t - \tau)}{4\pi D(t - \tau)} \left\{ \exp \left[-\frac{(\sigma - \sigma')^2 + (\zeta - \zeta')^2}{4D(t - \tau)} \right] \right. \\ &\quad \left. + \exp \left[-\frac{(\sigma - \sigma')^2 + (\zeta + \zeta')^2}{4D(t - \tau)} \right] \right\}, \end{aligned} \quad (16)$$

where H is the Heaviside step function. Differentiating with respect to ζ and setting $\zeta = 0$, the two terms contribute $(\zeta - \zeta')/[2D(t - \tau)]$ and $(\zeta + \zeta')/[2D(t - \tau)]$, which cancel identically, so Eq. (16) satisfies Eq. (15). The equal sign between the real and image contributions is the image rule for a Neumann boundary [30], and is what makes the reflection coefficient at the wall exactly unity: in the laboratory formulation the reflected signal returns without any Robin attenuation factor.

B. Dendrite as a moving oblique line in wall fixed coordinates

We translate the tip kinematics Eq. (8) into the wall fixed coordinates Eq. (13). Substituting $(X, Y) = (0, vt - s)$ gives

$$\sigma_R(s, t) = (vt - s) \cos \theta, \quad (17)$$

$$\zeta_R(s, t) = d_0 - (vt - s) \sin \theta = d_0 + (s - vt) \sin \theta. \quad (18)$$

The decisive observation is that both coordinates depend on s and t only through

$$u := s - vt. \quad (19)$$

In terms of u the image of the dendrite is the time independent oblique line

$$\sigma_D(u) = -u \cos \theta, \quad \zeta_D(u) = d_0 + u \sin \theta, \quad (20)$$

the infinite straight line of slope $-\tan \theta$ with respect to the wall normal, passing through the fixed point $(\sigma, \zeta) = (0, d_0)$, which is the foot of the perpendicular from the initial tip onto the wall. The variable u converts the moving dendrite into a static line, and relegates the time dependence of the absorbing support to the specification of the portion of that line which is physically active at any given time.

That portion is easy to identify. The physical arc length range $s > 0$ becomes $u > -vt$ under Eq. (19). The instantaneous tip sits at $u = -vt$ and moves to the left at constant speed v ; trailing material points correspond to larger

u . The spacetime support of any density associated with the dendrite is therefore the wedge

$$\Omega_+ := \{(u, t) : t > 0, u > -vt\}, \quad (21)$$

on which the wedge Wiener-Hopf transform of Sec. V will be defined. The kinematic relation Eq. (10) is recovered automatically by evaluating $\zeta_D(u)$ at the instantaneous tip $u = -vt$, with no frozen offset assumption.

C. Single-layer representation and the moving endpoint boundary integral equation

We represent the concentration field as a single-layer potential supported on the moving dendrite [31]. Let $\mu(s, t)$ denote the line density on $\Gamma_D(t)$, parametrized by the laboratory arc length s behind the tip. Then

$$c(\sigma, \zeta, t) = \int_0^t \int_0^\infty G_N(\sigma, \zeta, t; \sigma_R(s', \tau), \zeta_R(s', \tau), \tau) \times \mu(s', \tau) ds' d\tau, \quad (22)$$

which by construction satisfies Eq. (14) in the liquid, Eq. (15) on the wall, and Eq. (2) at $t = 0$, because each of these properties holds for G_N . The remaining condition to enforce is the Dirichlet element Eq. (9) on the dendrite.

Imposing that element at a point on $\Gamma_D(t)$ labeled by $s > 0$, with $(\sigma, \zeta) = (\sigma_R(s, t), \zeta_R(s, t))$, and changing the inner integration variable in Eq. (22) from s' to $u' = s' - v\tau$ at fixed τ , with unit Jacobian and support $u' > -v\tau$, the single-layer representation evaluated on the dendrite takes the form

$$c_{LC} = \int_0^t d\tau \int_{-v\tau}^\infty K_\theta(u, t; u', \tau) v(u', \tau) du', \quad (23)$$

for $u = s - vt > -vt$, where

$$v(u', \tau) := \mu(u' + v\tau, \tau) \quad (24)$$

is the density in the fixed line variable, and $K_\theta(u, t; u', \tau)$ is the value of G_N with both source and field points on the dendrite line Eq. (20) at parameters u' and u , respectively. Equation (23) is the moving endpoint boundary integral equation of the laboratory problem: a Volterra equation in time with a lower limit $u' = -v\tau$ that moves to the left at constant speed, with unknown v supported on Ω_+ . The physical density is recovered by

$$\mu(s, t) = v(s - vt, t), \quad s > 0, \quad (25)$$

and the field itself by substituting μ into Eq. (22).

The kernel K_θ inherits its structure from the image representation Eq. (16). Because G_N is a sum of two Gaussians, the kernel splits as

$$K_\theta(u, t; u', \tau) = K_\theta^{\text{dir}}(u - u'; t - \tau) + K_\theta^{\text{ref}}(u, u'; t - \tau), \quad (26)$$

a direct part from the real source and a reflected part from the image source, with qualitatively different analytic structures.

For the direct part, using Eq. (20) gives $\sigma_D(u) - \sigma_D(u') = -(u - u') \cos \theta$ and $\zeta_D(u) - \zeta_D(u') = (u - u') \sin \theta$, so that

$$(\sigma - \sigma')^2 + (\zeta - \zeta')^2 = (u - u')^2 (\cos^2 \theta + \sin^2 \theta) = (u - u')^2. \quad (27)$$

The distance between two points on the dendrite is exactly the difference of their fixed line coordinates, irrespective of θ , because the dendrite is a straight line. The direct kernel is therefore the one-dimensional free space heat kernel along the dendrite arc length,

$$K_\theta^{\text{dir}}(u - u'; t - \tau) = \frac{H(t - \tau)}{4\pi D(t - \tau)} \exp \left[-\frac{(u - u')^2}{4D(t - \tau)} \right], \quad (28)$$

a convolution in both $u - u'$ and $t - \tau$ separately. This is the object on which the scalar Wiener-Hopf factorization of Sec. VI will act.

For the reflected part, the image of the source has wall fixed coordinates $(\sigma_D(u'), -\zeta_D(u'))$, so $\sigma - \sigma'$ is unchanged from the direct case and contributes $(u - u')^2 \cos^2 \theta$, but the ζ contribution is $\zeta_D(u) + \zeta_D(u') = 2d_0 + (u + u') \sin \theta$. Collecting,

$$(\sigma - \sigma')^2 + (\zeta + \zeta')^2 = (u - u')^2 \cos^2 \theta + (2d_0 + (u + u') \sin \theta)^2, \quad (29)$$

which, unlike Eq. (27), is not a function of $u - u'$ alone. The reflected kernel is therefore

$$K_\theta^{\text{ref}}(u, u'; t - \tau) = \frac{H(t - \tau)}{4\pi D(t - \tau)} \times \exp \left[-\frac{(u - u')^2 \cos^2 \theta + (2d_0 + (u + u') \sin \theta)^2}{4D(t - \tau)} \right], \quad (30)$$

and its dependence on $u + u'$, weighted by $\sin \theta$, is the analytic expression of the obliquity of the wall. Geometrically, the combination $u + u'$ encodes the fact that the distance from a source point to the mirror image of a field point changes not only with their relative separation but also as the whole configuration slides along the oblique dendrite. A pure convolution in $u - u'$ would require the distance to depend only on the relative separation, which happens only at $\theta = 0$, where the cross term collapses to the constant $(2d_0)^2$. For any $\theta > 0$ the reflected kernel is a genuine two variable kernel, whose (u, u') dependence makes the corresponding operator a Hankel [32] rather than a Toeplitz operator on the half line $u > -vt$.

Equation (23), together with the kernel splitting Eq. (26) and the explicit pieces Eqs. (28) and (30), is the exact starting point for the rest of the analysis. It is a Volterra integral equation in time with moving lower endpoint, convolution in the direct part, and Hankel structure in the reflected part. The moving lower limit obstructs any ordinary one-sided Laplace transform in u , and the Hankel “flipped” coupling $u + u'$ obstructs any scalar Wiener-Hopf reduction at fixed p . The first obstruction is removed in Sec. V by the wedge transform, which absorbs the moving endpoint into the analyticity strip of a single two variable transform. The second is the subject of Sec. IV, where the Hankel coupling is exposed in spectral

form and shown to have a rank one structure at each wave number.

IV. EXACT WIENER-HOPF-HANKEL STRUCTURE OF THE MOVING ENDPOINT PROBLEM

The boundary integral equation Eq. (23) is exact but carries two obstructions to a scalar Wiener-Hopf treatment [33]. The moving lower limit $u' > -v\tau$ prevents any ordinary one-sided Laplace transform in u from commuting with the Volterra structure in t ; this is handled in Sec. V by the wedge transform. At fixed time, the spatial kernel Eq. (26) is not of difference type, because the reflected kernel Eq. (30) depends on u and u' through both $u - u'$ and $u + u'$. This second obstruction is the subject of the present section. We compute the Laplace-Fourier symbol of the direct kernel and its canonical factorization, derive the spectral representation of the reflected kernel through the half-space image Green function and expose its rank one factorization at each wave number, and close with the two limiting cases $\theta \rightarrow 0$ and $d_0 \rightarrow \infty$.

A. Obstruction to a scalar reduction

A scalar Wiener-Hopf reduction [26,34] of Eq. (23) at fixed p would require the spatial kernel to be translation invariant in $u - u'$, so that a single Fourier or bilateral Laplace transform in the difference variable would diagonalize the operator. The direct kernel Eq. (28) is a pure convolution and meets this requirement. The reflected kernel Eq. (30) does not: its Gaussian exponent contains

$$(u - u')^2 \cos^2 \theta + (2d_0 + (u + u') \sin \theta)^2, \quad (31)$$

and the expansion of the second term,

$$(2d_0 + (u + u') \sin \theta)^2 = 4d_0^2 + 4d_0(u + u') \sin \theta + (u + u')^2 \sin^2 \theta, \quad (32)$$

produces three contributions that are not of difference type. The constant $4d_0^2$ is a pure attenuation factor that will reappear as the round trip exponential $e^{-2d_0\kappa}$. The linear term $4d_0(u + u') \sin \theta$ shifts the exponent proportionally to both points and cannot be removed by any translation. The quadratic term $(u + u')^2 \sin^2 \theta$ couples the two points symmetrically. All three vanish at $\theta = 0$, so the reflected kernel is translation invariant if and only if the wall is parallel; thus, the obliquity is the origin of a Hankel “flipped” $u + u'$ coupling in this problem. The structural phenomenon of a scalar Wiener-Hopf problem acquiring a Hankel coupling through the geometry of an oblique boundary is familiar from the operator theoretic treatment of elliptic wedge diffraction problems, where the name Wiener-Hopf-Hankel was introduced for the Helmholtz equation on a rectangular wedge with mixed boundary data [28,29]; in the present parabolic setting, the Hankel coupling arises from a different geometric source, the oblique angle of a planar reflector relative to a moving absorbing line, but the operator structure is the same. The remedy here is not to discard the coupling but to absorb it into an operator whose spectral structure is rank one at each wave number [28], which is the content of the next two subsections.

B. Laplace-Fourier symbol of the direct kernel and its canonical factorization

At fixed p with $\text{Re } p > 0$, the direct kernel Eq. (28) depends only on $\xi = u - u'$ and $\Delta = t - \tau$, and its Laplace-Fourier transform

$$\widehat{K}^{\text{dir}}(k, p) := \int_0^\infty d\Delta e^{-p\Delta} \int_{-\infty}^\infty d\xi e^{ik\xi} K_\theta^{\text{dir}}(\xi; \Delta) \quad (33)$$

is computed in two elementary steps. The inner Gaussian integral is

$$\int_{-\infty}^\infty \frac{e^{ik\xi - \xi^2/(4D\Delta)}}{4\pi D\Delta} d\xi = \frac{e^{-Dk^2\Delta}}{\sqrt{4\pi D\Delta}}, \quad (34)$$

obtained by completing the square in the exponent. The remaining time integral reduces to

$$\widehat{K}^{\text{dir}}(k, p) = \int_0^\infty \frac{e^{-(p+Dk^2)\Delta}}{\sqrt{4\pi D\Delta}} d\Delta = \frac{1}{\sqrt{4\pi D}} \sqrt{\frac{\pi}{p+Dk^2}}, \quad (35)$$

by the standard Laplace integral $\int_0^\infty \Delta^{-1/2} e^{-a\Delta} d\Delta = \sqrt{\pi/a}$ with $a = p + Dk^2$, which converges because $\text{Re } a > 0$. Introducing the spectral function

$$\kappa(k, p) := \sqrt{k^2 + \frac{p}{D}}, \quad (36)$$

on the principal branch with $\text{Re } \kappa(k, p) > 0$ on the working half plane, Eq. (35) simplifies to

$$\widehat{K}^{\text{dir}}(k, p) = \frac{1}{2D\kappa(k, p)}. \quad (37)$$

The radical carries no $-ivk/D$ shift because the laboratory equation is the plain heat equation, so the branch points of the direct symbol will sit on the real axis rather than being rotated off it. The prefactor carries no $1/\sin \theta$ weight of the kind produced by the gauge reduction of the frozen offset problem, and the entire θ dependence is carried by the reflected operator.

In the spectral variable q conjugate to u through a bilateral Laplace transform, related to k by $k \mapsto -iq$,

$$\kappa(k, p)|_{k=-iq} = \sqrt{-q^2 + \frac{p}{D}} =: \Gamma(q, p), \quad (38)$$

with the branch of Γ selected so that $\text{Re } \Gamma(q, p) > 0$ on the working strip, and the direct symbol becomes

$$\mathcal{K}_0(q, p) := \widehat{K}^{\text{dir}}(-iq, p) = \frac{1}{2D\Gamma(q, p)}. \quad (39)$$

It has two square-root branch points at

$$q_\pm(p) := \pm \sqrt{\frac{p}{D}} \quad (40)$$

and is analytic and nonvanishing on the complement of any straight cut Σ_p joining them. We take $\Sigma_p = \{q = t\sqrt{p/D} : t \in [-1, 1]\}$, the canonical Schwarz reflective choice. On any vertical strip interior to $-\text{Re}\sqrt{p/D} < \text{Re } q < \text{Re}\sqrt{p/D}$ the symbol admits the canonical scalar factorization

$$\mathcal{K}_0(q, p) = \mathcal{K}_{0,+}(q, p) \mathcal{K}_{0,-}(q, p), \quad (41)$$

with explicit factors

$$\mathcal{K}_{0,+}(q, p) = \sqrt{\frac{\sqrt{p/D}}{\sqrt{p/D} - q}}, \quad (42)$$

$$\mathcal{K}_{0,-}(q, p) = \frac{1}{2D\sqrt{p/D}} \sqrt{\frac{\sqrt{p/D}}{\sqrt{p/D} + q}}. \quad (43)$$

The factor $\mathcal{K}_{0,+}$ has its sole singularity at $q = +\sqrt{p/D}$ and is analytic and nonvanishing in $\text{Re} q < \text{Re}\sqrt{p/D}$; $\mathcal{K}_{0,-}$ has its sole singularity at $q = -\sqrt{p/D}$ and is analytic and nonvanishing in $\text{Re} q > -\text{Re}\sqrt{p/D}$. Both are therefore simultaneously analytic on the working strip, as required for the scalar Wiener-Hopf sequential inversion of Sec. VI. To verify Eq. (41) explicitly one writes

$$\begin{aligned} \mathcal{K}_{0,+}(q, p)\mathcal{K}_{0,-}(q, p) &= \frac{1}{2D\sqrt{p/D}} \sqrt{\frac{(\sqrt{p/D})^2}{(\sqrt{p/D} - q)(\sqrt{p/D} + q)}}, \end{aligned} \quad (44)$$

and simplifies the radicand using $(\sqrt{p/D} - q)(\sqrt{p/D} + q) = p/D - q^2 = \Gamma(q, p)^2$, which yields

$$\mathcal{K}_{0,+}(q, p)\mathcal{K}_{0,-}(q, p) = \frac{1}{2D\sqrt{p/D}} \cdot \frac{\sqrt{p/D}}{\Gamma(q, p)} = \frac{1}{2D\Gamma(q, p)}, \quad (45)$$

in agreement with Eq. (39). The normalization is fixed by $\mathcal{K}_{0,+}(0, p) = 1$, and correspondingly $\mathcal{K}_{0,-}(0, p) = 1/(2D\sqrt{p/D}) = \mathcal{K}_0(0, p)$. Equations (42) and (43) replace the gauge weighted factors of the frozen offset formulation, whose branch points sit at $q = -\delta_\theta \pm \lambda(p)$ with $\delta_\theta = v \cos^2 \theta / (2D)$; here the branch points sit symmetrically on the real axis at $\pm\sqrt{p/D}$ with no obliquity dependence.

C. Half-space image representation and the rank one form of the reflected kernel

The Laplace-Fourier symbol of the reflected kernel Eq. (30) is more delicate, because the u and u' dependencies are not of difference type and a direct bilateral Laplace transform in $u - u'$ fails. A direct time Laplace using $\int_0^\infty t^{-1} e^{-at-b/t} dt = 2K_0(2\sqrt{ab})$ with $a = p$ and b equal to Eq. (31) divided by $4D$ would produce a Bessel function of an argument depending on both $u \pm u'$, which does not expose the rank one structure. The profitable route exploits the fact that, by its derivation in Sec. III C, the reflected kernel is the restriction of the image piece of G_N to two points on the dendrite line, and uses the Fourier-Laplace representation of that image piece with respect to σ and t .

Taking the Laplace transform in time and the Fourier transform in $\sigma - \sigma'$ of the image piece of Eq. (16), the partial transform $\bar{G}^{\text{image}}(k, \zeta, \zeta'; p)$ satisfies

$$(p + Dk^2 - D\partial_\zeta^2)\bar{G}^{\text{image}} = 0, \quad \zeta, \zeta' > 0, \quad (46)$$

with unit source at $\zeta = \zeta'$ and Neumann condition at $\zeta = 0$. Writing the solution as a superposition of $e^{\pm\kappa(k, p)\zeta}$ and imposing decay at infinity, the source condition, and the Neumann

image rule yields

$$\bar{G}^{\text{image}}(k, \zeta, \zeta'; p) = \frac{e^{-\kappa(k, p)(\zeta + \zeta')}}{2D\kappa(k, p)}, \quad (47)$$

which depends on ζ and ζ' only through their sum. Inverting the σ Fourier transform,

$$\bar{G}^{\text{image}}(\sigma, \zeta; \sigma', \zeta'; p) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(\sigma - \sigma')} e^{-\kappa(k, p)(\zeta + \zeta')}}{2D\kappa(k, p)}, \quad (48)$$

the standard half-space image representation. Restricting Eq. (48) to the dendrite line Eq. (20) at parameters u and u' , and using the evenness of the image symbol in k to fix the sign convention in the σ exponent,

$$\bar{K}_\theta^{\text{ref}}(u, u'; p) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(\sigma_D(u') - \sigma_D(u))} e^{-\kappa(k, p)(\zeta_D(u) + \zeta_D(u'))}}{2D\kappa(k, p)}. \quad (49)$$

The k integration converges because the decay factor $\exp[-\text{Re}\kappa(\zeta_D(u) + \zeta_D(u'))]$ is strictly positive: $\zeta_D(u) + \zeta_D(u') = 2d_0 + (u + u') \sin \theta > 0$ on the physical interval $0 < t < t_*$ by Eq. (11).

Substituting the explicit dendrite coordinates Eq. (20) into the exponents of Eq. (49) and using $\sigma_D(u) = -u \cos \theta$ and $\zeta_D(u) = d_0 + u \sin \theta$ gives

$$\sigma_D(u') - \sigma_D(u) = (u - u') \cos \theta, \quad (50)$$

$$\zeta_D(u) + \zeta_D(u') = 2d_0 + (u + u') \sin \theta, \quad (51)$$

and the full exponent expands as

$$\begin{aligned} ik(\sigma_D(u') - \sigma_D(u)) - \kappa(k, p)(\zeta_D(u) + \zeta_D(u')) \\ = -2d_0\kappa(k, p) + ik \cos \theta u - ik \cos \theta u' \\ - \sin \theta \kappa(k, p) u - \sin \theta \kappa(k, p) u'. \end{aligned} \quad (52)$$

Collecting by powers of u and u' ,

$$\begin{aligned} -2d_0\kappa(k, p) - [\sin \theta \kappa(k, p) - i \cos \theta k]u \\ - [\sin \theta \kappa(k, p) + i \cos \theta k]u', \end{aligned} \quad (53)$$

and introducing

$$A_\theta(k, p) := \sin \theta \kappa(k, p) - i \cos \theta k, \quad (54)$$

$$B_\theta(k, p) := \sin \theta \kappa(k, p) + i \cos \theta k, \quad (55)$$

the coefficient of u in Eq. (53) is $-A_\theta(k, p)$ and the coefficient of u' is $-B_\theta(k, p)$. Substituting into Eq. (49), the time Laplace reflected kernel takes the exact form

$$\bar{K}_\theta^{\text{ref}}(u, u'; p) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{-2d_0\kappa(k, p)}}{2D\kappa(k, p)} e^{-A_\theta(k, p)u} e^{-B_\theta(k, p)u'}. \quad (56)$$

This is the central structural result of the section. At each k the integrand is a rank one product in (u, u') , and the two variable structure is generated entirely by the k integration against the wall attenuation weight $e^{-2d_0\kappa}/(2D\kappa)$. Physically, the reflected signal returns through a single image source whose contribution at each wave number is a plane wave along the wall tangent times an exponential decay encoding the round trip distance $2d_0$.

The wall factor $e^{-2d_0\kappa(k,p)}$ carries the Laplace variable through $\kappa = \sqrt{k^2 + p/D}$, so it is a retardation and not only an attenuation. Inverting $e^{-2d_0\sqrt{p/D}}$ gives a one-sided kernel supported at positive times and weighted toward $t \sim 4d_0^2/D$, the diffusive round trip. The reflected contribution arriving at the tip at time t was therefore emitted around $t - 4d_0^2/D$, and at obliquity, where the offset closes as $d(t) = d_0 - vt \sin \theta$, it was emitted when the gap was wider than it is on arrival. The wall acts on the trajectory rather than on the instantaneous position, and the frozen offset approximation amounts to dropping the p dependence of κ in this exponential, which removes the delay and leaves a static image at the present separation.

The spectral combinations differ by

$$B_\theta(k, p) - A_\theta(k, p) = 2i \cos \theta k, \quad (57)$$

and sample different portions of the k spectrum whenever $\cos \theta \neq 0$, which is what makes the operator Hankel rather than diagonal. On the principal branch $\text{Re} \kappa(k, p) > 0$ for real k and $\text{Re} p > 0$, so

$$\text{Re} A_\theta(k, p) = \text{Re} B_\theta(k, p) = \sin \theta \text{Re} \kappa(k, p) > 0 \quad (58)$$

for $0 < \theta < \pi/2$, and both exponentials in Eq. (56) decay along the positive direction of their arguments, which is the condition that makes the plus projections of Sec. V well defined. The denominator combination

$$B_\theta(k, p) + A_\theta(k', p) = \sin \theta (\kappa(k, p) + \kappa(k', p)) + i \cos \theta (k - k') \quad (59)$$

has strictly positive real part for all real k, k' and $\text{Re} p > 0$, again because $\text{Re} \kappa > 0$ on the principal branch. Its zero set is the resonance locus of the reflected operator and represents the wall induced screening modes; the nonintersection of that locus with the real integration contour is what guarantees $L^2(\mathbb{R}, dk)$ boundedness of the reflected branch operators of Sec. V.

D. Limiting cases

Two limiting geometries reduce the problem: an isolate dendrite limit, achieved when the wall is infinitely far, and a parallel wall limit, arising when the dendrite is orthogonal to the wall's normal. Both cases are proved in full detail in Appendix D. Removing the wall, $d_0 \rightarrow \infty$ at fixed obliquity, leaves an isolated dendrite in free space. Since $\text{Re} \kappa(k, p) \geq \sqrt{\text{Re} p/D}$ uniformly in k for $\text{Re} p > 0$,

$$|e^{-2d_0\kappa(k,p)}| \leq e^{-2d_0\sqrt{\text{Re} p/D}}, \quad (60)$$

so the reflected weight in Eq. (56) vanishes uniformly at fixed p , the reflected branch operators vanish exponentially in Hilbert-Schmidt norm, and the density, concentration field and process zone functional converge to their free space counterparts. The limit supplies the reference $\Psi_{\text{iso}}(t)$ against which the sign of the wall correction is measured throughout Sec. IX. Equation (60) also fixes how far the wall is felt: on the diffusive timescale $t \sim d_0/v$ the attenuation is e^{-2d_0/ℓ_D} up to algebraic factors, so the influence of the wall is exponentially small in the offset measured in diffusion lengths and carries no algebraic far field tail. Wall proximity is accordingly a

local effect with a finite range, which is why the correction computed in Sec. IX is negligible beyond about two diffusion lengths and reaches tens of percent only within one. Setting the wall parallel to the growth direction, $\theta \rightarrow 0$ at fixed offset, leaves a dendrite advancing alongside a wall at constant separation. Here $A_0 = -ik$ and $B_0 = +ik$, so the two exponentials of Eq. (56) combine into $e^{ik(u-u')}$, the Hankel coupling disappears, and the reflected operator becomes a pure convolution with symbol $e^{-2d_0\kappa}/(2D\kappa)$. The same symbol follows directly from Eq. (30) at $\theta = 0$, where the Gaussian exponent reduces to $(u - u')^2 + 4d_0^2$, so the limit is available without passing through the branch representation. Direct and reflected contributions add

$$\mathcal{K}_0^{\text{tot}}(q, p)|_{\theta=0} = \frac{1 + e^{-2d_0\Gamma(q,p)}}{2D\Gamma(q, p)}, \quad (61)$$

and what remains is a scalar Wiener-Hopf equation with no auxiliary branch trace and no Fredholm problem. By the image rule Eq. (61) is the symmetric two dendrite symbol at spacing $h = 2d_0$, so the all angle framework is calibrated at $\theta = 0$ against the standard scalar Wiener-Hopf problem for two symmetrically placed dendrites. Its lowest pole lies at $|k| = \pi/(2d_0)$, and the associated screening length $2d_0/\pi$ is the scale over which the reflected depletion decorrelates from the dendrite's own. Section VIII G takes this as the process zone length, so the parallel limit fixes the averaging scale of the matching variable rather than leaving it free. Reflection is absent in the first limit and present at a fixed separation in the second, where $d(t) = d_0$, $t_* = \infty$ and $\text{Pe}_\theta = 0$, so the returned signal is generated at the separation at which it is received. The parallel limit is the exact fixed gap endpoint of the all angle theory. It removes the oblique branch coupling along with the closing gap kinematics, so the dynamical effect of the changing separation is isolated instead by comparing the exact oblique solution with the frozen offset construction at the same obliquity, as in Sec. IX B.

At fixed p in the right half plane, Eq. (23) therefore carries a Wiener-Hopf-Hankel structure that splits into a scalar Wiener-Hopf direct part with the explicit factorization Eqs. (42) and (43), and a Hankel reflected part with the rank one factorization Eq. (56) into exponentials $e^{-A_\theta u}$ and $e^{-B_\theta u'}$ weighted by $e^{-2d_0\kappa}/(2D\kappa)$. The rank one structure couples the unknown density only to a single auxiliary spectral trace rather than to its full two-dimensional spectrum, which is the reason the whole problem reduces to an upper triangular branch matrix rather than an intractable two-dimensional Hankel problem. The remaining obstruction, the moving lower limit $u' > -v\tau$, is handled in the next section by the wedge transform.

V. EXACT TRANSFORMED PROBLEM AND BRANCH MATRIX FORMULATION

The spectral analysis of Sec. IV was carried out at fixed p and made no use of the temporal history. What remains is to convert Eq. (23) into an operator equation for a single transformed unknown, on a domain that respects the wedge structure $\Omega_+ = \{(u, t) : t > 0, u > -vt\}$ of Sec. III B. The device is a single two variable transform [21–23] whose analyticity domain absorbs both the $u > -vt$ half line at each

time and the temporal Laplace decay, so that the moving endpoint becomes an analyticity condition on the transformed unknown. The rank one form Eq. (56) then translates into an operator equation in which the reflected contribution couples the unknown only to its trace along a single curve in the spectral plane, giving the upper triangular structure that the sequential inversion of Sec. VI will exploit.

A. Wedge transform and its analyticity strip

Let $v(u, t)$ be the fixed line density on the wedge Ω_+ introduced in Eq. (24), extended by zero to $u < -vt$. The wedge transform is the single two variable integral

$$\Phi_+(q, p) := \int_0^\infty dt e^{-pt} \int_{-vt}^\infty du e^{-qu} v(u, t). \quad (62)$$

This is not a composition of two one variable transforms: the inner limit depends on the outer variable, and Eq. (62) is intrinsically two-dimensional. Its convergence requires two separate conditions dictated by the wedge geometry. To identify them, change variables from u to $s = u + vt$ at fixed t , with $du = ds$ and lower limit $s = 0$, so that Eq. (62) becomes

$$\Phi_+(q, p) = \int_0^\infty dt e^{-(p-vq)t} \int_0^\infty ds e^{-qs} \mu(s, t), \quad (63)$$

where $\mu(s, t) = v(s - vt, t)$ is the physical density from Eq. (25). The inner Laplace integral in s converges provided

$$\text{Re} q > 0, \quad (64)$$

and the outer integral in t converges provided

$$\text{Re}(p - vq) > 0. \quad (65)$$

Together, Eqs. (64) and (65) define the working strip

$$\mathcal{W} := \{(q, p) \in \mathbb{C}^2 : \text{Re} q > 0, \text{Re}(p - vq) > 0\}, \quad (66)$$

on which the wedge transform is jointly analytic in (q, p) . The strip is genuinely two-dimensional, constrained to $\text{Re} p > v \text{Re} q > 0$ with boundary planes $\text{Re} q = 0$ and $\text{Re} p = v \text{Re} q$. Equation (63) identifies $p - vq$ as the kinematic Laplace parameter conjugate to t at fixed q , and the wedge transform as a two parameter family of one-sided Laplace-Laplace transforms of $\mu(s, t)$ in which the advance of the tip is encoded through the shift $p \rightarrow p - vq$. The moving endpoint $u = -vt$ has been absorbed into the joint analyticity condition on (q, p) .

Applied to the constant right-hand side c_{LC} of Eq. (23), supported on Ω_+ through $\mathbf{1}_{\{u > -vt\}}$, the wedge transform is computed directly. The inner u integral at fixed t is

$$\int_{-vt}^\infty e^{-qu} du = \frac{e^{qvt}}{q}, \quad (67)$$

valid for $\text{Re} q > 0$, and the outer t integral gives

$$G_+(q, p) := c_{\text{LC}} \int_0^\infty e^{-(p-vq)t} dt = \frac{c_{\text{LC}}}{q(p - vq)}, \quad (68)$$

valid for $(q, p) \in \mathcal{W}$. The transformed element has a simple pole at $q = 0$ from the spatial Laplace integral, and a second at $p - vq = 0$ from the temporal Laplace integral. The second is the kinematic factor encoding the advancing tip: the laboratory counterpart of the gauge shifted pole $q = \alpha \sin \theta$

of the frozen offset formulation, its position depending on v rather than on any θ -dependent gauge constant. Both poles lie outside $\mathcal{W}$, as they must.

B. Wedge transform of the direct operator and the complementary minus function

The direct kernel Eq. (28) is a pure convolution in $u - u'$ and $t - \tau$ separately, so that

$$[\mathcal{W}^{\text{dir}} v](u, t) := \int_0^t d\tau \int_{-\infty}^\infty du' K_\theta^{\text{dir}}(u - u'; t - \tau) v(u', \tau) \quad (69)$$

on the zero extended density is translation invariant in u at fixed time lag and in t at fixed spatial offset. Its full bilateral Laplace in u and one-sided Laplace in t therefore factor into the product of the symbol Eq. (37) with the corresponding transform of v . Working with the spectral variable q in the strip Eq. (66), the bilateral spatial Laplace of the zero extended v at fixed t is not Φ_+ itself but Φ_+ plus a complementary contribution from the half line $u < -vt$ on which v vanishes:

$$\int_{-\infty}^\infty e^{-qu} v(u, t) du = \int_{-vt}^\infty e^{-qu} v(u, t) du + 0, \quad (70)$$

which reduces the bilateral Laplace of the zero extension to the wedge Laplace at fixed t . Integrating against $e^{-pt} dt$ and using Eq. (62),

$$(\mathcal{W}^{\text{dir}} v)^\wedge(q, p) = \mathcal{K}_0(q, p) \Phi_+(q, p) + \Phi_-(q, p), \quad (71)$$

where $\mathcal{K}_0(q, p) = 1/(2D\Gamma(q, p))$ is the direct symbol Eq. (39), and $\Phi_-(q, p)$ is the wedge transform of the complementary contribution needed to convert the wedge integral into a bilateral one for the subsequent Wiener-Hopf projection. Φ_- is the unknown analytic extension of $\mathcal{K}_0 \Phi_+$ through the left boundary $\text{Re} q = 0$ of the strip, generated by the direct convolution acting on the wedge support; it is determined in Sec. VI simultaneously with Φ_+ as part of the scalar Wiener-Hopf solution. The direct operator remains multiplicative in q after transformation, with the same symbol $\mathcal{K}_0$ of Sec. IV.

C. Wedge transform of the reflected operator and the emergence of the branch trace

The reflected operator

$$[\mathcal{R}v](u, t) := \int_0^t d\tau \int_{-v\tau}^\infty du' K_\theta^{\text{ref}}(u, u'; t - \tau) v(u', \tau) \quad (72)$$

cannot be treated in the same way because K_θ^{ref} is not translation invariant in $u - u'$. We work instead from the time Laplaced kernel Eq. (56), which at each k is a rank one product of $e^{-A_\theta u}$ and $e^{-B_\theta u'}$ weighted by $e^{-2d_0 k}/(2Dk)$. The plan is to substitute this rank one representation into the wedge transform of Eq. (72), interchange orders of integration, and recognize the (u', τ) integral as a wedge transform in its own right evaluated at a specific point of the spectral plane.

Extend v by zero to $u < -v\tau$ at each τ , so the source side integration in Eq. (72) may be taken over $\mathbb{R}$. Writing

$$(\mathcal{R}v)^\wedge(q, p) = \int_0^\infty dt e^{-pt} \int_{-\infty}^\infty du e^{-qu} [\mathcal{R}v](u, t), \quad (73)$$

substituting Eq. (72), and exchanging the order of integration from $\int_0^\infty dt \int_0^t d\tau$ to $\int_0^\infty d\tau \int_\tau^\infty dt$, the temporal exponential splits as $e^{-pt} = e^{-p\tau} e^{-p(t-\tau)}$ and the inner t integral becomes the time Laplace of the reflected kernel at fixed u, u' , which is exactly Eq. (56):

$$(\mathcal{R}v)^\wedge(q, p) = \int_0^\infty d\tau e^{-p\tau} \int_{-\infty}^\infty du' v(u', \tau) \times \int_{-\infty}^\infty du e^{-qu} \bar{K}_\theta^{\text{ref}}(u, u'; p). \quad (74)$$

Substituting Eq. (56) and exchanging the k integration with the u and u' integrations, the innermost u integral is

$$\int_{-\infty}^\infty e^{-(q+A_\theta(k,p))u} du, \quad (75)$$

which as a bilateral integral does not converge absolutely. This is where the Wiener-Hopf plus projection intervenes: v is supported on $u > -v\tau$, and the natural object at fixed k is not the bilateral integral but its plus side half line part, representing the Laplace of $e^{-A_\theta u}$ restricted to the physical support. The rigorous plus projection, constructed in Sec. VI through an explicit strip Cauchy contour, replaces Eq. (75) by

$$\begin{aligned} \mathcal{P}_+ \left[\int_{-\infty}^\infty e^{-(q+A_\theta(k,p))u} du \right] \\ = \int_0^\infty e^{-(q+A_\theta(k,p))u} du = \frac{1}{q + A_\theta(k, p)}, \end{aligned} \quad (76)$$

which converges absolutely on the working strip because $\text{Re}(q + A_\theta(k, p)) = \text{Re}q + \sin \theta \text{Re}\kappa(k, p) > 0$ by Eqs. (64) and (58).

With the inner u integral replaced by Eq. (76), the plus projected wedge transform of Eq. (72) becomes

$$(\mathcal{P}_+(\mathcal{R}v)^\wedge)(q, p) = \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{e^{-2d_0\kappa(k,p)}}{2D\kappa(k, p)} \frac{1}{q + A_\theta(k, p)} \times \int_0^\infty d\tau e^{-p\tau} \int_{-\infty}^\infty du' e^{-B_\theta(k,p)u'} v(u', \tau). \quad (77)$$

The bracketed integral in the second line is the key object. Because v is zero extended to $u' < -v\tau$, the integration on $u' \in \mathbb{R}$ at fixed τ equals the integration on the physical wedge $u' > -v\tau$, and the resulting double integral is the wedge transform of v evaluated not at q but at the spectral argument $q = B_\theta(k, p)$. Comparing with Eq. (62),

$$\begin{aligned} \Phi_B(k, p) &:= \Phi_+(B_\theta(k, p), p) \\ &= \int_0^\infty d\tau e^{-p\tau} \int_{-v\tau}^\infty du' e^{-B_\theta(k,p)u'} v(u', \tau), \end{aligned} \quad (78)$$

a well-defined analytic function of k and p whenever $(B_\theta(k, p), p)$ lies in $\mathcal{W}$. On the real k contour this requires $\text{Re}B_\theta(k, p) > 0$, which follows from Eq. (58), and $\text{Re}(p - vB_\theta(k, p)) > 0$, which holds for $\text{Re}p$ sufficiently large and

is taken as part of the working assumption on the Laplace contour. Equation (78) is the branch trace: the restriction of Φ_+ to the one-dimensional curve $\{q = B_\theta(k, p) : k \in \mathbb{R}\}$ in $\mathcal{W}$.

Substituting Eq. (78) into Eq. (77),

$$(\mathcal{P}_+(\mathcal{R}v)^\wedge)(q, p) = \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{e^{-2d_0\kappa(k,p)}}{2D\kappa(k, p)} \frac{\Phi_B(k, p)}{q + A_\theta(k, p)}. \quad (79)$$

Equation (79) is the exact transformed expression of the Hankel coupling of Sec. IV. Read at fixed q , the reflected contribution does not multiply Φ_+ by a scalar symbol the way the direct contribution does. It samples the unknown on the single spectral branch $q = B_\theta(k, p)$, integrates the sampled values against $e^{-2d_0\kappa}/(2D\kappa)$, and redistributes the result through the Cauchy kernel $1/(q + A_\theta(k, p))$. The sampling curve is the image $\Gamma_B(p)$ of the real k axis under B_θ , and the redistribution curve is $\Gamma_A(p)$ on the field side. The identification of Φ_B as the wedge transform of v at $q = B_\theta(k, p)$ is the precise sense in which the moving endpoint $u' = -v\tau$ has been absorbed into the analyticity strip of Φ_+ .

D. The exact transformed operator equation

Combining the direct action Eq. (71), the reflected action Eq. (79), and the transformed element Eq. (68), Eq. (23) becomes

$$\begin{aligned} \mathcal{K}_0(q, p)\Phi_+(q, p) + \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{e^{-2d_0\kappa(k,p)}}{2D\kappa(k, p)} \frac{\Phi_B(k, p)}{q + A_\theta(k, p)} \\ + \Phi_-(q, p) = \frac{c_{\text{LC}}}{q(p - vq)}, \end{aligned} \quad (80)$$

valid for $(q, p) \in \mathcal{W}$. Every object in Eq. (80) is explicit: $\mathcal{K}_0$ from Eq. (39), A_θ, B_θ from Eqs. (54)–(55), the wall attenuation $e^{-2d_0\kappa}$, and the kinematic factor $p - vq$ in the transformed element. The only unknowns are $\Phi_+(q, p)$, its branch trace $\Phi_B(k, p)$ along $\Gamma_B(p)$, and the complementary minus function $\Phi_-(q, p)$. The equation is not scalar Wiener-Hopf because of the integral term: the unknown appears simultaneously as $\Phi_+(q, p)$ on the body of the strip and as $\Phi_B(k, p) = \Phi_+(B_\theta(k, p), p)$ on a codimension one curve. This is the exact analytic expression of the reflected coupling, and the task of the next section is to disentangle the two by evaluating Eq. (80) at carefully chosen points of the spectral plane.

E. Geometry of the spectral branches

Before carrying out that evaluation we need to know where $\Gamma_A(p)$ and $\Gamma_B(p)$ lie. By definition,

$$\begin{aligned} \Gamma_A(p) &:= \{A_\theta(k, p) : k \in \mathbb{R}\}, \\ \Gamma_B(p) &:= \{B_\theta(k, p) : k \in \mathbb{R}\} \end{aligned} \quad (81)$$

are the images of the real k axis under Eqs. (54) and (55). Writing $\kappa(k, p) = U(k, p) + iV(k, p)$ with $U = \text{Re}\kappa > 0$ on the principal branch,

$$A_\theta(k, p) = \sin \theta U(k, p) + i(\sin \theta V(k, p) - \cos \theta k), \quad (82)$$

$$B_\theta(k, p) = \sin \theta U(k, p) + i(\sin \theta V(k, p) + \cos \theta k). \quad (83)$$

Both curves share the common real part $\sin \theta U(k, p)$, strictly positive for $\theta \in (0, \pi/2)$ and real k . The curves lie entirely in

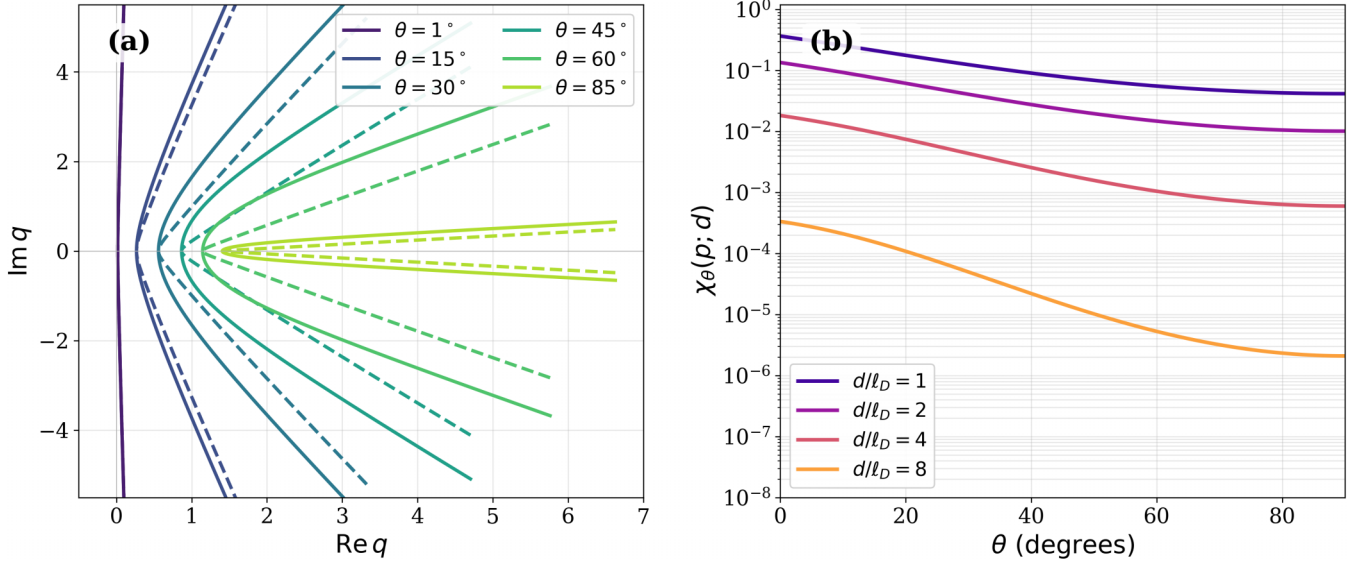


FIG. 3. Spectral structure of the wedge-transform branch problem at $p = 1$, $D = 1$, $v = 2$, $\ell_D = 0.5$. (a) Branch traces $\Gamma_A(p)$ (solid) and $\Gamma_B(p)$ (dashed) in the q plane, parametrized by $A_\theta, B_\theta = \sin \theta \kappa(k, p) \mp i \cos \theta k$ for real k , at $\theta = 1^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 85^\circ$. They locate the evaluation points for the reflected term in the Wiener-Hopf reconstruction of $\Phi_+(q, p)$. As $\theta \rightarrow \pi/2$ the traces coalesce on $q = \kappa(k, p)$; as $\theta \rightarrow 0$ they peel symmetrically into the upper and lower half-planes, recovering the translation-invariant parallel wall limit. (b) Single-bounce wall coupling defined as $\chi_\theta(p; d) = |\rho_\theta(0, p) e^{-2d\kappa_\theta(0, p)}|$, where $\rho_\theta(k, p) = 1/[2D\kappa(k, p)]$ is the spectral weight of the reflected kernel [Eq. (56)], and $\kappa_\theta(0, p) = \kappa(0, p) = \sqrt{p/D}$. χ_θ measures the amplitude of a single diffusive round trip from the dendrite to the wall and back at wave number $k = 0$. It serves as the natural figure of merit for the reflected branch operators R_{AB} and R_{BB} ; its exponential smallness in d/ℓ_D (here plotted vs versus θ at $d/\ell_D = 1, 2, 4, 8$) bounds the Hilbert–Schmidt norm of R_{BB} [Eq. (120)] and thereby controls both the convergence rate of the Neumann series [Eq. (132)] and the truncation order required in the Padé–Abrahams linear system, as we discuss in Appendix B.

the open right half plane $\text{Re } q > 0$ and touch the imaginary axis only at $\theta = 0$: the oblique wall pushes the sampling curves strictly into the interior of the working strip, and no contour indentation at $\text{Re } q = 0$ is needed.

The large $|k|$ asymptotics follow from $\kappa(k, p) \sim |k|$:

$$A_\theta(k, p) \sim |k|(\sin \theta - i \cos \theta \operatorname{sgn} k), \quad |k| \rightarrow \infty, \quad (84)$$

$$B_\theta(k, p) \sim |k|(\sin \theta + i \cos \theta \operatorname{sgn} k), \quad |k| \rightarrow \infty, \quad (85)$$

so both curves are asymptotic to rays through the origin at angles $\pm(\pi/2 - \theta)$ to the positive real q axis. For small θ the rays are nearly vertical and the curves approximate the imaginary axis; for θ near $\pi/2$ they fold down onto the positive real axis and approach each other. At $\theta = 0$, $A_0 = -ik$ and $B_0 = +ik$, so both curves collapse onto the imaginary axis with opposite orientations; at $\theta = \pi/2$, $A_{\pi/2} = B_{\pi/2} = \kappa(k, p)$, so they collapse onto the single curve $\Gamma_A = \Gamma_B = \{\kappa(k, p) : k \in \mathbb{R}\}$ in the right half plane.

Both branch maps inherit the branch point structure of κ . Since $\kappa(k, p) = \sqrt{k^2 + p/D}$, its branch points in the k plane are the roots of

$$k^2 + \frac{p}{D} = 0, \quad (86)$$

namely,

$$k_\pm^{\text{br}}(p) = \pm i \sqrt{\frac{p}{D}}. \quad (87)$$

These lie on the imaginary k axis and move away from the real axis as $\text{Re } p$ increases, so no real k ever coincides with

a branch point. Under Eqs. (54) and (55) they are mapped to corresponding points on Γ_A and Γ_B , and these are the only branch points of the composed maps on the complex k plane.

The two curves are plotted in Fig. 3(a) for a representative range of obliquities at $p = 1$. Figure 3(b) shows the single-bounce wall coupling $\chi_\theta(p; d)$ whose exponential smallness in d/ℓ_D is the quantitative content of the compactness statement of Sec. VI.

The critical points, where the local derivative vanishes and the branch map fails to be locally invertible, are found from

$$\partial_k A_\theta(k, p) = \sin \theta \frac{k}{\kappa(k, p)} - i \cos \theta = 0, \quad (88)$$

which rearranges to $\sin \theta k = i \cos \theta \kappa(k, p)$. Squaring yields $\sin^2 \theta k^2 = -\cos^2 \theta (k^2 + p/D)$, which simplifies to $k^2 = -\cos^2 \theta (p/D)$, so

$$k_\pm^{\text{cr}}(p) = \pm i \cos \theta \sqrt{\frac{p}{D}}. \quad (89)$$

The critical points lie on the imaginary k axis, at fraction $\cos \theta$ of the way from the origin to the branch points Eq. (87), and coincide with them only at $\theta = 0$. The same calculation for B_θ gives the same critical points, by the symmetry $B_\theta(k, p) = A_\theta(-k, \bar{p})$. The local inverse of A_θ is therefore well defined on any simply connected domain in q that avoids the images of Eqs. (87) and (89) under A_θ , and similarly for B_θ . These images are the only obstructions to local invertibility, and they reappear as the branch geometry controlling the Cagniard-de Hoop contour deformation in Sec. VI. Both the branch points

and the critical points sit on the imaginary k axis, rather than on shifted parallel contours, because $\kappa(k, p)$ is even in k : a frozen offset formulation would use $\sqrt{k^2 - ivk/D + p/D}$, whose branch points lie off the imaginary axis and whose critical points carry an extra $\beta_\theta = v/(2D)$ shift.

F. Resonance locus and the nonvanishing condition

The denominator $B_\theta(k, p) + A_\theta(k', p)$ that will arise in the reflected branch operators defines the resonance locus

$$\mathcal{M}_\theta(p) := \{(k, k') \in \mathbb{C}^2 : B_\theta(k, p) + A_\theta(k', p) = 0\}, \quad (90)$$

and the absence of singularities on the real contour is equivalent to $\mathcal{M}_\theta(p)$ not meeting $\mathbb{R}^2$. Using Eqs. (54) and (55),

$$\begin{aligned} B_\theta(k, p) + A_\theta(k', p) \\ = \sin \theta (\kappa(k, p) + \kappa(k', p)) + i \cos \theta (k - k'), \end{aligned} \quad (91)$$

and taking real parts for real k, k' and $\text{Re} p > 0$,

$$\begin{aligned} \text{Re}[B_\theta(k, p) + A_\theta(k', p)] \\ = \sin \theta (\text{Re}[\kappa(k, p)] + \text{Re}[\kappa(k', p)]) > 0, \end{aligned} \quad (92)$$

because $\sin \theta > 0$ for $\theta \in (0, \pi/2)$ and $\text{Re} \kappa > 0$ on the principal branch. The denominator is therefore never zero on the real contour:

$$\mathcal{M}_\theta(p) \cap \mathbb{R}^2 = \emptyset. \quad (93)$$

The wall induced modes represented by the resonance locus are therefore screened modes living off the real axis, and the integrals defining the reflected branch operators are nonsingular for Fredholm theory.

The two limiting cases sharpen this. At $\theta = 0$, Eq. (91) collapses to $i(k - k')$, whose vanishing is

$$k = k', \quad (\theta = 0), \quad (94)$$

so the parallel wall is the degenerate case where the resonance locus collapses onto the real diagonal and the Hankel coupling becomes convolutional, as anticipated in Sec. IV. At $\theta = \pi/2$ the two maps coincide and the resonance condition becomes $\kappa(k, p) + \kappa(k', p) = 0$, which has no solution on the principal sheet because both summands have positive real part, so the resonance locus is empty. For small θ , writing $k' = k + \delta$ and expanding Eq. (91) to first order using $A_\theta = -ik + \theta\kappa + O(\theta^2)$, $B_\theta = ik + \theta\kappa + O(\theta^2)$,

$$B_\theta(k, p) + A_\theta(k', p) = -i\delta + \theta(\kappa(k, p) + \kappa(k', p)) + O(\theta^2), \quad (95)$$

and setting the right-hand side to zero,

$$\delta = -2i\theta \kappa(k, p) + O(\theta^2). \quad (96)$$

The oblique resonance locus is the diagonal shifted into the complex plane by an amount of order θ , with imaginary shift $2\theta\kappa(k, p)$. The exponential $e^{-2d_0\kappa}$ evaluated on the locus therefore acquires an extra damping $\exp(-4d_0\theta \text{Re} \kappa)$ at small θ , the geometric content of the screening picture in the near parallel regime.

G. Branch matrix problem

We now carry out the evaluation announced at the end of Sec. IV. Equation (80) contains the unknown $\Phi_+(q, p)$ at

generic q and at the specific argument $q = B_\theta(k', p)$ inside the reflected integral. Evaluating Eq. (80) at $q = A_\theta(k, p)$ for real k pulls the unknown onto $\Gamma_A(p)$ and yields an equation involving the branch traces

$$\begin{aligned} \Phi_A(k, p) &:= \Phi_+(A_\theta(k, p), p), \\ \Phi_B(k, p) &:= \Phi_+(B_\theta(k, p), p), \end{aligned} \quad (97)$$

together with the branch pullbacks of the direct symbol and the transformed element,

$$\begin{aligned} K_A(k, p) &:= \mathcal{K}_0(A_\theta(k, p), p), \\ K_B(k, p) &:= \mathcal{K}_0(B_\theta(k, p), p), \end{aligned} \quad (98)$$

$$G_A(k, p) := \frac{c_{\text{LC}}}{A_\theta(k, p) [p - v A_\theta(k, p)]}, \quad (99)$$

$$G_B(k, p) := \frac{c_{\text{LC}}}{B_\theta(k, p) [p - v B_\theta(k, p)]}. \quad (100)$$

Evaluation at $q = A_\theta(k, p)$ converts the direct term into $K_A(k, p)\Phi_A(k, p)$, the minus function into $\Phi_{-,A}(k, p) := \Phi_-(A_\theta(k, p), p)$, and the reflected integral into

$$\begin{aligned} (\mathcal{R}_{AB}\Phi_B)(k, p) \\ := \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \frac{e^{-2d_0\kappa(k', p)}}{2D\kappa(k', p)} \frac{\Phi_B(k', p)}{A_\theta(k, p) + A_\theta(k', p)}, \end{aligned} \quad (101)$$

the substitution being $q \rightarrow A_\theta(k, p)$ in the Cauchy denominator of Eq. (79). The upper branch equation is therefore

$$K_A(k, p)\Phi_A(k, p) + (\mathcal{R}_{AB}\Phi_B)(k, p) + \Phi_{-,A}(k, p) = G_A(k, p). \quad (102)$$

Similarly, evaluating Eq. (80) at $q = B_\theta(k, p)$ converts the direct term into $K_B(k, p)\Phi_B(k, p)$, the minus function into $\Phi_{-,B}$, and the reflected integral into

$$\begin{aligned} (\mathcal{R}_{BB}\Phi_B)(k, p) \\ := \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \frac{e^{-2d_0\kappa(k', p)}}{2D\kappa(k', p)} \frac{\Phi_B(k', p)}{B_\theta(k, p) + A_\theta(k', p)}, \end{aligned} \quad (103)$$

with denominator given by Eq. (91), whose nonvanishing on the real contour was just established. The lower branch equation is

$$K_B(k, p)\Phi_B(k, p) + (\mathcal{R}_{BB}\Phi_B)(k, p) + \Phi_{-,B}(k, p) = G_B(k, p). \quad (104)$$

The structural feature that makes the sequential inversion possible is that the reflected term in Eq. (80) samples the unknown only through $\Phi_B(k', p)$, regardless of the spectral argument at which the equation is evaluated. This is a direct consequence of Eq. (56): the source side exponential is $e^{-B_\theta(k', p)u'}$, not $e^{-A_\theta(k', p)u'}$, so the source side integration produces the wedge transform at $q = B_\theta(k', p)$ independently of the field side projection. Both Eqs. (102) and (104) contain Φ_B inside their integrals, and neither contains Φ_A . Equation (104) is therefore a closed equation for Φ_B alone, which once solved feeds into Eq. (102) as a known inhomogeneity. Assembling the two as a matrix system,

$$\begin{pmatrix} K_A & \mathcal{R}_{AB} \\ 0 & K_B + \mathcal{R}_{BB} \end{pmatrix} \begin{pmatrix} \Phi_A \\ \Phi_B \end{pmatrix} + \begin{pmatrix} \Phi_{-,A} \\ \Phi_{-,B} \end{pmatrix} = \begin{pmatrix} G_A \\ G_B \end{pmatrix}, \quad (105)$$

the zero in the lower left entry confirms that the reflected operator never samples Φ_A . Equation (105) is the exact matrix Wiener-Hopf problem of the laboratory reflection geometry. It is not a problem with a 2×2 scalar symbol: K_A and K_B are scalar multiplication operators, whereas $\mathcal{R}_{AB}$ and $\mathcal{R}_{BB}$ are genuine integral operators with the kernels Eqs. (101) and (103). The upper triangular form reduces the whole system to a scalar Fredholm equation for Φ_B followed by a scalar Wiener-Hopf equation for Φ_A in Sec. VI, and is what allows the oblique reflection problem to be solved exactly at all angles.

H. Parallel wall limit as a consistency check

We verify that the upper triangular problem Eq. (105) collapses to the expected scalar Wiener-Hopf equation as $\theta \rightarrow 0$, recovering the symmetric pair problem at spacing $h = 2d_0$. In this limit $A_0(k, p) = -ik$ and $B_0(k, p) = +ik$, so $\Phi_B(k, p) = \Phi_+(ik, p)$ is the boundary value of Φ_+ on the imaginary q axis from the right. Changing variables in the reflected integral of Eq. (80) from k to $q' = ik$, with $dk = -i dq'$, $\kappa(k, p) \rightarrow \Gamma(q', p)$ by Eq. (38), and $q + A_\theta(k, p) \rightarrow q - q'$, the reflected integral becomes

$$\int_{\Lambda_0} \frac{-i dq'}{2\pi} \frac{e^{-2d_0\Gamma(q', p)}}{2D\Gamma(q', p)} \frac{\Phi_+(q', p)}{q - q'}, \quad (106)$$

where Λ_0 is the imaginary q' axis oriented upwards from the right half plane. Closing the contour in the half plane interior to the working strip and using Cauchy's formula, the integral collapses to the residue at $q' = q$:

$$\frac{e^{-2d_0\Gamma(q, p)}}{2D\Gamma(q, p)} \Phi_+(q, p), \quad (107)$$

and Eq. (80) reduces at $\theta = 0$ to

$$\frac{1 + e^{-2d_0\Gamma(q, p)}}{2D\Gamma(q, p)} \Phi_+(q, p) + \Phi_-(q, p) = \frac{c_{LC}}{q(p - vq)}. \quad (108)$$

The scalar symbol

$$\mathcal{K}_0^{\text{tot}}(q, p)|_{\theta=0} = \frac{1 + e^{-2d_0\Gamma(q, p)}}{2D\Gamma(q, p)} \quad (109)$$

is exactly the total symbol of the parallel wall problem Eq. (61), equivalent by the image rule to the symmetric two dendrite Wiener-Hopf problem at spacing $h = 2d_0$. The upper triangular matrix problem therefore reduces to a scalar Wiener-Hopf equation in the parallel limit, with the correct symbol, and the construction of the present section is calibrated against an independently known [35] result at $\theta = 0$. For $\theta > 0$ the Hankel coupling prevents this collapse, and the full triangular system Eq. (105) must be inverted sequentially, which is the subject of the next section. For reference and completion, the parallel wall solution and the steady state solutions both for parallel and oblique walls are given in Appendix C.

The moving endpoint is now an analyticity condition on a single transformed unknown, and the two-dimensional Hankel coupling has become a 2×2 branch matrix that is upper triangular.

VI. EXACT SEQUENTIAL INVERSION OF THE TRANSFORMED BRANCH PROBLEM

The upper triangular matrix problem Eq. (105) is the exact Wiener-Hopf-Hankel equation of the laboratory reflection problem in transformed variables. Its triangularity follows because the reflected integral samples the unknown through the trace Φ_B regardless of the point at which Eq. (80) is evaluated, so the lower row of Eq. (105) is a closed equation for Φ_B alone.

We exploit this to invert the system sequentially, without factorizing any 2×2 operator matrix in closed form, as the Wiener-Hopf matrix does not seem reducible. Instead, as outlined in Sec. VG, the lower branch is treated as a Fredholm equation of the second kind in the plus sector, whose well-posedness follows from the compactness of $\mathcal{R}_{BB}$ on the working strip and the nonvanishing of the resonance denominator Eq. (93). Once Φ_B is known, it enters the upper row as a known inhomogeneity, and what remains is a scalar Wiener-Hopf equation for Φ_A , which can be solved by the explicit factorizations given in Eqs. (42) and (43). The complementary minus functions $\Phi_{-,A}$, $\Phi_{-,B}$ are not prescribed data; they are unknowns dual to Φ_A , Φ_B generated by the half line restriction, and are eliminated throughout by plus projection. The reconstruction of $\Phi_+(q, p)$ on the body of the working strip follows by an explicit strip Cauchy integral, and the physical density $\mu(s, t)$ is recovered by inverting the wedge transform.

A. Analytic setting and the working strip at fixed p

Let us fix p with $\text{Re } p > 0$ and work in the q plane at that p . The direct Eq. (39) has branch points at $q = \pm\sqrt{p/D}$ from Eq. (40). We take the cut to be

$$\Sigma_p := \{q = t\sqrt{p/D} : t \in [-1, 1]\}, \quad (110)$$

that is, the canonical Schwarz reflective choice [36]. The transformed element $G_+(q, p) = c_{LC}/[q(p - vq)]$ from Eq. (68) has simple poles at $q = 0$ and $q = p/v$, and the wedge analyticity condition Eq. (66) requires $\text{Re } q > 0$ and $\text{Re } q < \text{Re}(p/v)$. The intersection of these constraints with the complement of Eq. (110) is the vertical strip

$$\mathcal{S}_p := \left\{ q \in \mathbb{C} : 0 < \text{Re } q < \min\left(\text{Re}\frac{p}{v}, \text{Re}\sqrt{p/D}\right) \right\}, \quad (111)$$

which is nonempty for every p with $\text{Re } p > 0$. On $\mathcal{S}_p$, $\mathcal{K}_0(q, p)$ is analytic and nonvanishing, $G_+(q, p)$ is analytic, and the branch curves $\Gamma_A(p)$, $\Gamma_B(p)$ (the traces of which are represented in Fig. 6) from Eq. (81) lie strictly inside $\mathcal{S}_p$ by Eqs. (82) and (83) whenever $\theta \in (0, \pi/2)$, provided the cut Eq. (110) has been chosen so as not to intersect $\Gamma_A(p) \cup \Gamma_B(p)$. This is immediate in the parameter regime of interest, $d_0 \sim \ell_D$ with θ away from the degenerate limits.

The branch pullbacks of the direct symbol,

$$\begin{aligned} K_A(k, p) &= \mathcal{K}_0(A_\theta(k, p), p), \\ K_B(k, p) &= \mathcal{K}_0(B_\theta(k, p), p), \end{aligned} \quad (112)$$

are therefore single valued and nonvanishing on the real k axis. The reflected branch operators $\mathcal{R}_{AB}$, $\mathcal{R}_{BB}$ defined

by Eqs. (101)–(103) are bounded integral operators on $L^2(\mathbb{R}, dk)$: their kernels are continuous on the real (k, k') plane because the resonance denominator is never zero there by Eq. (93), and they decay exponentially as $|k'| \rightarrow \infty$ through the wall attenuation factor $e^{-2d_0\kappa(k', p)}$ and the algebraic decay of $1/\kappa(k', p)$. Writing $\kappa(k', p) \sim |k'|$ at large $|k'|$, the kernel of $\mathcal{R}_{BB}$ is dominated by $e^{-2d_0|k'|}/(2D|k'|)$ times a bounded factor, so $\mathcal{R}_{BB}$ is bounded on $L^2(\mathbb{R})$ and therefore compact, with operator norm bounded above by

$$\|\mathcal{R}_{BB}\|_{\text{HS}}^2 = \iint \left| \frac{e^{-2d_0\kappa(k', p)}}{2D\kappa(k', p)} \frac{1}{B_\theta(k, p) + A_\theta(k', p)} \right|^2 dk dk' \quad (113)$$

which decays exponentially as $d_0\text{Re}\sqrt{p/D} \rightarrow \infty$. The same bound applies to $\mathcal{R}_{AB}$, *mutatis mutandis*. Under these hypotheses, $K_B^{-1}\mathcal{R}_{BB}$ is compact on $L^2(\mathbb{R}, dk)$, and the Fredholm alternative [37] applies to Eq. (104): either the homogeneous equation $(K_B + \mathcal{R}_{BB})\Phi_B = 0$ has only the trivial solution, in which case the inhomogeneous equation has a unique solution for every right-hand side, or -1 belongs to the spectrum of $K_B^{-1}\mathcal{R}_{BB}$ and solvability is governed by orthogonality to a finite dimensional kernel. The values of p at which this degeneracy occurs are the wall induced screening poles, corresponding to the zeros of $\det(I + K_B^{-1}\mathcal{R}_{BB})$; they lie off the Bromwich contour in the regime of physical interest.

B. Strip Cauchy projection and the annihilation of the minus functions

The minus side functions Φ_- , $\Phi_{-,A}$, $\Phi_{-,B}$ in Eqs. (80) and (105) are not prescribed data: they are the unknowns dual to Φ_+ , Φ_A , Φ_B generated by the half line Wiener-Hopf decomposition, and in the standard scalar machinery they are eliminated by Cauchy projection onto the plus sector. The same elimination must be carried out here, and because the working strip is genuinely two-sided, the appropriate projection is the strip Cauchy projection defined on a vertical contour interior to $\mathcal{S}_p$ rather than on the real axis. Choose γ with

$$0 < \gamma < \min\left(\text{Re}\frac{p}{v}, \text{Re}\sqrt{p/D}\right), \quad (114)$$

so that $\Lambda_\gamma := \{q \in \mathbb{C} : \text{Re}q = \gamma\}$, oriented upwards, lies strictly inside $\mathcal{S}_p$. For any $f(q, p)$ analytic on a neighborhood of Λ_γ in $\mathcal{S}_p$ and decaying sufficiently fast as $|\text{Im}q| \rightarrow \infty$ for the Cauchy integral to converge, the plus and minus projections are

$$(\mathcal{P}_+f)(q, p) := \frac{1}{2\pi i} \int_{\Lambda_\gamma} \frac{f(q', p)}{q' - q} dq', \quad \text{Re}q < \gamma', \quad (115)$$

$$(\mathcal{P}_-f)(q, p) := -\frac{1}{2\pi i} \int_{\Lambda_{\gamma''}} \frac{f(q', p)}{q' - q} dq', \quad \text{Re}q > \gamma'', \quad (116)$$

with $0 < \gamma'' < \gamma < \gamma' < \min(\text{Re}(p/v), \text{Re}\sqrt{p/D})$ and both contours in $\mathcal{S}_p$. By the standard Sokhotski-Plemelj relations for a strip contour [38], $\mathcal{P}_+f$ extends analytically into

$\text{Re}q < \gamma'$, $\mathcal{P}_-f$ into $\text{Re}q > \gamma''$, and on the overlap

$$(\mathcal{P}_+f)(q, p) + (\mathcal{P}_-f)(q, p) = f(q, p), \quad \gamma'' < \text{Re}q < \gamma'. \quad (117)$$

Equation (117) is the standard additive splitting of f into a part analytic to the left of $\Lambda_{\gamma'}$ and a part analytic to the right of $\Lambda_{\gamma''}$, the strip analog of the real line Hilbert transform decomposition. We use Φ_+ for the part analytic to the right of the strip contour and Φ_- for the part analytic to the left, in agreement with the convention that Φ_+ is the wedge transform of the dendrite density.

The decisive property is the annihilation identity

$$\mathcal{P}_+[\Phi_-(q, p)] = 0, \quad (118)$$

which follows from Eq. (115) whenever Φ_- is analytic throughout the half plane to the left of $\Lambda_{\gamma'}$: closing the Cauchy contour in that half plane picks up no residue. The same identity applied to the branch pullbacks gives $\mathcal{P}_+\Phi_{-,A} = 0$ and $\mathcal{P}_+\Phi_{-,B} = 0$ once these are interpreted as functions of k composed with the branch maps, and the strip Cauchy projection of the branch rows of Eq. (105) kills the minus side contributions outright. This reduces the triangular matrix problem to two projected scalar equations.

C. Lower branch as a projected Fredholm equation of the second kind

Applying the strip Cauchy projection to Eq. (104), $\Phi_{-,B}$ is annihilated by Eq. (118) and

$$\begin{aligned} \mathcal{P}_+[K_B(\cdot, p)\Phi_B(\cdot, p)](k) + \mathcal{P}_+[(\mathcal{R}_{BB}\Phi_B)(\cdot, p)](k) \\ = \mathcal{P}_+[G_B(\cdot, p)](k). \end{aligned} \quad (119)$$

Since $K_B(k, p)$ is analytic and nonvanishing on the working strip, we may divide Eq. (104) through by K_B before projecting. The normalized equation is

$$\begin{aligned} \Phi_B(k, p) + (K_B^{-1}\mathcal{R}_{BB}\Phi_B)(k, p) + (K_B^{-1}\Phi_{-,B})(k, p) \\ = (K_B^{-1}G_B)(k, p), \end{aligned} \quad (120)$$

and applying $\mathcal{P}_+$ yields

$$\Phi_B(k, p) + \mathcal{P}_+[K_B^{-1}\mathcal{R}_{BB}\Phi_B](k, p) = \mathcal{P}_+[K_B^{-1}G_B](k, p). \quad (121)$$

In obtaining Eq. (121) we have used that Φ_B is a plus function, so $\mathcal{P}_+\Phi_B = \Phi_B$ in the sense of Eq. (117), and the annihilation $\mathcal{P}_+[K_B^{-1}\Phi_{-,B}] = 0$, which holds because $K_B^{-1}\Phi_{-,B}$ is a minus function on the working strip whenever K_B^{-1} and $\Phi_{-,B}$ are; the former follows from the analyticity of $\mathcal{K}_{0,+}^{-1}$ and $\mathcal{K}_{0,-}^{-1}$ on the appropriate half strips, and the latter is the defining property of $\Phi_{-,B}$.

Introducing

$$\Phi_B^{(0)}(k, p) := \mathcal{P}_+[K_B^{-1}G_B](k, p), \quad (122)$$

$$(\mathcal{C}_{\theta, d_0, p}\psi)(k) := \mathcal{P}_+[K_B^{-1}\mathcal{R}_{BB}\psi](k, p), \quad (123)$$

Eq. (121) reads

$$(I + \mathcal{C}_{\theta, d_0, p})\Phi_B(\cdot, p) = \Phi_B^{(0)}(\cdot, p), \quad (124)$$

an exact Fredholm equation of the second kind with compact perturbation $\mathcal{C}_{\theta,d_0,p}$, by the Hilbert–Schmidt estimate Eq. (113) and the boundedness of K_B^{-1} on the real k line. Provided -1 is not in the spectrum of $\mathcal{C}_{\theta,d_0,p}$, the equation has a unique solution for every plus side forcing, given by the Neumann series

$$\Phi_B(k, p) = \sum_{n=0}^{\infty} (-1)^n (\mathcal{C}_{\theta,d_0,p}^n \Phi_B^{(0)})(k, p), \quad (125)$$

which converges absolutely whenever $\|\mathcal{C}_{\theta,d_0,p}\| < 1$, and by analytic continuation in p for a larger range of parameters. In resolvent form,

$$\Phi_B(k, p) = \Phi_B^{(0)}(k, p) - \int_{-\infty}^{\infty} R_{\theta,d_0,p}(k, k') \Phi_B^{(0)}(k', p) dk', \quad (126)$$

where $R_{\theta,d_0,p}$ is the Fredholm resolvent kernel,

$$R_{\theta,d_0,p}(k, k') = \sum_{n=1}^{\infty} (-1)^{n-1} (\mathcal{C}_{\theta,d_0,p}^n)(k, k'), \quad (127)$$

in the regime where the series converges. The wall induced screening modes of Sec. VIA are the values of p at which this resolvent acquires a pole, equivalently the zeros of the Fredholm determinant

$$\Delta_{\theta,d_0}(p) := \det(I + \mathcal{C}_{\theta,d_0,p}), \quad (128)$$

whose location controls the latest time at which the inversion produces a meaningful physical density. For $d_0 \sim \ell_D$ and the Bromwich contour used in Sec. VIF, the zeros of Eq. (128) lie well to the left of the contour, and the Neumann series Eq. (125) converges rapidly because the operator norm of $\mathcal{C}_{\theta,d_0,p}$ is controlled by Eq. (113) with an exponentially small upper bound.

Finally, for practical or numerical applications, the explicit form of the projected forcing Eq. (122) is necessary. Using Eq. (100) and the factorization Eq. (41) pulled back to $q = B_{\theta}(k, p)$, we get

$$\begin{aligned} & K_B^{-1}(k, p) G_B(k, p) \\ &= \frac{c_{\text{LC}}}{K_{B,+}(k, p) K_{B,-}(k, p) B_{\theta}(k, p) [p - v B_{\theta}(k, p)]}, \end{aligned} \quad (129)$$

and the projection $\mathcal{P}_+$ is evaluated by closing the strip Cauchy contour in the half plane to the right of $\Lambda_{\gamma'}$ and collecting residues at $q = 0$ and $q = p/v$ from the q plane factorization of Eq. (129), with $K_{B,-}$ interpreted through Eq. (43) composed with $B_{\theta}(k, p)$. The result is a fully explicit meromorphic function of k on the working strip, which serves as the initial guess $\Phi_B^{(0)}$ for the Neumann iteration Eq. (125).

D. Upper branch as a scalar Wiener-Hopf problem with known reflected forcing

Once Φ_B is known, the reflected contribution to the upper branch becomes a fully known function of k . Define

$$F_A(k, p) := G_A(k, p) - (\mathcal{R}_{AB} \Phi_B)(k, p), \quad (130)$$

so that Eq. (102) reduces to the scalar equation

$$K_A(k, p) \Phi_A(k, p) + \Phi_{-,A}(k, p) = F_A(k, p), \quad (131)$$

in which the only unknown is Φ_A , together with its dual minus function $\Phi_{-,A}$ to be eliminated by projection. Equation (131) is a standard scalar Wiener-Hopf equation in k with forcing F_A , solved by the canonical factorization Eq. (41) applied to the branch pullback. Let

$$K_{A,\pm}(k, p) := \mathcal{K}_{0,\pm}(A_{\theta}(k, p), p) \quad (132)$$

be the pullbacks of Eqs. (42) and (43) under A_{θ} , so that

$$K_A(k, p) = K_{A,+}(k, p) K_{A,-}(k, p). \quad (133)$$

By the analyticity properties of $\mathcal{K}_{0,\pm}$ on the two half strips bordering the working strip, and because $A_{\theta}(k, p)$ lies strictly inside the working strip for real k by Eq. (82), $K_{A,+}$ is analytic and nonvanishing on the half strip to the right of $\Lambda_{\gamma'}$ and $K_{A,-}$ is analytic and nonvanishing on the half strip to the left of $\Lambda_{\gamma''}$, both in the sense of the branch pullback through A_{θ} .

Divide Eq. (131) by $K_{A,-}(k, p)$. Using Eq. (133), the left-hand side becomes

$$K_{A,+}(k, p) \Phi_A(k, p) + \frac{\Phi_{-,A}(k, p)}{K_{A,-}(k, p)}, \quad (134)$$

and the right-hand side is $F_A(k, p)/K_{A,-}(k, p)$. Applying $\mathcal{P}_+$: the first term is the product of two plus functions $K_{A,+} \Phi_A$ and is itself a plus function, so $\mathcal{P}_+[K_{A,+} \Phi_A] = K_{A,+} \Phi_A$; the second term is the product of the minus function $\Phi_{-,A}$ with the minus function $K_{A,-}^{-1}$ and is a minus function, so $\mathcal{P}_+[\Phi_{-,A} K_{A,-}^{-1}] = 0$ by Eq. (118). The projected equation is

$$K_{A,+}(k, p) \Phi_A(k, p) = \mathcal{P}_+ \left[\frac{F_A(\cdot, p)}{K_{A,-}(\cdot, p)} \right](k), \quad (135)$$

and dividing by $K_{A,+}$, which is nonvanishing on the relevant half strip,

$$\Phi_A(k, p) = \frac{1}{K_{A,+}(k, p)} \mathcal{P}_+ \left[\frac{G_A - \mathcal{R}_{AB} \Phi_B}{K_{A,-}} \right](k, p). \quad (136)$$

This is the exact upper branch trace. The plus projection is evaluated by Eq. (115) on a vertical contour interior to the working strip. The sequential inversion of Eq. (105) is therefore complete: Φ_B is given by the projected Fredholm equation Eq. (124) and its Neumann–resolvent solution Eqs. (125) and (126), and Φ_A by the explicit scalar Wiener-Hopf formula Eq. (136) once Φ_B is known.

E. Reconstruction of the transformed density on the body of the strip

The branch traces Φ_A, Φ_B determine $\Phi_+(q, p)$ on the two curves $\Gamma_A(p), \Gamma_B(p)$, but not at a generic point of the body of the working strip. To recover Φ_+ in the interior of $\mathcal{S}_p$ we return to Eq. (80). With $\Phi_B(k, p)$ known, the reflected term becomes a known function of (q, p) ,

$$R(q, p) := \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{-2d_0\kappa(k,p)}}{2D\kappa(k,p)} \frac{\Phi_B(k, p)}{q + A_{\theta}(k, p)}, \quad (137)$$

and Eq. (80) reduces to the scalar Wiener-Hopf equation

$$\mathcal{K}_0(q, p) \Phi_+(q, p) + \Phi_-(q, p) = \frac{c_{LC}}{q(p - vq)} - R(q, p), \quad (138)$$

on $\mathcal{S}_p$, with $\mathcal{K}_0$ from Eq. (39) and factorization Eqs. (42) and (43). The standard scalar Wiener-Hopf procedure applied to Eq. (138), exactly as in the derivation of Eq. (136), produces

$$\begin{aligned} \Phi_+(q, p) &= \mathcal{K}_{0,+}^{-1}(q, p) \mathcal{P}_+ \\ &\times \left[\mathcal{K}_{0,-}^{-1}(q, p) \left(\frac{c_{LC}}{q(p - vq)} - R(q, p) \right) \right], \end{aligned} \quad (139)$$

valid for every $q \in \mathcal{S}_p$ and p in the right half plane. The projection is evaluated by Eq. (115) acting on the q argument of the bracketed expression. Equation (139) gives Φ_+ on the full body of the strip, not only on the two branches, and is the analytic object from which every physical observable is built.

F. Double inversion via Cagniard-de Hoop and the laboratory density

The physical density on the fixed line is recovered from Φ_+ by inverting the wedge transform Eq. (62). Since the transform is a single two variable integral, not a composition, its inverse is the iterated Bromwich formula

$$v(u, t) = \frac{1}{(2\pi i)^2} \int_{\gamma_p - i\infty}^{\gamma_p + i\infty} dp e^{pt} \int_{\gamma_q - i\infty}^{\gamma_q + i\infty} dq e^{qu} \Phi_+(q, p), \quad (140)$$

with (γ_q, γ_p) chosen jointly inside the working strip Eq. (66), that is, $\gamma_q > 0$ and $\gamma_p > v\gamma_q$. The laboratory density is recovered by the shift $u = s - vt$,

$$\mu(s, t) = v(s - vt, t), \quad s > 0, \quad 0 < t < t_*, \quad (141)$$

in agreement with Eq. (25). Equation (140) is a genuine double inversion in p and q jointly: the integrand depends on both variables through $\mathcal{K}_0$, through $c_{LC}/[q(p - vq)]$, and through $R(q, p)$, and these do not factor as a product. The natural analytic route is to evaluate the inner q integral first, at fixed p on the outer Bromwich contour, by contour deformation onto $\Gamma_A(p) \cup \Gamma_B(p)$. This is possible because $\Phi_+(q, p)$ is analytic throughout the working strip by Eq. (139), and because the spectral branches lie strictly inside $\mathcal{S}_p$. The contribution from the deformed contour is a sum of residues at the poles of $\Phi_+(q, p)$ in the deformation region plus the integrals along the branch curves. We note that when $\Phi_+(q, p)$ is rational in q , as is the case after deriving the Padé-Abrahams approximant surrogate in Appendix B, the deformed q integral collapses to a finite sum of residues, but that is not the case for the general solution.

Two equivalent analytic strategies are available for the outer p integral. The first is the Cagniard-de Hoop transformation [39,40], which introduces

$$v_C := \sqrt{p/D}, \quad p = Dv_C^2, \quad dp = 2Dv_C dv_C, \quad (142)$$

and rewrites the Bromwich integral as

$$\mu(s, t) = \frac{D}{\pi i} \int_{B'_C} e^{Dv_C^2 t} \mu_L(s, Dv_C^2) v_C dv_C, \quad (143)$$

where B'_C is the image of the Bromwich contour under $p = Dv_C^2$ and $\mu_L(s, p)$ is the Laplace domain laboratory density obtained by carrying out the inner q integral of Eq. (140) and applying Eq. (141). The square-root branch cut at $q = \pm\sqrt{p/D}$ is replaced by a pair of branch points of $v_C(p)$ on the imaginary v_C axis, and B'_C can be deformed onto $v_C = i\tau + 0^+$, $\tau \in \mathbb{R}$, without crossing any singularity other than the wall induced screening modes Eq. (128), which lie off the contour. Combining conjugate boundary values across the imaginary axis and using Schwarz's reflection principle $\text{Re}[(i\tau)\mu_L] = -\tau \text{Im} \mu_L$,

$$\mu(s, t) = \frac{2D}{\pi} \text{Re} \int_0^\infty e^{-D\tau^2 t} \mu_L(s, -D\tau^2 + i0^+) (i\tau) d\tau, \quad (144)$$

the exact Cagniard-de Hoop representation. Equation (144) is convenient when closed form expressions for μ_L are available on the imaginary p axis, as happens in the Padé-Abrahams reduction of the next section: the imaginary v_C contour avoids the exponential blow up of the Bromwich integrand at large $|\text{Im} p|$, and reveals the physical time dependence through the Gaussian factor $e^{-D\tau^2 t}$.

The Cagniard-de Hoop approach is useful to achieve an analytically rich solution, but if one must integrate Eq. (144) numerically, it is problematic as many poles in μ_L are bound to lie along the contour of integration. Thus, the second strategy via the Talbot contour [41,42], which deforms the Bromwich contour into an open shaped curve (parabola, hyperbola, co-tangent, etc) in the left half-plane wrapping around the cut Eq. (110) and the screening poles while driving the integrand to zero exponentially along its asymptotic arms. Talbot quadrature achieves exponential convergence in the number of nodes [42] and is used here for the isolated dendrite reference, because it does not require the closed form knowledge of μ_L on the imaginary axis that Cagniard exploits, and it controls the growth of the integrand uniformly. For the wall problem the outer integral is evaluated instead by the Stehfest algorithm [43], which samples μ_L only at real positive $p = n \log 2/t$. The reason is the wall induced screening poles Eq. (128). Their positions depend on d_0 and θ through the Fredholm determinant and must be located before a Talbot contour can be placed to wrap around them, whereas a real sampling path never approaches them and the branch maps A_θ and B_θ can be tracked by continuation in p along it. The density decays monotonically behind the tip and carries no oscillation, which is the regime in which Stehfest is reliable: at $N = 14$ nodes the result is converged to four decimal places, and the value is unchanged from $N = 10$ upwards.

We note that the contour strategies are analytically equivalent: they give the same value for Eq. (140) whenever the Bromwich contour can be deformed onto either without crossing a singularity, the criterion being the absence of wall induced screening poles between the Bromwich line and the deformed contour. In the parameter regime of the present work, this criterion is met automatically, and all three routes produce the same $\mu(s, t)$ to any prescribed tolerance.

Once $\mu(s, t)$ is known either analytically or through numerical reconstruction, the concentration field in the wall half plane is recovered by substituting it into Eq. (22). The

single-layer integral is evaluated in wall fixed coordinates using Eq. (16), and the result is an explicit quadrature in (s', τ) over the wedge Ω_+ . No additional inversion is required: μ is the full analytic content of the transport problem, and the field is a direct integral of μ against a known Gaussian kernel. The analytic chain from Eq. (23) to $\mu(s, t)$ is complete, and every step is an exact equality on the working strip, without approximation beyond the choices of branch and contour made in Sec. VI A. The rational reduction of Appendix B replaces the Fredholm equation of Sec. VI C by a finite dimensional linear system, turning the exact analytic construction of the present section into a closed form algebraic replacement amenable to direct evaluation.

The chain just described is analytically complete but not algebraically closed: the lower branch trace solves a Fredholm equation whose kernel contains the radical $\kappa(k, p) = \sqrt{k^2 + p/D}$, and the upper branch requires a canonical factorization evaluated by contour integration. Both obstructions are removed by replacing κ with a rational approximant of controllable order, via the Padé-Abrahams approach originally developed for elastodynamics [24,25,27]. As applied to our problem, this Padé-Abrahams algebraic approximation is derived in Appendix B; here we summarize its findings and uses. Indeed, written as a function of $z = k^2$, the radical is a Stieltjes function on the plane cut along $z \in (-\infty, -\lambda^2]$ with $\lambda = \sqrt{p/D}$, so its $[n-1/n]$ Padé approximant has the canonical partial fraction form

$$\kappa_n(k, p) = \lambda + \sum_{j=1}^n \frac{r_j(p) k^2}{k^2 + \alpha_j(p)^2}, \quad (145)$$

with $\alpha_j^2 > 0$, $r_j > 0$ and $\sum_j r_j = 1$, the last condition enforcing the correct $\kappa \sim |k|$ growth. The coefficients follow at each Laplace node from a Hankel linear system and a companion matrix eigendecomposition, both $O(n^3)$. Equation (145) is even in k , so the conjugation symmetry $B_\theta(k, p) = A_\theta(-\bar{k}, \bar{p})$ between the two branch maps survives the approximation, and its only singularities are the simple poles $k = \pm i\alpha_j(p)$ on the imaginary axis. Consequently the branch maps become rational, the reflected kernel becomes finite rank and the Fredholm equation of Sec. VI C collapses to an $n \times n$ linear system, the canonical factorization reduces to sorting polynomial roots by half plane, and the Cauchy projection becomes residue arithmetic. The only approximation in the entire chain is Eq. (145); every subsequent step is an exact equality within the rational reduction, and the error decays exponentially in n by the convergence theorem for Stieltjes Padé approximants, reaching the floor set by the outer Bromwich quadrature at $n \simeq 12$, as can be seen in Fig. 16 in the Appendix B. The construction of the Padé-Abrahams approximant therefore provides an accurate and cheap means of evaluating the outer field at a fraction of the cost of evaluating the exact solutions. Nevertheless, all results reported in the sequel are computed with the exact solver of this section; the rational reduction is used only as an independent check and as a fast surrogate.

VII. WALL-CORRECTED CONCENTRATION FIELD

The exact branch matrix problem of Sec. V, sequentially inverted by the procedure of Sec. VI, gives the wall-corrected

line density $\mu(s, t; d_0, \theta)$ on the physical dendrite and, through the single-layer representation of Sec. III C, the concentration field $c(x, y, t; d_0, \theta)$ everywhere in the laboratory liquid. In this section we examine the structure of this solution as a function of the obliquity angle, the initial wall offset, and the elapsed time. The aim is to convert the operator theory of the previous sections into a quantitative picture of the field around an oblique dendrite near a wall, and to extract from that picture a small number of compact field-level scalars that summarize the wall feedback in a form that can be used as the basis for the matching argument of the next section.

All numerical results in this section, and in the second results section that follows the process zone interlude, are computed by the closed form Wiener-Hopf reconstruction of Sec. VI, with Stehfest quadrature [43] for the time inversion in p and the analytic residue sum of Eqs. (140) and following for the wedge inversion in q . The implementation is described in detail in Appendix A, where we also describe an independent direct boundary integral solver for the half-space problem of Sec. III, which we use throughout to cross validate the closed form solver. The agreement between the two solvers is at the level of one part in a thousand on the integrated line density across the entire parameter range examined here, and is reported in Appendix A. Unless otherwise stated all results are computed in the standard units $D = 1$, $v = 2$, $c_{LC} = 1$, for which the natural diffusion length is $\ell_D = D/v = 1/2$ and the natural diffusion time is $t_D = D/v^2 = 1/4$.

A. Concentration field around an oblique dendrite near a wall

We begin with the qualitative picture. Figure 4(a) shows the wall-corrected concentration field $c(x, y, t; d_0, \theta)$ for a dendrite at obliquity $\theta = \pi/6$, initial offset $d_0 = 1$, and evaluation time $t = 0.4$. The three panels of the figure show, in order, the wall-corrected concentration field itself, the difference between the wall-corrected field and the corresponding isolated single dendrite field at the same instant (solved by the author in Ref. [35]), and the line density $\mu(s, t; d_0, \theta)$ along the dendrite together with its isolated counterpart. The wall-corrected field shows the expected pile up of solute on the side of the dendrite that faces the wall, and the difference field isolates this pile up as a localized positive lobe whose support is set by the geometric tip to wall distance and whose amplitude grows as the distance shrinks. The line density panel shows that the wall-corrected μ is uniformly larger than the isolated μ over the entire arc length range, with the largest enhancement occurring near the tip where the matching to the inner solvability problem is performed.

Figures 4(b) and 4(c) show the corresponding fields at obliquities $\theta = \pi/4$ and $\theta = \pi/3$, respectively, with the same initial offset and evaluation time. The qualitative structure is preserved across the three angles. The amplitude of the difference lobe in the central panel and the gap between the wall-corrected and isolated traces in the right panel both decrease monotonically as the obliquity increases. The grazing case $\theta = \pi/6$ produces the largest wall feedback because the wall and the dendrite are nearly aligned, so that the diffusion plume has the longest available path along the wall before being absorbed by the dendrite shadow. The near normal case $\theta = \pi/3$ produces the smallest wall feedback because the wall sits ahead of the tip rather than alongside it, so that

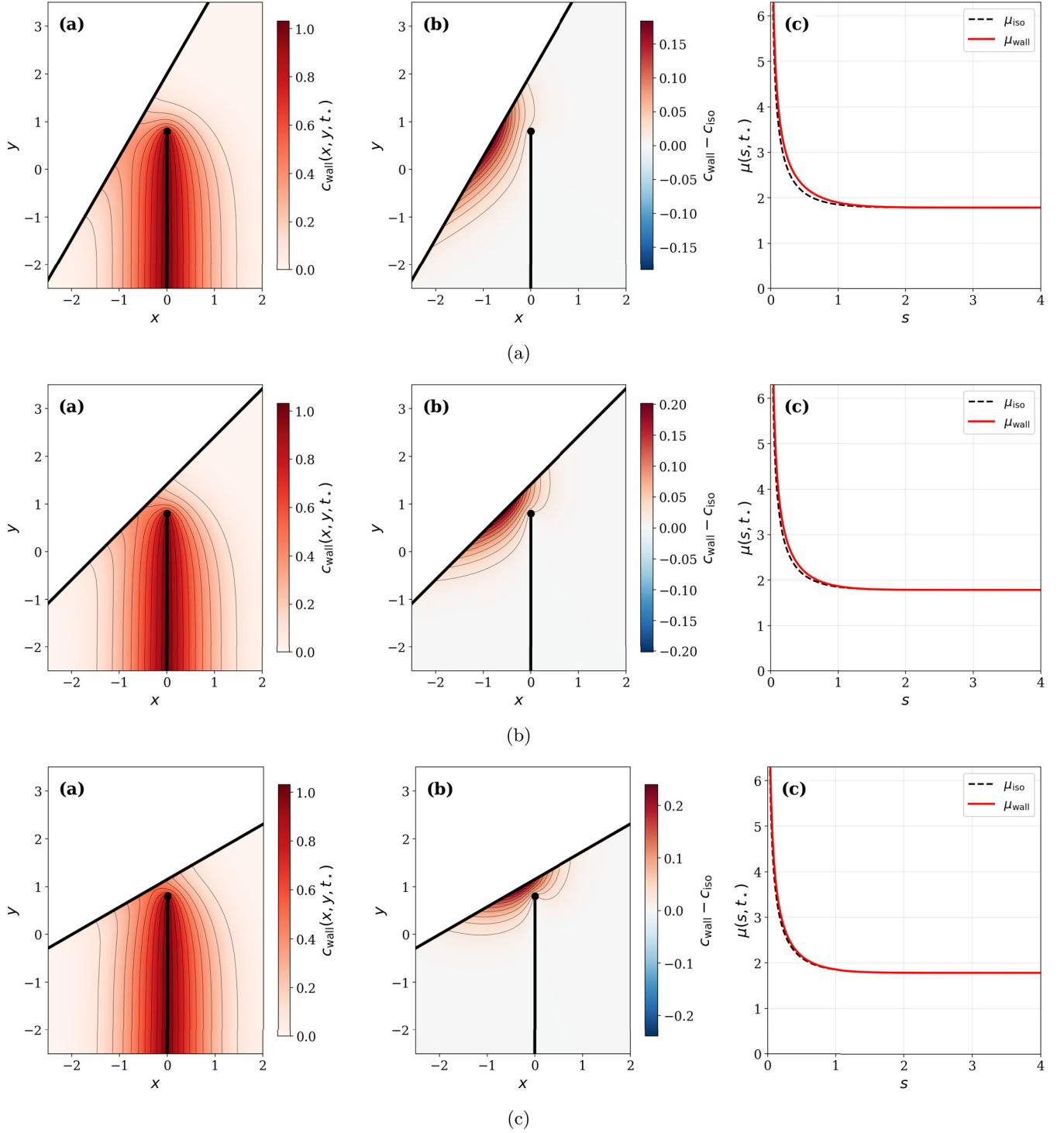


FIG. 4. Wall-corrected concentration field at $d_0 = 1$, $t = 0.4$ for three obliquities. In each row: (left) the field $c(x, y, t)$ with isoconcentration contours; (middle) the difference $c_{\text{wall}} - c_{\text{iso}}$, showing the wall induced pile up between dendrite and wall; (right) the line density $\mu(s, t)$ (solid) against the isolated reference (dashed). (a) $\theta = \pi/6$, (b) $\theta = \pi/4$, and (c) $\theta = \pi/3$.

the reflected plume must travel against the growth direction to reach the dendrite and is correspondingly attenuated. The intermediate case $\theta = \pi/4$ interpolates smoothly between the two extremes. These three figures together establish the angular hierarchy of the wall feedback in the cleanest possible way, by holding the kinematic parameters fixed and varying only the obliquity.

B. Time evolution of the line density as the tip approaches the wall

The figures of the previous subsection show the wall-corrected field at a single instant. The unsteady character of the laboratory problem is best seen by examining how $\mu(s, t)$ evolves as the tip closes in on the wall. Figure 5 shows $\mu(s, t; d_0, \theta)$ at three obliquities $\theta \in \{\pi/6, \pi/4, \pi/3\}$

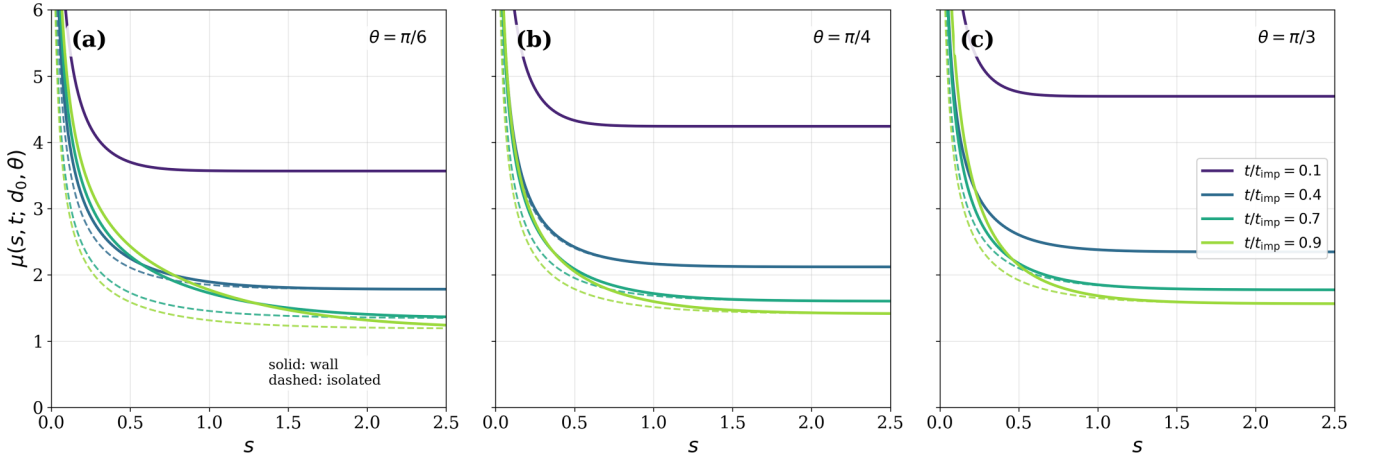


FIG. 5. Time evolution of $\mu(s, t; d_0, \theta)$ at $d_0 = 1$ for $\theta = \pi/6$ (a), $\pi/4$ (b), $\pi/3$ (c), at four common fractions $f = t/t_{\text{imp}}(\theta)$ of the kinematic contact time. Wall-corrected curves solid, isolated reference dashed. Sampling at common f fixes $d(t_*) = d_0(1 - f)$ across the angles.

at $d_0 = 1$, sampled at four representative fractions $f \in \{0.1, 0.4, 0.7, 0.9\}$ of the kinematic contact time

$$t_{\text{imp}}(\theta) = \frac{d_0}{v \sin \theta}, \quad (146)$$

which is the laboratory time at which the tip would reach the wall if continued at constant velocity along the original growth direction. The instantaneous wall distance at evaluation time $t_* = f t_{\text{imp}}$ is then $d(t_*) = d_0(1 - f)$, which is independent of the obliquity. Sampling at common fractions of the contact time therefore separates the kinematic dependence on θ , which is absorbed in the choice of t_* , from the spectral dependence on θ , which remains as the only difference between the three panels. In each panel the wall-corrected curves are drawn solid and the isolated reference at the same evaluation time is drawn dashed in the same color. At $f = 0.1$ the wall has had little time to project a coherent feedback onto the dendrite and the wall-corrected and isolated curves are essentially indistinguishable across the three panels. At $f = 0.4$ the wall feedback has begun to develop, and the wall-corrected curves sit visibly above the isolated reference at the smallest obliquities. By $f = 0.7$ the wall feedback is fully developed, with the wall-corrected μ exceeding the isolated value by a substantial fraction over the entire arc length range. At $f = 0.9$, immediately before the kinematic contact, the wall feedback dominates the line density at small s . The progressive lifting of the wall-corrected curves above their isolated references is the unsteady signature of the wall projected onto the dendrite, and its angular ordering, with the largest lift at $\theta = \pi/6$ and the smallest at $\theta = \pi/3$, recovers the angular hierarchy already seen in the static figures.

C. Scalar-field-level diagnostics

The figures of the previous two subsections give a complete visual account of the wall feedback but produce no compact summary that can be quoted as a number. We now extract two scalar measurements from the wall-corrected solution, both defined directly from the field and the line density without reference to any inner matching variable, which together compress the angular and temporal dependence of the wall feedback into a single figure. The first is the integrated line

density excess

$$\mathcal{Q}_\mu(t; d_0, \theta) := \int_0^{s_{\text{max}}} [\mu_{\text{wall}}(s, t; d_0, \theta) - \mu_{\text{iso}}(s, t)] ds, \quad (147)$$

where the upper integration limit s_{max} is chosen large enough that the integrand is numerically zero by the time the limit is reached. The second is the maximum reflected concentration excess

$$\Delta c_{\text{max}}(t; d_0, \theta) := \max_{(x,y)} [c_{\text{wall}}(x, y, t; d_0, \theta) - c_{\text{iso}}(x, y, t)], \quad (148)$$

the maximum over the laboratory liquid of the wall to isolated difference. Both quantities are strictly positive for any nonzero wall feedback and vanish in the isolated limit $d_0 \rightarrow \infty$. The line density excess summarizes the contribution of the wall to the integrated tip drive, and the field excess summarizes the strength of the local pile up that the dendrite generates against the no flux boundary.

Figure 6 shows the time evolution of these two scalars at $d_0 = 1$ for the three obliquities $\theta \in \{\pi/6, \pi/4, \pi/3\}$, with the time axis normalized by $t_{\text{imp}}(\theta)$ so that the three trajectories occupy the same range $t/t_{\text{imp}} \in [0, 1]$ and the angular comparison is direct. Both scalars rise smoothly from zero at small t/t_{imp} , where the wall has had no time to project a feedback, to a finite limit just before kinematic contact. The angular ordering is the same in both panels, with $\theta = \pi/6$ the largest and $\theta = \pi/3$ the smallest, and the gap between the curves is systematic. At $t/t_{\text{imp}} = 0.85$, the integrated line density excess at the grazing angle has reached $\mathcal{Q}_\mu \simeq 1.2$ in the standard units, against $\mathcal{Q}_\mu \simeq 0.4$ at the steepest angle, a factor of three between the extremes. At the same fraction of the contact time, the maximum reflected concentration excess is $\Delta c_{\text{max}} \simeq 0.38$ at the grazing angle and $\Delta c_{\text{max}} \simeq 0.32$ at the steepest, against the bulk concentration scale $c_{\text{LC}} = 1$. The two scalars compress the descriptive content of the previous two subsections into a single figure, and they will be the natural reference for the matching argument of Sec. VIII.

D. Kinematic collapse at fixed instantaneous wall distance

The two diagnostics of the previous subsection mix the obliquity dependence of the wall feedback with the kinematic

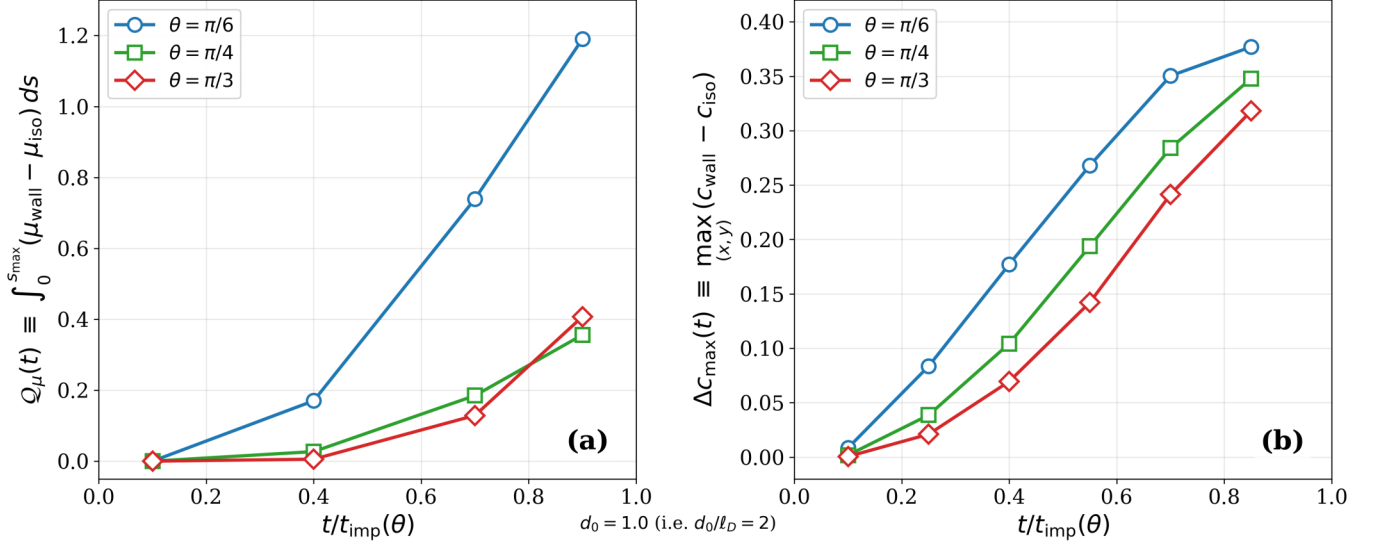


FIG. 6. Scalar-field diagnostics at $d_0 = 1$ for three obliquities versus $t/t_{\text{imp}}(\theta)$. (a) integrated line density excess Q_μ , Eq. (147); (b) maximum reflected concentration excess Δc_{max} , Eq. (148). Both rise monotonically with the angular ordering $\pi/6 > \pi/4 > \pi/3$.

dependence of the instantaneous wall distance $d(t) = d_0 - vt \sin \theta$. To isolate the spectral dependence of the wall coupling on the obliquity, we now hold both the evaluation time t_* and the instantaneous wall distance $d(t_*)$ fixed and vary only θ , by choosing the initial offset as $d_0(\theta) = d_{\text{fixed}} + v t_* \sin \theta$ for each angle. With this construction, every member of the family of wall-corrected solutions is evaluated at the same instant of laboratory time and at the same geometric tip to wall distance, and the only quantity that changes from one member to the next is the obliquity of the wall as seen from the dendrite.

Figure 9(a) shows the resulting family of wall to isolated line density ratios $\mu_{\text{wall}}(s, t_*)/\mu_{\text{iso}}(s, t_*)$ at $t_* = 0.5$ and $d(t_*)/\ell_D = 1$, for 10 obliquities in $\{10^\circ, 15^\circ, 20^\circ, 25^\circ, 30^\circ, 40^\circ, 50^\circ, 60^\circ, 70^\circ, 80^\circ\}$. The family is smoothly ordered in θ , with each curve carrying a single broad maximum in s that drifts continuously from a position near $s = 0.6$ at the smallest obliquity to a position near $s = 0$ at the largest. No two curves cross each other, and the entire family interpolates monotonically between a strongly wall-enhanced grazing limit and a near isolated normal limit. The clean ordering shows that the laboratory wall problem is, at fixed instantaneous geometry, a one-parameter family in the obliquity, with the spectral coupling of the wall acting as a smooth single valued function of θ . There is no anomalous angle, no special configuration where the family crosses itself, and no signature of any operator theoretic structure beyond the smooth angular drift of the branch traces of Sec. V. This is a positive structural result on its own: the laboratory problem reduces, at fixed instantaneous geometry, to a clean competition between geometry and obliquity with no hidden dynamical complication.

E. Effective range of the wall feedback

The figures of the previous subsections all use moderately small initial offsets $d_0 \sim 1$ –1.5 in the standard units, where the wall feedback is at the order of 10 percent or larger and is therefore visible in the line density and in the field. For

larger initial offsets the wall feedback is small and the isolated single dendrite solution is essentially valid. We now ask how rapidly the feedback decays with d_0 , both as a quantitative test of the laboratory theory in the perturbative regime and as a practical engineering criterion for when the much simpler isolated theory can be substituted for the laboratory theory without loss of accuracy.

The natural scalar diagnostic for this purpose is the relative line density excess

$$\mathcal{E}(d_0; \theta, t_*) := \max_s \frac{\mu_{\text{wall}}(s, t_*; d_0, \theta) - \mu_{\text{iso}}(s, t_*)}{\mu_{\text{iso}}(s, t_*)}, \quad (149)$$

which measures the largest relative departure of the wall-corrected line density from the isolated reference at evaluation time t_* . By construction $\mathcal{E} \rightarrow 0$ as $d_0 \rightarrow \infty$ at fixed t_* , and the speed of the decay carries information about the effective range of the wall feedback. Figure 7 shows $\mathcal{E}(d_0; \theta, t_* = 0.4)$ on a logarithmic vertical scale for the three obliquities $\theta \in \{\pi/6, \pi/4, \pi/3\}$, with d_0 measured in units of the diffusion length ℓ_D . The decay is approximately exponential in d_0/ℓ_D , with a characteristic length of order ℓ_D across all three angles, consistent with the round trip diffusion length argument that the reflected plume must travel a distance of order $2d_0$ from the dendrite to the wall and back before contributing to the line density at the dendrite. The horizontal dashed line at $\mathcal{E} = 10^{-2}$ marks a one percent threshold below which the isolated solution is essentially valid; the three angles cross this threshold at $d_0/\ell_D \simeq 2.4$ to 2.6, with a mild angular dependence that places the grazing and steepest angles slightly above and the intermediate $\theta = \pi/4$ slightly below. The angular dependence of the threshold is nonmonotone in θ , reflecting the competition between the long feedback path that grazing incidence provides and the short kinematic time that steep incidence requires. Beyond $d_0/\ell_D \simeq 5$ the relative excess falls below 10^{-6} and the laboratory and isolated solutions are numerically identical to the floor of the closed form solver.

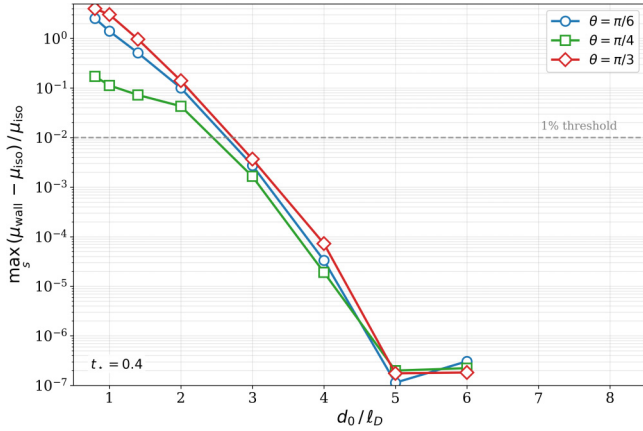


FIG. 7. Effective range of the wall feedback. Relative line density excess $\mathcal{E}(d_0)$, Eq. (149), at $t_* = 0.4$ versus d_0/ℓ_D for three obliquities. The decay is approximately exponential with characteristic length ℓ_D . The dashed line marks the one percent threshold; the angles cross it at $d_0/\ell_D \simeq 2.4$ to 2.6 .

Thus, for $d_0 \gtrsim 5\ell_D$ the wall is invisible and the isolated single dendrite theory of the literature can be used with no detectable error, while for $d_0 \lesssim 2\ell_D$ the wall is the dominant correction and the theory here presented is mandatory. The crossover between the two regimes is narrow and is well captured by the closed form reconstruction.

F. Padé-Abrahams surrogate

For applications where the cost of the closed form Wiener-Hopf reconstruction becomes prohibitive, the rational surrogate of Appendix B can be substituted for the exact branch reconstruction. Figure 8 shows that at order $n = 12$ the surrogate reproduces both the line density $\mu(s, t)$ and the concentration field $c(x, y, t)$ to better than 5×10^{-5} in relative error throughout the parameter range examined here. For the purposes of a detailed wall-dendrite analysis under discussion here, the closed form reconstruction is computationally fast enough that there is no operational reason to prefer the

Abrahams surrogate, and all the figures in this section are computed from the exact reconstruction; however, as can be seen in Fig. 8, the Abrahams approximant offers remarkable accuracy and, as discussed in Appendixes A and B, is computationally simpler to implement.

G. Absence of observable resonance signatures

The spectral analysis of Sec. V identified a rich branch trace structure for the wedge transformed reflection operator. It is natural to ask whether the stationary phase critical points of the trace functions $\Gamma_A(k, p)$ and $\Gamma_B(k, p)$ produce resonant features in the inverted line density at special angles where the local geometry of the branches admits constructive interference. The kinematic collapse subsection above already provides part of the answer: the family of wall-corrected line density ratios at fixed instantaneous geometry is a smooth monotone function of θ with no anomalous features. We complete the answer here by inspecting the spectral representation directly.

Figure 9(b) shows the absolute value $|\Phi_+(q, p)|$ of the closed form Wiener-Hopf reconstruction of Eq. (140) on the probe line $q = \gamma + i\omega$ at fixed real Laplace parameter $p = 5$, which is a representative Stehfest node in the production scheme. Seventeen obliquities in the range 5° to 85° are overlaid. The spectral envelope is a smoothly ordered family of bell shaped curves with no satellite features, no curve crossings, and no pole collisions, with the amplitude scaling smoothly from ~ 24 at the smallest obliquity to ~ 2 at the largest. Direct inspection of the pole locations $\{q_k\}$ and residues $\{B_k\}$ extracted from the closed form reconstruction confirms that every individual mode of the representation drifts smoothly with θ , with the dominant residue being the angle independent forcing pole and the subdominant residues decaying smoothly through several orders of magnitude as the obliquity is increased. The branch trace structure of Sec. V therefore does not project onto observable resonances in the line density or in the spectral representation, and is fully absorbed into the inner Fredholm solve before any of it can reach the macroscopic observable.

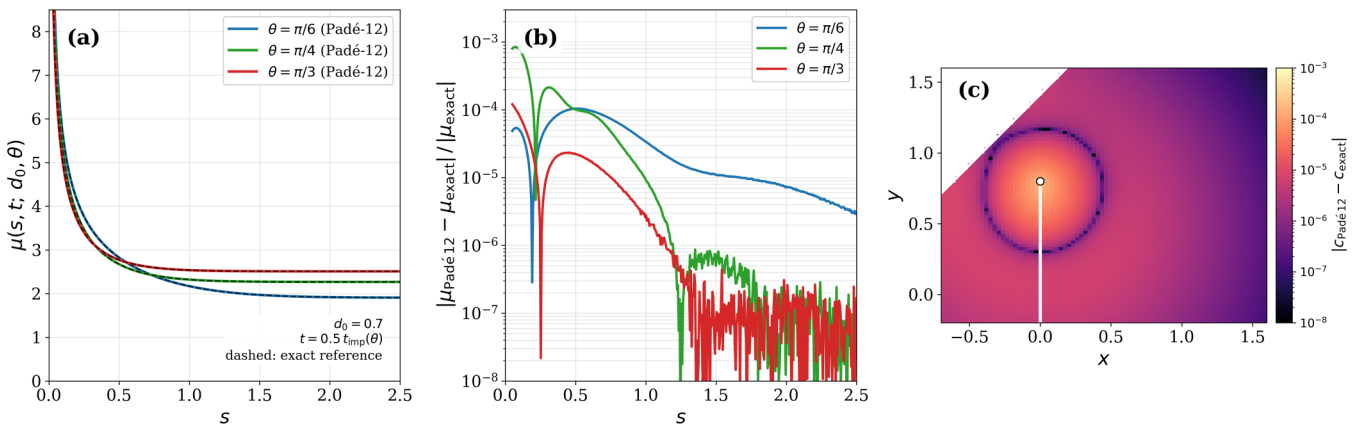


FIG. 8. Padé-Abrahams surrogate at order $n = 12$ against the exact closed form reconstruction. (a) Line density at $d_0 = 0.7$, $t_* = \frac{1}{2}t_{\text{imp}}(\theta)$, surrogate solid and exact dashed; (b) relative error of the surrogate on a log scale; (c) absolute error of the reconstructed concentration field at $\theta = \pi/4$, $d_0 = 1$, $t = 0.4$, on a log color scale. Field error stays below 2×10^{-4} .

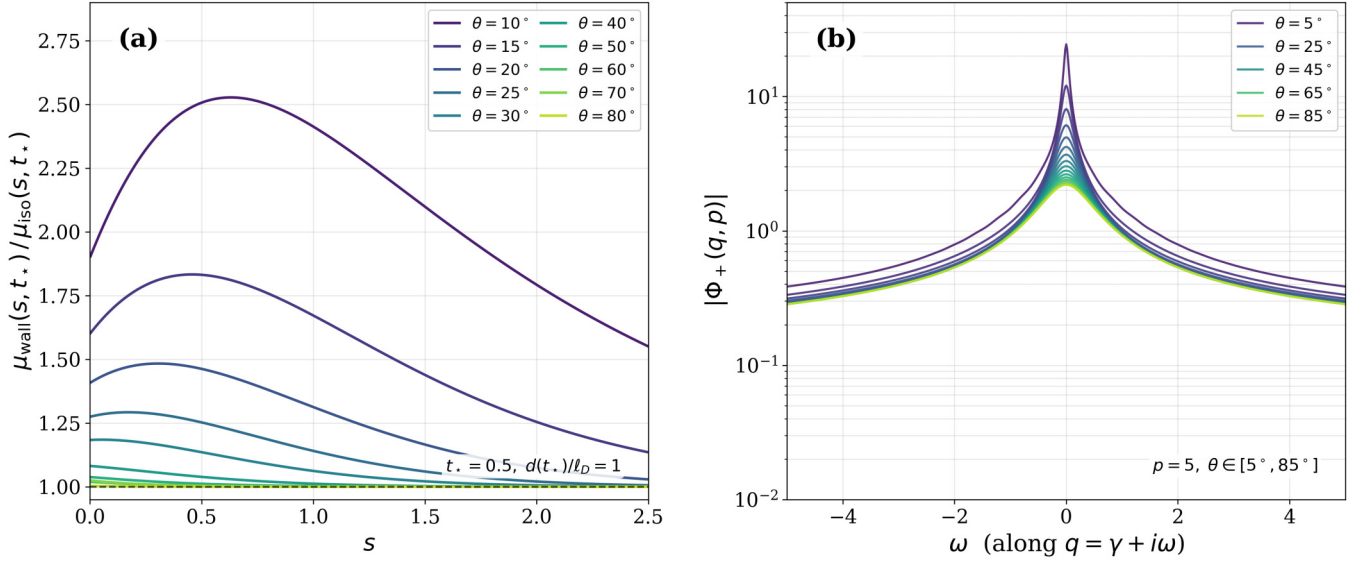


FIG. 9. Search for spectral resonance signatures at fixed $t_* = 0.5$ and $d(t_*)/\ell_D = 1$, with $d_0(\theta)$ adjusted to hold the instantaneous geometry constant. (a) wall to isolated line density ratio for 10 obliquities; (b) absolute value of the closed form representation $|\Phi_+(q, p)|$ on the probe line $q = \gamma + i\omega$ at $p = 5$ for seventeen obliquities. Both panels show smoothly ordered families, with a localized bump in $\mu(s, t)$ near $s \simeq 0.5$ at the smallest obliquities marking the first return of the reflected depletion from the wall; the bump smears out at larger θ as the image source moves off the dendrite axis and the return spreads along the arc length.

H. Postcontact tip continuation

At the instant of contact, $t = t_{\text{imp}}(\theta)$, the laboratory model breaks down: the tip reaches the wall and the constant velocity straight line assumption no longer holds. One can still ask what the concentration field predicts about the natural continuation direction of a tip that moves with the local gradient. The no flux condition forces ∇c to be tangent to the wall, so the only available continuation direction is along the wall tangent $\pm(\sin \theta, \cos \theta)$. The symmetry between the two orientations is broken by the dendrite shadow: the existing dendrite has

depleted the solute behind the tip, so the backward wall direction is more shadowed than the forward one, and the gradient of the undercooling is larger in the forward direction. The resulting turning angle, measured from $\hat{y}$ to $(\sin \theta, \cos \theta)$, is exactly θ ; it is not the billiard reflection 2θ that an elastic analogy would suggest, but the wall tangent itself.

Figure 10 verifies this directly. Figure 10(a) shows the wall-corrected field at $t = 0.95 t_{\text{imp}}$ for $\theta = \pi/4$, $d_0 = 1$, $v = 2$, with the contact point marked by a star and the predicted postcontact direction overlaid. The contours are visibly

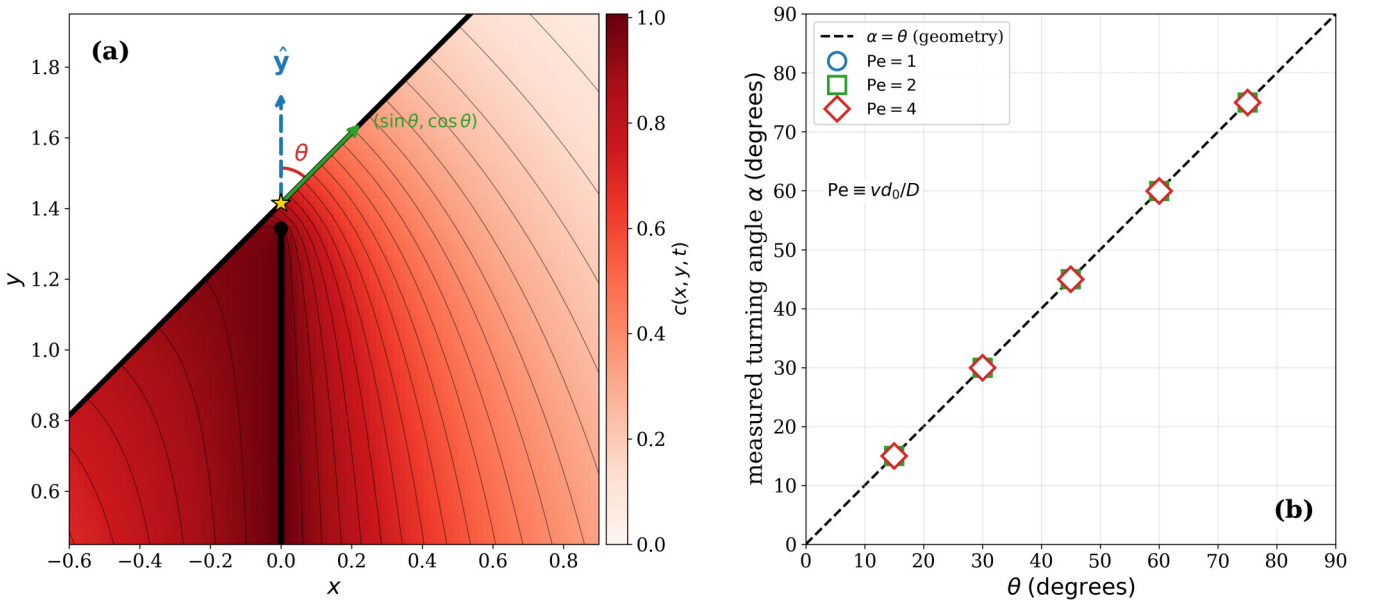


FIG. 10. Postcontact tip continuation. (a) Concentration field at $t = 0.95 t_{\text{imp}}$ for $\theta = \pi/4$, $d_0 = 1$, $v = 2$, with contact point (gold star), blocked direction $\hat{y}$ (dashed), predicted postcontact direction $(\sin \theta, \cos \theta)$ (solid), and turning angle θ marked. (b) Measured turning angle α versus θ for three Peéclet numbers; all 15 measurements lie on $\alpha = \theta$ (dashed).

compressed on the forward side and spread out on the backward side, where the dendrite shadow has depleted the solute. Figure 10(b) measures α by sampling c on either side of the contact point along the wall tangent, at five obliquities $\theta \in \{15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ\}$ and three Péclet numbers $Pe = vd_0/D \in \{1, 2, 4\}$. Every measurement lies on $\alpha = \theta$. The independence from Pe follows directly from the no flux condition: the wall tangent is geometric and the sign of the tangential gradient is fixed by the symmetric dendrite shadow, neither of which depends on any dynamical parameter.

This tangent selection falls out of the same solver that produces the field results of this section, without any additional assumption beyond the no flux condition. The actual evolution past contact is controlled by the inner sharp interface problem, with anisotropy and curvature selection that lie outside the scope of this paper; the statement here is about the direction the concentration field would select for a gradient driven tip in the isotropic limit. In a real alloy the transport field bias toward the wall tangent competes with the preferred crystallographic growth axes of the solid. When the wall obliquity places the tangent direction far from a low-angle orientation, the conflict between the diffusive gradient and the anisotropic capillarity is severe, and the physical response is likely to be tip splitting, side branching, or the nucleation of a misoriented grain rather than a smooth isotropic curve. The isotropic prediction here that $\alpha = \theta$ is therefore best read as the geometric bias that the transport field applies to the tip at contact, not as a literal trajectory: it identifies the direction the outer field would select in the absence of crystalline preference, and thereby sets the reference against which the anisotropic deflection is measured.

VIII. PROCESS ZONE FUNCTIONAL AND THE CRITICAL WALL OFFSET

The sequential inversion of Sec. VI delivers the wall-corrected line density $\mu(s, t)$ on the dendrite of the advancing dendrite. This is the full analytic content of the transport problem, but it is not yet the physical object that decides whether the wall sustains the tip. To make that decision we need an observable built from μ that communicates the outer transport field to the inner sharp interface problem in whose terms tip selection is formulated. The present section identifies that observable as a finite integral of μ over a small region behind the tip, analyses the characteristic times governing its departure from the isolated dendrite, and extracts the critical wall offset beyond which the wall correction is too weak to sustain advance.

A. Why the tip does not see the wall directly

The dendrite tip is a tiny, coherent selection structure. Its radius ρ is set by the balance between capillarity and solvability on a length scale much smaller than anything else in the problem, and in particular much smaller than d_0 or $\ell_D = D/v$. On the scale of the tip, the rest of the dendrite, the wall, and the outer depletion field look like infinity. The inner tip problem is therefore local, quasistationary, and universal: it is the same Ivantsov solvability problem that a solitary dendrite would solve, and it is insensitive to the existence of the wall in any direct sense. Whatever the wall does, it does not enter the inner

equations that decide how the tip curves and advances on its own scale.

What the wall changes is the outer transport field, and specifically the rate at which fresh liquid can flow into the inner region. The inner selection problem is matched to the outer field not through a pointwise value but through this supply rate [44,45], set by the gradient of concentration established in a small neighborhood of the tip on a length scale intermediate between the capillary length and ℓ_D . An isolated dendrite builds that gradient from self-depletion alone; a dendrite close to an oblique wall builds it from self-depletion minus the reflected return of its own signal, which reduces the available supply.

The obvious candidate coupling variable is the pointwise amplitude of $\mu(s, t)$ as $s \downarrow 0$, or the coefficient $A(t)$ of the leading $1/\sqrt{s}$ singularity in

$$\mu(s, t) = \frac{A(t)}{\sqrt{s}} + B(t) + O(\sqrt{s}), \quad s \downarrow 0. \quad (150)$$

This coefficient is the natural scalar associated with the tip in any boundary integral representation, but it is useless as a competitive variable: $A(t)$ is universal, determined entirely by the local sharp interface problem at the tip, and carries no information about the wall. The leading singularity is a kinematic consequence of the Dirichlet condition at the endpoint, and any two dendrites with the same internal selection parameters carry the same $A(t)$ regardless of how much external interaction they experience. Appendix A 4 confirms this on the computed solution: the reconstructed $\mu(s, t)\sqrt{s}$ collapses onto a common plateau for an isolated dendrite and for wall offsets $d_0/\ell_D = 2, 3$ and 4, with the fitted amplitudes agreeing to six significant figures. The reflected kernel carries $e^{-2d_0\Gamma(q,p)}$ with $\text{Re}\Gamma \sim |q|$ at large $|q|$, so the large $|q|$ limit that fixes $A(t)$ is controlled by the direct symbol alone. The competitive information lives in the subleading coefficient $B(t)$ and more generally in the short distance structure of μ on the process zone of length ℓ behind the tip. The correct coupling variable is a finite integral of μ over that zone, weighted to regularise the singular leading term and keep $B(t)$ as a genuine competitively sensitive contribution.

B. Exact process zone functional

The exact process zone feeding functional (originally defined for the interaction between two dendrites in Ref. [35]) of the reflection problem is

$$\Psi_{\text{ex}}(\ell, t; d_0, \theta) := \frac{1}{2\sqrt{\ell}} \int_0^\ell \mu(s, t; d_0, \theta) ds, \quad (151)$$

with $\ell > 0$ the process zone length and the prefactor $1/(2\sqrt{\ell})$ dictated by dimensional and asymptotic considerations. Substituting Eq. (150) and using

$$\frac{1}{2\sqrt{\ell}} \int_0^\ell \frac{ds}{\sqrt{s}} = 1, \quad \frac{1}{2\sqrt{\ell}} \int_0^\ell ds = \frac{\sqrt{\ell}}{2}, \quad (152)$$

one finds

$$\begin{aligned} \Psi_{\text{ex}}(\ell, t; d_0, \theta) &= A(t; d_0, \theta) \\ &\quad + \frac{1}{2}B(t; d_0, \theta)\sqrt{\ell} + O(\ell), \quad \ell \downarrow 0. \end{aligned} \quad (153)$$

The leading term is exactly the universal tip amplitude A , so Ψ_{ex} limits onto the pointwise amplitude as $\ell \downarrow 0$ and inherits the correct leading behavior of the Dirichlet singularity. The next term is proportional to $\sqrt{\ell}$ and weighted by the subleading coefficient B , which is where the competitive information about the wall lives, and it enters at a rate set by the process zone length rather than the tip scale. The construction is linear in μ and respects superposition: writing $\mu = \mu_{\text{dir}} + \mu_{\text{ref}}$, the functional splits as $\Psi_{\text{ex}} = \Psi_{\infty} + \Psi_{\text{ref}}$, and Ψ_{ref} is the direct measure of how much the wall has reduced the feeding of the tip.

The prefactor $1/(2\sqrt{\ell})$ has dimensions of inverse square root of length and makes Ψ_{ex} dimensionally a concentration, the same as A , which is what makes it a legitimate proxy for the effective supersaturation seen by the tip. Without it, $\int_0^\ell \mu ds$ would be a concentration times a length and could not couple directly to the dimensionless Ivantsov relation [2] of the sharp interface selection problem. This is not a convention: the prefactor is what converts an amount of depleted material into a local supersaturation, which is the physical currency of dendrite selection. The inverse square-root structure of the leading coefficient $A(t)$ and the role of the process zone length ℓ as the regularizing scale of the matching are the diffusion analogues of the inverse square-root stress singularity at a crack tip and of the fracture process zone, respectively, in the static and dynamic fracture mechanics literature [46,47].

The analogy is not merely dimensional. In fracture mechanics the stress intensity factor couples the outer elastic field to the inner cohesive zone through a scalar that is insensitive to the detailed asymmetry of the applied loading, because the process zone is small enough that the incoming field is effectively homogenized over it. The same mechanism operates here: the capillary length d_0^{cap} is orders of magnitude smaller than the scale over which the wall induced asymmetry of the concentration field varies, so the immediate tip region homogenizes the incoming flux over the process zone ℓ , and the scalar coupling of Ψ_{ex} to the symmetric Ivantsov relation is a physically sound homogenization rather than an uncontrolled approximation. The error of the scalar reduction is of order d_0^{cap}/ℓ , which is negligible in any regime where the sharp interface description itself is valid.

C. Wedge transform representation

For numerical evaluation and for the analytic discussion below, it is convenient to express Ψ_{ex} directly in terms of the wedge transform $\Phi_+(q, p)$ rather than reconstruct $\mu(s, t)$ first. Changing variables in Eq. (151) from s to $u = s - vt$, so the integration runs from $u = -vt$ to $u = \ell - vt$, and substituting the inverse wedge transform Eq. (140), the inner u integral against e^{qu} is elementary and evaluates to $(e^{q(\ell-vt)} - e^{-qv t})/q = e^{-qv t}(e^{q\ell} - 1)/q$. Pulling the factor $e^{-qv t}$ into the p integrand, where it combines with e^{pt} into $e^{(p-vq)t}$,

$$\begin{aligned} \Psi_{\text{ex}}(\ell, t; d_0, \theta) &= \frac{1}{2\sqrt{\ell}} \frac{1}{(2\pi i)^2} \iint e^{(p-vq)t} \frac{e^{q\ell} - 1}{q} \Phi_+(q, p) dq dp, \end{aligned} \quad (154)$$

with the double contour taken in the working strip $\gamma_q > 0$, $\text{Re}(\gamma_p - v\gamma_q) > 0$. Equation (154) is the exact evaluation of the process zone functional directly from the transformed density. The kinematic phase is $e^{(p-vq)t}$, not e^{pt} : the advancing tip enters through the shift $p \rightarrow p - vq$, which is the signature of the wedge analyticity condition and encodes the fact that as the tip moves, the region from which feeding is drawn moves with it. The spectral envelope $(e^{q\ell} - 1)/q$ is the transform of the indicator of the process zone in the fixed line variable. The wall geometry enters only through $\Phi_+(q, p)$, which carries d_0 and θ through the branch maps and wall attenuation of earlier sections; there is no gauge factor, no Robin reflection coefficient, and no frozen offset parameter.

D. Reflected return time, kinematic shrinkage, and the oblique Péclet number

The most important physical quantity of this section is not a formula but the comparison of two timescales. The reflected contribution to Ψ_{ex} does not act instantaneously: the depletion must diffuse to the wall, bounce, and return, and the round trip time on the diffusive scale is

$$t_{\text{ref}} \sim \frac{(2d_0)^2}{D} = \frac{4d_0^2}{D}. \quad (155)$$

Before t_{ref} the wall is effectively invisible, Ψ_{ex} reduces to its isolated value Ψ_{∞} , and the wall plays no role. After t_{ref} the reflected depletion has matured, and the feeding begins to be reduced. The quadratic dependence of t_{ref} on the round trip distance $2d_0$ is the generic signature of any diffusive return process and carries no information about obliquity.

The second timescale is kinematic. While the depletion travels out and back, the tip itself moves, and the offset shrinks. At any time $t < t_*$, $d(t) = d_0 - vt \sin \theta$ from Eq. (10), which reaches zero at

$$t_* = \frac{d_0}{v \sin \theta}. \quad (156)$$

The kinematic shrinkage time t_* and the diffusive return time t_{ref} are independent physical scales, and the whole structure of the reflected competition is controlled by their ratio

$$\frac{t_{\text{ref}}}{t_*} = \frac{4d_0^2/D}{d_0/(v \sin \theta)} = \frac{4vd_0 \sin \theta}{D} = 4 \text{Pe}_\theta, \quad (157)$$

where

$$\text{Pe}_\theta := \frac{vd_0 \sin \theta}{D} \quad (158)$$

is the oblique Péclet number built from the wall kinematics. Equation (157) is the whole physical content of the laboratory versus frozen offset distinction.

When $\text{Pe}_\theta \ll 1$, the diffusive round trip is much shorter than the kinematic shrinkage, so the reflected depletion matures essentially instantaneously on the scale of offset evolution. The wall is effectively stationary during the diffusive return, the reflected signal arrives while the offset is still close to d_0 , and the frozen offset treatment at $d = d_0$ is accurate at leading order. The small Pe_θ limit is the regime in which the old literature on parallel and weakly oblique walls is quantitatively correct.

When $\text{Pe}_\theta \gtrsim 1$, the two timescales are comparable and the picture changes qualitatively. The offset shrinks appreciably during the diffusive return, so the reflected signal arrives at a tip that is now closer to the wall than the tip that emitted it, and the reflected amplitude is systematically larger than at frozen d_0 . The feeding deficit accumulates with a memory of the whole trajectory from d_0 down to $d(t)$, weighted by the diffusive return kernel over the history. Neither frozen offset treatment reproduces this: $d = d_0$ underestimates the reflected amplitude because it uses the largest offset throughout, and $d = d(t)$ overestimates it because it uses the smallest. The laboratory formulation is the only one that captures the history exactly, and $\text{Pe}_\theta \gtrsim 1$ is the regime in which the difference becomes quantitatively important.

Thus, in directional solidification at prescribed pulling speed v , Pe_θ is fixed by materials and geometry, and decides whether a simple frozen analysis suffices. For typical metallic systems with $D \sim 10^{-9} \text{ m}^2/\text{s}$ and $v \sim 10^{-4} \text{ m/s}$, $\ell_D = D/v \sim 10 \mu\text{m}$, and for offsets of tens of micrometres and obliquities of tens of degrees one finds Pe_θ of order unity. The regime in which the laboratory formulation is a faithful description is therefore not a mathematical curiosity but the regime of primary dendrite arrays growing against grain boundaries or container walls, which is the setting of most practical interest.

E. From feeding to tip speed and selected radius

Ψ_{ex} is built from the outer transport field and is not yet a growth law. The physical chain that converts it into a statement about the tip speed V and selected radius ρ is the standard sharp interface matching of the outer feeding to the inner Ivantsov-solvability problem, and it is where the wall actually affects the growth. The inner problem solves for the tip speed and radius of a parabolic needle in a uniform supersaturation (cf. Ref. [9]) through

$$\Delta = I(P), \quad \rho^2 V = \frac{2Dd_0^{\text{cap}}}{\sigma^*}, \quad (159)$$

with Δ the effective supersaturation at the tip, $P = \rho V/(2D)$ the tip Péclet number, I the Ivantsov function of a parabolic needle in two dimensions, d_0^{cap} the chemical capillary length, and σ^* the solvability selection constant. In the presence of a wall, Δ is not externally imposed but induced by the outer transport field at the process zone, which is what Ψ_{ex} measures. The matching is

$$\Delta_{\text{ex}}(t; d_0, \theta, \ell) = \Lambda_m \Psi_{\text{ex}}(\ell, t; d_0, \theta), \quad (160)$$

with Λ_m the dimensionless matching constant whose value depends on the process zone geometry and is the same for all dendrites by universality of the tip singularity structure.

Substituting Eq. (160) into Eq. (159) and eliminating P ,

$$V_{\text{ex}}(t; d_0, \theta, \ell) = \frac{2Dd_0^{\text{cap}}}{\sigma^*} [I^{-1}(\Lambda_m \Psi_{\text{ex}})]^2, \quad (161)$$

$$\rho_{\text{ex}}(t; d_0, \theta, \ell) = \frac{d_0^{\text{cap}}}{\sigma^* I^{-1}(\Lambda_m \Psi_{\text{ex}})}. \quad (162)$$

Physically, Eqs. (161) and (162) show that the wall does not enter either equation directly: it enters only through the single scalar Ψ_{ex} , and everything else is the isolated dendrite

selection law. When Ψ_{ex} falls, the Péclet number falls with it, the tip speed falls as its square, and the tip radius rises as its reciprocal. A dendrite growing toward an oblique wall therefore simultaneously slows down and thickens, which is a distinctive morphological signature of reflected competition that distinguishes it from competition mediated by an externally imposed supersaturation drop. Both effects trace to the same reduction of feeding, and their relative magnitudes are set by the shape of the Ivantsov function near the operating point.

The sensitivity of V_{ex} to reductions of Ψ_{ex} is strongest for slow tips in weakly supersaturated baths, where I' is small and even a modest wall induced feeding deficit produces a disproportionately large reduction of the tip speed; at the moderate supersaturations of most practical interest the response is finite and order unity in appropriate dimensionless units.

F. Exact critical wall offset

A dendrite cannot sustain steady advance for arbitrarily small feeding. Equation (159) admits a physically meaningful solution only for supersaturations at which $I^{-1}(\Delta)$ is well defined and positive, which sets a minimum feeding level

$$\Psi_c := \frac{1}{\Lambda_m} I(0^+) \quad (163)$$

below which the tip Péclet vanishes and steady advance is impossible. Physically, when the feeding drops to Ψ_c the effective supersaturation can no longer maintain the Gibbs–Thomson undercooling of the curved tip against capillary resistance, and the tip arrests. For a prescribed evaluation time t_* , obliquity θ , and inner threshold Ψ_c , the critical initial wall offset $d_{0,c}(\theta)$ is defined implicitly by

$$\Psi_{\text{ex}}(t_*; d_{0,c}, \theta) = \Psi_c. \quad (164)$$

Because the late-time wall correction increases as the initial offset is reduced, offsets $d_0 < d_{0,c}(\theta)$ place the tip within the wall-assisted regime, where $\Psi_{\text{ex}} > \Psi_c$ and the mobility law admits a steadily advancing tip. For $d_0 > d_{0,c}(\theta)$ the wall return is too weak to raise the matching variable above threshold and the tip remains subcritical. The quantity $d_{0,c}$ is therefore the outer boundary of the wall-assisted survival zone. The dependence on obliquity reflects both the geometric dependence of the wall attenuation $e^{-2d_0\kappa(k,p)}$ on the oblique branch geometry of Sec. V and the kinematic shrinkage of the offset encoded in the wedge transform.

The critical offset $d_{0,c}(\theta)$ should not be confused with the parallel wall spacing of earlier literature. The parallel result $d_{0,c}(0)$ is the limit of Eq. (168) as $\theta \rightarrow 0$ at fixed d_0 , and agrees with the symmetric pair problem at spacing $h = 2d_{0,c}(0)$, as verified in Sec. IV. For $\theta > 0$ the oblique criterion departs from the parallel one, and the departure is not small when $\text{Pe}_\theta \gtrsim 1$, for the reasons given in Sec. VIII D. Any attempt to apply a frozen offset parallel wall result to a genuinely oblique geometry underestimates the critical offset in this regime, because it misses the history by which the reflected depletion was built up on a shrinking offset.

Equation (168) is a prediction for a single tip in two dimensions at prescribed velocity, and comparison with measured primary spacings in cast alloys is not direct: an observed

spacing reflects competition within the array as well as proximity to the boundary, and the present criterion addresses only the latter. The testable content is instead the sign reversal of the wall correction along an individual approach trajectory. Beyond about two diffusion lengths the correction is suppressive and below one percent; within one it is enhancing and reaches tens of percent at grazing incidence. Resolving that sequence requires the tip position and velocity to be followed continuously along the approach, which a post mortem spacing measurement cannot supply.

G. Natural choice of the process zone scale

It remains to specify the process zone length ℓ at which Eq. (151) and the offset criterion are evaluated. On dimensional grounds the only intrinsic length built from the transport parameters alone is $\ell_D = D/v$, and in the absence of any other constraint the process zone should be set to a number of order unity times ℓ_D . In a problem with an external competitor, however, a second length enters through the screening of the dominant reflected mode, and this length rather than ℓ_D fixes the process zone scale. The physical reason is straightforward: the tip averages the outer feeding over the region from which the inner growth zone draws supply, which in an isolated dendrite has size ℓ_D , but in the presence of a wall the reflected depletion decorrelates from the self depletion beyond the screening length of the dominant wall induced mode. Averaging over a region longer than the screening length would wash out the competitive signal rather than enhance it, so the process zone length should be chosen equal to that screening length.

In the parallel wall limit, the dominant reflected mode is the first pole of the convolutional symbol Eq. (61), whose lowest pole on the imaginary Γ axis corresponds to an imaginary wave number k of magnitude $\pi/(2d_0)$, and whose screening length is therefore

$$\ell_0^* := \frac{2d_0}{\pi}. \quad (165)$$

This is a factor $2/\pi$ smaller than the round trip distance that appears in the diffusive return time, and the physical interpretation is that the tip averages the feeding over a length comparable to the offset itself, which is the scale on which the parallel wall modifies the concentration field perceptibly. In the oblique case, the dominant reflected pole is the lowest zero of the Fredholm determinant $\Delta_{\theta,d_0}(p)$ of Eq. (128) at the evaluation Laplace parameter, and the corresponding screening length depends on θ through the branch geometry of the spectral curves. Denoting it ℓ_θ^* , we take

$$\ell = c_\ell \ell_\theta^*, \quad c_\ell = O(1), \quad (166)$$

with $c_\ell = 1$ adopted hereafter unless otherwise stated. This anchors the process zone scale to the dominant reflected screening mode rather than leaving it as a free parameter, and is the same self consistent prescription used in the free space symmetric pair problem: under the image rule of Sec. IV, which maps the parallel wall at offset d_0 onto the symmetric pair at spacing $h = 2d_0$, Eq. (165) is exactly the dominant screening length h/π of that problem [35]. Equation (166) is not a fit: it is determined by the spectrum of the lower branch

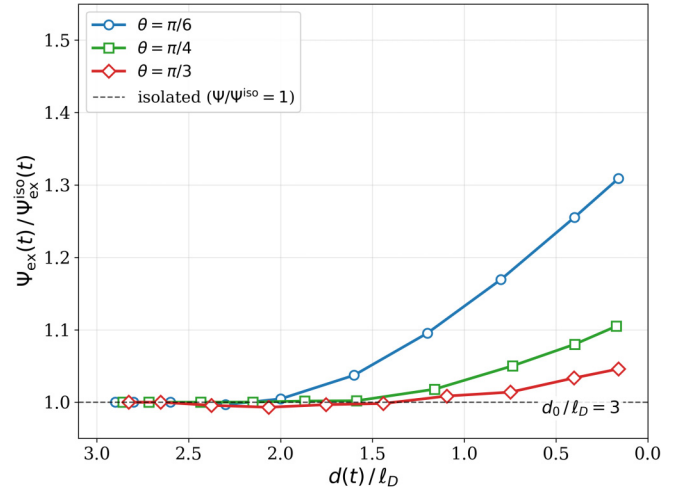


FIG. 11. Enhancement factor $\Psi_{ex}(t)/\Psi_{ex}^{iso}(t)$ versus the instantaneous wall distance $d(t)/\ell_D$, along the trajectory of an oblique dendrite at $d_0/\ell_D = 3$, for three obliquities. The horizontal axis is inverted so that time runs left to right.

Fredholm operator, which is itself determined by d_0 and θ , so the offset criterion Eq. (168) becomes a closed statement with no adjustable parameters beyond those fixed by the materials data of the sharp interface selection problem. Appendix A 4 shows that varying c_ℓ over an order of magnitude changes the magnitude of the wall correction but leaves its sign, its decay shape and the offset beyond which it vanishes unaltered.

H. Range of validity of the transport memory mechanism

The late-time recovery and enhancement rest on the reflected field reaching the tip while retaining information about an earlier, wider gap. The signal arriving at time t was emitted at $t - t_{ref}$ with $t_{ref} \simeq 4d^2/D$, and it carries a geometry distinct from the instantaneous one only if the gap has changed appreciably over that interval, that is if t_{ref} is comparable to the closing time $d/(v \sin \theta)$. Writing the instantaneous oblique Péclet number as $Pe_\theta(t) = v d(t) \sin \theta / D$, the two timescales balance at $4Pe_\theta \sim 1$, so the mechanism operates in a band of order unity in Pe_θ rather than in a limit. The band is bounded on both sides, and for different reasons. At small Pe_θ the reflected return is quasistatic: the signal completes its round trip before the geometry has moved, carries the present gap rather than a wider one, and the correction reduces to the instantaneous blockage that the frozen offset construction describes. At large Pe_θ the round trip is not completed within the trajectory available, so the wall feedback has no time to develop and the correction stays small rather than becoming strongly suppressive. Both conditions are satisfied over the portion of the parameter range in which the late-time recovery is observed. The weak suppression at $d/\ell_D \gtrsim 2$ in Fig. 11 is the small Pe_θ edge of the band, where the return is still quasistatic and the reflected depletion acts as blockage. Two idealizations bound the result independently of Pe_θ . The dendrite is represented as a line of zero thickness, which requires $\rho \ll d$; as the tip approaches contact this fails, and the last fraction of the trajectory, $d \lesssim \rho$, lies outside the theory. The tip is also treated as isotropic, and anisotropic

surface energy enters the inner solvability problem, modifying both the mobility relation of Sec. VIII E and the postcontact continuation direction, which in the isotropic limit is the wall tangent. The restriction is weaker for cubic alloys, where the anisotropy is small and the isotropic solvability constant is a reasonable approximation, than for hexagonal alloys such as magnesium, where the basal to prismatic asymmetry is large and the selected growth direction is set by the crystallography rather than by the transport field. Quantitative application to the latter would require the anisotropic inner problem. For the prescribed straight line trajectory considered here, neither modifies the outer transport solution. Both would affect its coupling to a self-consistently evolving tip.

Thus, the transport problem is solved at prescribed constant v , and the wall-corrected tip velocity is obtained afterwards by feeding Ψ_{ex} into the mobility relation of Sec. VIII E. The tip does not accelerate inside the transport calculation. What the exact solution delivers is the outer feeding along a prescribed trajectory, and the velocity inferred from it is the instantaneous drive that this feeding would sustain, not the solution of a coupled moving boundary problem in which the trajectory responds to the drive. The neglected feedback is of definite sign. A tip whose drive is lifted advances faster, closes the gap sooner, and therefore has less time to accumulate the retarded return that produced the lift. The enhancement reported here is accordingly an upper bound on what a self-consistent calculation would give, and the crossing distances quoted in Sec. IX A are lower bounds on the gap at which the correction changes sign. Resolving this requires the trajectory to be advanced with the feeding recomputed at each step, which the present formulation supports but which lies outside the scope of an exact solution.

IX. IMPLICATIONS OF THE WALL-CORRECTED FEEDING FOR AN OBLIQUE TIP NEAR A WALL

The process zone functional $\Psi_{\text{ex}}(\ell, t; d_0, \theta)$ introduced in the previous section acts as the matching variable that converts the wall-corrected field of Sec. VII into an outer-to-inner coupling for the sharp interface problem, and we interpret it throughout as the local diffusive flux delivered to the tip over a process zone of length ℓ .

The central result of this section inverts the naive frozen offset picture. A quasistatic analysis of the wall problem at the instantaneous tip to wall distance would predict that reflected starvation depresses the local feeding; the exact unsteady result is the opposite. The laboratory wall-corrected Ψ_{ex} falls marginally below the isolated reference while the wall is distant and then rises above it as the gap closes, so that the wall *lifts* the flux available to the approaching tip through the later part of the approach, while tempering the divergent enhancement that a frozen reading at small instantaneous distance would give. There is a close analogy with the image force on an elastodynamically moving dislocation near a free surface: the time-dependent elastic field carries an enhanced attractive contribution that the elastostatic image construction misses entirely [48,49], and an analogous mechanism operates here for a diffusive field reflected off a closing wall. Combined with the tip mobility law of Sec. VIII, the wall-enhanced Ψ_{ex} yields a wall lifted tip drive for an individual dendrite

in the geometry studied here, with the inner problem entering parametrically through the threshold Ψ_c . Standard units $D = 1$, $v = 2$, $c_{LC} = 1$ and the natural process zone scale $\ell = \ell_D = D/v$ are used throughout.

A. Wall enhancement of the matching variable along the approach trajectory

The cleanest way of studying the wall feedback at the level of the matching variable is the dimensionless ratio $\Psi_{\text{ex}}(t)/\Psi_{\text{ex}}^{\text{iso}}(t)$, in which the laboratory feeding at evaluation time t is normalized by the isolated single dendrite feeding at the same instant. The ratio is unity when the wall has no effect, falls below unity where the reflected depletion blocks the feeding, and rises above unity when the retarded return lifts it. Figure 11 shows this ratio as a function of the instantaneous wall distance $d(t)/\ell_D$ along the trajectory of a dendrite at fixed initial offset $d_0/\ell_D = 3$ for three obliquities $\theta \in \{\pi/6, \pi/4, \pi/3\}$. The horizontal axis is inverted so that time runs from right to left as the tip closes in on the wall, with the dendrite starting at $d = d_0$ at $t = 0$ and reaching $d = 0$ at the kinematic contact.

The three trajectories share the same qualitative structure. Each curve dips marginally below the isolated reference over $1.5 \lesssim d/\ell_D \lesssim 2.7$, where the reflected depletion returns with little retardation and acts as ordinary geometric blockage. This is the behavior a quasistatic reading would predict, and it is present, but it never exceeds one percent. Each curve then crosses unity and rises monotonically as d/ℓ_D decreases further, reaching its maximum just before the kinematic contact at $d = 0$. The crossing occurs at $d/\ell_D \simeq 2.0$ at the grazing angle $\theta = \pi/6$, at $d/\ell_D \simeq 1.55$ at $\theta = \pi/4$ and at $d/\ell_D \simeq 1.5$ at $\theta = \pi/3$, so the grazing trajectory both crosses earliest and gains most. The maximum enhancement is approximately 31% at the grazing angle $\theta = \pi/6$, approximately 11% at the intermediate angle $\theta = \pi/4$, and approximately 4% at the steepest angle $\theta = \pi/3$, exceeding the preceding suppression by factors of roughly 30, 10, and 4, respectively. The wall correction is therefore of both signs along the approach, but the two phases are not comparable in magnitude: the suppressive phase is a percent level effect and the enhancement is a tens of percent effect at grazing incidence. The angular hierarchy is the same as in the line density figures of Sec. VII, but the matching variable compresses the spatial information of $\mu(s, t)$ into a single scalar per (t, d_0, θ) , so that the hierarchy can be read in one glance. The departure of the wall-corrected flux from the isolated reference is rapidly suppressed beyond roughly two diffusion lengths in the parameter range shown, and is of order 10 percent or larger only when the tip is within one diffusion length of the wall. The fact that the curves do not collapse onto a single envelope when plotted against d/ℓ_D alone is the unsteady signature of the laboratory problem: the enhancement at a given instantaneous wall distance depends on the history of approach, not only on the present geometry.

B. Inversion of the naive frozen offset picture

The wall enhancement of the previous subsection invites two opposite physical readings, and Fig. 12 shows that the laboratory result selects neither of them and instead

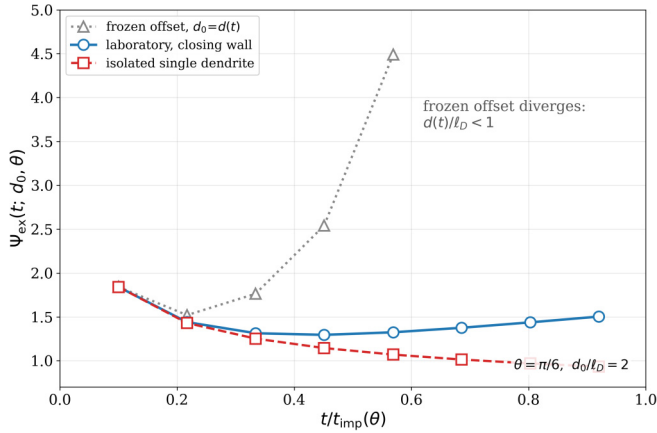


FIG. 12. The matching variable $\Psi_{\text{ex}}(t)$ along the trajectory of a closing wall dendrite at $\theta = \pi/6$, $d_0/\ell_D = 2$ (laboratory, blue), against the isolated single dendrite reference at the same evaluation time (red dashed) and against the frozen offset prediction obtained by holding the wall at the instantaneous distance $d(t)$ for the entire elapsed history (grey dotted). The frozen offset curve is shown only where it is reliably defined and diverges as $d(t)/\ell_D \rightarrow 0$.

occupies a clean intermediate regime. The figure displays $\Psi_{\text{ex}}(t)$ at $\theta = \pi/6$, $d_0/\ell_D = 2$, along three different reconstructions of the wall problem at the same evaluation times. The first reconstruction is the laboratory closing wall calculation, in which the wall is fixed in the laboratory frame and the dendrite approaches it at constant velocity with the kinematic shrinkage $d(t) = d_0 - vt \sin \theta$ taken into account self consistently inside the wedge transform. The second reconstruction is the isolated single dendrite reference $\Psi_{\text{ex}}^{\text{iso}}(t)$, with no wall present, evaluated at the same elapsed time. The third reconstruction is the frozen offset prediction $\Psi_{\text{ex}}^{\text{frozen}}(t)$, in which the wall is held at the instantaneous geometric distance $d(t)$ for the entire elapsed history; this is the prediction that an experimentalist with access to the static wall problem alone would write down by reading the present geometry off a snapshot.

The three curves disagree in physically illuminating ways. At early times, $t/t_{\text{imp}} \lesssim 0.2$, the wall is far enough away that all three curves agree to within a percent, and the natural plume decay of an isolated dendrite dominates the time evolution. At intermediate times, $0.2 \lesssim t/t_{\text{imp}} \lesssim 0.6$, the laboratory and isolated curves separate as the wall feedback begins to develop, with the laboratory curve sitting above the isolated reference but well below the frozen offset prediction. At late times, $t/t_{\text{imp}} \gtrsim 0.6$, the frozen offset prediction grows without bound as $d(t)/\ell_D \rightarrow 0$ and the wedge transform analyticity strip degenerates, while the laboratory curve continues to sit above the isolated reference but stays bounded. The isolated reference itself decays monotonically through the entire trajectory. The laboratory curve is the only one of the three that is nonmonotonic in time: it follows the isolated decay at early times, reaches a minimum at intermediate times where the natural plume decay and the wall enhancement are in balance, and rises again at late times as the wall enhancement takes over.

There are two points to remark. First, the laboratory result inverts the naive frozen offset reading at the level of

magnitude. The frozen offset prediction overshoots the laboratory feeding by a factor of two at $d(t)/\ell_D \simeq 1$ and diverges further as the tip closes in, because it gives the wall infinite time to project its feedback at the present geometry. The laboratory result correctly accounts for the finite history of the closing wall and produces a tempered, finite enhancement. Second, and more importantly, the laboratory result inverts the frozen offset reading at the level of sign of the time derivative. At late times, the laboratory feeding is *rising* with time, while the isolated reference is falling. An observer looking only at the isolated dendrite would conclude that the local flux available to the tip is monotonically decaying as the natural plume disperses; an observer with access to the laboratory result sees instead that the wall recovery overtakes the natural decay in the late phase of the approach and that the local flux available to the tip increases as the contact becomes imminent. This is the central physical result of the section, and it is a genuine consequence of the laboratory formulation. No frozen offset reading of the wall problem would have predicted it.

C. Wall lifting of the supercritical margin

Combining the wall enhancement of Ψ_{ex} with the tip mobility law derived in Sec. VIII gives an immediate prediction for the time evolution of the tip drive of an individual dendrite in the laboratory wall geometry. The mobility law of the previous section maps the matching variable monotonically onto a tip velocity above an inner threshold Ψ_c and gives no steady tip below it; the wall lifted Ψ_{ex} of the present section therefore corresponds to a wall lifted tip drive whenever Ψ_{ex} exceeds the threshold. Rather than translate the matching variable into an absolute tip velocity, which would require committing to the constant of proportionality of the inner closure, we work directly with the supercritical margin

$$\Delta_{\Psi}(t; d_0, \theta, \Psi_c) := \max(0, \Psi_{\text{ex}}(t; d_0, \theta) - \Psi_c), \quad (167)$$

the surplus of the matching variable above the inner threshold, clipped at zero where the threshold is not met. The supercritical margin is a property of the outer calculation and the inner threshold alone, with no further inner closure required, and it is monotonically related to the tip velocity by the mobility law of Sec. VIII.

Figure 13 shows the supercritical margin along the same trajectories as Fig. 11, with the time on the horizontal axis rather than the wall distance. Figure 13(a) compares the three obliquities at a single threshold, with the isolated reference shown as a dashed black curve. The three wall-corrected trajectories follow the isolated decline almost exactly during the early phase of the approach, where the wall is too far away to matter. As the tip closes in, the wall-corrected curves progressively peel off from the isolated reference and decay more slowly. At the grazing angle $\theta = \pi/6$ the deviation is large enough that the supercritical margin plateaus and recovers in the late phase of the trajectory, just before the kinematic contact, in keeping with the nonmonotonic time evolution of Ψ_{ex} documented in Fig. 12. At the intermediate and steepest angles the wall correction slows the isolated decline but does not reverse it within the kinematic time available.

Figure 13(b) shows the same supercritical margin at the grazing angle for three threshold values Ψ_c , sweeping the

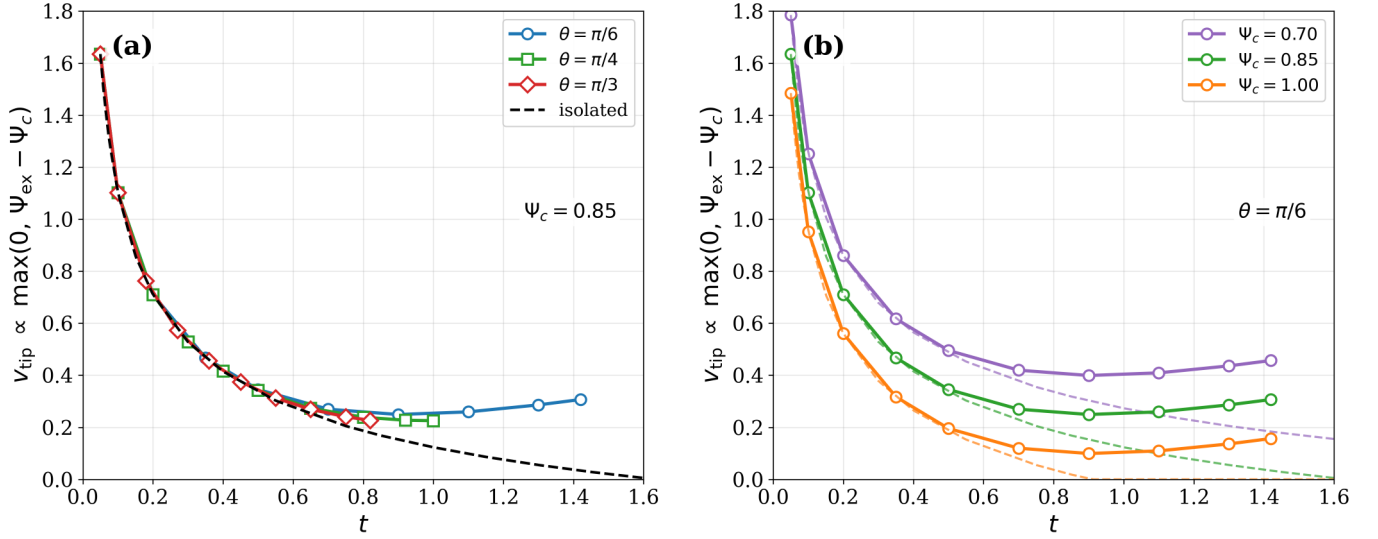


FIG. 13. Supercritical margin of the matching variable above the inner threshold, Eq. (167), along the same trajectories as Fig. 11. Panel (a) shows three obliquities at a single threshold Ψ_c , with the isolated reference dashed; the wall lifted curves peel off from the isolated decline as the tip closes in on the wall, with the largest lift at the grazing angle. Panel (b) shows the grazing angle at three threshold values, demonstrating that the wall lifting is robust to the choice of Ψ_c .

inner parameter while holding the outer trajectory fixed. The wall lifting feature is robust across the threshold range: at every threshold the wall-corrected trajectory decays more slowly than the isolated reference, and at every threshold the late phase of the trajectory shows a measurable lifting of the supercritical margin. The amplitude of the lifting is largest at the highest threshold, where the isolated margin has dropped closer to extinction by the time the wall feedback develops, and where any wall correction therefore represents a relatively larger fraction of the surviving margin. In all three cases the wall-corrected curve sits above the isolated reference through the late phase of the trajectory, the early suppressive phase being too small at the percent level to register in the supercritical margin.

The combined physical statement is that, taking the tip mobility law of Sec. VIII as given, the laboratory wall problem predicts a wall lifted tip drive for an individual dendrite approaching an oblique wall, with the strongest lifting occurring at grazing incidence and at high inner thresholds. The frozen offset intuition that local saturation should depress the tip drive is at later times contradicted by the exact unsteady result.

D. Critical wall offset for inner threshold crossing

The wall lifting of the matching variable allows us to identify, for each obliquity angle and each inner threshold, the largest initial offset at which the wall feedback alone is sufficient to push Ψ_{ex} above Ψ_c at evaluation time t_* . We define this threshold crossing distance as

$$d_{0,c}(\theta; \Psi_c, t_*) := \{d_0 : \Psi_{\text{ex}}(\ell_D, t_*; d_0, \theta) = \Psi_c\}, \quad (168)$$

the offset at which the wall-corrected matching variable equals the inner threshold at the chosen evaluation time. For $d_0 < d_{0,c}(\theta)$ the wall-enhanced matching variable lies above the threshold and the tip mobility law of Sec. VIII gives a steady tip; for $d_0 > d_{0,c}(\theta)$ the wall feedback is insufficient to

reach the threshold and the tip mobility law gives no steady tip at evaluation time. The function $d_{0,c}(\theta)$ is therefore the boundary, in the space of initial offsets, of the regime in which the wall feedback succeeds in pushing the tip above the inner threshold. It is a property of the one tip plus wall problem that the present formalism solves, and we make no claim about its meaning for arrays of dendrites; we return briefly to this scope question in the closing remarks.

Figure 14 shows $d_{0,c}(\theta)/\ell_D$ for nine obliquities in the range $\theta \in [15^\circ, 80^\circ]$ at four representative thresholds $\Psi_c \in \{1.40, 1.70, 2.10, 2.80\}$, all evaluated at $t_* = 0.4$ and at the natural process zone scale $\ell = \ell_D$. The four threshold values are above the isolated single dendrite feeding at the same evaluation time, $\Psi_{\text{ex}}^{\text{iso}}(t_*) \simeq 1.27$, so that all four require a positive wall contribution to be reached and all four admit a

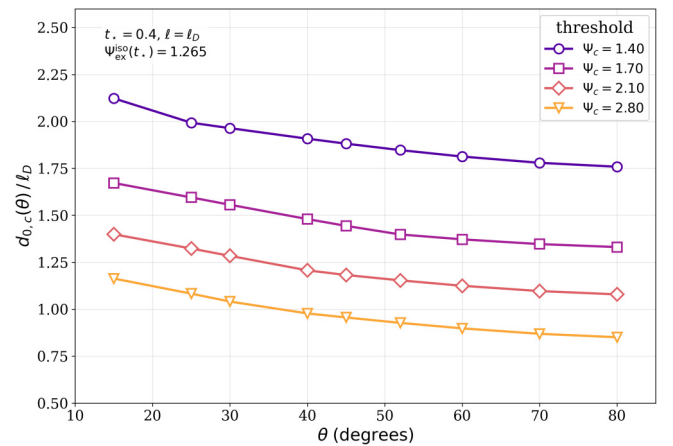


FIG. 14. Critical initial offset $d_{0,c}(\theta)/\ell_D$ extracted from the threshold crossing condition $\Psi_{\text{ex}}(\ell_D, t_*; d_{0,c}, \theta) = \Psi_c$ at $t_* = 0.4$, for nine obliquities and four threshold values $\Psi_c \in \{1.40, 1.70, 2.10, 2.80\}$. The four curves are monotonically decreasing in θ and quasiparallel on the linear scale.

finite critical offset across the entire angular range. Each curve is monotonically decreasing in θ , from a maximum at the grazing angle to a minimum at the steepest angle, with the four curves quasiparallel on the linear scale used in the figure. At the mildest threshold $\Psi_c = 1.40$ the threshold crossing zone extends from $d_{0,c}/\ell_D \simeq 2.1$ at $\theta = 15^\circ$ to $d_{0,c}/\ell_D \simeq 1.8$ at $\theta = 80^\circ$, an angular variation of approximately 20%. At the strongest threshold $\Psi_c = 2.80$ the zone shrinks to $d_{0,c}/\ell_D \simeq 1.2$ at $\theta = 15^\circ$ and $d_{0,c}/\ell_D \simeq 0.85$ at $\theta = 80^\circ$, an angular variation of approximately 37%. Higher thresholds compress the threshold crossing zone in absolute terms but make the obliquity dependence relatively more important, because the wall has to do more work against a higher threshold and the angular geometry becomes the controlling factor.

The quasiparallel structure of the four curves on the linear scale suggests that the obliquity shape $d_{0,c}(\theta)/d_{0,c}(\pi/4)$ is close to a universal function across thresholds. The departures from exact universality reflect the threshold-dependent sensitivity of the inverse map $\Psi_{\text{ex}}(d_0) \mapsto d_0$ and are quantitatively small. The practical content of the figure is that the threshold crossing offset for an oblique tip approaching a wall lies in the range one to two diffusion lengths for thresholds of physical interest, with a mild but systematic dependence on the obliquity angle that can be read directly off the figure for any threshold within the displayed range.

Thus, the wall correction changes sign along the approach: a suppression of at most one percent while the wall is distant, followed by an enhancement of up to 31 percent as contact nears, producing a nonmonotonic time evolution qualitatively different from the monotone decay of the isolated dendrite. The suppressive phase is what a quasistatic reading predicts throughout, and the physical content of the exact solution is that the unsteady dynamics overcome and reverse it. Combined with the tip mobility law of Sec. VIII, this gives a wall lifted tip drive that is largest at grazing incidence and robust to the choice of inner threshold, with a threshold crossing distance $d_{0,c}(\theta; \Psi_c)$ in the range one to two diffusion lengths. All four statements are confined to the single tip geometry studied here. A frozen offset reading of the wall problem would predict the opposite, assigning an arrest mechanism to the close wall regime; it fails both in magnitude, by overshooting the laboratory feeding, and in the sign of its time derivative, by predicting depression rather than enhancement. The unsteady history of the closing wall is essential, and no snapshot at any instantaneous distance reproduces it. The natural extension is to a dense array of dendrites approaching an oblique wall, in which the wall-enhanced feeding of a dominant tip would interact with the intertip competition of the free space array; that calculation requires a separate two tip plus wall treatment and is left for future work.

X. CONCLUSIONS

This work has solved the transient diffusion problem for an oblique dendrite advancing past a fixed impermeable wall, retaining the closing gap exactly. Relative to an isolated dendrite at the same elapsed time, the reflected depletion field initially suppresses the process zone feeding, though by a margin small compared with the later enhancement. Indeed, as the gap between dendrite and wall closes, the solutions

achieved here show that the suppression is overcome, the correction crosses the isolated dendrite reference, and eventually the feeding becomes substantially enhanced before impact, most strongly at grazing incidence. Thus, the wall is shown to enhance, rather than arrest, dendrite growth. The reversal occurs because diffusion carries the rejected solute signal to the wall and back over a time of order $4d^2/D$. The gap narrows during that interval, so the field returning to the tip retains information about an earlier, wider separation, while the feeding of the isolated dendrite is itself decaying. The returned field can therefore overtake the evolving isolated reference while remaining a reflected depletion field. This process is revealed to be a memory effect: the wall acts on the history of the trajectory rather than on the instantaneous geometry alone. Previous analytical treatments of this geometry, reliant on steady state accounts, contain no approach history and cannot represent the crossover.

We have solved the problem exactly by removing its two obstructions in turn. In the laboratory frame the dendrite occupies a half line whose endpoint moves, which denies an ordinary one-sided transform, and the obliquity makes the reflected kernel depend on $u + u'$ as well as on $u - u'$, which denies the difference kernel a scalar Wiener-Hopf reduction requires. We have shown that a single two variable wedge transform absorbs the moving endpoint into an analyticity condition on the transformed unknown, and the reflected kernel proves to be rank one at each wave number, being the restriction of a single image source, so the Hankel part couples the density to one auxiliary spectral trace rather than to its full two-dimensional spectrum. What remains is a 2×2 branch matrix, upper triangular and invertible sequentially: a Fredholm equation for the lower branch trace followed by a scalar Wiener-Hopf factorization for the upper. The exact relation $d(t) = d_0 - vt \sin \theta$ is the only kinematic input and the offset is frozen at no stage. We have validated the reconstruction against an independent direct boundary integral implementation, at a cost no larger than a careful frozen offset calculation. The Wiener-Hopf-Hankel structure is known from elliptic wedge diffraction, where the Hankel coupling arises from mixed boundary data on a stationary wedge; here it arises from the angle between a planar reflector and a moving absorber in a parabolic problem, and the moving endpoint is carried exactly. We are not aware of a previous Wiener-Hopf treatment of a solidification problem, nor of any analytical treatment of dendrite-wall interaction that retains the moving gap.

The frozen (steady state) construction overestimates the influence of the near contact geometry and therefore diverges as the offset tends to zero, whereas the true retarded feeding remains bounded because it is dominated by contributions generated at finite earlier separations. Near contact the frozen approximation double counts the near field depletion, leading to an $\mathcal{O}(1)$ overshoot relative to the laboratory feeding, while the exact solution remains controlled by the delayed return of the broader diffusive cloud. The sign change reverses the qualitative prediction for the tip drive in precisely the regime, d_0 of order ℓ_D and Pe_θ of order unity, in which competitive array selection against walls and grain boundaries takes place in directional solidification. Second, near contact the frozen approximation double counts the near field depletion, leading

to an $\mathcal{O}(1)$ overshoot relative to the laboratory feeding, while the exact solution remains controlled by the delayed return of the broader diffusive cloud.

Two consequences of the wall-corrected matching variable follow directly from the exact result. Combining the laboratory feeding with the tip mobility law of the inner selection problem, the wall lifts the tip drive of an oblique dendrite through the later part of its approach, and inverting the lifting condition gives a critical wall offset $d_{0,c}(\theta; \Psi_c)$ in the range of one to two diffusion lengths for the inner thresholds of physical interest. This critical offset is a property of the one tip plus wall problem and does not by itself determine array spacing; the corresponding question for a dense array requires a separate two tip plus wall calculation, to which the same wedge Wiener-Hopf machinery is directly applicable. The second consequence concerns the moment of contact itself. At the instant of contact, the no flux condition forces the local concentration gradient to be tangent to the wall, and the asymmetry of the dendrite shadow selects the forward wall tangent as the unique direction in which the gradient drives outward growth. The original primary tip is extinguished at contact, but the wall-corrected concentration field selects the wall tangent itself, making angle θ with the original growth direction, as the natural direction for a postcontact daughter branch in the isotropic limit. The selection follows from the no flux boundary condition alone and has been verified directly across a range of obliquities and Péclet numbers. The breaking of isotropy by crystalline anisotropy in the inner sharp interface problem would in general modify the geometric prediction, and tracing that modification is a natural direction for further work.

The methodology developed here, the wedge Wiener-Hopf-Hankel formulation (with a Padé-Abrahams rational reduction if so wished), is not restricted to the single tip geometry of the present paper. The same operator structure arises whenever a half line absorber interacts with a planar reflector at an oblique angle, and the sequential triangular inversion that exploits the rank one form of the reflected kernel carries over to any problem with that geometry, whether the underlying equation is the heat equation, the modified Helmholtz equation, or any other second order elliptic or parabolic operator whose free space Green function admits a Fourier-Laplace representation. The extension to two competing tips at different offsets from the same wall requires only the addition of a second absorbing half line to the single-layer representation, which doubles the size of the branch matrix but preserves its triangular structure, and the resulting two tip plus wall problem is the natural next step toward a full understanding of competitive array selection near oblique boundaries. The closed form solver provides a fast and analytically transparent reference for that calculation, and the broader expectation is that the wall-enhanced feeding of a dominant tip will favor competitive starvation of a subordinate neighbor in regimes where the corresponding free space competition is marginal, opening a direct route to a spacing law for primary dendrite arrays growing against grain boundaries and container walls.

On the experimental side, the wall correction is testable in transparent analog solidification systems and by *in situ* synchrotron radiography of aluminum alloys, both of which follow a single tip continuously through the approach. A

positive confirmation would establish that the unsteady reflected history, rather than the instantaneous geometry, is the controlling variable in wall mediated dendrite competition, and would validate the process zone matching framework as a quantitative tool for predicting array selection in confined solidification geometries.

DATA AVAILABILITY

The data that support the findings of this article are available from the authors upon reasonable request.

APPENDIX A: NUMERICAL IMPLEMENTATION

The figures of Secs. VII and IX are produced by two independent solvers. The first is the closed form Wiener-Hopf reconstruction of Sec. VI, which is the production tool for all results in the paper. The second is a direct boundary integral solver for the half-space problem of Sec. III, used as an independent reference for cross validation. This Appendix documents both, in the order in which they enter the data flow.

1. Closed form Wiener-Hopf reconstruction

a. Time inversion in p . The wedge transformed branch system of Eq. (105) is solved at a discrete set of nodes p_n on the Bromwich contour and the line density $\mu(s, t; d_0, \theta)$ is reconstructed by inverse Laplace transform in p . We use the Gaver-Stehfest scheme [43] of order $N = 14$ throughout. At evaluation time t the Stehfest weights V_i for $i = 1, \dots, N$ are evaluated once via the closed form combinatorial expression, and the inverse transform is the linear combination

$$\mu(s, t) = \frac{\ln 2}{t} \sum_{i=1}^N V_i \bar{\nu}(s, p_i), \quad p_i = \frac{i \ln 2}{t}, \quad (\text{A1})$$

where $\bar{\nu}(s, p)$ is the wedge inverted physical density at the real Laplace node p_i . Stehfest is preferred over Talbot quadrature because the nodes lie on the positive real axis, where the analyticity strip (66) of the wedge transform is broadest and the branch tracking of the radical $\kappa(k, p)$ is unambiguous. Convergence in N is monitored on the integrated line density and saturates by $N \simeq 10$ at all parameters used in the paper. The choice $N = 14$ provides a comfortable margin without entering the regime where the weights V_i themselves become numerically large.

b. Branch tracking. At each Stehfest node p_i the radical $\kappa(k, p_i) = \sqrt{k^2 + p_i/D}$ and the branch related quantities $A(k, p_i)$, $B(k, p_i)$ of Eqs. (54) and (55) must be evaluated on a grid of complex k . Because p_i is real and positive, the radical is single valued on the entire physical sheet and no branch tracking is required at the level of p . At the level of k , the Sommerfeld branch cuts of κ are taken along the imaginary axis, and the principal value $\kappa(k, p_i) > 0$ is selected by the convention $\arg \kappa \in (-\pi/2, \pi/2]$, which is enforced by a continued analytic continuation procedure that uses the previous value as a phase reference whenever the natural branch of square root would cross the cut. This precaution is necessary only for complex k off the real axis, where the discretization of the wedge contour visits the complex strip $0 < \text{Im} q < \sigma(p_i)$.

c. Lower branch Fredholm equation. The lower branch trace $\Phi_B(k, p_i)$ is computed by Nyström collocation of the lower branch Fredholm equation (121) on a real k grid of $n_k = 201$ points spanning $k \in [-k_{\max}, k_{\max}]$ with $k_{\max} = 25$ in the standard units. The grid is uniform, the trapezoidal weights are corrected at the endpoints, and the kernel is regularized by the natural decay scale $\exp(-2d_0 \kappa(k, p_i))$, beyond which the kernel is numerically zero. The discretized system is well conditioned at all parameters of physical interest, with condition number below 10^3 at $d_0 \sim \ell_D$ and decreasing as d_0 increases. The compactness of the reflected operator $\mathcal{R}_{BB}$ in the working strip (66) ensures that no real k pole is encountered in the Fredholm solve, and that no contour indentation is required.

d. Upper branch scalar Wiener-Hopf. Given $\Phi_B(k, p_i)$, the upper branch trace $\Phi_A(k, p_i)$ is obtained from the scalar Wiener-Hopf equation (102) via the factorization $K_0(q, p) = K_0^+(q, p) K_0^-(q, p)$ of the direct symbol. The factorization is performed by Cauchy integration of $\log K_0$ on a contour interior to the working strip; in the implementation, the Cauchy integral is evaluated by a fast Fourier transform on the strip parametrized as $q = \gamma + i\omega$, with a Tukey window [50] of fractional width 0.10 applied at the contour endpoints to suppress aliasing. The plus projection of the right-hand side is performed by the same FFT routine, with explicit pole removal at the kinematic poles $q = A_\theta$ and $q = B_\theta$ to avoid the spurious contribution from the principal value at the singularities.

e. Wedge inversion in q and the residue sum. The wedge inversion of Eq. (140) is performed by closed form residue summation rather than by numerical contour integration. The output of the upper branch solve is a sequence of poles q_k and residues B_k such that the analytic representation

$$\Phi_+(q, p_i) = \sum_k \frac{B_k \sqrt{\sigma(p_i) + q}}{q - q_k} \quad (\text{A2})$$

holds on the working strip, with $\sigma(p_i)$ the wedge strip parameter. The physical density at arc length u is then

$$\bar{v}(u, p_i) = \sum_k B_k e^{q_k u} \operatorname{erfcx}(-\sqrt{\sigma(p_i) + q_k} \sqrt{u}), \quad (\text{A3})$$

which is the analytic Laplace inversion of the term $B_k \sqrt{\sigma + q}/(q - q_k)$ evaluated at $u > 0$. The scaled complementary error function $\operatorname{erfcx}(z) = e^{z^2} \operatorname{erfc}(z)$ is used in place of the standard erfc to avoid catastrophic cancellation when the argument is large; the implementation uses the Faddeeva package for complex arguments. Each pole contributes independently and the sum converges rapidly because the residues $|B_k|$ decay smoothly with the index k , as documented in Sec. VII G of the main text.

f. Reconstruction of the line density. The physical line density $\mu(s, t; d_0, \theta)$ on the dendrite is recovered from the wedge density $\bar{v}(u, p_i)$ via the kinematic shift $u = s - vt$, and from the time inverted sum (A1). The arc length grid is uniform with $n_u = 1600$ points covering $u \in [0, 4]$ in the standard units, which is well beyond the support of the kinematic shift at all evaluation times of interest. Interpolation onto the physical s grid is performed by linear interpolation, which is sufficient because μ is a smooth function of s away from the tip singularity at $s = 0$. The tip singularity itself is

not resolved by the uniform grid; quantities that involve the singular integral, such as the integrated line density excess $\mathcal{Q}_\mu$ of Eq. (147) and the process zone functional Ψ_{ex} of Eq. (151), are computed via a $v = \sqrt{s}$ change of variables that absorbs the singularity into the integration measure.

g. Padé-Abrahams rational reduction. The radical $\kappa(k, p) = \sqrt{k^2 + p/D}$ that appears in the direct symbol, in the spectral combinations A_θ, B_θ of Eqs. (54) and (55), and in the wall attenuation $e^{-2d_0 \kappa}$ is the only nonrational object of the production pipeline, and it is rationalized once at each Stehfest node by the Padé-Abrahams construction of Appendix B. The construction is the standard Stieltjes Padé approximant of order $[n - 1/n]$ to the function $h(z) = \sqrt{z + \lambda^2}$ in the variable $z = k^2$, with $\lambda = \sqrt{p_i/D}$, computed through the three step algorithm of Appendix B 2. The first $2n$ Taylor coefficients of $h(z)$ at $z = 0$ are evaluated in closed form from Eq. (B8); the Hankel linear system (B10) is solved by direct LU factorization for the denominator coefficients $q_0, \dots, q_{n-1}$; and the roots $z_j(p_i) = -\alpha_j(p_i)^2$ of the denominator polynomial Q_n are extracted by a single companion matrix eigendecomposition. The residues $r_j(p_i)$ are then read off from Eq. (B14), and the resulting structured rational form

$$\kappa_n(k, p_i) = \sqrt{p_i/D} + \sum_{j=1}^n \frac{r_j(p_i) k^2}{k^2 + \alpha_j(p_i)^2} \quad (\text{A4})$$

is by construction even in k , matches κ at $k = 0$ exactly, satisfies the consistency condition $\sum_j r_j(p_i) = 1$ to working precision, and converges to κ exponentially in n on compact sets in the k plane away from the imaginary axis branch points. The exponent $1/2$ that appears in the canonical Wiener-Hopf factors (B38) and (B39) is handled by sorting the roots of the polynomial (B37) by sign of imaginary part, which is the closed form factorization of any rational symbol with no real zeros or poles.

The Stieltjes property of h guarantees that the poles $\alpha_j(p_i)^2$ are real and strictly positive and that the residues $r_j(p_i)$ are real and strictly positive, so no pole collision and no spurious cancellation can occur at any order n . This is the key structural reason for using the Stieltjes-Padé algorithm rather than a least squares collocation on a grid of real k points: the latter does not preserve the Stieltjes structure, can place poles arbitrarily close to the real axis when n is increased, and degrades the conditioning of the downstream Fredholm assembly. The Stieltjes-Padé construction preserves the structure exactly, so that the rational branch maps $A_\theta^{(n)}, B_\theta^{(n)}$ of Eqs. (B5) and (B6) inherit a clean pole geometry on the imaginary k axis and the linear system (B30) for the rationalized lower branch trace is well conditioned at every n and every p_i used in the production runs.

By the homogeneity property recorded at the end of Appendix B 2, the dimensionless constants $\hat{\alpha}_j$ and $\hat{r}_j$ defined by $\alpha_j(p_i) = \lambda(p_i) \hat{\alpha}_j$ and $r_j(p_i) = \hat{r}_j$ are universal: they do not depend on p_i . They are therefore computed once at the start of each run, at unit λ , and the rescaling $\alpha_j(p_i) = \lambda(p_i) \hat{\alpha}_j$ at each Stehfest node is the only Padé level operation performed inside the time inversion loop. The total cost of the Padé step at each p_i is therefore a single multiplication of the precomputed $\hat{\alpha}_j$ by $\lambda(p_i)$, which is negligible compared to the cost of the Fredholm assembly and solve. The production

order used in all the figures of Secs. VII and IX is $n = 12$, at which the exponential convergence of the Stieltjes-Padé approximant has saturated the floor set by the Stehfest quadrature of the outer inversion and the rationalization contributes nothing to the total numerical error of the line density. The Padé-Abrahams reduction is therefore not an approximation to be checked against the exact construction but the closed form algebraic form of the exact construction itself, and the chain from the wedge transformed branch matrix problem (105) to the inverse wedge transform (B46) is executed in fully closed form arithmetic at every Stehfest node.

2. Direct boundary integral solver

The direct solver implements the half-space single-layer boundary integral equation of Sec. IIIC in real space, with the wall enforced by the method of images and the moving endpoint of the dendrite handled by an explicit time stepping scheme. The aim of the direct solver is independence from the closed form transform pipeline, so that any agreement between the two methods constitutes a genuine cross-check. We must point out that the direct solver is considerably costlier to run, and prone to considerably instabilities not present in the analytical case.

a. Spatial discretization. The dendrite is parametrized by arc length $s \in [0, s_{\max}]$ with $n_s = 70$ collocation points distributed on a stretched grid that resolves the tip singularity at $s = 0$ via a $\sqrt{s}$ substitution. The boundary integral kernel of Eq. (30), including the image term that enforces the no flux condition on the wall, is evaluated by closed form integration of the heat kernel against the local linear interpolant of $v(s, t)$ over each panel. The closed form panel integration uses the E_n family of exponential integrals [51], evaluated by Taylor expansion near the tip and by the standard recurrence elsewhere.

b. Time discretization. The dendrite line density at successive time steps is related by an implicit step formulation, in which the unknown v^m at step m satisfies a linear system whose right-hand side depends on the history $v^{m'}$ for $m' < m$. The time grid has $n_t = 24$ steps spanning $[0, t_{\text{eval}}]$ in the production runs, and the closed form panel integration over each time interval $[t_{m-1}, t_m]$ uses the same E_n Taylor expansion as the spatial step. The implicit step is solved by direct LU factorization of the local matrix at each step, which is small enough that the cost is dominated by the assembly of the history convolution rather than by the linear solve.

c. Field reconstruction. The two-dimensional concentration field $c(x, y, t)$ on a regular 80×80 patch around the dendrite is reconstructed from the line density history v^m via the same single-layer representation, with the panel integration performed by the same E_n family. The reconstructed field takes the value c_{LC} at the dendrite collocation points up to spatial discretization error, which is the sole consistency check on the reconstruction. The field is not used for any production observable in Secs. VII and IX; its sole role is to enable the cross validation described below.

3. Cross validation

The closed form Wiener-Hopf solver and the direct boundary integral solver are compared on the line density

$\mu(s, t; d_0, \theta)$ at a grid of parameters spanning the physical range used in the paper, with $\theta \in \{\pi/6, \pi/4, \pi/3\}$, $d_0 \in \{0.5, 1.0, 1.5, 2.0\}$, and $t \in \{0.1, 0.2, 0.4, 0.6\}$, all in the standard units. The two solvers agree to a relative error of better than 10^{-3} on the line density at every collocation point, and to better than 10^{-4} on the integrated line density. The discrepancy is dominated by the time discretization of the direct solver, which converges as $1/n_t^2$, and is reduced below 10^{-4} at $n_t = 48$ for the integrated line density. The closed form solver is taken as the reference throughout the paper. The cross validation is performed once and is not repeated for each figure, but the comparison data are available on request.

The agreement between the two independent solvers, on a quantity that probes the entire data flow of both pipelines, is the most important quality control we have. It establishes that the closed form Wiener-Hopf reconstruction and the direct boundary integral solver are not coupled by any shared error mode, and that the figures presented in the main text reflect a genuine solution of the laboratory wedge problem rather than a numerical artefact of either approach.

4. Process zone length and the resolution of the tip singularity

The length ℓ enters only the functional Eq. (151) and not the transport problem, so the density is unchanged as ℓ varies and the dependence is obtained by re-evaluating the quadrature on a single solution at each configuration. Figure 15(a) shows $\mu(s, t)\sqrt{s}$ for an isolated dendrite and for wall offsets $d_0/\ell_D = 2, 3$, and 4 at $\theta = \pi/6$ and $t = 0.10$. The four curves are indistinguishable, and the amplitudes obtained by fitting Eq. (150) on the resolved range agree to six significant figures, $A = 1.522198$ for the isolated dendrite and 1.522253 , 1.522198 , and 1.522198 at the three offsets. The residual at the closest offset is the exponential remainder expected from the wall attenuation $e^{-2d_0\Gamma(q,p)}$, which suppresses the reflected contribution in the large $|q|$ limit that fixes $A(t)$, so the leading tip amplitude is controlled by the direct symbol alone and the wall enters through $B(t)$ and the structure behind the tip. The reconstruction resolves arc lengths down to a floor set by the truncation of the wave-number grid at $k_{\max}$. The density is built from a spectral representation sampled on $|k| \leq k_{\max}$, so features shorter than $\sim 1/k_{\max}$ are absent from the reconstruction, and with $k_{\max} = 30$ this places the floor at $s \simeq 0.03$, marked in Fig. 15(a). Below it the computed $\mu\sqrt{s}$ departs from its plateau and neither A nor a process zone average is meaningful, the departure being a property of the truncation rather than of the density. Refining the arc length quadrature does not move the floor, which responds only to $k_{\max}$, and the fits above are taken above it. Figure 15(b) shows the relative wall correction for $c_\ell \in \{0.4, 1.0, 3.0\}$, all of which place ℓ above the floor. The magnitude depends on the window: at $d_0/\ell_D = 0.8$ the correction is 0.31 , 0.23 and 0.11 for the three values, decreasing as the window extends behind the tip and takes in the region where the wall-corrected and isolated densities approach a common bulk asymptote. The correction is positive throughout, decays to zero beyond about two diffusion lengths, and both the decay shape and that distance are unchanged across the range. The value $c_\ell = 1$ adopted in the main text therefore fixes a normalization of

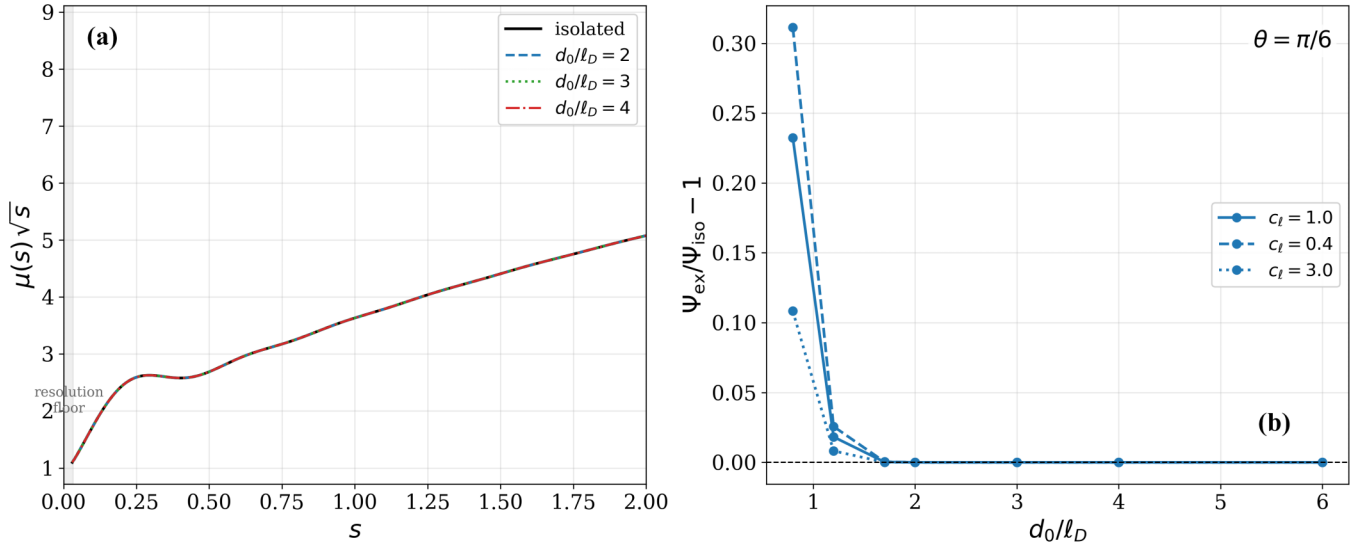


FIG. 15. (a) Reconstructed $\mu(s, t)\sqrt{s}$ at $\theta = \pi/6$ and $t = 0.10$ for an isolated dendrite and for wall offsets $d_0/\ell_D = 2, 3$ and 4 ; the curves coincide and the fitted tip amplitudes agree to six significant figures. The shaded band marks the resolution floor $s \simeq 1/k_{\max}$ set by truncation of the wave-number grid. (b) Relative wall correction $\Psi_{\text{ex}}/\Psi_{\text{iso}} - 1$ against initial offset at $\theta = \pi/6$, for three values of the process zone constant c_ℓ of Eq. (166).

the magnitude and none of the qualitative conclusions, which follow from the difference $B_{\text{wall}} - B_{\text{iso}}$ rather than from the window length.

APPENDIX B: PADÉ-ABRAHAMS RATIONAL REDUCTION OF THE BRANCH INVERSION

The exact sequential inversion of Sec. VI is analytically complete but not algebraically closed. The lower branch trace Φ_B is defined as the solution of the projected Fredholm equation Eq. (121), whose kernel contains the nonrational $\kappa(k, p) = \sqrt{k^2 + p/D}$ and the spectral combinations A_θ, B_θ of Eqs. (54)–(55), and its solution is obtained through a Neumann resolvent series or a Nyström discretization of the continuous k integration. The upper branch trace Φ_A then requires the canonical factorization of $K_A = K_0(A_\theta(k, p), p)$, again nonrational in k through κ , and the Cauchy projection has to be evaluated by contour integration rather than by residue arithmetic. All these steps are feasible numerically, but none is algebraic. The Padé-Abrahams construction [24,25] converts the whole chain into closed form algebra by replacing the single nonrational object that causes the difficulty, the radical $\kappa(k, p)$, by a rational function of k of controllable order. Once κ is rationalized, the branch maps A_θ, B_θ become rational in k at fixed p , the composite direct symbol K_A becomes the product of a rational function with its image under a rational transformation, the reflected kernel becomes a finite rank kernel whose integral equation collapses to a linear system, and the canonical Wiener-Hopf factorization reduces to sorting zeros and poles by half plane. The sequential chain of Sec. VI is then executed in closed form on the rational reduction, and the only approximation is the choice of rational approximant to κ ; every subsequent step is an exact equality, and the error relative to the exact benchmark of Sec. VI is controlled by the Padé approximant's order. The

laboratory formulation simplifies the Padé-Abrahams reduction substantially compared to the gauge formulation. In the latter, the relevant radical is $\kappa_\theta(k, p) = \sqrt{(k - i\beta_\theta)^2 + \lambda^2}$ with $\beta_\theta = v/(2D)$ and $\lambda(p)^2 = p/D + \beta_\theta^2 \cos^2 \theta$, whose branch points sit on a shifted contour parallel to the imaginary k axis and whose rational approximation must track that shift. In the laboratory formulation, the relevant radical is simply $\kappa(k, p) = \sqrt{k^2 + p/D}$ from Eq. (36), even in k , with branch points at $k = \pm i\sqrt{p/D}$ on the imaginary axis, and whose rational approximation is a rational function of k^2 with real positive pole locations. All the computations below exploit this even parity.

1. Rational approximation of the radical

The starting point is that $\kappa(k, p)$ is an analytic function of $z := k^2$ away from the single branch point at $z = -p/D$, with cut along $z \in (-\infty, -p/D]$. Setting $\lambda := \sqrt{p/D}$ on the principal branch with $\text{Re} \lambda > 0$,

$$h(z) := \sqrt{z + \lambda^2} \quad (\text{B1})$$

is a Stieltjes function on the cut plane $\mathbb{C} \setminus (-\infty, -\lambda^2]$: Its derivative admits the integral representation

$$h'(z) = \frac{1}{2\sqrt{z + \lambda^2}} = \frac{1}{2\pi} \int_{\lambda^2}^{\infty} \frac{d\sigma}{(z + \sigma)\sqrt{\sigma - \lambda^2}}, \quad (\text{B2})$$

the Cauchy integral of the jump of h' across its branch cut. The Padé approximant of order $[n - 1/n]$ to such a Stieltjes function has the canonical structure of a sum of n simple poles on the negative real z axis with strictly positive residues, and converges to h exponentially fast in n on compact sets away from the cut. In partial fraction form,

$$\kappa_n(k, p) := \lambda + \sum_{j=1}^n \frac{r_j(p)k^2}{k^2 + \alpha_j(p)^2}, \quad (\text{B3})$$

with $\alpha_j(p)^2 > 0$ and $r_j(p) > 0$ on the principal branch. The constant term λ matches κ at $k = 0$, and the rational remainder $k^2 \sum_j r_j / (k^2 + \alpha_j^2)$ vanishes at $k = 0$ and tends to $\sum_j r_j$ as $|k| \rightarrow \infty$. The consistency condition $\sum_j r_j(p) = 1$, needed to match the leading $\kappa(k, p) \sim |k|$ behavior, is automatically satisfied by the Stieltjes Padé construction. Two features of Eq. (B3) control everything that follows. First, κ_n is even in k , so substitutions into A_θ and B_θ preserve the conjugation symmetry $B_\theta(k, p) = \overline{A_\theta(-\bar{k}, \bar{p})}$. Second, the poles of κ_n in the complex k plane are

$$k_j^\pm(p) = \pm i\alpha_j(p), \quad j = 1, \dots, n, \quad (\text{B4})$$

located symmetrically on the imaginary axis. The rational branch maps

$$A_\theta^{(n)}(k, p) := \sin \theta \kappa_n(k, p) - i \cos \theta k, \quad (\text{B5})$$

$$B_\theta^{(n)}(k, p) := \sin \theta \kappa_n(k, p) + i \cos \theta k, \quad (\text{B6})$$

inherit these poles, and computing the zeros and poles of the Wiener-Hopf factors of $K_A^{(n)}$, treated in Appendix B 5, reduces entirely to polynomial root finding once the $\alpha_j(p)$ are known.

2. Computation of the Padé coefficients by the Hankel linear system

An explicit algorithm for $\alpha_j(p)$ and $r_j(p)$ at any prescribed p with $\text{Re } p > 0$ is the standard Padé algorithm applied to the Stieltjes function Eq. (B1) in $z = k^2$, in three steps: Taylor coefficients of h at $z = 0$, a Hankel linear system for the denominator polynomial, and companion matrix eigendecomposition for the roots and residues. For the first step, expand

$$h(z) = \lambda \sqrt{1 + \frac{z}{\lambda^2}} = \lambda \sum_{m=0}^{\infty} \binom{1/2}{m} \frac{z^m}{\lambda^{2m}} = \sum_{m=0}^{\infty} h_m z^m, \quad (\text{B7})$$

with explicit Taylor coefficients

$$h_0 = \lambda,$$

$$h_m = \lambda \binom{1/2}{m} \lambda^{-2m} = \frac{(-1)^{m-1} (2m-2)!}{2^{2m-1} m! (m-1)!} \lambda^{1-2m} \quad (\text{B8})$$

with $m \geq 1$. The radius of convergence of Eq. (B7) is λ^2 , the distance from $z = 0$ to the branch point, but the Padé approximant converges on a much larger domain than the Taylor series, essentially the cut plane $\mathbb{C} \setminus (-\infty, -\lambda^2]$. Generate the first $2n$ Taylor coefficients $h_0, h_1, \dots, h_{2n-1}$ from Eq. (B8) at $O(n)$ cost. For the second step, the denominator of the $[n-1/n]$ Padé approximant is

$$Q_n(z; p) = \sum_{i=0}^n q_i(p) z^i, \quad q_n(p) = 1, \quad (\text{B9})$$

whose coefficients $q_0, \dots, q_{n-1}$ are determined by requiring that $Q_n(z; p)h(z)$ match a polynomial of degree $n-1$ in its lowest $2n$ coefficients. The Cauchy product condition that the coefficients of $z^n, \dots, z^{2n-1}$ vanish gives

$$\sum_{i=0}^{n-1} q_i(p) h_{n+m-i} = -h_{n+m}, \quad m = 0, 1, \dots, n-1, \quad (\text{B10})$$

with $q_n = 1$ moved to the right-hand side. This is a Hankel linear system with matrix $H_{mi} = h_{n+m-i}$ and right-hand side $-h_{n+m}$. The matrix is the moment matrix of the Stieltjes measure associated with h and is positive definite by the Stieltjes moment property, so Eq. (B10) has a unique solution stable under small perturbations. Gaussian elimination costs $O(n^3)$, negligible for the moderate $n \in \{4, 8, 16\}$ used in practice. For the third step, compute the roots of

$$Q_n(z; p) = \prod_{j=1}^n (z - z_j(p)) \quad (\text{B11})$$

by forming the companion matrix

$$C_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & -q_0 \\ 1 & 0 & \cdots & 0 & -q_1 \\ 0 & 1 & \cdots & 0 & -q_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -q_{n-1} \end{pmatrix}, \quad (\text{B12})$$

and computing its eigenvalues by QR or any backward stable eigensolver, at $O(n^3)$ cost. By the Stieltjes property the eigenvalues $z_j(p)$ lie on the negative real axis, so $z_j(p) = -\alpha_j(p)^2$ with $\alpha_j(p) > 0$, and these are the pole locations of Eq. (B3). The residues are obtained by partial fraction expansion:

$$\begin{aligned} r_j(p) &= \lim_{z \rightarrow z_j(p)} \frac{(z - z_j(p))[h(z) - \lambda]}{z/[z - z_j(p)]} \\ &= -z_j(p) \frac{P_{n-1}(z_j(p); p)}{Q'_n(z_j(p); p)} \cdot \frac{1}{z_j(p)}, \end{aligned} \quad (\text{B13})$$

which simplifies to

$$r_j(p) = \frac{P_{n-1}(z_j(p); p)}{Q'_n(z_j(p); p)}, \quad (\text{B14})$$

with P_{n-1} the numerator polynomial determined by

$$P_{n-1}(z; p) = [Q_n(z; p)h(z)]_{\leq n-1}, \quad (\text{B15})$$

the truncation of the Taylor product to its first n terms. Evaluating at $z = z_j(p)$,

$$P_{n-1}(z_j(p); p) = \sum_{i=0}^{n-1} \left[\sum_{i'=0}^i q_{i'}(p) h_{i-i'} \right] z_j(p)^i, \quad (\text{B16})$$

and the derivative is

$$Q'_n(z_j(p); p) = \prod_{\ell \neq j} (z_j(p) - z_\ell(p)), \quad (\text{B17})$$

all factors of which are real and nonzero by the Stieltjes property. Equation (B14) is therefore well defined and gives $r_j(p) > 0$ for every j . The consistency condition $\sum_j r_j(p) = 1$ is then automatic and provides a useful diagnostic: the computed r_j should sum to unity to working precision. Steps 1–3 constitute a complete $O(n^3)$ algorithm. For fixed n it is a rational function of $\lambda = \sqrt{p/D}$ by Eq. (B8) and the linearity of Eq. (B10), so $\alpha_j(p)$ and $r_j(p)$ depend on p through λ alone as $\alpha_j(p) = \lambda \hat{\alpha}_j$, $r_j(p) = \hat{r}_j$, with dimensionless universal constants $\hat{\alpha}_j, \hat{r}_j$ independent of p . These are computed once at the start of the numerical evaluation and reused for every p , so only a trivial rescaling is required at each Talbot

node. The approximant Eq. (B3) takes the explicit closed form

$$\kappa_n(k, p) = \lambda \left[1 + \sum_{j=1}^n \frac{\hat{r}_j (k/\lambda)^2}{(k/\lambda)^2 + \hat{\alpha}_j^2} \right], \quad (\text{B18})$$

manifestly homogeneous of degree 1 in (k, λ) and the form used below.

3. Rational branch maps and their poles in the k plane

With Eq. (B18) in hand, the branch maps Eqs. (B5) and (B6) become explicit rational functions of k . Setting

$$D_n(k, p) := \prod_{j=1}^n (k^2 + \alpha_j(p)^2), \quad (\text{B19})$$

a polynomial of degree $2n$ in k with roots at Eq. (B4), and multiplying Eq. (B3) by D_n ,

$$\kappa_n(k, p) D_n(k, p) = \lambda D_n(k, p) + k^2 \sum_{j=1}^n r_j(p) \prod_{\ell \neq j} (k^2 + \alpha_\ell(p)^2), \quad (\text{B20})$$

whose right-hand side is a polynomial $N_n(k, p)$ of degree $2n$. Thus,

$$\kappa_n(k, p) = \frac{N_n(k, p)}{D_n(k, p)}, \quad (\text{B21})$$

with both N_n, D_n explicit polynomials of degree $2n$ computable from the α_j, r_j . Substituting Eq. (B21) into Eqs. (B5) and (B6) and clearing denominators,

$$A_\theta^{(n)}(k, p) = \frac{\sin \theta N_n(k, p) - i \cos \theta k D_n(k, p)}{D_n(k, p)} = \frac{\mathcal{A}_n(k, p)}{D_n(k, p)}, \quad (\text{B22})$$

$$B_\theta^{(n)}(k, p) = \frac{\sin \theta N_n(k, p) + i \cos \theta k D_n(k, p)}{D_n(k, p)} = \frac{\mathcal{B}_n(k, p)}{D_n(k, p)}, \quad (\text{B23})$$

with numerators $\mathcal{A}_n, \mathcal{B}_n$ polynomials of degree $2n + 1$ and common denominator D_n of degree $2n$. The rational branch maps are therefore meromorphic in k , with poles at $k = \pm i\alpha_j$ and otherwise analytic. Computing the poles in k of quantities such as $1/(q + A_\theta^{(n)}(k, p))$ for given complex q reduces to the polynomial equation

$$\mathcal{A}_n(k, p) + q D_n(k, p) = 0, \quad (\text{B24})$$

of degree $2n + 1$ in k , with coefficients that are explicit linear combinations of those of $\mathcal{A}_n$ and D_n . Its roots are obtained by companion matrix eigendecomposition at $O(n^3)$ cost; the roots are complex in general and come in conjugate pairs when q is real, because the coefficients of $\mathcal{A}_n$ and D_n are real or purely imaginary in a definite pattern dictated by the parity of κ_n . The same procedure applied to $\mathcal{B}_n(k, p) + q D_n(k, p) = 0$ gives the poles of $1/(q + B_\theta^{(n)}(k, p))$. Every pole location required in the rationalized scheme is therefore a root of a polynomial whose coefficients are explicit combinations of $\lambda, \sin \theta, \cos \theta, q$, and the precomputed universal constants $\hat{\alpha}_j, \hat{r}_j$.

4. Rationalized lower branch as a finite linear system

We can then apply the rationalization to Eq. (121). The reflected operator $\mathcal{R}_{BB}$ of Eq. (103) contains three nonrational quantities: $e^{-2d_0\kappa(k', p)}$, $1/(2D\kappa(k', p))$, and $1/(B_\theta(k, p) + A_\theta(k', p))$. The second and third become rational in k' through $\kappa_n, A_\theta^{(n)}, B_\theta^{(n)}$, while the wall exponential retains its exponential dependence and must be handled separately. The Padé-Abrahams device is to rationalize the combined quantity $e^{-2d_0\kappa(k', p)}/(2D\kappa_n(k', p))$ as a rational function of k' with its own pole set on the imaginary axis, by a second Padé step of controlled order. At the Padé pole locations $k' = \pm i\alpha_j(p)$, the rationalized radical takes the finite value

$$\kappa_n(\pm i\alpha_j(p), p) = \lambda \left[1 - \sum_{\ell \neq j} \frac{\hat{r}_\ell \hat{\alpha}_j^2}{\hat{\alpha}_\ell^2 - \hat{\alpha}_j^2} \right], \quad (\text{B25})$$

obtained from Eq. (B18) by taking the regular part of the partial fraction at the double pole, and the exponential weight at each such pole becomes a concrete number $e^{-2d_0\lambda\beta_j}$ with β_j the bracket in Eq. (B25). Distributing these weights as residues of

$$W_n(k', p) := \sum_{j=1}^n \frac{w_j(p)}{k'^2 + \alpha_j(p)^2}, \quad w_j(p) := \frac{r_j(p) e^{-2d_0\lambda\beta_j(p)}}{2D\kappa_n(i\alpha_j(p), p)} \quad (\text{B26})$$

gives the rationalized reflected weight in the same $2n$ pole form as κ_n , with the wall attenuation baked into $w_j(p)$. The reflected kernel of Eq. (103) is then

$$\begin{aligned} & \frac{e^{-2d_0\kappa_n(k', p)}}{2D\kappa_n(k', p)} \frac{1}{B_\theta^{(n)}(k, p) + A_\theta^{(n)}(k', p)} \\ & \approx \frac{W_n(k', p)}{B_\theta^{(n)}(k, p) + A_\theta^{(n)}(k', p)}, \end{aligned} \quad (\text{B27})$$

and the whole operator becomes the finite rank

$$(\mathcal{R}_{BB}^{(n)}\psi)(k, p) = \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \frac{W_n(k', p) \psi(k', p)}{B_\theta^{(n)}(k, p) + A_\theta^{(n)}(k', p)}. \quad (\text{B28})$$

The integrand is rational in k' with explicit pole structure: simple poles at $k' = \pm i\alpha_j(p)$ from W_n , poles at the $2n + 1$ roots of the polynomial $\mathcal{A}_n(k', p) + B_\theta^{(n)}(k, p)D_n(k', p)$ from the resonance denominator in k' , and no other singularities. The integrand decays faster than $1/|k'|^2$ at large $|k'|$ through the $1/|k'|^2$ factor in W_n and the bounded reflected denominator, so the integral along the real k' axis may be closed in either half plane without contribution from the arc at infinity, and equals the residue sum in the enclosed half plane. Closing upward, the poles at $k' = i\alpha_j(p)$ from W_n contribute residues depending linearly on ψ evaluated there, while the poles from the reflected denominator contribute residues algebraic in ψ at the root locations of $\mathcal{A}_n(k', p) + B_\theta^{(n)}(k, p)D_n(k', p)$. The reflected integral becomes a finite sum of evaluations of ψ at a fixed pole set, and no continuous integration remains. The projected Fredholm equation Eq. (121) therefore reduces to a finite linear system once the unknown is evaluated at the same pole set and these evaluations are treated as the independent variables. Setting

$$\Phi_B^{(n),j}(p) := \Phi_B^{(n)}(i\alpha_j(p), p), \quad j = 1, \dots, n, \quad (\text{B29})$$

and substituting the residue evaluation of Eq. (B28) into Eq. (121) at each collocation point $k = i\alpha_i(p)$,

$$\sum_{j=1}^n [\delta_{ij} + M_{ij}(p, d_0, \theta)] \Phi_B^{(n),j}(p) = F_i(p, d_0, \theta), \quad (\text{B30})$$

with $i = 1, \dots, n$ and matrix entries

$$M_{ij}(p, d_0, \theta) = \frac{w_j(p)}{2 K_B^{(n)}(i\alpha_i(p), p) [B_\theta^{(n)}(i\alpha_i(p), p) + A_\theta^{(n)}(i\alpha_i(p), p)]} \quad (\text{B31})$$

computed directly from the Padé data. The right-hand side F_i is the value of $K_B^{(n),-1} G_B^{(+)}$ at the same collocation points, evaluated in closed form through Eq. (100). Equation (B30) is an $n \times n$ complex linear system costing $O(n^3)$ per Talbot node by direct factorization. Once solved, the $\Phi_B^{(n),j}(p)$ determine $\Phi_B^{(n)}(k, p)$ globally on the k plane, as the unique rational function of the prescribed structure taking the computed values at the collocation points.

5. Rationalized upper branch and the factorization by roots

Once $\Phi_B^{(n)}$ is known from Eq. (B30), the reflected forcing on the upper branch,

$$F_A^{(n)}(k, p) := G_A^{(n)}(k, p) - (\mathcal{R}_{AB}^{(n)} \Phi_B^{(n)})(k, p), \quad (\text{B32})$$

is a meromorphic function of k with closed form pole structure. The element contribution $G_A^{(n)}$ follows from Eq. (99) by $A_\theta \rightarrow A_\theta^{(n)}$ and reduction to the common denominator D_n :

$$G_A^{(n)}(k, p) = \frac{c_{LC} D_n(k, p)^2}{\mathcal{A}_n(k, p) [p D_n(k, p) - v \mathcal{A}_n(k, p)]}, \quad (\text{B33})$$

with poles at the roots of

$$\mathcal{A}_n(k, p) = 0, \quad p D_n(k, p) - v \mathcal{A}_n(k, p) = 0, \quad (\text{B34})$$

of degrees $2n+1$. Each is solved by companion matrix eigendecomposition at $O(n^3)$ per Talbot node. The resulting root sets are disjoint from the real k axis in the regime of interest, because the exact analogues $A_\theta(k, p) = 0$ and $p - v A_\theta(k, p) = 0$ have no real roots for real k and $\text{Re } p > 0$. The reflected contribution $(\mathcal{R}_{AB}^{(n)} \Phi_B^{(n)})(k, p)$ is obtained by applying the same residue evaluation as in Eq. (B28) to the off diagonal kernel Eq. (101), with the known $\Phi_B^{(n)}$ replacing the unknown, giving a rational function of k whose poles are the roots of a further polynomial equation $\mathcal{A}_n(k, p) + B_\theta^{(n)}(k', p) D_n(k, p) = 0$ for each $k' = i\alpha_j(p)$, again by companion matrix eigendecomposition. The direct upper branch symbol $K_A^{(n)}(k, p) = \mathcal{K}_0(A_\theta^{(n)}(k, p), p)$ is rational in k once $A_\theta^{(n)}$ is. Substituting Eq. (B22) into Eq. (39),

$$K_A^{(n)}(k, p) = \frac{1}{2D \Gamma(A_\theta^{(n)}(k, p), p)} = \frac{1}{2D \sqrt{p/D - (\mathcal{A}_n(k, p)/D_n(k, p))^2}}, \quad (\text{B35})$$

which simplifies to

$$K_A^{(n)}(k, p) = \frac{D_n(k, p)}{2D \sqrt{(p/D) D_n(k, p)^2 - \mathcal{A}_n(k, p)^2}}, \quad (\text{B36})$$

with the square-root branch chosen to agree with the exact factor on the working strip. The radicand

$$\Pi_n(k, p) := (p/D) D_n(k, p)^2 - \mathcal{A}_n(k, p)^2 \quad (\text{B37})$$

is a polynomial in k of degree $4n+2$, with real coefficients (for real p/D) up to an overall phase inherited from the parity of $\mathcal{A}_n$, and its roots are the zeros of $K_A^{(n)}$. These are computed by a single companion matrix eigendecomposition of Π_n at each p , automatically sorted by sign of imaginary part. The canonical Wiener-Hopf factorization $K_A^{(n)} = K_{A,+}^{(n)} K_{A,-}^{(n)}$ is obtained by splitting the roots of Π_n and D_n between the upper and lower half k planes. The classical fact for rational symbols is that a rational function of k with no real roots or poles factorizes canonically as a product of two rational functions, one analytic and nonvanishing in the upper half plane and the other in the lower. Let $\{a_m^+(p)\}$, $\{a_m^-(p)\}$ be the roots of $\Pi_n = 0$ with $\text{Im } a_m^+ > 0$ and $\text{Im } a_m^- < 0$, and $\{b_n^+(p)\}$, $\{b_n^-(p)\}$ the analogues for D_n . Then

$$K_{A,+}^{(n)}(k, p) = C_+(p) \frac{\prod_m (k - a_m^+(p))^{-1/2}}{\prod_n (k - b_n^+(p))^{-1}}, \quad (\text{B38})$$

$$K_{A,-}^{(n)}(k, p) = C_-(p) \frac{\prod_m (k - a_m^-(p))^{-1/2}}{\prod_n (k - b_n^-(p))^{-1}}, \quad (\text{B39})$$

with $C_\pm(p)$ chosen so that $K_{A,+}^{(n)} K_{A,-}^{(n)} = K_A^{(n)}$ as meromorphic functions of k , equivalent by elementary bookkeeping to $C_+(p) C_-(p) = (2D)^{-1}$ up to a sign. The $-1/2$ exponents on the $a_m^\pm$ factors are inherited from the outer square root in Eq. (B36), and make the factorization multi valued with branches fixed by the half plane analyticity requirement.

6. Residue calculus for the plus projection and the branch trace

The plus projection required by Eq. (136) acts on

$$Q_A^{(n)}(k, p) := \frac{F_A^{(n)}(k, p)}{K_{A,-}^{(n)}(k, p)}, \quad (\text{B40})$$

a known rational function of k with explicit poles. Its poles in the lower half plane come from two sources: the lower half plane poles of $F_A^{(n)}$, which are the lower half plane subset of the roots of Eqs. (B34) together with the analogues for the reflected contribution, and the simple zeros $a_m^-(p)$ of $K_{A,-}^{(n)}$, at which $F_A^{(n)}$ does not vanish so that $1/K_{A,-}^{(n)}$ has a pole. Collecting these into $\{\zeta_j^A(p)\}_{j=1}^{N_A}$ after discarding accidental cancellations, the plus projection becomes

$$\mathcal{P}_+[Q_A^{(n)}(\cdot, p)](k) = \sum_{j=1}^{N_A} \frac{R_j^A(p)}{k - \zeta_j^A(p)}, \quad (\text{B41})$$

with $R_j^A(p)$ given by

$$R_j^A(p) = \frac{\text{Res}_{k=\zeta_j^A(p)} F_A^{(n)}(k, p)}{K_{A,-}^{(n)}(\zeta_j^A(p), p)}, \quad (\text{B42})$$

when $\zeta_j^A(p)$ is inherited from $F_A^{(n)}$, and by

$$R_j^A(p) = \frac{F_A^{(n)}(a_m^-(p), p)}{(K_{A,-}^{(n)})'(a_m^-(p), p)}, \quad (\text{B43})$$

when $\zeta_j^A(p) = a_m^-(p)$ is a lower half plane zero of $K_{A,-}^{(n)}$, with the derivative computed from Eq. (B39). Substituting Eq. (B41) into Eq. (136),

$$\Phi_A^{(n)}(k, p) = \frac{1}{K_{A,+}^{(n)}(k, p)} \sum_{j=1}^{N_A} \frac{R_j^A(p)}{k - \zeta_j^A(p)}, \quad (\text{B44})$$

which is the exact upper branch trace for the rational reduction. Every object on the right-hand side is a concrete algebraic quantity computed from the Padé data, and the only continuous operations at this stage are the polynomial root computations of Appendix B 5.

7. Meromorphic reconstruction in the q plane and inverse transforms

The transformed density $\Phi_+^{(n)}(q, p)$ is reconstructed on the body of the working strip by the strip Cauchy projection Eq. (139), with the known reflected contribution $R(q, p)$ now given by a finite sum of explicit residues from the rationalized lower branch. In partial fraction form,

$$\Phi_+^{(n)}(q, p) = \mathcal{K}_{0,+}^{-1}(q, p) \left[\frac{c_{\text{LC}}}{q(p - vt) \mathcal{K}_{0,-}(0, p)} - \sum_{\ell=1}^{N_\ell} \frac{\rho_\ell(p, d_0, \theta)}{q + A_\theta^{(n)}(k_\ell(p), p)} \right], \quad (\text{B45})$$

where the sum runs over the finite pole set $\{k_\ell(p)\}$ of the reflected contribution inherited from Eq. (B4) and the roots of the reflected polynomial equations of Appendix B 4, and the residues ρ_ℓ depend linearly on the $\Phi_B^{(n),j}(p)$ through Eq. (B31). Equation (B45) is meromorphic in q on the working strip, and all its singularities are rational poles inherited from the Padé branch maps and the polynomial root computations. No continuous contour integration in q is required. The inverse wedge transform Eq. (140) applied to Eq. (B45) is carried out in closed form on the spatial side. The inner q contour is deformed onto the rational pole geometry of $\Phi_+^{(n)}(q, p)$ and the integral collapses to a finite residue sum:

$$v^{(n)}(u, t) = \frac{1}{2\pi i} \int_{\gamma_p - i\infty}^{\gamma_p + i\infty} dp e^{pt} \sum_{\ell} r_\ell(p) e^{q_\ell(p)u}, \quad (\text{B46})$$

with $\{q_\ell(p)\}$ the complete rational pole set of $\Phi_+^{(n)}(q, p)$, namely, the images under $A_\theta^{(n)}$ and $B_\theta^{(n)}$ of the k plane pole sets computed above, and $r_\ell(p)$ the corresponding residues, both available in closed form at every p . The remaining p integral is the outer Bromwich or Talbot inversion of Sec. VIF. The laboratory density is recovered by

$$\mu^{(n)}(s, t) = v^{(n)}(s - vt, t), \quad (\text{B47})$$

the rationalized analog of Eq. (141). The process zone functional of Sec. VIII, an integral of $\mu^{(n)}$ against a simple indicator over the process zone, becomes a finite sum of

closed form contributions once Eq. (B46) is substituted and the u integral is evaluated term by term:

$$\int_{-vt}^{\ell-vt} e^{q_\ell(p)u} du = \frac{e^{q_\ell(p)(\ell-vt)} - e^{-q_\ell(p)vt}}{q_\ell(p)}, \quad (\text{B48})$$

an algebraic function of p through the Padé data alone, inverted in p by the same outer Talbot quadrature as the density. The only approximation enters at the single step Eq. (B3) in which the exact radical is replaced by the finite Padé approximant $\kappa_n(k, p)$, and the error decays exponentially in n on compact sets away from the branch points, by the classical convergence theorem for Stieltjes Padé approximants. Every subsequent step [the rationalization of the branch maps, the reduction of the Fredholm equation to Eq. (B30), the factorization of $K_A^{(n)}$ by sorting polynomial roots, the plus projection by Eq. (B41), the partial fraction reconstruction Eq. (B45), and the inverse wedge transform Eq. (B46)] is an exact equality within the rational reduction. The computational cost at each Talbot node is dominated by three $O(n^3)$ operations: the Hankel solve Eq. (B10), the companion matrix eigendecomposition for the polynomial root computations, and the linear solve Eq. (B30), plus $O(n)$ residue evaluations.

For the practical orders $n \in \{2, 4, 8, 12\}$, the total cost per Talbot node is of the order of a few hundred microseconds on a modern processor, and the error relative to the exact benchmark of Sec. VI is dominated by the residual Padé approximation error alone, every other component being treated exactly. Increasing n reduces this residual error without altering any other ingredient, and is the natural knob by which the reduction is tuned to any prescribed tolerance. The convergence is illustrated in Fig. 16, which shows that the rational reduction reaches the floor set by the outer Bromwich quadrature already at $n \simeq 12$ on the representative configuration $\theta = \pi/4, d_0 = 1$.

APPENDIX C: STEADY TRANSLATING LIMITS AND THE PARALLEL WALL REDUCTION

The unsteady laboratory wedge construction of Sec. III through Appendix B solves the moving endpoint problem at finite time and at all obliquities, and produces a transformed density $\Phi_+(q, p)$ on the two-dimensional working strip (66). The natural question of whether the same construction admits a uniformly translating steady state, in which the line density on the dendrite becomes time independent in the comoving coordinate $u = s - vt$, requires a separate analytical treatment. The reason is that the steady regime is not the final value limit $p \rightarrow 0$ of the unsteady wedge transform: the direct symbol $\mathcal{K}_0(q, p) = 1/[2D\Gamma(q, p)]$ at $p = 0$ becomes $1/[2D\sqrt{-q^2}]$, which loses the finite diffusive scale and collapses to a degenerate object on the working strip. The unsteady wedge transform is the correct tool for the closing wall problem of the main text, but it is not the natural transform in which to formulate a uniformly translating steady state. To access the steady regime, we therefore re pose the problem directly in the comoving frame, in which the field satisfies the stationary convection diffusion equation, and we Laplace transform only in the dendrite coordinate. The resulting steady reduction is structurally cleaner than the unsteady one, and

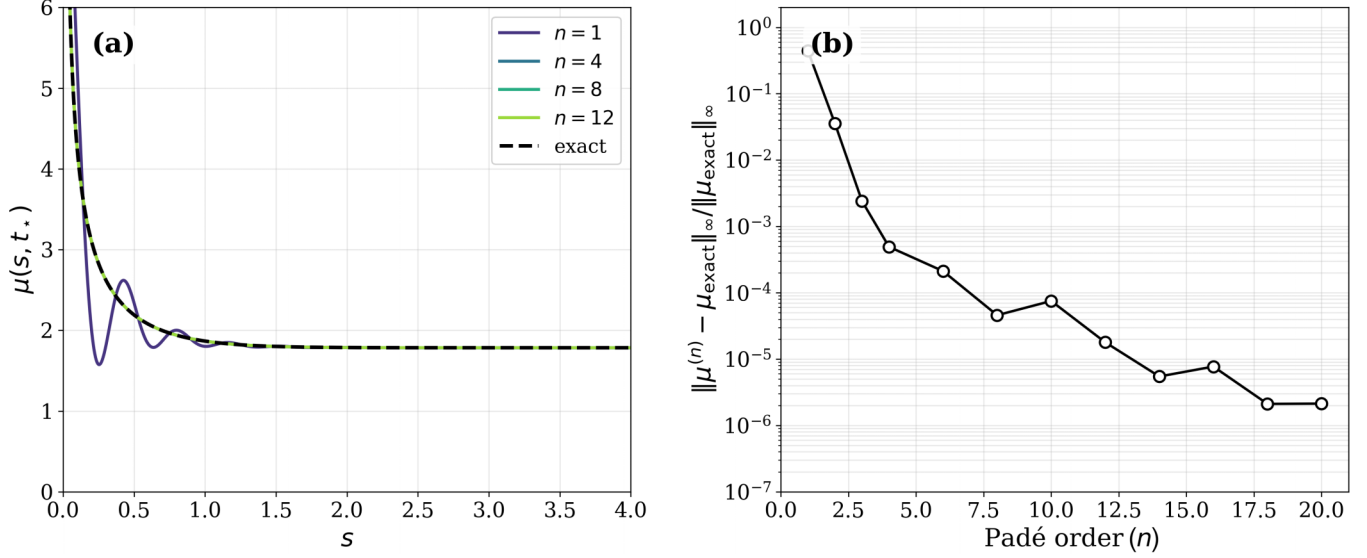


FIG. 16. Convergence of the Padé-Abrahams rational reduction at $\theta = \pi/4$, $d_0 = 1$, $t_* = 0.4$, $D = 1$, $v = 2$, $c_{\text{LC}} = 1$. (a) Line density $\mu(s, t_*)$ at Padé orders $n = 1, 4, 8, 12$ against the exact solution of Sec. VI. (b) Relative max-norm error across $s \in [0.1, 3.5]$ versus n , converging exponentially at about one decade per four orders and saturating near 2×10^{-6} at $n \gtrsim 18$; by $n = 12$ the two solutions are visually indistinguishable in panel (a). The same pattern holds across $\theta \in [\pi/6, \pi/3]$ and $d_0 \in [0.5, 1.5]$ (not shown).

at $\theta = 0$ it reduces to a scalar Wiener-Hopf problem whose canonical factorization can be written down explicitly.

1. Steady comoving formulation

Let $c^{\text{st}}(\sigma, \zeta)$ denote the time independent concentration field in the wall fixed coordinates (13) of Sec. III A, expressed as a function of comoving coordinates aligned with the steadily translating tip. In the steady regime the field obeys the stationary convection diffusion equation

$$-v \partial_s c^{\text{st}} + D \Delta c^{\text{st}} = 0, \quad (\text{C1})$$

in which s is the dendrite arc length behind the tip. The wall and dendrite boundary conditions are the steady analogues of the unsteady ones. The single-layer representation produces the steady boundary integral equation

$$c_{\text{LC}} = \int_0^\infty K_\theta^{\text{st}}(s, s') \mu^{\text{st}}(s') ds', \quad s > 0, \quad (\text{C2})$$

with K_θ^{st} the steady Green kernel of Eq. (C1) in the half plane. After the standard exponential gauge transform $c^{\text{st}} = e^{vs/(2D)} \tilde{c}^{\text{st}}$, Eq. (C1) becomes a modified Helmholtz equation. Its spectral representation yields, after Laplace transform in s ,

$$\Gamma_{\text{st}}(q) := \sqrt{q \left(q + \frac{1}{\ell_D} \right)}, \quad \ell_D = \frac{D}{v}, \quad (\text{C3})$$

and the steady direct symbol

$$\kappa_0^{\text{st}}(q) = \frac{1}{2D \Gamma_{\text{st}}(q)} = \frac{1}{2D \sqrt{q(q + 1/\ell_D)}}. \quad (\text{C4})$$

The symbol has square-root branch points at

$$q = 0, \quad q = -\frac{1}{\ell_D}, \quad (\text{C5})$$

with branch cut taken along $[-1/\ell_D, 0]$. The natural domain of analyticity is the open half plane $\text{Re} q > 0$. A canoni-

cal Wiener-Hopf factorization is obtained by assigning one branch point to each factor,

$$\kappa_{0,+}^{\text{st}}(q) = \frac{1}{\sqrt{2D}} \frac{1}{\sqrt{q + 1/\ell_D}}, \quad \kappa_{0,-}^{\text{st}}(q) = \frac{1}{\sqrt{2D}} \frac{1}{\sqrt{q}}, \quad (\text{C6})$$

with square-root branches chosen consistently with the cut $[-1/\ell_D, 0]$, so that $\kappa_{0,+}^{\text{st}} \kappa_{0,-}^{\text{st}} = \kappa_0^{\text{st}}$ on the overlap region. For $\theta > 0$, the reflected contribution retains the same upper triangular Wiener-Hopf-Hankel structure as in the unsteady problem, and no scalar reduction is available.

2. Steady oblique Wiener-Hopf-Hankel reduction

For $\theta \in (0, \pi/2)$, the steady reflected kernel is not of difference type in the dendrite coordinate s , for exactly the same geometric reason as in the unsteady laboratory problem: reflection at an oblique wall couples the source and field points through the sum $s + s'$ rather than through the difference $s - s'$ alone. The steady problem is therefore not a scalar Wiener-Hopf problem at nonzero obliquity, but a steady Wiener-Hopf-Hankel problem with the same upper triangular branch structure as in Sec. V, now at fixed translating state rather than in the moving wedge geometry.

After the exponential gauge $c^{\text{st}} = e^{vs/(2D)} \tilde{c}^{\text{st}}$, the steady equation reduces to a modified Helmholtz equation with screening length $2\ell_D$, and the reflected half plane Green kernel admits the spectral representation

$$K_\theta^{\text{ref,st}}(s, s') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{-2d_0 \kappa_{\text{st}}(k)}}{2D \kappa_{\text{st}}(k)} e^{-A_\theta^{\text{st}}(k)s} e^{-B_\theta^{\text{st}}(k)s'}, \quad (\text{C7})$$

with $s, s' > 0$, where

$$\kappa_{\text{st}}(k) := \sqrt{k^2 + \frac{1}{4\ell_D^2}}, \quad (\text{C8})$$

and the steady spectral combinations are

$$A_\theta^{\text{st}}(k) := \sin \theta \kappa_{\text{st}}(k) - i \cos \theta k, \quad (\text{C9})$$

$$B_\theta^{\text{st}}(k) := \sin \theta \kappa_{\text{st}}(k) + i \cos \theta k. \quad (\text{C10})$$

For real k and $0 < \theta < \pi/2$,

$$\text{Re} A_\theta^{\text{st}}(k) = \text{Re} B_\theta^{\text{st}}(k) = \sin \theta \kappa_{\text{st}}(k) > 0, \quad (\text{C11})$$

so both branch curves lie strictly in the open half plane $\text{Re} q > 0$. At $\theta = 0$ one recovers

$$A_0^{\text{st}}(k) = -ik, \quad B_0^{\text{st}}(k) = ik, \quad (\text{C12})$$

and the Hankel coupling collapses to convolution, as expected.

Let

$$\Phi_+^{\text{st}}(q; d_0, \theta) := \int_0^\infty e^{-qs} \mu^{\text{st}}(s; d_0, \theta) ds, \quad \text{Re} q > 0, \quad (\text{C13})$$

denote the steady one-sided Laplace transform of the line density, and let $\Phi_-^{\text{st}}(q; d_0, \theta)$ denote the complementary minus function generated by the Wiener-Hopf splitting on the overlap strip $-1/\ell_D < \text{Re} q < 0$. Then the steady transformed boundary integral equation takes the exact form

$$\begin{aligned} \mathcal{K}_0^{\text{st}}(q) \Phi_+^{\text{st}}(q; d_0, \theta) + \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{e^{-2d_0\kappa_{\text{st}}(k)}}{2D\kappa_{\text{st}}(k)} \frac{\Phi_+^{\text{st}}(B_\theta^{\text{st}}(k); d_0, \theta)}{q + A_\theta^{\text{st}}(k)} \\ + \Phi_-^{\text{st}}(q; d_0, \theta) = \frac{C_{\text{st}}}{q}, \quad \text{Re} q > 0, \end{aligned} \quad (\text{C14})$$

with $\mathcal{K}_0^{\text{st}}$ given by Eq. (C4). The reflected term is no longer multiplicative in q , but its dependence on the unknown enters only through the single trace of Φ_+^{st} along the branch $q = B_\theta^{\text{st}}(k)$.

Evaluating Eq. (C14) on the two steady branch curves,

$$\Gamma_A^{\text{st}} := \{A_\theta^{\text{st}}(k) : k \in \mathbb{R}\}, \quad \Gamma_B^{\text{st}} := \{B_\theta^{\text{st}}(k) : k \in \mathbb{R}\}, \quad (\text{C15})$$

and introducing the branch traces

$$\begin{aligned} \Phi_A^{\text{st}}(k; d_0, \theta) &:= \Phi_+^{\text{st}}(A_\theta^{\text{st}}(k); d_0, \theta), \\ \Phi_B^{\text{st}}(k; d_0, \theta) &:= \Phi_+^{\text{st}}(B_\theta^{\text{st}}(k); d_0, \theta), \end{aligned} \quad (\text{C16})$$

together with

$$K_A^{\text{st}}(k) := \mathcal{K}_0^{\text{st}}(A_\theta^{\text{st}}(k)), \quad K_B^{\text{st}}(k) := \mathcal{K}_0^{\text{st}}(B_\theta^{\text{st}}(k)), \quad (\text{C17})$$

$$G_A^{\text{st}}(k) := \frac{C_{\text{st}}}{A_\theta^{\text{st}}(k)}, \quad G_B^{\text{st}}(k) := \frac{C_{\text{st}}}{B_\theta^{\text{st}}(k)}, \quad (\text{C18})$$

one obtains the steady upper triangular branch system

$$\begin{aligned} K_A^{\text{st}}(k) \Phi_A^{\text{st}}(k; d_0, \theta) + (\mathcal{R}_{AB}^{\text{st}} \Phi_B^{\text{st}})(k; d_0, \theta) + \Phi_{-,A}^{\text{st}}(k; d_0, \theta) \\ = G_A^{\text{st}}(k), \end{aligned} \quad (\text{C19})$$

$$\begin{aligned} K_B^{\text{st}}(k) \Phi_B^{\text{st}}(k; d_0, \theta) + (\mathcal{R}_{BB}^{\text{st}} \Phi_B^{\text{st}})(k; d_0, \theta) + \Phi_{-,B}^{\text{st}}(k; d_0, \theta) \\ = G_B^{\text{st}}(k), \end{aligned} \quad (\text{C20})$$

where

$$\begin{aligned} (\mathcal{R}_{AB}^{\text{st}} \Phi_B^{\text{st}})(k; d_0, \theta) \\ = \int_{-\infty}^\infty \frac{dk'}{2\pi} \frac{e^{-2d_0\kappa_{\text{st}}(k')}}{2D\kappa_{\text{st}}(k')} \frac{\Phi_B^{\text{st}}(k'; d_0, \theta)}{A_\theta^{\text{st}}(k) + A_\theta^{\text{st}}(k')}, \end{aligned} \quad (\text{C21})$$

$$(\mathcal{R}_{BB}^{\text{st}} \Phi_B^{\text{st}})(k; d_0, \theta):$$

$$= \int_{-\infty}^\infty \frac{dk'}{2\pi} \frac{e^{-2d_0\kappa_{\text{st}}(k')}}{2D\kappa_{\text{st}}(k')} \frac{\Phi_B^{\text{st}}(k'; d_0, \theta)}{B_\theta^{\text{st}}(k) + A_\theta^{\text{st}}(k')}. \quad (\text{C22})$$

For real k, k' , the denominator of Eq. (C22) has strictly positive real part,

$$\text{Re}[B_\theta^{\text{st}}(k) + A_\theta^{\text{st}}(k')] = \sin \theta (\kappa_{\text{st}}(k) + \kappa_{\text{st}}(k')) > 0, \quad (\text{C23})$$

so the lower branch operator is nonsingular on the real contour and defines a compact Fredholm perturbation of the multiplication operator K_B^{st} .

The steady oblique problem is therefore solved sequentially, exactly as in the unsteady laboratory theory. First, the lower row (C20) is a closed Fredholm equation of the second kind for Φ_B^{st} . Once Φ_B^{st} is known, the reflected term in Eq. (C14) becomes a known inhomogeneity and the remaining equation for Φ_+^{st} is a scalar steady Wiener-Hopf problem with direct symbol $\mathcal{K}_0^{\text{st}}(q)$. Using the canonical factorization (C6), the exact reconstructed solution may be written as

$$\begin{aligned} \Phi_+^{\text{st}}(q; d_0, \theta) \\ = \frac{1}{\mathcal{K}_{0,+}^{\text{st}}(q)} \mathcal{P}_+^{\text{st}} \left[\frac{\frac{C_{\text{st}}}{q} - \int_{-\infty}^\infty \frac{dk}{2\pi} \frac{e^{-2d_0\kappa_{\text{st}}(k)}}{2D\kappa_{\text{st}}(k)} \frac{\Phi_B^{\text{st}}(k; d_0, \theta)}{q + A_\theta^{\text{st}}(k)}}{\mathcal{K}_{0,-}^{\text{st}}(q)} \right] (q), \end{aligned} \quad (\text{C24})$$

valid on the half plane $\text{Re} q > 0$, with $\mathcal{P}_+^{\text{st}}$ the same strip Cauchy projection as in Eq. (C31). The inverse Laplace transform in q may then be reduced by the steady Cagniard-de Hoop deformation. Since $\text{Re} A_\theta^{\text{st}}(k) > 0$ for real k , the Bromwich contour may be pushed onto the image contour Γ_A^{st} , and with the parametrization $q = A_\theta^{\text{st}}(k)$ one obtains the exact real-space representation

$$\begin{aligned} \mu^{\text{st}}(s; d_0, \theta) &= \frac{1}{2\pi i} \int_{\Gamma_A^{\text{st}}} e^{qs} [\Phi_+^{\text{st}}]_{\Gamma_A^{\text{st}}}(q; d_0, \theta) dq \\ &= \frac{1}{2\pi i} \int_{-\infty}^\infty e^{A_\theta^{\text{st}}(k)s} [\Phi_+^{\text{st}}]_{\Gamma_A^{\text{st}}} \\ &\quad \times (A_\theta^{\text{st}}(k); d_0, \theta) A_\theta^{\text{st}'}(k) dk, \end{aligned} \quad (\text{C25})$$

where $[\Phi_+^{\text{st}}]_{\Gamma_A^{\text{st}}}$ denotes the jump of the reconstructed transform across the Cagniard image. Equivalently, using the Schwarz reflection,

$$\mu^{\text{st}}(s; d_0, \theta) = \frac{1}{\pi} \text{Im} \left[\int_0^\infty e^{A_\theta^{\text{st}}(k)s} \Phi_A^{\text{st}}(k; d_0, \theta) A_\theta^{\text{st}'}(k) dk \right], \quad (\text{C26})$$

with the understanding that Φ_A^{st} is the boundary value obtained from the sequential branch solution above.

Equations (C25) and (C26) are the steady oblique analogues of the unsteady Cagniard-de Hoop inversion of the main text. Thus, although the oblique steady problem is not a one step scalar Wiener-Hopf reduction, its real-space inversion is nevertheless reducible to a single real spectral path once the lower branch trace has been solved.

3. Steady parallel wall scalar Wiener-Hopf reduction

At $\theta = 0$, the spectral combinations reduce to $A_0^{\text{st}}(k) = -ik$ and $B_0^{\text{st}}(k) = ik$, and the reflected kernel becomes convolutional. The steady operator equation collapses to the scalar Wiener-Hopf equation

$$\frac{1 + e^{-2d_0 \Gamma_{\text{st}}(q)}}{2D \Gamma_{\text{st}}(q)} \Phi_+^{\text{st}}(q; d_0) + \Phi_-^{\text{st}}(q; d_0) = \frac{C_{\text{st}}}{q}, \quad (\text{C27})$$

on the half plane $\text{Re} q > 0$. The total symbol

$$\mathcal{K}_0^{\text{tot, st}}(q; d_0) = \frac{1 + e^{-2d_0 \Gamma_{\text{st}}(q)}}{2D \Gamma_{\text{st}}(q)} \quad (\text{C28})$$

admits a canonical factorization obtained by additive splitting of its logarithm on the overlap strip $-1/\ell_D < \text{Re} q < 0$. Writing

$$\log \mathcal{K}_0^{\text{tot, st}}(q; d_0) = \log \mathcal{K}_0^{\text{st}}(q) + \log[1 + e^{-2d_0 \Gamma_{\text{st}}(q)}], \quad (\text{C29})$$

the second term is split by the standard Cauchy construction on this strip. Denoting the resulting functions by $S_{\pm}(q; d_0)$, the canonical factors are

$$\begin{aligned} \mathcal{K}_{0,+}^{\text{tot, st}}(q; d_0) &= \mathcal{K}_{0,+}^{\text{st}}(q) e^{S_+(q; d_0)}, \\ \mathcal{K}_{0,-}^{\text{tot, st}}(q; d_0) &= \mathcal{K}_{0,-}^{\text{st}}(q) e^{S_-(q; d_0)}. \end{aligned} \quad (\text{C30})$$

The Wiener-Hopf solution is then

$$\Phi_+^{\text{st}}(q; d_0, 0) = \frac{1}{\mathcal{K}_{0,+}^{\text{tot, st}}(q; d_0)} \mathcal{P}_+^{\text{st}} \left[\frac{C_{\text{st}}/q}{\mathcal{K}_{0,-}^{\text{tot, st}}(q; d_0)} \right] (q), \quad (\text{C31})$$

where $\mathcal{P}_+^{\text{st}}$ denotes the Cauchy projection defined on the steady overlap strip $-1/\ell_D < \text{Re} q < 0$.

4. Inverse Laplace reconstruction in real space

The steady line density on the dendrite is recovered from the transformed solution (C31) by the inverse Laplace transform

$$\mu^{\text{st}}(s; d_0, 0) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{qs} \Phi_+^{\text{st}}(q; d_0, 0) dq, \quad s > 0, \quad (\text{C32})$$

with $\gamma > 0$ chosen in the steady analyticity half plane $\text{Re} q > 0$. Substituting Eq. (C31) gives the exact real-space reconstruction

$$\begin{aligned} \mu^{\text{st}}(s; d_0, 0) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{qs} \frac{1}{\mathcal{K}_{0,+}^{\text{tot, st}}(q; d_0)} \mathcal{P}_+^{\text{st}} \left[\frac{C_{\text{st}}/q}{\mathcal{K}_{0,-}^{\text{tot, st}}(q; d_0)} \right] dq. \end{aligned} \quad (\text{C33})$$

Equation (C33) is the exact inversion formula for the steady parallel wall problem. It is the steady analog of the double Bromwich reconstruction of the unsteady laboratory wedge problem, with the time Laplace inversion absent because the steady co moving formulation carries only the dendrite variable. The contour in Eq. (C32) may be deformed to the left, around the branch cut of $\Gamma_{\text{st}}(q) = \sqrt{q(q + 1/\ell_D)}$ along the interval $[-1/\ell_D, 0]$, provided the boundary values of the projected term in Eq. (C31) are taken consistently on

the two sides of the cut. This gives the equivalent cut representation

$$\begin{aligned} \mu^{\text{st}}(s; d_0, 0) &= \frac{1}{2\pi i} \int_{-1/\ell_D}^0 e^{qs} [\Phi_+^{\text{st}}(q + i0; d_0, 0) - \Phi_+^{\text{st}}(q - i0; d_0, 0)] dq, \end{aligned} \quad (\text{C34})$$

or, equivalently,

$$\mu^{\text{st}}(s; d_0, 0) = \frac{1}{\pi} \int_{-1/\ell_D}^0 e^{qs} \text{Im}[\Phi_+^{\text{st}}(q - i0; d_0, 0)] dq, \quad (\text{C35})$$

where $s > 0$, and the second form assumes the standard Schwarz reflection symmetry of the boundary values on the cut. Equations (C32)–(C35) are the correct real-space inversions of the steady transformed density. In general, the jump of Φ_+^{st} across the cut is not available in elementary closed form because it contains the nonlocal projection term in Eq. (C31). The natural exact representation is therefore the Bromwich integral (C33), or equivalently the cut integral (C34), rather than a further elementary reduction.

5. Asymptotic structure

The large s behavior of $\mu^{\text{st}}(s; d_0, 0)$ is governed by the rightmost branch point at $q = 0$, which yields the leading algebraic tail

$$\mu^{\text{st}}(s; d_0, 0) \sim \frac{B_0(d_0)}{\sqrt{s}}, \quad s \rightarrow \infty, \quad (\text{C36})$$

while the branch point at $q = -1/\ell_D$ contributes an exponentially damped correction of order $e^{-s/\ell_D}/\sqrt{s}$. The small s behavior is determined by the large $|q|$ asymptotics of Eq. (C33) and has the universal inverse square-root form

$$\mu^{\text{st}}(s; d_0, 0) \sim \frac{A^{\text{st}}(d_0)}{\sqrt{s}}, \quad s \rightarrow 0^+, \quad (\text{C37})$$

in which the leading coefficient $A^{\text{st}}(d_0)$ is determined by the value of the splitting functions $S_{\pm}$ at infinity along the Bromwich contour, together with the constants in the canonical factors (C6) and (C30). The two limits $\theta \rightarrow 0$ and $t \rightarrow \infty$ do not commute at the level of the intermediate constructions, but agree at the level of the final scalar Wiener-Hopf solution, which serves as the natural calibration point at $\theta = 0$ for the full unsteady oblique theory of the main text.

APPENDIX D: FINITE-TIME ISOLATED DENDRITE AND PARALLEL WALL LIMITS

These limits concern the finite-time laboratory operator of Secs. III to VI at fixed Laplace parameter p , and are distinct from the uniformly translating steady limits of Appendix C: No final value limit $p \rightarrow 0$ is taken. Throughout, $a(p) := \sqrt{p/D}$.

1. Isolated dendrite limit

For real $p > 0$ the radical is real and positive with $\kappa(k, p) = \sqrt{k^2 + a^2} \geq a(p) > 0$, so that

$$e^{-2d_0\kappa(k, p)} \leq e^{-2d_0a(p)} \quad (\text{D1})$$

uniformly in $k \in \mathbb{R}$, which is Eq. (60) specialized to the real axis.

Fix an obliquity $\theta \in (0, \pi/2)$; the endpoint $\theta = \pi/2$ is treated directly at the level of the physical operator, since the two branches coincide there and the branch representation is not used. The reflected branch operators of Sec. V have kernels

$$r_{XB}^{(d_0)}(k, k'; p) = \frac{1}{2\pi} \frac{e^{-2d_0\kappa(k', p)}}{2D\kappa(k', p)} \frac{1}{X_\theta(k, p) + A_\theta(k', p)}, \quad (\text{D2})$$

with $X = A, B$. Equations (54) and (55) give

$$\text{Re}[X_\theta(k, p) + A_\theta(k', p)] = \sin \theta [\kappa(k, p) + \kappa(k', p)], \quad (\text{D3})$$

with $X=A, B$, so that $|X_\theta(k, p) + A_\theta(k', p)| \geq \sin \theta [\kappa(k, p) + \kappa(k', p)]$. Bounding $\kappa(k, p) + \kappa(k', p) \geq \kappa(k, p)$ in one factor and using $\int_{\mathbb{R}} dk/(k^2 + a^2) = \pi/a$ for both integrations, the Hilbert-Schmidt norm of either reflected branch operator satisfies

$$\|\mathcal{R}_{XB}\|_{\text{HS}} \leq \frac{e^{-2d_0a(p)}}{4Da(p)\sin \theta}, \quad (\text{D4})$$

with $X = A, B$. The operator norm is bounded by the Hilbert-Schmidt norm, so both reflected branch operators vanish exponentially as $d_0 \rightarrow \infty$, uniformly for θ in any closed subinterval of $(0, \pi/2)$. The bound degrades as $\theta \rightarrow 0$, which is the analytic signature of the branch degeneration of Appendix D 3.

Convergence of the transformed solution follows by stability of the isolated Wiener-Hopf problem under an exponentially small reflected perturbation. Let $\mathcal{L}_{\theta, p}^{(0)}$ denote the projected isolated operator and $\mathcal{E}_{\theta, d_0, p}$ the pair of reflected branch operators appearing in Eq. (124) and in the upper equation. Away from the isolated poles $\mathcal{L}_{\theta, p}^{(0)}$ is invertible, while Eq. (D4) gives $\|\mathcal{E}_{\theta, d_0, p}\| = O(e^{-2d_0a(p)})$, so the resolvent identity yields

$$\Phi_+(q, p; d_0, \theta) = \Phi_+^{\text{iso}}(q, p) + O(e^{-2d_0\sqrt{p/D}}) \quad (\text{D5})$$

on compact subsets of the working strip, with Φ_+^{iso} the solution of the scalar problem carrying the direct symbol $\mathcal{K}_0$ of Eq. (39) alone.

Eq. (D5) is proved at real $p > 0$, where Eq. (D3) holds. Both sides are analytic in p on the working strip, so the estimate extends to the Talbot contour of Sec. VIF by analytic continuation, with $a(p)$ replaced by $\text{Re}\sqrt{p/D}$, which remains positive there. Further, the estimate holds at every real $p > 0$, and hence at each node of the Stehfest inversion used for the wall problem. Thus, on the one hand, finite Stehfest sum preserves the limit, and dominated convergence on the wedge inversion contour gives; on the other hand, dominated convergence on the wedge and Talbot inversion contours. Both then give

$$\mu(s, t; d_0, \theta) \longrightarrow \mu_{\text{iso}}(s, t) \quad (\text{D6})$$

for every fixed $T < \infty$ and $0 \leq t \leq T$, on compact intervals of arc length, and the single-layer reconstruction gives $c \rightarrow c_{\text{iso}}$ on compact subsets of the liquid domain. Since the process zone functional is a bounded linear functional of the density on $s \in [0, \ell]$,

$$|\Psi_{\text{ex}}(\ell, t; d_0, \theta) - \Psi_{\text{iso}}(\ell, t)| \leq \frac{1}{2\sqrt{\ell}} \int_0^\ell |\mu(s, t; d_0, \theta) - \mu_{\text{iso}}(s, t)| ds \longrightarrow 0. \quad (\text{D7})$$

The isolated dendrite solution is therefore the limit of the transformed density, the line density, the reconstructed concentration field and the process zone functional.

2. Parallel wall limit

At $\theta \rightarrow 0$,

$$A_0(k, p) = -ik, \quad B_0(k, p) = +ik, \quad (\text{D8})$$

and substituting into Eq. (56) the two exponentials combine into $e^{ik(u-u')}$, which depends only on the difference variable, thereby losing the Hankel coupling. The reflected kernel collapses to the convolutional form

$$\bar{K}_0^{\text{ref}}(u, u'; p) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{-2d_0\kappa(k, p)}}{2D\kappa(k, p)} e^{ik(u-u')}, \quad (\text{D9})$$

with Fourier symbol

$$\hat{K}_0^{\text{ref}}(k, p) = \frac{e^{-2d_0\kappa(k, p)}}{2D\kappa(k, p)}. \quad (\text{D10})$$

The same symbol follows from Eq. (30) at $\theta = 0$ without passing through the branch representation. The Gaussian exponent reduces to $(u - u')^2 + 4d_0^2$ divided by $4D\Delta$, and the Fourier transform in $u - u'$ gives

$$\int_{-\infty}^{\infty} \frac{e^{ik\xi - (\xi^2 + 4d_0^2)/(4D\Delta)}}{4\pi D\Delta} d\xi = \frac{e^{-Dk^2\Delta - d_0^2/(D\Delta)}}{\sqrt{4\pi D\Delta}}, \quad (\text{D11})$$

followed by the time Laplace transform evaluated using

$$\int_0^\infty t^{-1/2} e^{-at - b/t} dt = \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}}, \quad (\text{D12})$$

at $a = p + Dk^2$, $b = d_0^2/D$, so that $\sqrt{ab} = d_0\kappa(k, p)$ and

$$\int_0^\infty \frac{e^{-(p+Dk^2)\Delta - d_0^2/(D\Delta)}}{\sqrt{4\pi D\Delta}} d\Delta = \frac{e^{-2d_0\kappa(k, p)}}{2D\kappa(k, p)}, \quad (\text{D13})$$

in agreement with Eq. (D10). This second route takes the limit in the physical kernel before the branch equations are formed, and is therefore unaffected by the degeneration established in Appendix D 3.

Direct and reflected contributions add to

$$\hat{K}_0^{\text{tot}}(k, p) = \frac{1 + e^{-2d_0\kappa(k, p)}}{2D\kappa(k, p)}, \quad (\text{D14})$$

which in the q variable is Eq. (61). The wedge transform of the boundary integral equation is then the scalar Wiener-Hopf equation

$$\mathcal{K}_{\parallel}(q, p; d_0) \Phi_+^{\parallel}(q, p) + \Phi_-^{\parallel}(q, p) = \frac{c_{\text{LC}}}{q(p - vq)}, \quad (\text{D15})$$

in which the branch trace is not an independent unknown and the Fredholm lower row problem of Sec. VI does not arise.

The scalar symbol factorizes directly. Writing $\mathcal{K}_{\parallel} = \mathcal{K}_0 F_{\parallel}$ with $F_{\parallel}(q, p; d_0) = 1 + e^{-2d_0 \Gamma(q, p)}$, and noting that $\text{Re} \Gamma > 0$ on every closed substrip of the working strip so that $|e^{-2d_0 \Gamma}| < 1$ there, $F_{\parallel}$ is nonvanishing and admits an analytic logarithm. If $\log F_{\parallel} = S_+ + S_-$ is its additive Cauchy splitting on the working strip, then

$$\mathcal{K}_{\parallel,+} = \mathcal{K}_{0,+} e^{S_+}, \quad \mathcal{K}_{\parallel,-} = \mathcal{K}_{0,-} e^{S_-} \quad (\text{D16})$$

is a canonical factorization, and the transformed density is

$$\Phi_{\parallel,+}^{\parallel}(q, p) = \frac{1}{\mathcal{K}_{\parallel,+}(q, p)} \mathcal{P}_+ \left[\frac{c_{\text{LC}}/[q(p - vq)]}{\mathcal{K}_{\parallel,-}(q, p)} \right] (q). \quad (\text{D17})$$

At $\theta = 0$ the kinematics degenerate to $d(t) = d_0$, $t_{\star} = \infty$ and $\text{Pe}_{\theta} = 0$: reflection is retained, but the separation does not change while the reflected signal is in transit, so Eq. (D17) is the exact finite-time fixed gap response. By the Neumann image construction this limit is equivalent to a symmetric pair of identical dendrites at spacing $h = 2d_0$, so the all angle framework is calibrated at $\theta = 0$ against the standard scalar Wiener-Hopf problem for two symmetrically placed dendrites.

3. Relation between the two limits

At the level of the physical operator the limits commute. Letting $d_0 \rightarrow \infty$ in Eq. (61) returns the isolated scalar symbol $\mathcal{K}_0 = 1/(2D\Gamma)$, and removing the wall from the oblique

problem annihilates the reflected branch operators by Eq. (D4) irrespective of θ , leaving the same direct symbol. Hence,

$$\lim_{d_0 \rightarrow \infty} \lim_{\theta \rightarrow 0} \mathcal{L}_{\theta, d_0} = \mathcal{L}_{\text{iso}} = \lim_{\theta \rightarrow 0} \lim_{d_0 \rightarrow \infty} \mathcal{L}_{\theta, d_0}, \quad (\text{D18})$$

with $\mathcal{L}_{\theta, d_0}$ the full boundary integral operator.

The branch representation does not remain regular as $\theta \rightarrow 0$. The two branch curves collapse onto the imaginary q axis with opposite orientations. In this limit the kernel of $\mathcal{R}_{BB}$ develops a Cauchy singularity on the diagonal $k' = k$, while that of $\mathcal{R}_{AB}$ develops the corresponding singularity on the antidiagonal $k' = -k$. The loss of uniformity is reflected by the factor $1/\sin \theta$ in Eq. (D4). The singularity belongs to the representation and not to the operator, and its resolution is the scalar convolution symbol Eq. (D14), which is why the parallel limit is taken in the kernel by Eqs. (D11) to (D13) rather than in the branch matrix.

The two limits remove different ingredients that make the Wiener-Hopf-Hankel problem challenging. The isolated limit removes reflection and supplies the reference $\Psi_{\text{iso}}(t)$ against which the sign of the wall correction is measured. The parallel wall limit retains reflection but removes both the closing gap kinematics and the oblique branch coupling, and therefore provides the exact fixed gap endpoint of the all angle theory. The specifically dynamical effect of the changing separation is isolated instead by comparing the exact oblique solution with the frozen offset construction at the same obliquity, as in Sec. IX B.

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