Upper bound on locally extractable energy from entangled pure state under feedback control

Kanji Itoh [1,*] Yusuke Masaki [1] and Hiroaki Matsueda [1,2]
[1]*Department of Applied Physics, Graduate School of Engineering, Tohoku University, Sendai 980-8579, Japan*
[2]*Center for Science and Innovation in Spintronics, Tohoku University, Sendai 980-8577, Japan*

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In order to investigate locally extractable energy from multipartite entangled pure states, we introduce effective thermodynamics by defining an entanglement-based effective temperature. Based on the effective thermodynamics, we derive an upper bound on extractable energy with feedback control from a subsystem under a local Hamiltonian. The inequality that gives the upper bound corresponds to the second law of information thermodynamics in our effective thermodynamics. In addition, we derive a more general bound that is determined only by an initial state and the local Hamiltonian. This bound gives an explicit relationship between the extractable energy and the entanglement structure of the initial state. Using the entanglement-based effective temperature, we discuss how our energy extraction protocol on pure states is related to ordinary information thermodynamic processes and show the consistency of our results with previous information thermodynamics. We also investigate the tightness of the upper bounds and show that the bounds can be achieved in a simple example.

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I. INTRODUCTION

In light of the rapid progress in quantum information technologies, there is an urgent need to elucidate the energy efficiency of quantum information processing [1]. This challenge is important not only for applications in quantum technologies, but also for fundamental physics, as it requires deep understanding of the relationship between energy and quantum information.

The relationship between energy and information is one of the key issues in modern thermodynamics. In 2008, Sagawa and Ueda derived the second law of information thermodynamics [2]. This law extends the applicability of the second law of thermodynamics to processes that involve measurement and feedback control. The extended law shows that feedback control can improve work extraction by exploiting information gain from the measurement. In the case of an isothermal process, the second law of information thermodynamics is expressed as

$$W_{\mathrm{ext}} \leqslant -\Delta F + \frac{1}{\beta} I_{\mathrm{QC}}, \tag{1}$$

where W_{ext} is the extracted work, ΔF is the change in the Helmholtz free energy of the system, β is the inverse temperature, and I_{QC} is the quantum-classical (QC) mutual information (or Groenewold-Ozawa information) defined as the information gain by measuring the system [3,4]. Although the inequality (1) was derived for quantum systems, the setup does not explicitly include entanglement, which is a necessary resource for quantum information processing. Around the last decade, as extensions of Eq. (1), several information thermodynamic inequalities have been derived in entangled systems [5–8]. These inequalities give quantitative relations between energy and quantum information in entangled systems. In the setup of the above studies, work is extracted from systems at some specific temperatures.

On the other hand, *local* energy extraction from multipartite quantum systems has also been studied in various setups [9–17]. In such situations, temperatures of local systems cannot be specified because local systems are not in equilibrium. For such situations, general and quantitative energy-information relations, such as the second law of information thermodynamics, have not been established. Establishing such a relationship leads to a new theory of energy efficiency in various entanglement-based quantum protocols.

In this paper, we show that locally extractable energy from a multipartite quantum system is bounded with quantum information derived from entanglement. In our setup, the whole system is in an entangled pure state as an initial state, and energy is extracted by measurement and feedback control performed on some parts of the system. In entangled quantum systems, even if a whole system is not thermally fluctuating, i.e., the whole system is in a pure state, a subsystem can be in a mixed state. Conventional information thermodynamics is not applicable to such local systems as mentioned in the previous paragraph. However, in this study by defining an entanglement-based effective temperature of a local system instead of the real temperature, we make it possible to describe the energy extraction as a nonequilibrium process at the effective temperature. Based on the effective temperature, we introduce an effective thermodynamics and derive two upper bounds on the extractable energy. One bound can be seen as the second law of information thermodynamics in our effective thermodynamics. The other bound is looser, but provides

*Contact author: kanji.ito.q4@dc.tohoku.ac.jp

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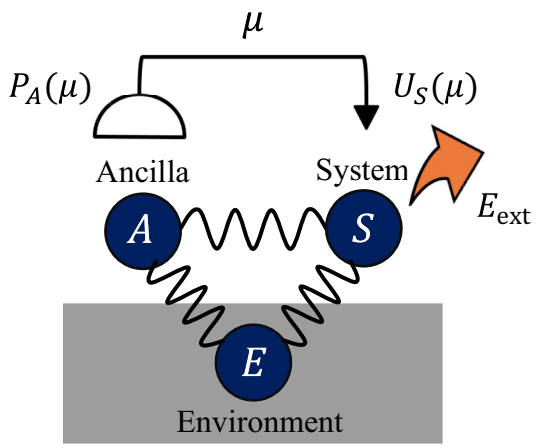


FIG. 1. Illustration of the setup. The whole system is initially in a entangled pure state, and is divided into the system S, ancilla A, and environment E. As the energy extraction protocol, projective measurement $P_A(\mu)$ is performed on A, and feedback unitary operation $U_S(\mu)$ is performed on S.

an explicit relation between energy and entanglement structure of the initial state of the energy extraction process. We also discuss how our results are related to previous information thermodynamics by the effective temperature, and show that our bounds are consistent with the previous information thermodynamic inequality derived in Ref. [5]. In addition, we examine the tightness of the bounds with a simple four-qubit example. Our results show that entanglement can play the role of thermal fluctuations in conventional information thermodynamics, and open the door to entanglement-based information thermodynamics without heat baths.

II. SETUP

Figure 1 schematically shows our setup. We divide the whole system into a system S, an ancilla A, and an environment E. The energy extraction protocol consists of measurement and feedback control. First, a projective measurement $P_A(\mu)$ is performed on the ancilla A, and the measurement result μ is obtained. Next, a feedback unitary operation $U_S(\mu)$ is performed on the system S, and the amount of energy $E_{\rm ext}$ is extracted from S. In the protocol, no operation is performed on the environment E. However, the environment E is necessary to define temperature of our effective thermodynamics.

At the beginning of the energy extraction process, the state of the whole system $\rho^{\rm i}_{\rm SAE}$ is an entangled pure state, i.e., $\rho^{\rm i}_{\rm SAE} \equiv |\psi\rangle_{\rm SAE}\langle\psi|_{\rm SAE}$, where $|\psi\rangle_{\rm SAE}$ is an entangled state vector. In this case, the system S is in a mixed state due to the entanglement. The von Neumann entropy of this mixed state represents information derived only from entanglement and is called entanglement entropy. The definition of the entanglement entropy of the system S is given by $S(\rho^{\rm i}_S) \equiv -{\rm tr}\rho^{\rm i}_S \log \rho^{\rm i}_S$, where $\rho^{\rm i}_S \equiv {\rm tr}_{AE}\rho^{\rm i}_{\rm SAE}$ is the reduced density matrix of $|\psi\rangle_{\rm SAE}$ on the system S. Here and hereafter, $S(\rho)$ denotes the von Neumann entropy of a density matrix ρ.

After the measurement with a result μ, the state of the whole system is written as $\rho^{\rm m}_{\rm SAE}(\mu) = \frac{1}{p_\mu}P_A(\mu)\rho^{\rm i}_{\rm SAE}P_A(\mu)$, where $p_\mu \equiv \langle\psi|P_A(\mu)|\psi\rangle_{\rm SAE}$ is the probability of obtaining a result μ. The reduced density matrix of the system S after obtaining the result μ is expressed by $\rho^{\rm m}_S(\mu) = {\rm tr}_{AE}\rho^{\rm m}_{\rm SAE}(\mu)$. In our setup, the information gain of the system S due to the measurement is calculated as the average reduction of the entanglement entropy: $I_{\rm QC} \equiv S(\rho^{\rm i}_S) - \sum_\mu p_\mu S(\rho^{\rm m}_S(\mu))$.

Next, the feedback unitary operation $U_S(\mu)$ is performed on the system S. The final state of the whole system is represented by $\rho^{\rm f}_{\rm SAE} = \sum_\mu U_S(\mu)P_A(\mu)\rho^{\rm i}_{\rm SAE}P_A(\mu)U^\dagger_S(\mu)$. Thus, the final state of the system S is given by $\rho^{\rm f}_S = {\rm tr}_{AE}\rho^{\rm f}_{\rm SAE} = \sum_\mu p_\mu U_S(\mu)\rho^{\rm m}_S(\mu)U^\dagger_S(\mu)$. The extracted energy is given by the energy reduction of the system S during the protocol

$$E_{\rm ext} \equiv E^{\rm i}_S - E^{\rm f}_S, \tag{2}$$

where $E^{\rm i}_S \equiv {\rm tr}\rho^{\rm i}_S H_S$ and $E^{\rm f}_S \equiv {\rm tr}\rho^{\rm f}_S H_S$ are the energy expectation values of the initial and final states of S, respectively, and H_S is the local Hamiltonian of S at the beginning and the end of the protocol. Note that the *maximally* extracted energy by feedback unitary operations under a fixed projective measurement is called daemonic ergotropy [18]. In this article, we will derive *upper bounds* on the extracted energy. As will be mentioned later, when our bound can be achieved, the bound is equal to the daemonic ergotropy; in this case, our bound gives information thermodynamic expression of the daemonic ergotropy for pure states.

III. EFFECTIVE THERMODYNAMICS AND UPPER BOUNDS

Let us introduce an effective thermodynamics for entangled pure states. In our effective thermodynamics, the energy extraction protocol is considered as a nonequilibrium process of the system S at an effective temperature, which is defined by the entanglement between the system S and the environment E. In the following, we derive a second law-like inequality of the effective thermodynamics.

First, we define the effective thermal equilibrium state. As mentioned above, the system S is in a mixed state due to the entanglement. Thus, we consider the system S to be at an effective temperature $1/\beta_{\rm eff} > 0$, for which we define the thermal equilibrium state as

$$\sigma_S \equiv \frac{1}{Z}e^{-\beta_{\rm eff}H_S}, \tag{3}$$

where $Z \equiv {\rm tr}\ e^{-\beta_{\rm eff}H_S}$ is the partition function. The thermal equilibrium state σ_S depends only on $\beta_{\rm eff}$, which is defined by the entanglement between the system S and the environment E. We postpone the explicit definition of $\beta_{\rm eff}$ until Eq. (12).

Next, we rewrite the extracted energy $E_{\rm ext}$ with quantities based on the effective thermodynamics. In the effective thermodynamics, the initial state $\rho^{\rm i}_S$ and the final state $\rho^{\rm f}_S$ of the protocol are considered as nonequilibrium states at the effective temperature $\beta_{\rm eff}$. Thus, we define nonequilibrium free energy [19] of the states ρ^α_S for $\alpha = {\rm i}, {\rm f}$ as

$$\mathcal{F}(\rho^\alpha_S; H_S) \equiv E^\alpha_S - \frac{1}{\beta_{\rm eff}}S(\rho^\alpha_S) = F(\sigma_S) + \frac{1}{\beta_{\rm eff}}D(\rho^\alpha_S\|\sigma_S), \tag{4}$$

where $F(\sigma_S) \equiv -\frac{1}{\beta_{\rm eff}}\log Z$ is the Helmholtz free energy of σ_S and $D(\rho^\alpha_S||\sigma_S)$ is the Kullback-Leibler (KL) divergence

(or quantum relative entropy). Using the nonequilibrium free energy, $E_{\rm ext}$ defined by Eq. (2) can be rewritten as

$$E_{\rm ext} = -\Delta\mathcal{F}_S - \frac{1}{\beta_{\rm eff}}\Delta S_S, \tag{5}$$

where $\Delta\mathcal{F}_S \equiv \mathcal{F}(\rho_S^{\rm f}; H_S) - \mathcal{F}(\rho_S^{\rm i}; H_S)$ is the change in the nonequilibrium free energy and $\Delta S_S \equiv S(\rho_S^{\rm f}) - S(\rho_S^{\rm i})$ is the change in the von Neumann entropy. Note that ΔS_S is always zero when $U_S(\mu)$ is not feedback control, i.e., $U_S(\mu)$ is independent of μ.

Let us now move on to the derivation of upper bounds on $E_{\rm ext}$. First, from the positivity of KL divergence $D(\rho_S^{\rm f}||\sigma_S)$, we obtain

$$-\Delta\mathcal{F}_S \leqslant \mathcal{F}\big(\rho_S^{\rm i}; H_S\big) - F(\sigma_S). \tag{6}$$

The equality holds if and only if $\rho_S^{\rm f} = \sigma_S$. Next, we focus on the entropy term in Eq. (5). Using the convexity of entropy, we have

$$S\big(\rho_S^{\rm f}\big) \geqslant \sum_\mu p_\mu S\big(\rho_S^{\rm m}(\mu)\big) \geqslant E_{\rm F}^{S-E}\big(\rho_{SE}^{\rm i}\big), \tag{7}$$

where $\rho_{SE}^{\rm i} \equiv {\rm tr}_A \rho_{\rm SAE}^{\rm i}$ and $E_{\rm F}^{S-E}(\rho_{SE}^{\rm i})$ is entanglement of formation [20], representing mixed state entanglement between the system S and the environment E. In our setup, $E_{\rm F}^{S-E}(\rho_{SE}^{\rm i})$ can be expressed as

$$E_{\rm F}^{S-E}\big(\rho_{SE}^{\rm i}\big) \equiv \min_{\{P_A(\mu)\}} \sum_\mu p_\mu S\big(\rho_S^{\rm m}(\mu)\big), \tag{8}$$

where the minimization is performed over all sets of possible orthogonal projection operators. This minimization is equivalent to optimizing the representation of the mixed state $\rho_{SE}^{\rm i}$ by mixture of pure states: $\sum_\mu p_\mu |\phi_\mu\rangle_{SE}\langle\phi_\mu|_{SE} = \rho_{SE}^{\rm i}$, where we put $|\psi\rangle_{\rm SAE} = \sum_\mu \sqrt{p_\mu}|a_\mu\rangle_A \otimes |\phi_\mu\rangle_{SE}$ and $P_A(\mu) = |a_\mu\rangle_A\langle a_\mu|_A$. Note that $\{|\phi_\mu\rangle\}$ need not be an orthogonal basis set. From Eq. (7), we obtain

$$-\Delta S_S \leqslant I_{\rm QC} \leqslant \mathcal{E}, \tag{9}$$

where we define the new quantity $\mathcal{E}$ which is defined as the difference between the total entanglement of S and the $S-E$ entanglement in the initial state

$$\mathcal{E} \equiv S\big(\rho_S^{\rm i}\big) - E_{\rm F}^{S-E}\big(\rho_{SE}^{\rm i}\big). \tag{10}$$

This definition is similar to the definition of entanglement information in Ref. [5]. The relationship between $\mathcal{E}$ and the entanglement information will be explained in the next section. Note that $\mathcal{E}$ does not strictly represent the $S-A$ entanglement because $S(\rho_S^{\rm i})$ also includes not only the bipartite entanglement of $S-A$ and $S-E$, but also the tripartite entanglement. Plugging inequalities (6) and (9) into Eq. (5) yields upper bounds on $E_{\rm ext}$:

$$\begin{aligned} E_{\rm ext} &\leqslant \mathcal{F}\big(\rho_S^{\rm i}; H_S\big) - F(\sigma_S) + \frac{1}{\beta_{\rm eff}} I_{\rm QC} \\ &\leqslant \mathcal{F}\big(\rho_S^{\rm i}; H_S\big) - F(\sigma_S) + \frac{1}{\beta_{\rm eff}}\mathcal{E}. \end{aligned} \tag{11}$$

The first bound in Eq. (11) depends on the choice of the set of projection operators $\{P_A(\mu)\}$ as well as $I_{\rm QC}$. When this bound can be achieved, the bound is the daemonic ergotropy itself. The second bound in Eq. (11) is determined only by the initial state $|\psi\rangle_{\rm SAE}$ and the local Hamiltonian H_S. This bound is achievable if and only if $D(\rho_S^{\rm f}||\sigma_S) = 0$ and $-\Delta S_S = I_{\rm QC} = \mathcal{E}$ hold.

Here, we define the effective temperature $1/\beta_{\rm eff}$ such that the bound is tight and independent of local operations $P_A(\mu)$ and $U_S(\mu)$. For this purpose, we minimize the second bound in Eq. (11) with respect to $1/\beta_{\rm eff}$, which leads to the condition

$$S(\sigma_S) = E_{\rm F}^{S-E}\big(\rho_{SE}^{\rm i}\big). \tag{12}$$

This condition is also the necessary condition for achieving the second bound in Eq. (11). The condition (12) uniquely determines $1/\beta_{\rm eff}$, except for the cases where S and E are not entangled or maximally entangled. This is because $S(\sigma_S)$ decreases monotonically with respect to $\beta_{\rm eff}$, and the value of $E_{\rm F}^{S-E}$ is always included in the range of $S(\sigma_S)$: the range of $S(\sigma_S)$ is $(0, \log(\dim \sigma_S))$, while $E_{\rm F}^{S-E}(\rho_{SE}^{\rm i})$ satisfies $0 < E_{\rm F}^{S-E}(\rho_{SE}^{\rm i}) < \min\{\log(\dim \rho_S^{\rm i}), \log \rho_E^{\rm i}\} \leqslant \log(\dim \sigma_S)$, where $\rho_E^{\rm i} \equiv {\rm tr}_{SA}\rho_{\rm SAE}^{\rm i}$. Therefore, there exists only one value of $\beta_{\rm eff}$ satisfying Eq. (12) in this case (see also Fig. 3 in Appendix A). For the cases where the $S-E$ entanglement is zero or maximal, we define the effective temperature as 0 and $+\infty$, respectively, so that the consistency of $E_{\rm F}^{S-E}(\rho_{SE}^{\rm i}) \to 0$ and $E_{\rm F}^{S-E}(\rho_{SE}^{\rm i}) \to \min\{\log(\dim \rho_S^{\rm i}), \log \rho_E^{\rm i}\}$ is guaranteed. Under the above definition of $\beta_{\rm eff}$, $E_{\rm ext}$ reaches the second bound in Eq. (11) only if the eigenvalues of $\rho_S^{\rm m}(\mu)$ that maximize $I_{\rm QC}$ are independent of the measurement results μ. The proof of this is given in Appendix A. In Sec. V, we will show that $E_{\rm ext}$ can actually reach the bound in a simple model when the above condition is satisfied. The achievability of the bound is also confirmed in a quantum spin model that is transformed into a model of Majorana fermions [21].

Let us discuss the physical meaning of the bounds in Eq. (11). Using Eq. (4), the bounds can be rewritten more simply as

$$E_{\rm ext} \leqslant \frac{D\big(\rho_S^{\rm i}\big\|\sigma_S\big) + I_{\rm QC}}{\beta_{\rm eff}} \leqslant \frac{D\big(\rho_S^{\rm i}\big\|\sigma_S\big) + \mathcal{E}}{\beta_{\rm eff}}. \tag{13}$$

The KL divergence $D(\rho_S^{\rm i}||\sigma_S)$, which represents how different the initial state of the system S is from the equilibrium state σ_S, is a resource of the energy extraction. Besides, the information obtained from the measurement $I_{\rm QC}$ and $\mathcal{E}$ can improve the extracted energy. This situation is similar to the case in the previous studies of nonequilibrium information thermodynamic process [19,22]. Thus, the first bound in Eq. (13) [or Eq. (11)] can be considered as the second law of information thermodynamics in our effective thermodynamics.

Obviously from Eq. (10), the total entanglement of S in the initial state can be decomposed into the $S-E$ entanglement $E_{\rm F}^{S-E}(\rho_{SE}^{\rm i})$ and the remaining entanglement $\mathcal{E}$. The former entanglement is the origin of the effective temperature in the effective thermodynamics. On the other hand, the latter entanglement $\mathcal{E}$ can be considered as the informative resource for the energy extraction, because $\mathcal{E}$ is the upper bound on the information gain $I_{\rm QC}$ and $I_{\rm QC}$ can contribute positively to the energy extraction. Therefore, the second bound in Eq. (13) gives the explicit relationship between the extracted energy $E_{\rm ext}$ and the entanglement structure of the initial state $|\psi\rangle_{\rm SAE}$.

IV. RELATION WITH PREVIOUS INFORMATION THERMODYNAMICS

In this section, we discuss the relation between our results and previous information thermodynamics. In our setup, the initial state $\rho_S^{\rm i}$ is not in thermal equilibrium, and the temperature is not specified. On the other hand, ordinary information thermodynamic processes begin with thermal equilibrium states at some given temperatures. To relate our setup to an information thermodynamic process, we treat the state $\rho_S^{\rm i}$ not as an initial state, but as an intermediate state of an information thermodynamic process starting with a thermal equilibrium state. As such an information thermodynamic process, we consider a process examined in Ref. [5], and discuss the relationship and consistency between our results and the results in Ref. [5].

The process in Ref. [5] consists of the following steps on the composite system SAE from time $t=t_0$ to $t=t_{\rm f}$ via $t=t_{\rm i}$. At $t=t_0$, the system S is in a thermal equilibrium state $\rho_{S,{\rm eq}}^0$ at an inverse temperature β_0, i.e., $\rho_{S,{\rm eq}}^0=\frac{1}{Z_0}\exp[-\beta_0 H_S]$, where $Z_0\equiv{\rm tr}\,e^{-\beta_0 H_S}$ is the partition function. The initial state $\rho_{S,{\rm eq}}^0$ can be purified by adding the environment E. The ancilla A is initialized to a pure state, and is not correlated with S and E. From $t=t_0$ to $t=t_{\rm i}$, an unitary interaction V_{SA} is performed between the system S and the ancilla A. From $t=t_{\rm i}$ to $t=t_{\rm f}$, a projective measurement $P_A(\mu)$ with a result μ is performed on A, and a feedback unitary operation $U_S(\mu)$ is performed on S.

In the above process, the purified state of SE is entangled, and the unitary interaction V_{SA} distributes the S–E entanglement to the whole system SAE. At $t=t_{\rm i}$, the whole system SAE is in an entangled pure state, and the process from $t=t_{\rm i}$ to $t=t_{\rm f}$ is the same as our energy extraction protocol described in Sec. II. Based on this observation, we designate the state of S at $t=t_{\rm i}$ and $t=t_{\rm f}$ as $\rho_S^{\rm i}$ and $\rho_S^{\rm f}$, respectively. The extracted energy from S in the entire process is given by the energy reduction of S from $t=t_0$ to $t=t_{\rm f}$, and is divided into the terms before and after $t=t_{\rm i}$:

$$E_{S,{\rm eq}}^0-E_S^{\rm f}=\left(E_{S,{\rm eq}}^0-E_S^{\rm i}\right)+E_{\rm ext}, \tag{14}$$

where $E_{S,{\rm eq}}^0\equiv{\rm tr}\rho_{S,{\rm eq}}^0 H_S$ is the energy expectation value of S at $t=t_0$. We emphasize that the final state $\rho_S^{\rm f}$ does not necessarily reach thermal equilibrium at a specific inverse temperature $\beta_{\rm f}$. For $\beta_{\rm f}\geqslant\beta_{\rm eff}$, $E_S^{\rm f}$ is bounded below by $E_{S,{\rm eq}}^{\rm f}\equiv{\rm tr}\rho_{S,{\rm eq}}^{\rm f}H_S$, i.e., the energy expectation value in thermal equilibrium state $\rho_{S,{\rm eq}}^{\rm f}\equiv\frac{1}{Z_{\rm f}}\exp[-\beta_{\rm f}H_S]$ with $Z_{\rm f}\equiv{\rm tr}\,e^{-\beta_{\rm f}H_S}$, as follows:

$$\begin{aligned}E_S^{\rm f}&=F\left(\rho_{S,{\rm eq}}^{\rm f}\right)+\frac{1}{\beta_{\rm f}}D\left(\rho_S^{\rm f}\middle\|\rho_{S,{\rm eq}}^{\rm f}\right)+\frac{S\left(\rho_S^{\rm f}\right)}{\beta_{\rm f}}\\&\geqslant F\left(\rho_{S,{\rm eq}}^{\rm f}\right)+\frac{S\left(\rho_{S,{\rm eq}}^{\rm f}\right)}{\beta_{\rm f}}=E_{S,{\rm eq}}^{\rm f},\end{aligned} \tag{15}$$

where $F(\rho_{S,{\rm eq}}^{\rm f})\equiv-\frac{1}{\beta_{\rm f}}\log Z_{\rm f}$ is the Helmholtz free energy at the inverse temperature $\beta_{\rm f}$, and we have used positivity of KL divergence as well as Eqs. (7) and (12), and $S(\sigma_S)\geqslant S(\rho_{S,{\rm eq}}^{\rm f})$ for $\beta_{\rm f}\geqslant\beta_{\rm eff}$. From Eqs. (14) and (15), we have

$$E_{\rm ext}\leqslant\left(E_{S,{\rm eq}}^0-E_{S,{\rm eq}}^{\rm f}\right)-\left(E_{S,{\rm eq}}^0-E_S^{\rm i}\right). \tag{16}$$

The equality holds only if $\beta_{\rm f}=\beta_{\rm eff}$, where the state $\rho_{S,{\rm eq}}^{\rm f}$ is equal to the thermal equilibrium state of our effective thermodynamics, σ_S. Note that Eq. (16) is not necessarily true for $\beta_{\rm f}<\beta_{\rm eff}$.

In the following, we show that Eq. (16) connects our bounds (13) with previous information thermodynamics, which assumes that initial and final states are in thermal equilibrium at some temperatures. Here, we consider the case where the final state $\rho_S^{\rm f}$ reaches thermal equilibrium state $\rho_{S,{\rm eq}}^{\rm f}$. In this case, from the non-isothermal version of the second law of information thermodynamics for the process in Ref. [5] (Theorem 1 in Ref. [5]), we have

$$\beta_0E_{S,{\rm eq}}^0-\beta_{\rm f}E_{S,{\rm eq}}^{\rm f}\leqslant\beta_0F\left(\rho_{S,{\rm eq}}^0\right)-\beta_{\rm f}F\left(\rho_{S,{\rm eq}}^{\rm f}\right)+I_{\rm E}, \tag{17}$$

where $F(\rho_{S,{\rm eq}}^0)\equiv-\frac{1}{\beta_0}\log Z_0$ is the Helmholtz free energy of $\rho_{S,{\rm eq}}^0$, and $I_{\rm E}$ is entanglement information given by $I_{\rm E}=S(\rho_{S,{\rm eq}}^0)-E_{\rm F}^{S-E}(\rho_{SE}^{\rm i})$. In the process in Ref. [5], the unitary interaction and projective measurement $P_A(\mu)V_{SA}$ are considered as the indirect measurement process of S, and $I_{\rm E}$ represents the maximum reduction in the von Neumann entropy of S during the indirect measurement process. On the other hand, $\mathcal{E}$ in Eq. (10) represents the maximum reduction in the von Neumann entropy of S due to the projective measurement $P_A(\mu)$, and thus $I_{\rm E}\neq\mathcal{E}$. Equation (17) can be rewritten as

$$E_{S,{\rm eq}}^0-E_{S,{\rm eq}}^{\rm f}\leqslant F\left(\rho_{S,{\rm eq}}^0\right)-F\left(\rho_{S,{\rm eq}}^{\rm f}\right)+\frac{S\left(\rho_{S,{\rm eq}}^0\right)}{\beta_0}-\frac{E_{\rm F}^{S-E}\left(\rho_{SE}^{\rm i}\right)}{\beta_{\rm f}}, \tag{18}$$

and $E_{S,{\rm eq}}^0-E_S^{\rm i}$ can be represented as

$$\begin{aligned}E_{S,{\rm eq}}^0-E_S^{\rm i}=&\left[F\left(\rho_{S,{\rm eq}}^0\right)+\frac{S\left(\rho_{S,{\rm eq}}^0\right)}{\beta_0}\right]\\&-\left[F\left(\rho_{S,{\rm eq}}^{\rm f}\right)+\frac{D\left(\rho_S^{\rm i}\middle\|\rho_{S,{\rm eq}}^{\rm f}\right)}{\beta_{\rm f}}+\frac{S\left(\rho_S^{\rm i}\right)}{\beta_{\rm f}}\right].\end{aligned} \tag{19}$$

By plugging Eqs. (18) and (19) into Eq. (16) with Eq. (10), we obtain Eq. (13) again for $\beta_{\rm f}=\beta_{\rm eff}$. Therefore, Eq. (13) is consistent with the previous information thermodynamic inequality in Ref. [5], and Eq. (16) provides the alternative proof of Eq. (13). We emphasize that the definition of the temperature (12) is essential to both the proofs. We also emphasize that $\beta_{\rm eff}$ is not just the value that yields tight upper bounds. Rather, it has clear physical meaning: the effective temperature $1/\beta_{\rm eff}$ is yielded by the entanglement between the system S and environment E.

V. EXAMPLE

Let us examine the tightness of the bound (13) with a simple example of a four-qubit system consisting of qubits 1, 2, 3, and 4. Energy is extracted from qubit 1 by a projective measurement performed on qubit 2 and a feedback unitary operation performed on qubit 1. Thus, qubits 1 and 2 correspond to the system S and the ancilla A, respectively, and qubits 3 and 4 correspond to the environment E. The local Hamiltonian of qubit 1 is defined by $H_S\equiv\sigma_1^z$, where σ_1^z is the z component of the Pauli matrices for qubit 1. We define the initial state $|\psi\rangle$

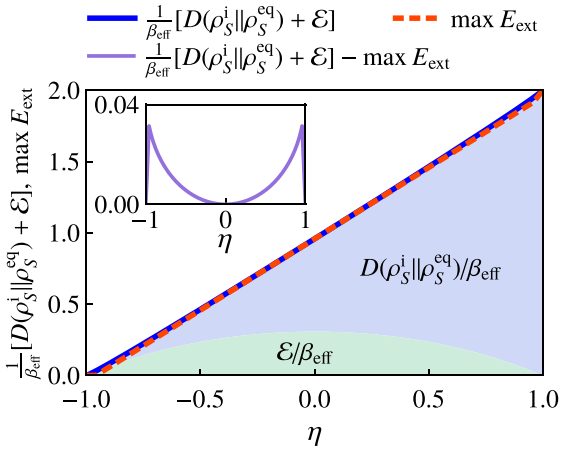


FIG. 2. Tightness of the second bound in Eq. (13). The bound is compared to the maximally extracted energy, and is displayed separately for the contributions of the free energy term $D(\rho_S^{\rm i}||\sigma_S)/\beta_{\rm eff}$ (the blue area) and information term $\mathcal{E}/\beta_{\rm eff}$ (the green area). The inset shows the difference between the upper bound and maximally extracted energy.

with a parameter η as follows:

$$|\psi\rangle \equiv \sqrt{\frac{1+\eta}{14}}(2|\uparrow\uparrow\uparrow\uparrow\rangle + |\uparrow\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\uparrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle) + \sqrt{\frac{1-\eta}{14}}(|\downarrow\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\downarrow\rangle - |\downarrow\downarrow\downarrow\uparrow\rangle - 2|\downarrow\downarrow\downarrow\downarrow\rangle), \tag{20}$$

where we use the abbreviated notation such as $|\uparrow\uparrow\uparrow\uparrow\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes |\uparrow\rangle_3 \otimes |\uparrow\rangle_4$. The spin bases of qubit i ($i = 1, \ldots, 4$), $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$, are the eigenstates of σ_i^z as $\sigma_i^z|\uparrow\rangle_i = |\uparrow\rangle_i$ and $\sigma_i^z|\downarrow\rangle_i = -|\downarrow\rangle_i$. Since the entanglement of qubit 1 vanishes for $\eta = \pm 1$, we consider $-1 < \eta < 1$. When $\eta = 0$, the eigenvalues of $\rho_S^{\rm m}(\mu)$ are independent of μ, i.e., the necessary condition for $E_{\rm ext}$ to reach the upper bound is satisfied. At this value of η, the reduced density matrix of Eq. (20) for qubit 1 is the highly symmetric state, which is invariant to any unitary operation: $\rho_S^{\rm i} = (|\uparrow\rangle_1\langle\uparrow|_1 + |\downarrow\rangle_1\langle\downarrow|_1)/2$. This symmetry is broken for $|\eta| > 0$.

Figure 2 shows the tightness of the second bound in Eq. (13) in the above model. Both the upper bound and the maximally extracted energy increase monotonically with respect to η. This is because the energy of the initial state of qubit 1 is higher for larger η. This increase of the initial state energy corresponds to the increase of the nonequilibrium free energy of the initial state in terms of the effective thermodynamics. In contrast to the η-dependence of the free energy term $D(\rho_S^{\rm i}||\sigma_S)/\beta_{\rm eff}$, the information term $\mathcal{E}/\beta_{\rm eff}$ decreases monotonically with $|\eta|$. As shown in the inset, the difference between the upper bound and the maximally extracted energy is always less than 0.04 even though the magnitude of the bound and the maximally extracted energy increase up to 2. Therefore, the bound is tight except for $\eta \approx -1$. In addition, the maximally extracted energy reaches the bound at $\eta = 0$. Thus, our bound is achievable for highly symmetric initial state in this entangled four qubit model. The details of the calculation are given in Appendix B.

VI. CONCLUSION

In order to investigate local energy extraction, we have defined an effective temperature and introduced effective thermodynamics for entangled multipartite pure states. Based on the effective thermodynamics, we have derived two upper bounds on locally extractable energy with feedback control. One bound corresponds to the second law of information thermodynamics in our effective thermodynamics, and the other bound shows the explicit relationship between energy and the entanglement structure of the initial state. We have confirmed the consistency of our results with previous information thermodynamic inequality in Ref. [5] by relating our energy extraction process to an information thermodynamic process starting from a thermal equilibrium state. We emphasize that even in this point of view the effective temperature introduced by Eq. (12) plays a crucial role in obtaining a tight bound on the extracted energy. We have also investigated the tightness of the bounds with a simple model, and showed that the bounds are achievable for highly symmetric initial states.

Our study goes beyond the previous framework of information thermodynamics in the presence of heat baths; the validity range of this framework has recently been verified [23]. Our results without heat baths open the door to entanglement-based information thermodynamics and lead to a fundamental theory for the energy efficiency of quantum information processing executed on entangled pure states.

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APPENDIX A: PROOF OF THE EQUALITY CONDITION

As mentioned in Sec. III, when the effective temperature $1/\beta_{\rm eff}$ is defined so that Eq. (12) holds, a necessary condition for $E_{\rm ext}$ to be reach the second bound in Eq. (11) [or Eq. (13)] is given by the following theorem.

Theorem A1. Let $\rho_S^{\rm m}(\mu)$ be post measurement states that maximizes $I_{\rm QC}$. Then $\max E_{\rm ext} = \frac{1}{\beta_{\rm eff}}[D(\rho_S^{\rm i}||\sigma_S) + \mathcal{E}]$ holds only if the eigenvalues of $\rho_S^{\rm m}(\mu)$ are independent of μ.

Proof. Suppose $\max E_{\rm ext} = \frac{1}{\beta_{\rm eff}}[D(\rho_S^{\rm i}||\sigma_S) + \mathcal{E}]$. Then the equalities in Eq. (7) hold, i.e., there exists $U_S(\mu)$ which satisfies the equality of $S(\rho_S^{\rm f}) \geqslant \sum_\mu p_\mu S(\rho_S^{\rm m}(\mu))$ for $\rho_S^{\rm m}(\mu)$ that maximizes $I_{\rm QC}$. This inequality is derived by using positivity of KL divergence:

$$\begin{aligned} S\big(\rho_S^{\rm f}\big) &= -{\rm tr}\rho_S^{\rm f}\log\rho_S^{\rm f} \\ &= -\sum_\mu p_\mu {\rm tr} U_S(\mu)\rho_S^{\rm m}(\mu)U_S^\dagger(\mu)\log\rho_S^{\rm f} \\ &\geqslant -\sum_\mu p_\mu {\rm tr} U_S(\mu)\rho_S^{\rm m}(\mu)U_S^\dagger(\mu) \\ &\quad \times \log U_S(\mu)\rho_S^{\rm m}(\mu)U_S^\dagger(\mu) \\ &= \sum_\mu p_\mu S\big(\rho_S^{\rm m}(\mu)\big), \end{aligned} \tag{A1}$$

where $\rho_S^{\rm f}=\sum_\mu p_\mu {\rm tr} U_S(\mu)\rho_S^{\rm m}(\mu)U_S^\dagger(\mu)$ is used to obtain the second line. In the third line, the equality holds if and only if $U_S(\mu)\rho_S^{\rm m}(\mu)U_S^\dagger(\mu)=\rho_S^{\rm f}$ for any μ. Since $U_S(\mu)$ does not change eigenvalues of $\rho_S^{\rm m}(\mu)$, the equality condition implies that the eigenvalues of $\rho_S^{\rm m}(\mu)$ are independent of μ. ■

Note that the converse of the above theorem is not necessarily true. If the eigenvalues of $\rho_S^{\rm m}(\mu)$ are independent of μ, there exist $P_A(\mu)$ and $U_S(\mu)$ which satisfy

$$S\big(\rho_S^{\rm f}\big)=\sum_\mu p_\mu S\big(\rho_S^{\rm m}(\mu)\big)=S\big(\rho_S^{\rm m}(\mu)\big) =E_{\rm F}^{S-E}\big(\rho_{SE}^{\rm i}\big)=S(\sigma_S). \tag{A2}$$

However, since $S(\rho_S^{\rm m}(\mu))=S(\sigma_S)$ does not imply that the eigenvalues of $\rho_S^{\rm m}(\mu)$ and σ_S are equal, Eq. (A2) does not imply $\rho_S^{\rm f}=\sigma_S$, which is the necessary condition for $E_{\rm ext}$ to reach the upper bound. It is an open question whether there exist initial states $|\psi\rangle_{\rm SAE}$ that cannot reach the upper bound and whose post-measurement states $\rho_S^{\rm m}(\mu)'$s have eigenvalues that are independent of the measurement results.

APPENDIX B: DETAILED ANALYSIS OF THE FOUR-QUBIT EXAMPLE

In this section, we investigate in what initial states the equality condition discussed in Appendix A is satisfied in a four-qubit example that is more general than the example in Sec. V. We also provide details of the calculation in Sec. V.

Here, we consider the same four-qubit system and the same local Hamiltonian H_S as in Sec. V, but we consider the following more general initial state:

$$\begin{aligned}|\psi\rangle_{\rm SAE}\equiv{}& a|\uparrow\uparrow\uparrow\uparrow\rangle_{1234}+b|\uparrow\uparrow\uparrow\downarrow\rangle_{1234}+c|\uparrow\downarrow\uparrow\uparrow\rangle_{1234}\\ &+d|\uparrow\downarrow\uparrow\downarrow\rangle_{1234}+e|\downarrow\uparrow\downarrow\uparrow\rangle_{1234}+f|\downarrow\uparrow\downarrow\downarrow\rangle_{1234}\\ &+g|\downarrow\downarrow\downarrow\uparrow\rangle_{1234}+h|\downarrow\downarrow\downarrow\downarrow\rangle_{1234}.\end{aligned} \tag{B1}$$

The initial state of qubit 1 is written as

$$\rho_S^{\rm i}={\rm tr}_{AE}|\psi\rangle_{\rm SAE}\langle\psi|_{\rm SAE} =\begin{pmatrix}a^2+b^2+c^2+d^2 & 0\\ 0 & e^2+f^2+g^2+h^2\end{pmatrix}. \tag{B2}$$

We chose the initial state so that there is no need of diagonalization in the computation of max $E_{\rm ext}$ and the upper bounds on $E_{\rm ext}$. In this model, projective measurement $P_A(\mu)$ and feedback unitary operation $U_S(\mu)$ is written as

$$P_A(\mu)=\tfrac{1}{2}[1+(-1)^\mu\vec{n}\cdot\vec{\sigma}] =\tfrac{1}{2}[1+(-1)^\mu(n_x\sigma_x+n_y\sigma_y+n_z\sigma_z)], \tag{B3}$$

$$\begin{aligned}U_S(\mu)&=\cos\theta_\mu+i\vec{u}_\mu\cdot\vec{\sigma}\sin\theta_\mu\\ &=\cos\theta_\mu+i(u_{x\mu}\sigma_x+u_{y\mu}\sigma_y+u_{z\mu}\sigma_z)\sin\theta_\mu,\end{aligned} \tag{B4}$$

where $\mu=0,1$ is the measurement results, and $\vec{n}\equiv(n_x,n_y,n_z)$ and $\vec{u}_\mu\equiv(u_{x\mu},u_{y\mu},u_{z\mu})$ are unit vectors, i.e., $n_x^2+n_y^2+n_z^2=1$ and $u_{x\mu}^2+u_{y\mu}^2+u_{z\mu}^2=1$.

Next, we examine the equality condition stated in the theorem in Appendix A. The post measurement state $\rho_S^{\rm m}(\mu)$ is computed as

$$\begin{aligned}\rho_S^{\rm m}(\mu)&=\frac{1}{p_\mu}{\rm tr}_{AE}P_A(\mu)|\psi\rangle_{\rm SAE}\langle\psi|_{\rm SAE}P_A(\mu)\\ &=\frac{1}{2p_\mu}[a^2+b^2+c^2+d^2+(-1)^\mu\{2(ac+bd)n_x\\ &\qquad+(a^2+b^2-c^2-d^2)n_z\}]|\uparrow\rangle_1\langle\uparrow|_1\\ &\quad+\frac{1}{2p_\mu}[e^2+f^2+g^2+h^2+(-1)^\mu\{2(eg+fh)n_x\\ &\qquad+e^2+f^2-g^2-h^2)n_z\}]|\downarrow\rangle_1\langle\downarrow|_1,\end{aligned} \tag{B5}$$

where p_μ is the probability of obtaining a result μ:

$$\begin{aligned}p_\mu&=\langle\psi|P_A(\mu)|\psi\rangle\\ &=\tfrac{1}{2}+(-1)^\mu\big[(ac+bd+eg+fh)n_x+\tfrac{1}{2}(a^2+b^2\\ &\quad-c^2-d^2+e^2+f^2-g^2-h^2)n_z\big].\end{aligned} \tag{B6}$$

Thus, if following equations hold, the eigenvalues of $\rho_S^{\rm m}(\mu)'$s are independent of μ for any measurement:

$$a^2+b^2+c^2+d^2=e^2+f^2+g^2+h^2=\tfrac{1}{2}, \tag{B7a}$$

$$ac+bd=-(eg+fh), \tag{B7b}$$

$$a^2+b^2-c^2-d^2=-(e^2+f^2-g^2-h^2). \tag{B7c}$$

If Eq. (B7a) holds, the initial state of qubit 1 is invariant under any unitary operation: $\rho_S^{\rm i}=(|\uparrow\rangle_1\langle\uparrow|_1+|\downarrow\rangle_1\langle\downarrow|_1)/2$. This suggests that the upper bound is tight for highly symmetric initial states.

In the following, we take the coefficients as $a/2=b=c=d=\sqrt{(1+\eta)/14}$ and $e=f=-g=-h/2=\sqrt{(1-\eta)/14}$ so that the initial state is equal to Eq. (20), and provide detailed calculations of the upper bound and the extractable energy in Sec. V. It is also shown analytically that the upper bound is achievable when Eq. (B7) holds. When $\eta=0$, the conditions of Eq. (B7) hold. First, we calculate maximally extracted energy max $E_{\rm ext}$. At the beginning of the energy extraction process, the reduced density matrix of qubit 1 is

$$\rho_S^{\rm i}={\rm tr}_{AE}|\psi\rangle_{\rm SAE}\langle\psi|_{\rm SAE}=\frac{1}{2}\begin{pmatrix}1+\eta & 0\\ 0 & 1-\eta\end{pmatrix}, \tag{B8}$$

and thus the energy expectation value in the initial state of qubit 1 is calculated as

$$E_S^{\rm i}\equiv{\rm tr}\rho_S^{\rm i}H_S=\eta. \tag{B9}$$

The post measurement state of qubit 1 with a result μ is

$$\rho_S^{\rm m}(\mu)=\frac{1}{2p_\mu}\begin{pmatrix}\frac{1+\eta}{14}[7+(-1)^\mu 3(2n_x+n_z)] & 0\\ 0 & \frac{1-\eta}{14}[7+(-1)^{\mu+1}3(2n_x+n_z)]\end{pmatrix}\begin{pmatrix}\lambda_\uparrow(\mu) & 0\\ 0 & \lambda_\downarrow(\mu)\end{pmatrix}, \tag{B10}$$

$$p_\mu = \frac{1}{14}[7 + (-1)^\mu 3\eta(2n_x + n_z)]. \tag{B11}$$

Thus, the maximally extracted energy can be calculated as

$$\begin{aligned}
\max E_{\rm ext} &= E_S^{\rm i} - \min_{\{P_A(\mu)\},\{U_S(\mu)\}} E_S^{\rm f} \\
&= \eta + \max_{\{P_A(\mu)\},\{U_S(\mu)\}} \left\{ -\sum_\mu p_\mu {\rm tr} U_S(\mu)\rho_S^{\rm m}(\mu)U_S^\dagger(\mu)H_S \right\} \\
&= \eta + \max_{\{P_A(\mu)\}} \sum_\mu p_\mu |\lambda_\uparrow(\mu) - \lambda_\downarrow(\mu)| \\
&= \eta + \frac{1}{14} \max_{\{P_A(\mu)\}} \{|7\eta + 3(2n_x + n_z)| + |7\eta - 3(2n_x + n_z)|\} \\
&= \begin{cases} 0 & \text{for } -1 < \eta \leqslant -\frac{3\sqrt{5}}{7}, \\ \eta + \frac{3\sqrt{5}}{7} & \text{for } -\frac{3\sqrt{5}}{7} \leqslant \eta \leqslant \frac{3\sqrt{5}}{7}, \\ 2\eta & \text{for } \frac{3\sqrt{5}}{7} \leqslant \eta < 1, \end{cases}
\end{aligned} \tag{B12}$$

where we used $-\sqrt{5} \leqslant 2n_x + n_z \leqslant \sqrt{5}$. The extracted energy is maximized when

$$\begin{cases} U_S(0) = U_S(1) = I_S & \text{for } -1 < \eta \leqslant -\frac{3\sqrt{5}}{7}, \\ 2n_x + n_z = -\sqrt{5}, U_S(0) = I_S, U_S(1) = \sigma_1^x, \text{ or } 2n_x + n_z = \sqrt{5}, U_S(0) = \sigma_1^x, U_S(1) = I_S & \text{for } -\frac{3\sqrt{5}}{7} \leqslant \eta \leqslant \frac{3\sqrt{5}}{7}, \\ U_S(0) = U_S(1) = \sigma_1^x & \text{for } \frac{3\sqrt{5}}{7} \leqslant \eta < 1. \end{cases} \tag{B13}$$

Note that when $-1 < \eta < -\frac{3\sqrt{5}}{7}$ and $\frac{3\sqrt{5}}{7} < \eta < 1$, the operations on qubit 1 that maximize $E_{\rm ext}$ is not feedback control. In this region, $\rho_S^{\rm i}$ is almost equal to $|\downarrow\rangle_1\langle\downarrow|_1$ or $|\uparrow\rangle_1\langle\uparrow|_1$, and thus the optimal operations for extracting energy is to do nothing or the spin flip. Next, we calculate the upper bound presented in Eq. (13). The entanglement entropy of the qubit 1 is

$$S\big(\rho_S^{\rm i}\big) = {\rm tr}\rho_S^{\rm i} \log \rho_S^{\rm i} = -\frac{1+\eta}{2}\log\frac{1+\eta}{2} - \frac{1-\eta}{2}\log\frac{1-\eta}{2}. \tag{B14}$$

After the measurement $P_A(\mu)$, the entanglement entropy of qubit 1 can decrease on average to

$$\begin{aligned}
\sum_\mu p_\mu S\big(\rho_S^{\rm m}(\mu)\big) &= -\sum_\mu p_\mu {\rm tr}\rho_S^{\rm m}(\mu)\log\rho_S^{\rm m}(\mu) \\
&= -\sum_\mu p_\mu [\lambda_\uparrow(\mu)\log\lambda_\uparrow(\mu) + \lambda_\downarrow(\mu)\log\lambda_\downarrow(\mu)] \\
&= -\frac{1+\eta}{2}\log\frac{1+\eta}{2} - \frac{1-\eta}{2}\log\frac{1-\eta}{2} - \frac{1}{14}(7 + 3(2n_x + n_z))\log(7 + 3(2n_x + n_z)) \\
&\quad - \frac{1}{14}(7 - 3(2n_x + n_z))\log(7 - 3(2n_x + n_z)) + \frac{1}{14}(7 + 3\eta(2n_x + n_z))\log(7 + 3\eta(2n_x + n_z)) \\
&\quad + \frac{1}{14}(7 - 3\eta(2n_x + n_z))\log(7 - 3\eta(2n_x + n_z)) \\
&\geqslant -\frac{1+\eta}{2}\log\frac{1+\eta}{2} - \frac{1-\eta}{2}\log\frac{1-\eta}{2} - \frac{1}{14}(7 + 3\sqrt{5})\log(7 + 3\sqrt{5}) - \frac{1}{14}(7 - 3\sqrt{5})\log(7 - 3\sqrt{5}) \\
&\quad + \frac{1}{14}(7 + 3\sqrt{5}\eta)\log(7 + 3\sqrt{5}\eta) + \frac{1}{14}(7 - 3\sqrt{5}\eta)\log(7 - 3\sqrt{5}\eta) \\
&= E_{\rm F}^{S-E}\big(\rho_{SE}^{\rm i}\big),
\end{aligned} \tag{B15}$$

where we used $-\sqrt{5} \leqslant 2n_x + n_z \leqslant \sqrt{5}$. From Eq. (B15), QC-mutual information $I_{\rm QC} \equiv S(\rho_S^{\rm i}) - \sum_\mu p_\mu S(\rho_A^{\rm m}(\mu))$ and the entanglement $\mathcal{E} \equiv S(\rho_S^{\rm i}) - E_{\rm F}^{S-E}(\rho_{SE}^{\rm i})$ can be computed as

$$\begin{aligned} I_{\rm QC} = & \tfrac{1}{14}(7+3(2n_x+n_z))\log(7+3(2n_x+n_z)) \\ & + \tfrac{1}{14}(7-3(2n_x+n_z))\log(7-3(2n_x+n_z)) \\ & - \tfrac{1}{14}(7+3\eta(2n_x+n_z))\log(7+3\eta(2n_x+n_z)) \\ & - \tfrac{1}{14}(7-3\eta(2n_x+n_z))\log(7-3\eta(2n_x+n_z)), \end{aligned} \tag{B16}$$

$$\begin{aligned} \mathcal{E} = & \tfrac{1}{14}(7+3\sqrt{5})\log(7+3\sqrt{5}) \\ & + \tfrac{1}{14}(7-3\sqrt{5})\log(7-3\sqrt{5}) \\ & - \tfrac{1}{14}(7+3\sqrt{5}\eta)\log(7+3\sqrt{5}\eta) \\ & - \tfrac{1}{14}(7-3\sqrt{5}\eta)\log(7-3\sqrt{5}\eta). \end{aligned} \tag{B17}$$

In our effective thermodynamics, the thermal equilibrium state σ_S is defined by Eq. (3). Thus, the von Neumann entropy of σ_S is

$$\begin{aligned} S(\sigma_S) &= -{\rm tr}\sigma_S \log \sigma_S \\ &= \frac{\log\left(1+e^{2\beta_{\rm eff}}\right)}{1+e^{2\beta_{\rm eff}}} + \frac{\log\left(1+e^{-2\beta_{\rm eff}}\right)}{1+e^{-2\beta_{\rm eff}}}, \end{aligned} \tag{B18}$$

and the KL divergence $D(\rho_S^{\rm i}||\sigma_S)$ is calculated as

$$\begin{aligned} D\left(\rho_S^{\rm i}||\sigma_S\right) &= {\rm tr}\rho_S^{\rm i}\log\rho_S^{\rm i} - {\rm tr}\rho_S^{\rm i}\log\sigma_S \\ &= -\log 2 + \frac{1+\eta}{2}\log(1+\eta)(1+e^{2\beta_{\rm eff}}) \\ &\quad + \frac{1-\eta}{2}\log(1-\eta)(1+e^{-2\beta_{\rm eff}}). \end{aligned} \tag{B19}$$

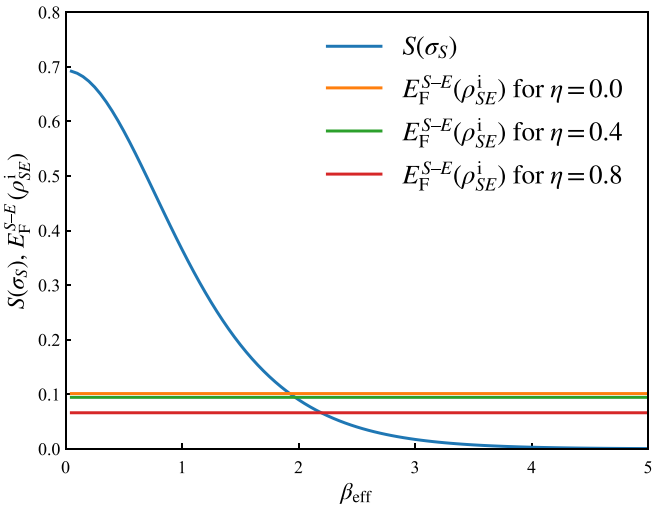


FIG. 3. Comparison between $S(\rho_S)$ and $E_{\rm F}^{S-E}(\rho_{SE}^{i})$. The value of $\beta_{\rm eff}$ is determined by the intersection points.

The effective inverse temperature $\beta_{\rm eff}$ can be determined numerically by Eq. (12) as shown in Fig. 3. When $\eta = 0$, $\beta_{\rm eff}$ can be calculated analytically as $\beta_{\rm eff} = \frac{1}{2}\log\frac{7+3\sqrt{5}}{7-3\sqrt{5}}$. At this value of η, $D(\rho_S^{\rm i}||\sigma_S) = -\log 2 + \log 7$, $\mathcal{E} = \log 2 - \log 7 + \frac{3\sqrt{5}}{14}\log\frac{7+3\sqrt{5}}{7-3\sqrt{5}}$, and thus $\frac{1}{\beta_{\rm eff}}[D(\rho_S^{\rm i}||\sigma_S) + \mathcal{E}] = \max E_{\rm ext} = \frac{3\sqrt{5}}{7}$. Therefore, in this model, $E_{\rm ext}$ reaches the upper bound if the eigenvalues of $\rho_S^{\rm m}(\mu)$ are independent of μ.

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