

Diversity-induced resonance at surfactant-laden interfaces

Stefano Scialla ^{*}*National Institute of Chemical Physics and Biophysics, R vala 10, Tallinn 15042, Estonia
and Department of Engineering, Universit  Campus Bio-Medico di Roma, Via  . del Portillo 21, 00128 Rome, Italy*

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Networks of coupled nonlinear elements can amplify weak periodic signals when unit heterogeneity is intermediate—a phenomenon known as diversity-induced resonance (DIR). We test this mechanism at surfactant-laden interfaces by modeling the interface as a two-dimensional lattice of laterally coupled patches obeying Frumkin adsorption-desorption kinetics, driven by sinusoidal modulation of the monomer concentration; diversity enters as patchwise variability in the adsorption drive. Using linear analysis and numerical simulations, we compute the first-harmonic gain of the mean surface coverage while sweeping diversity and other parameters, and connect the model to experimental observables. Intermediate diversity maximizes the collective response, increasing mean coverage by $\sim 3\text{--}4\times$ versus a homogeneous interface and improving robustness to frequency variations, consistent with DIR. Because the effect emerges from network coupling, the model requires no empirical time constants or *ad hoc* relaxation pathways.

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I. INTRODUCTION

Networks of coupled nonlinear units often display counterintuitive collective effects that are absent at the single-unit level. A prominent example is diversity-induced resonance (DIR) [1]: the enhancement of a network’s in-phase response to a weak periodic drive when the network units are heterogeneous and their degree of diversity has an intermediate magnitude. DIR is distinct from stochastic resonance [2], which relies on noise; in DIR, the amplification is deterministic and arises from parameter heterogeneity. Here, “diversity” refers to the spread of an intrinsic control parameter across units; depending on the physical realization this parameter may be, for instance, a barrier height, a bias or threshold (excitability) parameter, an electronic gain or conductance, an intrinsic relaxation time, or an effective adsorption propensity. Intuitively, if the population is nearly homogeneous, every unit responds similarly and a drive that is slightly too fast or too weak for the typical network element elicits only a modest, lagged response. At the opposite extreme, if diversity is very large, the population responds with a broad spread of phases and amplitudes, so partial cancellations reduce the macroscopic signal. Between these limits, at a suitable intermediate level of diversity, a subset of “responsive” units (made responsive by their intrinsic parameters) follows the drive with small lag and, through coupling, recruits their neighbors into a partially synchronized state, maximizing the coherent response of the network. This nonmonotonic dependence on diversity

is what we refer to as DIR [1]. In most DIR settings, the “response” is quantified by the amplitude of the first harmonic of a macroscopic output signal at the driving frequency—for example, the network-averaged voltage in electronic circuits, the population-averaged membrane potential in excitable media, or more generally the spatial or network average of the relevant state variable.

DIR has been reported and theorized in a variety of contexts, including biological networks, e.g., pancreatic β -cells, cardiac tissue, and neural ensembles [3–12]; opinion dynamics [13,14]; and coupled electronic oscillators [15].

It is helpful to review some specific examples in which heterogeneity in a dynamical parameter across network elements gives rise to DIR. In the original DIR study by Tessone *et al.* [1], each network node i is a bistable unit with a control parameter a_i that sets the barrier height of the local double-well potential acting on that element. Diversity is introduced as a distribution of a_i values across nodes. Physically, lower barriers (smaller $|a_i|$) imply a higher local susceptibility and a faster relaxation rate, so those units follow the drive with a smaller phase lag and, through all-to-all coupling, “pull” other units that, on their own, would not be able to follow the drive due to their higher potential barrier. The macroscopic response is quantified by the spectral amplification factor $\eta \equiv 4A_{\text{ext}}^{-2} |\langle X(t) e^{i\Omega t} \rangle|^2$, i.e., the squared modulus of the first harmonic of the population mean $X(t)$ normalized by the forcing amplitude A_{ext} ; η displays a clear nonmonotonic dependence on the width of the a_i distribution [1]. In the same paper, the authors show that this amplification mechanism can be extended to networks of excitable units.

In neuronal and pancreatic β -cell ensembles modeled by networks of FitzHugh-Nagumo units [16–18], diversity is typically implemented either in the bias current I_i or in a parameter a_i that sets the local excitability threshold for each unit i . Units closer to the excitability threshold are more susceptible and respond with larger amplitude and smaller phase

^{*}Contact author: stefano.scialla@unicampus.it

lag to a weak external input; via coupling, these “leader” units recruit neighbors, maximizing the first harmonic of the population-averaged membrane potential at an intermediate heterogeneity level, resulting in a DIR peak [4,5,12]. The directly measured observables are the first harmonic of the mean voltage and its phase lag relative to the drive.

There are also a few studies that focus on technical applications. For example, Yao and Zhan designed a lattice of coupled electronic circuit modules as a proposed platform to experimentally probe DIR [19]. In their setup, diversity is introduced by assigning each module a slightly different negative conductance, which effectively tunes the local gain and the transition from monostability to bistability. Under a weak periodic input, the amplitude of the network-level response, quantified by the mean-field (network-averaged) voltage, is maximized at an intermediate spread of the negative conductances, giving rise to a DIR peak.

Recent work has shown that DIR also appears in systems of mobile heterogeneous oscillators, where interactions depend on spatial proximity [15]. In these models, each unit is a bistable oscillator with unit-to-unit parameter variability (i.e., heterogeneity), and the agents perform random walks in a two-dimensional (2D) domain; coupling occurs within a finite vision radius. In addition to the usual DIR effect, even the resonance strength varies nonmonotonically with agent speed, with an optimal mobility that maximizes the first harmonic of the population mean. The authors observed the same effect with FitzHugh-Nagumo units [15].

Perc *et al.* explored resonance phenomena in soft-matter systems using a polymer-stabilized ferroelectric liquid-crystal cell as a representative platform [20]. There, both static and dynamic disorder modulate the effective electro-optic susceptibility, and the observable is the first harmonic of the liquid crystal’s response to an alternating current drive. While the mechanism analyzed is stochastic resonance rather than DIR, it demonstrates in a soft-matter context how controlled disorder can boost a macroscopic periodic response and thus provides a useful experimental analog for heterogeneity-assisted amplification [20].

Very recently, Patriarca *et al.* established a dynamical equivalence between DIR and resonant polymer translocation [6]. In that mechanical analog, heterogeneity in the network (e.g., a distribution of bias parameters across units) maps to parameters controlling the driven motion of a polymer over a periodic substrate; the resonant enhancement of translocation has the same mathematical origin as the DIR peak. This connection broadens the range of potential soft-matter and nanoscale applications of DIR.

Across all of the above-mentioned examples, the essential ingredients required to be able to observe DIR are (i) an external forcing, usually but not necessarily periodic; (ii) a network of nonlinear units, such as quartic or FitzHugh-Nagumo oscillators; (iii) unit-to-unit variability (diversity), quantified by the width of a distribution of a physical control parameter that modulates local susceptibility or relaxation and thus amplitude or phase lag; and (iv) a coupling mechanism that allows responsive units to influence their neighbors.

In the present study, we want to investigate whether the DIR framework can be applied to surfactant-laden fluid inter-

faces subject to periodic actuation, e.g., periodic refreshing or renewal of the subsurface solution, or controlled oscillations in the concentration or chemical potential, and whether, as a consequence, DIR is a relevant phenomenon in the context of interface adsorption processes. Such interfaces are well known to develop lateral heterogeneity—domains and patches differing in coverage, elasticity, or local adsorption propensities—under both equilibrium and driven conditions. For example, lateral heterogeneity at surfactant-laden interfaces has been observed by Brewster angle microscopy and fluorescence imaging [21–24], while oscillating drop or bubble and capillary-pressure tensiometry have been used to quantify the dilational response over broad frequency ranges [25–31].

Interfacial transport under time-dependent driving has traditionally been modeled at a lumped level through adsorption-desorption kinetics and analyzed via dynamic surface-tension or dilational-rheology measurements. In these models (and their mixture generalizations), kinetic rate limitations and cross-species coupling map directly onto the distribution of relaxation times and the frequency-dependent interfacial response. In particular, the free-energy–kinetic theory of Diamant and co-workers and subsequent treatments for mixtures provide a quantitative baseline for dynamic responses of multicomponent interfacial layers. This free-energy–kinetic framework rests on the classical lattice-gas picture of interfacial monolayers: in equilibrium it reduces to the Frumkin–Fowler–Guggenheim isotherm (Langmuir in the ideal limit), and in mixtures it includes cross-interaction terms yielding multicomponent Frumkin-type behavior [26,32–35].

On the other hand, spatially resolved descriptions capture lateral heterogeneity and domain formations that have been often observed. One route is lattice-gas–Ising-like modeling, where amphiphiles occupy a two-dimensional lattice with nearest-neighbor interactions; Monte Carlo and mean-field treatments recover Frumkin–Fowler–Guggenheim-type nonidealities and mosaic domains [22,36]. Another route is continuum phase-field (Cahn–Hilliard-type) modeling of monolayers and transfer processes, which embeds lateral diffusion, line tension, and pattern selection under spatial or temporal forcing [37,38]. A third, mesoscale approach uses lattice-Boltzmann or related lattice-gas schemes to study amphiphilic fluids with explicit interfacial structure [39,40]. Together, these studies motivate coarse-grained patch discretizations as a physically meaningful abstraction for driven, laterally heterogeneous interfaces.

However, neither classical dynamic adsorption nor spatially resolved interface models have addressed whether the combination of controlled diversity and lateral coupling, under the action of an external forcing, can lead to a coherent response of the macroscopic interface and consequent signal amplification.

Against this backdrop, our study introduces a minimal interfacial network model in which a two-dimensional lattice of laterally coupled patches follows microreversible Frumkin kinetics and is subjected to a sinusoidal concentration or chemical-potential drive. As detailed in Sec. II, diversity is introduced as patch-to-patch variability in the response to an external drive, and diffusive coupling enables synchronization across the lattice. This discrete formulation is the interfa-

cial analog of heterogeneous oscillator networks studied in complex systems physics, allowing us to investigate the occurrence of DIR in this different context.

Based on the above, we ask whether, and under what conditions, DIR emerges in this interfacial setting and how it depends on the key parameters that determine system behavior. We show both analytically and via numerical simulations that a suitable level of physicochemical diversity can maximize the coherent response of the interface.

The paper is structured as follows. After the Introduction, Sec. II presents the network model equations in a dimensional and dimensionless form, introduces the metric to measure network response, and describes the numerical methods used to integrate the equations. In Sec. III, after an analytic discussion of the model equations for both homogeneous and heterogeneous systems, we report the results of numerical simulations that quantify DIR for different network configurations and as a function of key system parameters. We also outline a potential experimental validation of the model and provide evidence of DIR in published experimental data. Finally, in Sec. IV, we present the key conclusions and discuss the limitations and future potential improvements of the model.

II. METHODS

A. Model

We model the interface as a square-tiled surface (Cartesian grid) made of N patches or cells identical in size, as schematized in Fig. 1(a). Each patch exchanges surfactant with a subsurface monomer reservoir and, at the same time, exchanges surfactant laterally along the interface with its four edge-sharing neighbors, to which it is therefore coupled, as shown in Fig. 1(b). The local surface coverage on patch i is

$$\theta_i(t) \in [0, 1], \quad \theta_i(t) = \Gamma_i(t)/\Gamma_{\text{sat}},$$

where Γ_i is the surfactant surface excess (Gibbs surface concentration) and Γ_{sat} is the saturation surface coverage.

Neglecting, for the moment, coupling between adjacent patches, we describe the adsorption-desorption dynamics at each patch i by a microreversible Frumkin rate law [34,35], according to the equation [41]

$$\dot{\theta}_i = k_a c(t) (1 - \theta_i) e^{-sA\theta_i} - k_d \theta_i e^{(1-s)A\theta_i}, \quad (1)$$

where k_a and k_d are the adsorption and desorption rate constants, respectively; A is the (dimensionless) Frumkin interaction parameter ($A < 0$ attractive, $A > 0$ repulsive), assumed to be identical for all patches; and $s \in [0, 1]$ is a symmetry factor that partitions the interaction-dependent free-energy contribution $A\theta_i$ between the forward and backward activation barriers. We assume $s = \frac{1}{2}$, which means that Frumkin interactions are equally split between adsorption and desorption [41–43]. It should be noted that since $c(t)$ has units of concentration and θ_i is dimensionless, the adsorption prefactor $k_a c(t)$ must have units of 1/time; therefore k_a has units of (concentration $\times$ time) $^{-1}$, whereas k_d has units of time $^{-1}$.

Equation (1) is independent of the side length ℓ of a square patch; however, for the Frumkin description to be valid, ℓ should be large compared with a molecular adsorption site so

that each patch contains many adsorption sites and the continuum coverage variable θ is meaningful. At the same time, ℓ should be small compared with the characteristic heterogeneity scale of the interface, so that each patch is approximately uniform while the lattice still resolves spatial patterns. In practice, we can assume ℓ lies between ~ 50 – 100 nm (ensuring $\gtrsim 10^4$ adsorption sites per patch for typical molecular areas 0.5 – 1 nm ^{2}) and ~ 1 – 5 μ m.

The adsorption term in Eq. (1) contains a time-dependent factor $c(t)$, which expresses an effective monomer concentration in the subsurface reservoir that feeds the interface [Figs. 1(a) and 1(b)]. In many adsorption-kinetics models this subsurface concentration is treated as constant [33,41,42,44]; here we allow it to be periodically modulated by an external drive, representing, for example, oscillatory renewal of the near-interface region by convection, or an effective modulation of the local chemical potential induced by interfacial deformation. Specifically, we assume

$$c(t) = c_0[1 + \rho \sin(\omega t)], \quad 0 \leq \rho < 1. \quad (2)$$

Here, c_0 is the baseline monomer concentration in the subsurface, while ρ and $\omega = 2\pi f$ are, respectively, the (dimensionless) relative amplitude and angular frequency of the sinusoidal forcing. When the solution is above the critical micelle concentration (CMC), c_0 denotes the monomer concentration available to the interface—often pinned near the CMC—while the modulation in Eq. (2) can be viewed as an effective time-periodic modulation of the subsurface reservoir activity (chemical potential).

In steady conditions $c(t) \equiv c_0$ and $\dot{\theta}_i = 0$, thus Eq. (1) reduces to the Frumkin isotherm,

$$\frac{\theta_i}{1 - \theta_i} = K c_0 e^{-A\theta_i}, \quad K = \frac{k_a}{k_d}, \quad (3)$$

which is independent of s . Here $K = k_a/k_d$ has units of (concentration) $^{-1}$ because k_a has units of (concentration $\times$ time) $^{-1}$ whereas k_d has units of time $^{-1}$; accordingly, Kc_0 is dimensionless.

It is convenient to transform Eq. (1) into a nondimensional version, by introducing the desorption timescale $\tau_d = 1/k_d$ and dimensionless time $\tau = t/\tau_d$, plus the following definitions:

$$\alpha \equiv \frac{k_a c_0}{k_d}, \quad \Omega \equiv \frac{\omega}{k_d} = \frac{2\pi f}{k_d}. \quad (4)$$

Then $c(\tau) = c_0[1 + \rho \sin(\Omega\tau)]$, and Eq. (1) becomes

$$\frac{d\theta_i}{d\tau} = \alpha[1 + \rho \sin(\Omega\tau)](1 - \theta_i) e^{-sA\theta_i} - \theta_i e^{(1-s)A\theta_i}. \quad (5)$$

The parameter $\alpha \equiv (k_a c_0)/k_d$ is a dimensionless adsorption drive (equivalently, $\alpha \equiv Kc_0$ with $K = k_a/k_d$). It measures the supply of adsorbable monomers relative to the intrinsic desorption rate, so a larger α biases the interface toward higher equilibrium coverage on the Frumkin isotherm. The quantity $\Omega \equiv \omega/k_d = 2\pi f/k_d$ is the dimensionless angular drive frequency: it compares the rate of the external forcing to the intrinsic relaxation set by k_d . Thus $\Omega \ll 1$ corresponds to quasiadiabatic forcing, $\Omega \sim 1$ to comparable timescales, and $\Omega \gg 1$ to fast forcing; in nondimensional time the drive

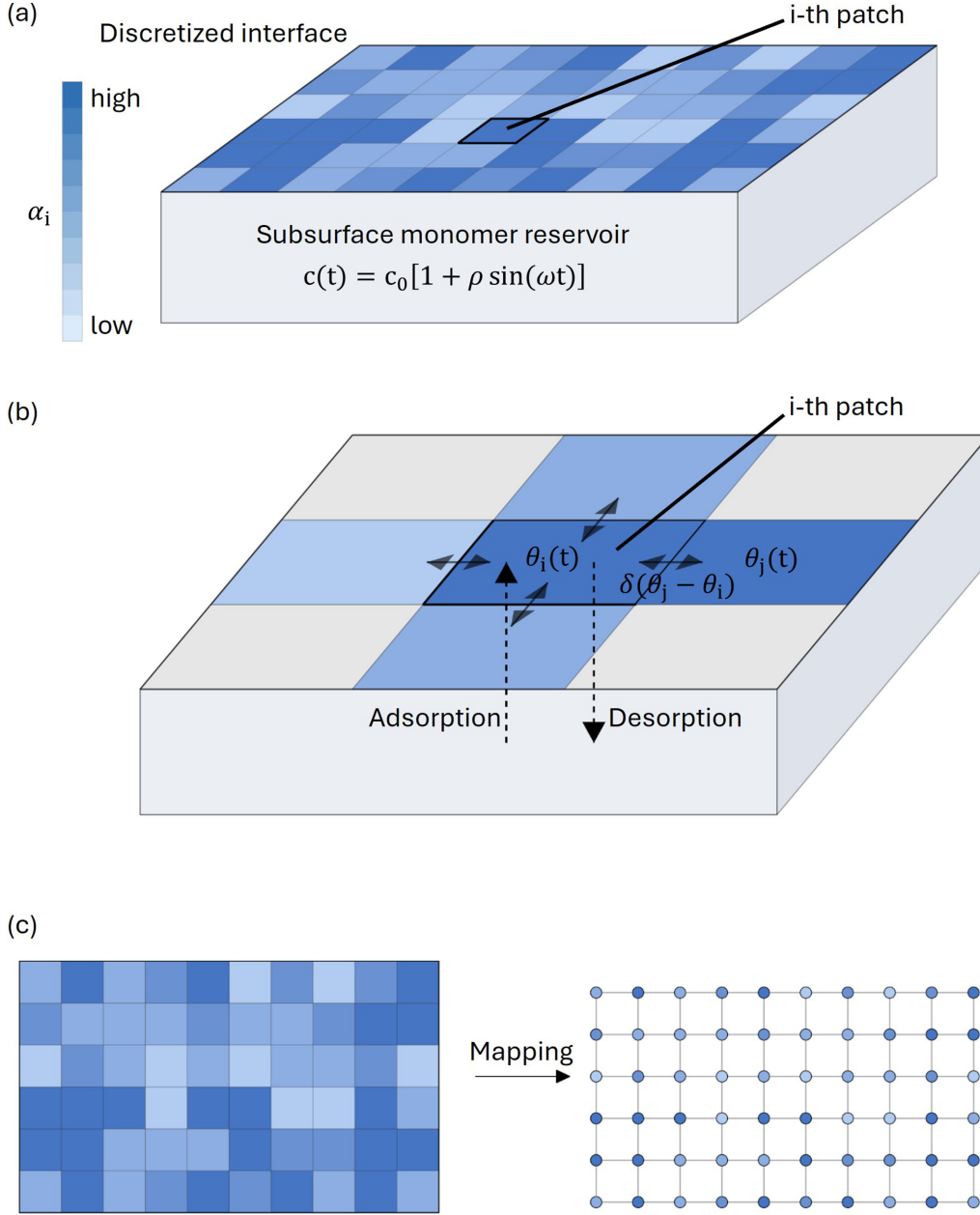


FIG. 1. Schematic of the patch model for a heterogeneous, surfactant-laden interface. (a) The interface is discretized into N square patches (blue tiles) in contact with a subsurface monomer reservoir with concentration $c(t) = c_0[1 + \rho \sin(\omega t)]$. Different shades of blue indicate patch-to-patch heterogeneity in local adsorption properties, represented through different α_i values (see color bar). The outlined i th patch marks the region zoomed in (b). (b) Zoom on the i th patch and its four nearest neighbors. The surface coverage on each patch, $\theta_i(t)$, evolves through adsorption-desorption exchange with the subsurface reservoir (dashed vertical arrows) and through lateral coupling of strength δ with adjacent patches [illustrated by the term $\delta(\theta_j - \theta_i)$ for a neighbor j]. (c) Mapping between the discretized interface and a square-lattice network representation: each patch corresponds to a node and links represent nearest-neighbor coupling; node color encodes local heterogeneity (α_i).

period is $T = 2\pi/\Omega$. Throughout, we will refer to α as the adsorption drive and to Ω as the drive frequency.

To account for lateral exchange along the interface, we couple each patch to its four edge-sharing neighbors, as illustrated in Fig. 1(b). We model this exchange as a diffusive (discrete-Laplacian) coupling term, which should be interpreted as an effective lateral transport operator capturing local exchange of adsorbate between neighboring patches. It is worth pointing out that in real interfaces, additional lateral

coupling channels (e.g., Marangoni-driven advection and interfacial viscoelastic stresses) may contribute beyond purely diffusive exchange, while in our simplified model we only take into account the latter. Thus we obtain

$$\begin{aligned} \frac{d\theta_i}{d\tau} = & \alpha[1 + \rho \sin(\Omega\tau)](1 - \theta_i) e^{-sA\theta_i} \\ & - \theta_i e^{(1-s)A\theta_i} + \delta \sum_{j \in \mathcal{N}(i)} (\theta_j - \theta_i). \end{aligned} \quad (6)$$

Here $\mathcal{N}(i)$ denotes the set of four nearest neighbors of patch i , and δ is a dimensionless coupling strength [e.g., $\delta \sim D_\theta/(k_d \ell^2)$ in terms of an effective lateral diffusivity D_θ and patch size ℓ]. It is interesting to note that the same structure can be viewed as a square-lattice network, as shown in Fig. 1(c). This is a convenient representation that enables a discussion of our model in more general terms, connecting it to the broader dynamical systems literature on driven, coupled nonlinear units where DIR has been observed.

In Eq. (6), the coupling term $\delta \sum_{j \in \mathcal{N}(i)} (\theta_j - \theta_i)$ is a discrete diffusion operator (graph Laplacian) acting on the coverage field, i.e., $\delta \sum_{j \in \mathcal{N}(i)} (\theta_j - \theta_i) = -\delta (L\theta)_i$, where L is the Laplacian of the chosen connectivity graph. In our formulation of the model, we take $\mathcal{N}(i)$ as the four edge-sharing neighbors of a square grid, which corresponds to the standard minimal stencil for an isotropic 2D Laplacian on a square tiling. However, the model is not restricted to this choice: other local stencils, including diagonal neighbors, or other tilings, e.g., hexagonal or triangular, can be used by changing $\mathcal{N}(i)$. When comparing different connectivities, it is often convenient to use a degree-normalized coupling, $\frac{\delta}{z} \sum_{j \in \mathcal{N}(i)} (\theta_j - \theta_i)$, so that δ retains the meaning of an effective lateral transport strength. Accordingly, we expect the qualitative DIR phenomenology to be robust to changes in the neighbor setting, while quantitative features, such as peak location or height, may depend on the Laplacian spectrum and effective coupling scale.

We now introduce diversity in the system described by Eq. (6), by assigning to each patch a different value of the adsorption drive, denoted as α_i , drawn from a suitable Gaussian or log-normal distribution. Equation (6) then becomes

$$\frac{d\theta_i}{d\tau} = \alpha_i [1 + \rho \sin(\Omega\tau)] (1 - \theta_i) e^{-sA\theta_i} - \theta_i e^{(1-s)A\theta_i} + \delta \sum_{j \in \mathcal{N}(i)} (\theta_j - \theta_i). \quad (7)$$

Physically, this corresponds to patchwise variability in the adsorption drive parameter $\alpha_i = K_i c_{0,i} = (k_{a,i} c_{0,i})/k_{d,i}$; see Fig. 1(a). Such variability can arise from (i) compositional heterogeneity in blends of surfactants with different adsorption propensities (e.g., head group structure or number of ethoxylated units and/or chain length); (ii) local solution conditions (ionic strength, pH, temperature, cosolvent or hydrotrope content) that modify k_a/k_d ; (iii) spatial variations of the subsurface monomer availability $c_{0,i}$ due to hydrodynamic renewal or micellar buffering (boundary-layer thickness differences, weak flows); and (iv) weak site-energy differences at the interface (roughness, co-adsorbed impurities). In the regime above the CMC, α_i heterogeneity can be interpreted as small patchwise offsets in the interfacial chemical potential, $\alpha_i = \bar{\alpha} \exp(\mu_i/k_B T)$.

The dynamics of the heterogeneous, periodically driven patch system (7) is the setting in which we seek diversity-induced resonance effects analogous to those previously reported in ensembles of driven, coupled nonlinear units. The square-lattice network analogy shown in Fig. 1(c) provides a direct conceptual bridge to those DIR studies while leaving the physical interpretation in terms of interfacial patches

explicit. It also allows the system to be naturally described using the language of network theory.

B. Metric for network response

We denote by $\mathcal{G}(\sigma; \Omega)$ the first-harmonic gain of the spatial mean surfactant surface coverage $\bar{\theta}(\tau)$ under a sinusoidal drive of angular frequency Ω and amplitude ρ . When the adsorption parameter is heterogeneous across sites, α_i 's are drawn from a suitable distribution (e.g., Gaussian or log-normal) with mean $\bar{\alpha}$ and standard deviation σ , where σ expresses the degree of diversity of the network. The homogeneous case, in which all α_i 's are identical, corresponds to $\mathcal{G}(\sigma = 0; \Omega)$. Formally, let

$$\begin{aligned} \bar{\theta}(\tau) &\equiv \frac{1}{N} \sum_{i=1}^N \theta_i(\tau), \\ T &\equiv \frac{2\pi}{\Omega}, \\ \langle \bar{\theta} \rangle_T &\equiv \frac{1}{T} \int_{\tau_0}^{\tau_0+T} \bar{\theta}(\tau) d\tau, \end{aligned}$$

and consider the steady (long-time) periodic state. We define the complex first-harmonic coefficient at the drive frequency as

$$A_1(\sigma; \Omega) \equiv \frac{2}{T} \int_{\tau_0}^{\tau_0+T} [\bar{\theta}(\tau) - \langle \bar{\theta} \rangle_T] e^{-i\Omega\tau} d\tau, \quad (8)$$

and set the first-harmonic gain to

$$\mathcal{G}(\sigma; \Omega) \equiv \frac{|A_1(\sigma; \Omega)|}{\rho}. \quad (9)$$

This definition is independent of the choice of the phase origin τ_0 once transients have decayed.

Equivalently, based on the real sine or cosine projections,

$$\begin{aligned} a(\sigma; \Omega) &= \frac{2}{T} \int_{\tau_0}^{\tau_0+T} [\bar{\theta}(\tau) - \langle \bar{\theta} \rangle_T] \sin(\Omega\tau) d\tau, \\ b(\sigma; \Omega) &= \frac{2}{T} \int_{\tau_0}^{\tau_0+T} [\bar{\theta}(\tau) - \langle \bar{\theta} \rangle_T] \cos(\Omega\tau) d\tau, \end{aligned} \quad (10)$$

one can write $A_1 = b - ia$, so that $|A_1| = \sqrt{a^2 + b^2}$ and

$$\mathcal{G}(\sigma; \Omega) \equiv \frac{|A_1(\sigma; \Omega)|}{\rho} = \frac{1}{\rho} \sqrt{a(\sigma; \Omega)^2 + b(\sigma; \Omega)^2}. \quad (11)$$

We use $\mathcal{G}$ as a metric for the network response to the external drive both in the context of the analytic discussion (Sec. III A) and in numerical simulations (Secs. II C and III B).

C. Numerical methods

To quantify the effect of diversity on the dynamics of our surfactant-laden interface model, we numerically integrate Eq. (7) on a periodic lattice of size $L_x \times L_y = 25 \times 40$ (so $N = L_x L_y = 1000$). Time integration, performed by a code written in JULIA [45], uses an implicit Rosenbrock method with adaptive stepping (RODAS5, as implemented in OrdinaryDiffEq.jl [46]), with tolerances `reltol` = 10^{-6} and `abstol` = 10^{-8} . Simulations start from weakly randomized initial conditions to seed nontrivial spatial structure for

the diffusive coupling. After discarding an initial transient, we record the spatial mean $\bar{\theta}(\tau) \equiv \frac{1}{N} \sum_{i=1}^N \theta_i(\tau)$ over a measurement window of length $T_{\text{meas}} = n_{\text{per}} T$ with $n_{\text{per}} \in \mathbb{N}$, where $T \equiv 2\pi/\Omega$ is the drive period. Denoting the time average over this window by

$$\langle \bar{\theta} \rangle_{T_{\text{meas}}} \equiv \frac{1}{T_{\text{meas}}} \int_{\tau_0}^{\tau_0 + T_{\text{meas}}} \bar{\theta}(\tau) d\tau, \quad (12)$$

we compute the complex first-harmonic coefficient exactly as defined in Sec. II B:

$$A_1(\sigma; \Omega) = \frac{2}{T_{\text{meas}}} \int_{\tau_0}^{\tau_0 + T_{\text{meas}}} (\bar{\theta}(\tau) - \langle \bar{\theta} \rangle_{T_{\text{meas}}}) e^{-i\Omega\tau} d\tau, \quad (13)$$

$$\mathcal{G}(\sigma; \Omega) = \frac{|A_1(\sigma; \Omega)|}{\rho}. \quad (13)$$

These integrals are approximated by the trapezoidal rule on a uniform output grid that includes both end points. Let $\tau_k = \tau_0 + k\Delta\tau$ for $k = 0, \dots, M-1$, with $\Delta\tau = T_{\text{meas}}/(M-1)$, and weights $w_0 = w_{M-1} = \frac{1}{2}$ and $w_k = 1$ otherwise. Then

$$\langle \bar{\theta} \rangle_{T_{\text{meas}}} \approx \frac{\Delta\tau}{T_{\text{meas}}} \sum_{k=0}^{M-1} w_k \bar{\theta}(\tau_k), \quad (14)$$

$$A_1(\sigma; \Omega) \approx \frac{2\Delta\tau}{T_{\text{meas}}} \sum_{k=0}^{M-1} w_k [\bar{\theta}(\tau_k) - \langle \bar{\theta} \rangle_{T_{\text{meas}}}] e^{-i\Omega\tau_k}. \quad (15)$$

Choosing $T_{\text{meas}} = n_{\text{per}} T$ makes the projection insensitive to the phase origin τ_0 in the steady periodic regime.

III. RESULTS

A. Analytic discussion

1. Homogeneous case

In the homogeneous system, where all α_i 's are identical, $\alpha_i = \alpha$ and $\theta_i = \theta$ ($i = 1, \dots, N$), the coupling term vanishes and we can study the single equation

$$\frac{d\theta}{d\tau} = \alpha[1 + \rho \sin(\Omega\tau)](1 - \theta) e^{-\frac{1}{2}A\theta} - \theta e^{\frac{1}{2}A\theta}. \quad (16)$$

In the absence of external forcing ($\rho = 0$), the fixed points θ^* of Eq. (16) satisfy the equation

$$\alpha(1 - \theta^*) e^{-\frac{A}{2}\theta^*} = \theta^* e^{\frac{A}{2}\theta^*} \implies \alpha e^{-\frac{A}{2}\theta^*} = \frac{\theta^*}{1 - \theta^*} e^{\frac{A}{2}\theta^*}. \quad (17)$$

From Eq. (17) we obtain

$$\ln \alpha = \ln \frac{\theta^*}{1 - \theta^*} + A\theta^*, \quad (18)$$

which is the Frumkin isotherm in nondimensional form. Its derivative, which we will use later, is therefore given by

$$\frac{d \ln \alpha}{d\theta} = \frac{1}{\theta(1 - \theta)} + A. \quad (19)$$

Linearizing about θ^* with $\theta = \theta^* + x$ and keeping first order in x and ρ [neglecting $O(\rho x)$] we obtain

$$\dot{x}(\tau) = -\lambda x(\tau) + \rho G \sin(\Omega\tau), \quad (20)$$

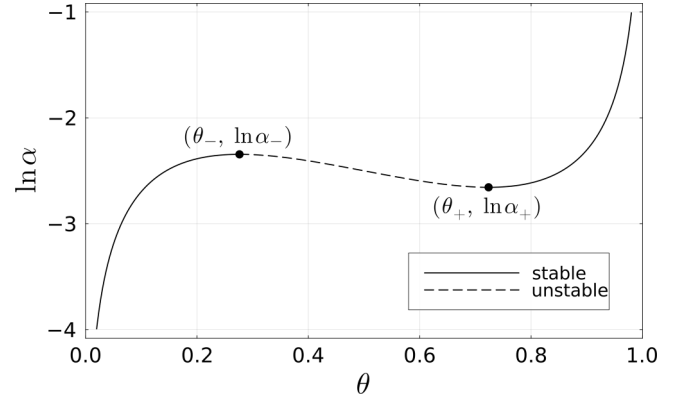


FIG. 2. Frumkin isotherm $\ln \alpha(\theta) = \ln(\theta/(1 - \theta)) + A\theta$ for $A = -5$. The outer branches are linearly stable (solid); the middle branch is unstable (dashed). Filled dots mark the turning points $(\theta_{\pm}, \ln \alpha_{\pm})$ where $d \ln \alpha / d\theta = 0$. θ is the surface coverage, α is the dimensionless adsorption drive [$\alpha \equiv (k_a c_0)/k_d$], and A is the Frumkin interaction parameter.

where

$$\lambda = e^{\frac{A}{2}\theta^*} \left(\frac{1}{1 - \theta^*} + A\theta^* \right), \quad G = \theta^* e^{\frac{A}{2}\theta^*}. \quad (21)$$

λ is the linear relaxation rate at fixed point θ^* . G converts a unit input amplitude ρ into an additive forcing $\rho G \sin(\Omega\tau)$ in the linearized equation at the operating point.

We solve Eq. (20) in the steady (long-time) periodic regime and write the solution as $x(\tau) = a \sin(\Omega\tau) + b \cos(\Omega\tau)$; the real Fourier coefficients are

$$a = \rho G \frac{\lambda}{\lambda^2 + \Omega^2}, \quad b = -\rho G \frac{\Omega}{\lambda^2 + \Omega^2}. \quad (22)$$

Using the definition in Eq. (9) with $\bar{\theta}(\tau) \equiv \theta(\tau)$ (all sites identical), the gain $\mathcal{G}$ for the homogeneous system is evaluated as

$$\mathcal{G}(0; \Omega) = \frac{1}{\rho} \sqrt{a^2 + b^2} = \frac{G(\theta^*; A)}{\sqrt{\lambda(\theta^*; A)^2 + \Omega^2}}. \quad (23)$$

In the low-frequency limit ($\Omega \ll \lambda$), Eq. (23) reduces to

$$\mathcal{G}(0; \Omega) \simeq \frac{G(\theta^*; A)}{\lambda(\theta^*; A)} = \frac{1}{\frac{1}{\theta^*(1 - \theta^*)} + A} = \frac{1}{\frac{d \ln \alpha}{d\theta}}, \quad (24)$$

where, in the last equality, we used the comparison with Eq. (19).

Upon plotting the curve $\ln \alpha(\theta)$ (see Fig. 2), we can observe that it is S shaped and has two folds at the turning points $\theta_- < \frac{1}{2} < \theta_+$, defined by

$$\left. \frac{d \ln \alpha}{d\theta} \right|_{\theta=\theta_t} = 0 \iff \frac{1}{\theta_t(1 - \theta_t)} + A = 0, \quad (25)$$

and equivalently characterized by the vanishing of the relaxation rate,

$$\lambda(\theta_t; A) = e^{\frac{A}{2}\theta_t} \left(\frac{1}{1 - \theta_t} + A\theta_t \right) = 0. \quad (26)$$

Here, with θ_i we have indicated the generic turning point $\theta_i \in \{\theta_-, \theta_+\}$. Equation (25) provides

$$\theta_{\pm} = \frac{1 \pm \sqrt{1 + \frac{4}{A}}}{2}, \quad \alpha_{\pm} = \frac{\theta_{\pm}}{1 - \theta_{\pm}} e^{A\theta_{\pm}}, \quad (27)$$

showing that $\theta_{\pm}$ are real for $A \leq -4$. It should also be noted that $\alpha_+ \leq \alpha_-$ for $A \leq -4$.

The turning points divide the isotherm into three segments, as shown in Fig. 2. The outer segments $\theta \in (0, \theta_-)$ and $\theta \in [\theta_+, 1)$ are linearly stable (here $\lambda > 0$), which means that small perturbations x decay exponentially, $x(\tau) \sim e^{-\lambda\tau}$, so $\theta(\tau)$ relaxes back to θ^* . Instead, the middle segment $\theta \in (\theta_-, \theta_+)$ is unstable (here $\lambda < 0$), meaning that small perturbations grow, so $\theta(\tau)$ moves away from θ^* toward another (stable) equilibrium.

Examining Eq. (24) together with the general expression (23), one sees that the gain of the homogeneous system grows as the linear relaxation rate $\lambda(\theta^*; A)$ decreases, and becomes very large when the operating point approaches a turning point from a stable branch ($\lambda \rightarrow 0^+$). This yields two qualitatively distinct amplification routes:

(i) When the span of the drive covers the entire bistable window, $\alpha(1 - \rho) \leq \alpha_+$ and $\alpha(1 + \rho) \geq \alpha_-$, the system switches branches once per half cycle. If the drive is slow enough for relaxation between crossings [e.g., $\Omega \lesssim \lambda_{\text{typ}}$, with $\lambda_{\text{typ}} := \min\{\lambda(\theta^*[\alpha(1 - \rho)]), \lambda(\theta^*[\alpha(1 + \rho)])\}$ being a representative rate along the two stable segments visited], the switching is coherent and produces large time-domain excursions.

(ii) Even when the drive does not straddle the unstable segment $[\alpha_+, \alpha_-]$, a large first-harmonic response arises if the driven trajectory $\alpha[1 + \rho \sin(\Omega\tau)]$ passes sufficiently close to a turning point while remaining on a stable branch. In this case $\mathcal{G}$ increases as λ decreases and, for fixed Ω , saturates at $\mathcal{G} \approx G(\theta^*; A)/\Omega$ when $\lambda \ll \Omega$.

In summary, the response peaks when the drive rate is comparable to the effective relaxation rate that controls the dominant contribution: the interbranch relaxation in route (i) or the near-fold trajectory in route (ii).

2. Heterogeneous case

Let α_i vary across sites, with $\mathbb{E}[\alpha_i] = \bar{\alpha}$ and $\text{Var}(\alpha_i) = \sigma^2$. Through the isotherm this induces equilibria θ_i^* and thus site-wise rates $\lambda_i = \lambda(\theta_i^*; A)$ and couplings $G_i = G(\theta_i^*; A)$. For each site, the small-signal coefficients (a_i, b_i) are given by (22) with $(\lambda, G) \mapsto (\lambda_i, G_i)$. Averaging yields

$$\bar{a} = \rho \left\langle G_i \frac{\lambda_i}{\lambda_i^2 + \Omega^2} \right\rangle, \quad \bar{b} = -\rho \left\langle G_i \frac{\Omega}{\lambda_i^2 + \Omega^2} \right\rangle, \quad (28)$$

and therefore, by (9),

$$\begin{aligned} \mathcal{G}(\sigma; \Omega) &= \frac{1}{\rho} \sqrt{\bar{a}^2 + \bar{b}^2} \\ &= \left[\left\langle G_i \frac{\lambda_i}{\lambda_i^2 + \Omega^2} \right\rangle^2 + \left\langle G_i \frac{\Omega}{\lambda_i^2 + \Omega^2} \right\rangle^2 \right]^{1/2}. \end{aligned} \quad (29)$$

The $\sigma = 0$ baseline (no diversity) corresponds to the homogeneous system, with

$$\begin{aligned} \ln \bar{\alpha} &= \ln \frac{\bar{\theta}^*}{1 - \bar{\theta}^*} + A\bar{\theta}^*, \\ \lambda_0 &\equiv \lambda(\bar{\theta}^*; A), \\ G_0 &\equiv G(\bar{\theta}^*; A). \end{aligned}$$

Writing $\lambda_i = \lambda_0 + \delta\lambda_i$ and $G_i = G_0 + \delta G_i$ with $\overline{\delta\lambda_i} = \overline{\delta G_i} = 0$ and $\overline{\delta\lambda_i^2} = O(\sigma^2)$, a second-order Taylor expansion of (28) gives

$$\begin{aligned} \bar{a} &\approx \rho \left(G_0 g_1(\lambda_0) + \frac{1}{2} G_0 g_1''(\lambda_0) \overline{\delta\lambda_i^2} \right), \\ \bar{b} &\approx -\rho \left(G_0 g_2(\lambda_0) + \frac{1}{2} G_0 g_2''(\lambda_0) \overline{\delta\lambda_i^2} \right), \end{aligned}$$

where

$$\begin{aligned} g_1(\lambda) &= \frac{\lambda}{\lambda^2 + \Omega^2}, \quad g_2(\lambda) = \frac{\Omega}{\lambda^2 + \Omega^2}, \\ g_1''(\lambda) &= \frac{2\lambda(\lambda^2 - 3\Omega^2)}{(\lambda^2 + \Omega^2)^3}, \quad g_2''(\lambda) = \frac{2\Omega(3\lambda^2 - \Omega^2)}{(\lambda^2 + \Omega^2)^3}. \end{aligned}$$

Since g_1 is convex for $\lambda \geq \sqrt{3}\Omega$ and g_2 for $\lambda \geq \Omega/\sqrt{3}$, a convenient sufficient condition placing both averages in a convex regime already at $\sigma = 0$ is

$$\lambda_0 \geq \sqrt{3}\Omega, \quad (30)$$

with $\lambda_0 \equiv \lambda(\bar{\theta}^*; A)$. Under (30), the first nontrivial correction to $\mathcal{G}$ is positive and of order $O(\sigma^2)$:

$$\mathcal{G}(\sigma; \Omega) = \mathcal{G}(0; \Omega) + c(\Omega) \overline{\delta\lambda_i^2} + o(\sigma^2), \quad c(\Omega) > 0,$$

where $c(\Omega) = \mathcal{G}(0; \Omega) (\lambda_0^2 - \Omega^2)/(\lambda_0^2 + \Omega^2)^2$.

As σ grows further, two things happen: (i) dephasing increases (many network elements enter $\lambda_i \lesssim \Omega$; phase lags $\rightarrow \pi/2$), which reduces the mean first harmonic, and (ii) non-linear clipping or hysteresis sets in for elements whose driven span intersects the unstable branch $[\alpha_+, \alpha_-]$, i.e., $\alpha_i(1 - \rho) < \alpha_+$ and $\alpha_i(1 + \rho) > \alpha_-$, reducing their first-harmonic amplitude below its linear-response value. The DIR peak arises from the initial enhancement of the gain followed by its reduction as σ increases.

It follows from the above analysis that there are two qualitative routes by which diversity can produce an interior maximum of the gain as σ increases (DIR), depending on how the driven trajectory samples the vicinity of the folds. The two routes are completely analogous to those seen in Sec. III A 1 for the homogeneous case.

(i) If the drive span covers the entire bistable window, $\bar{\alpha}(1 - \rho) \leq \alpha_+$ and $\bar{\alpha}(1 + \rho) \geq \alpha_-$, a non-negligible fraction of sites crosses from one stable branch to the other once per half cycle. When the external rate is not too fast relative to the relevant relaxation rate along the visited branches [e.g., $\Omega \lesssim \lambda_{\text{typ}}$, with $\lambda_{\text{typ}} := \min\{\lambda(\theta^*[\bar{\alpha}(1 - \rho)]), \lambda(\theta^*[\bar{\alpha}(1 + \rho)])\}$], switching remains coherent and yields a large excursion of the mean field. In the small- σ regime, Eq. (29) combined with the convexity condition $\lambda_0 \geq \sqrt{3}\Omega$ [Eq. (30)] guarantees a positive $O(\sigma^2)$ increment of $\mathcal{G}$, before dephasing and clipping or hysteresis eventually reduce the first harmonic at larger σ .

(ii) Even when the mean drive does not straddle $[\alpha_+, \alpha_-]$, diversity can populate sites whose operating points lie near a turning point while staying on a stable branch. For such sites $\lambda_i \rightarrow 0^+$, so in Eq. (29) the term $\langle G_i \Omega / (\lambda_i^2 + \Omega^2) \rangle$ grows as more weight accumulates at small λ_i ; in the quasiadiabatic regime this produces a large first harmonic that, for fixed Ω , saturates at $\sim \langle G_i \rangle / \Omega$ as $\lambda_i \ll \Omega$ (compare this with the homogeneous expression $\mathcal{G}(0; \Omega) = G / \sqrt{\lambda^2 + \Omega^2}$). Thus, for sufficiently small Ω , reducing ρ below full straddling while increasing σ to place a moderate fraction of sites near a fold can yield an even larger gain than in the switching case, before dephasing at very large σ diminishes the response.

B. Numerical simulations

1. Choice of simulation parameters

Unless otherwise noted, the following parameter values of Eq. (7) are used in all simulations.

As indicated in Sec. II A, we assume Frumkin interactions to be equally split between adsorption and desorption; therefore $s = \frac{1}{2}$.

To ensure that the turning points of the isotherm are real [Eq. (27)], we set the Frumkin interaction parameter $A = -5$. This is a standard strong-interaction choice just beyond the Fowler-Guggenheim threshold $|A| \gtrsim 4$ [47,48]. For this value of A , the turning points of the isotherm (Fig. 2) are

$$\begin{aligned}\theta_- &= \frac{1}{2} \left(1 - \sqrt{1 + \frac{4}{A}} \right) \approx 0.276, \\ \theta_+ &= \frac{1}{2} \left(1 + \sqrt{1 + \frac{4}{A}} \right) \approx 0.724,\end{aligned}$$

which represent mid-to-high interface coverages typical of efficient interfacial-tension reduction [44,49]. The corresponding $\alpha_{\pm}$ values are

$$\begin{aligned}\alpha_-(A) &= \frac{\theta_-}{1 - \theta_-} e^{A\theta_-} \approx 0.0959, \\ \alpha_+(A) &= \frac{\theta_+}{1 - \theta_+} e^{A\theta_+} \approx 0.0703.\end{aligned}$$

We set the mean adsorption drive at the midpoint,

$$\bar{\alpha} = \frac{1}{2}[\alpha_+(A) + \alpha_-(A)] \approx 0.0831,$$

and choose the drive amplitude ρ so that $[\bar{\alpha}(1 - \rho), \bar{\alpha}(1 + \rho)]$ contains the entire window with a 5% safety margin, yielding $\rho \approx 0.20$. This corresponds to a $\approx \pm 20\%$ oscillatory forcing of the adsorption drive about $\bar{\alpha}$, in line with amplitudes used in dynamic interfacial forcing experiments [50,51]. We set $\rho = 0.20$ in most simulations reported in the present work, except in Sec. III B 5, where we study the impact of smaller ρ values.

The adsorption parameters α_i , which encode diversity, are drawn from a log-normal distribution with $\ln \alpha_i \sim \mathcal{N}(\mu, \sigma^2)$, where $\mu = \ln \bar{\alpha} - \frac{1}{2}\sigma^2$, so that $\mathbb{E}[\alpha_i] = \bar{\alpha}$. Thus $\alpha_i > 0$, and we use σ as our diversity parameter.

The remaining parameters, i.e., the coupling constant δ and the driving frequency Ω , are varied along with diversity σ in the simulations to study their effect on network response.

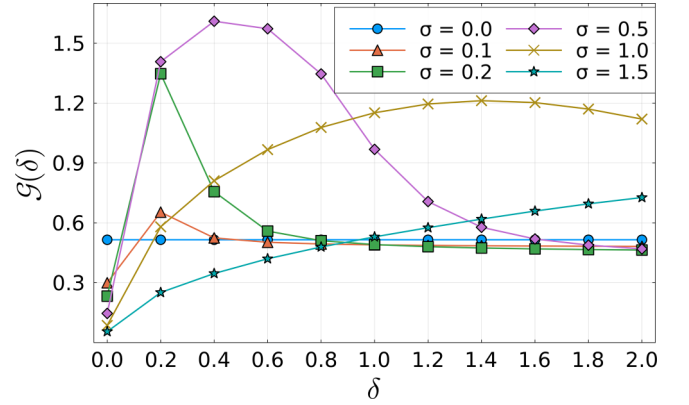


FIG. 3. Gain $\mathcal{G}$ as a function of the coupling constant δ for different values of diversity σ . The lines are guides for the eye.

2. Determining the optimal coupling strength

As an initial step, we compute the gain $\mathcal{G}$ as a function of the coupling strength $\delta \in [0, 2]$ across multiple diversity levels σ while holding all other parameters fixed, in order to select a single optimal value of δ for subsequent simulations. As shown in Fig. 3, for every $\sigma > 0$ there exists an optimal coupling $\delta^*(\sigma)$ that maximizes the gain $\mathcal{G}$, and δ^* tends to shift to higher values as σ increases. The corresponding peak gain $\mathcal{G}_{\max}(\sigma)$ increases with σ up to $\sigma = 0.5$ and then declines. This behavior is consistent with the fact that moderate coupling promotes synchronization and amplifies the global response, whereas overly strong coupling diminishes it as it makes the network excessively rigid.

Hereafter, we use $\delta = 0.40$ [the optimal coupling at $\sigma = 0.5$, $\delta^*(0.5)$] as the coupling strength that produces the most efficient network response in the studied σ range.

3. Gain as a function of diversity at different driving frequencies

With the coupling fixed to the above value $\delta = 0.40$, the simulation results reported in Fig. 4 quantify how the response gain $\mathcal{G}$ varies with the diversity σ for three driving frequencies, $\Omega = \{0.006, 0.06, 0.6\}$. To tie the dimensionless drive Ω to a physically meaningful actuation, we fix the mechanical drive frequency to $f = 1$ Hz (recall that

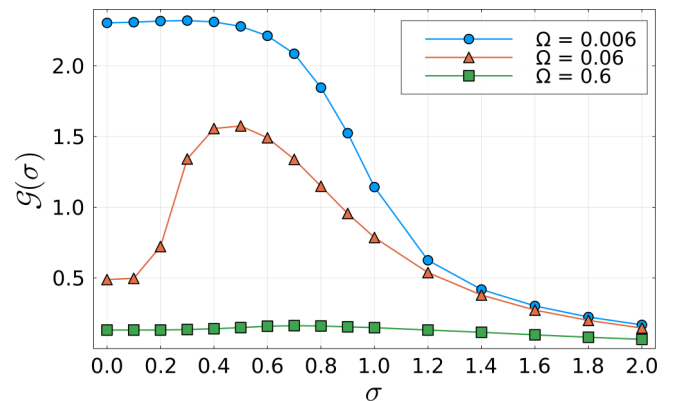


FIG. 4. Gain $\mathcal{G}$ as a function of diversity σ for different values of the driving frequency Ω . The lines are guides for the eye.

$\Omega \equiv \frac{\omega}{k_d} = \frac{2\pi f}{k_d}$). With this value of f , the three dimensionless drives $\Omega = \{0.006, 0.06, 0.6\}$ map to intrinsic relaxation rates $k_d = 2\pi f/\Omega \approx \{10^3, 10^2, 10^1\} \text{ s}^{-1}$. These rates span regimes that have been documented in the literature across different interfaces: rapid adsorption or relaxation at freshly created gas-liquid and liquid-liquid interfaces typically occurs on $\mathcal{O}(1\text{--}100)\text{ ms}$ to $\mathcal{O}(1)\text{ s}$, depending on surfactant chemistry and concentration [52–54], whereas at solid-liquid interfaces, attenuated total reflectance - Fourier transform infrared spectroscopy (ATR-FTIR) studies often report exchange or desorption on the order of 1 s^{-1} [55]. Thus our three k_d values ($\sim 10^3, 10^2, 10^1 \text{ s}^{-1}$) deliberately bracket fast, intermediate, and slower interfacial kinetics while keeping $f = 1 \text{ Hz}$ within the standard frequency band used in dynamic interfacial tension and oscillatory dilational interfacial rheology measurements [27,28].

The choice of three decade-spaced driving frequencies allows us to sample as many distinct dynamical regimes relative to the network's intrinsic relaxation time. Specifically, $\Omega = 0.006$ probes the quasiadiabatic limit, $\Omega = 0.06$ targets an intermediate regime where entrainment and DIR are expected to be most pronounced, and $\Omega = 0.6$ tests the fast-forcing regime where synchronization is more difficult and attenuation should dominate.

Looking at the plots in Fig. 4, two clear trends emerge. First, the diversity level σ^* that maximizes the gain $\mathcal{G}$ shifts to larger values as Ω increases. Second, the maximal attainable gain decreases markedly with Ω .

For the slow drive $\Omega = 0.006$, $\mathcal{G}(\sigma)$ is largest near homogeneity and then decays with increasing σ : it forms a broad plateau up to $\sigma^* = 0.3$ (where it reaches its maximum value) and then decreases roughly monotonically. Under an intermediate drive $\Omega = 0.06$, $\mathcal{G}$ exhibits a clear DIR peak: it rises from ~ 0.5 at $\sigma = 0$ to a distinct peak $\mathcal{G}_{\text{max}} \approx 1.55$ at $\sigma^* = 0.5$ (with a relative uplift of about 3 times versus $\sigma = 0$), and subsequently declines for larger σ . For the fast drive $\Omega = 0.6$, the response remains small overall; however, a shallow DIR peak can still be observed with a maximum at $\sigma^* = 0.7$, corresponding to a $\mathcal{G}$ increase of $\approx 20\%$ with respect to the homogeneous system ($\sigma = 0$).

These observations are consistent with the following picture. At very low $\Omega = 0.006$, the network can adiabatically follow the forcing; while a small degree of spatial disorder is not detrimental up to $\sigma^* = 0.3$, additional diversity primarily disrupts phase coherence and reduces the collective response. At intermediate $\Omega = 0.06$, the forcing is faster than the typical linear relaxation rate of a homogeneous population ($\Omega \gtrsim \lambda_0$), so a network made of identical units responds weakly and with a relatively large phase lag. Introducing a moderate degree of diversity creates a subpopulation of fast, highly susceptible units whose individual relaxation rate λ_i satisfies $\lambda_i \gtrsim \Omega$. These units are entrained by the drive with small lag and, through coupling, reduce the effective threshold of their neighbors. The resulting leader-follower cascades repeat once per cycle and compress the phase distribution, yielding a coherent macroscopic oscillation and the maximal gain $\mathcal{G}$ at $\sigma^* \approx 0.5$. For larger σ , the spread of amplitudes and phase lags becomes too broad: leaders are too sparse and many units lie outside the entrainable band ($\lambda_i \ll \Omega$), so coupling

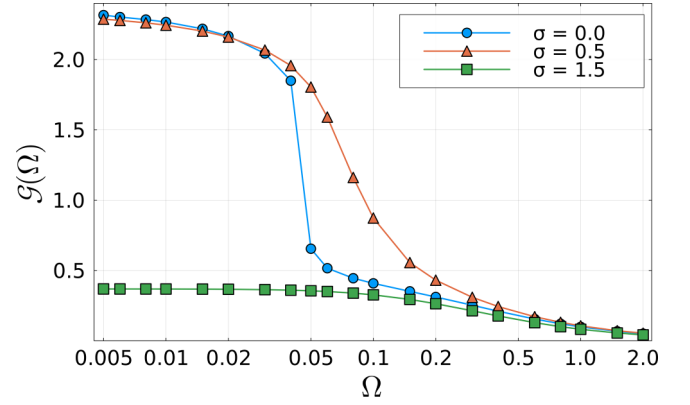


FIG. 5. Gain $\mathcal{G}$ as a function of the driving frequency Ω for different values of diversity σ . The x -axis scale is logarithmic. The lines are guides for the eye.

cannot recruit them within a single period. The resulting phase cancellation lowers $\mathcal{G}$, producing the right shoulder of the DIR curve. Finally, at high $\Omega = 0.6$, the drive is too rapid to entrain the network effectively, and heterogeneity can only partially compensate for this, so $\mathcal{G}$ remains low in absolute terms and shows a modest DIR pattern.

4. Gain as a function of the driving frequency at different diversity values

Complementary to the previous analysis of $\mathcal{G}(\sigma)$ at fixed Ω , we now examine how the response gain varies with the driving frequency for three representative diversity levels, $\sigma = \{0, 0.5, 1.5\}$, keeping the coupling fixed at $\delta = 0.40$. As before, $\Omega \equiv 2\pi f/k_d$ [Eq. (4)], and we fix $f = 1 \text{ Hz}$.

The three curves in Fig. 5 show a clear, frequency-dependent separation of behaviors across diversity levels. In the low-frequency (quasiadiabatic) regime, $\Omega \lesssim 2 \times 10^{-2}$, both the homogeneous network ($\sigma = 0$) and the moderately heterogeneous network ($\sigma = 0.5$) display large gains $\mathcal{G} \approx 2.2\text{--}2.3$, indicating near-adiabatic tracking of the forcing. In contrast, strong heterogeneity ($\sigma = 1.5$) suppresses the response to a low, nearly flat plateau $\mathcal{G} \approx 0.35$, consistent with phase dispersion dominating over any recruitment benefits at large σ .

At intermediate frequencies, $3 \times 10^{-2} \lesssim \Omega \lesssim 2 \times 10^{-1}$, the three curves differ most markedly. The homogeneous system exhibits a sharp fall, dropping from $\mathcal{G} \approx 1.8$ at $\Omega \approx 0.04$ to $\lesssim 0.6$ at $\Omega \approx 0.05$. Moderate diversity ($\sigma = 0.5$) substantially extends the high-gain band, increasing system robustness: $\mathcal{G}$ remains $\gtrsim 2$ up to $\Omega \approx 0.05$, is still ~ 1.6 at $\Omega \approx 0.07$, and stays close to ~ 1.0 at $\Omega = 0.1$. Thus, relative to $\sigma = 0$, $\sigma = 0.5$ improves the midband gain by factors of $\sim 2\text{--}3\times$. The highly diverse population ($\sigma = 1.5$) remains low throughout this range, with $\mathcal{G}$ hovering near $\sim 0.3\text{--}0.35$ before beginning a gradual decline. At high frequencies, $\Omega \gtrsim 0.5$, all curves decay toward small gains as the external drive outpaces the network's intrinsic relaxation; differences between $\sigma = 0$ and $\sigma = 0.5$ narrow, and $\sigma = 1.5$ remains the lowest. By $\Omega = 1$ one finds $\mathcal{G} \lesssim 0.1$ for all three cases.

These frequency-response trends align with the picture developed for diversity-induced recruitment. At low Ω , most

units can track the drive and coupling promotes phase alignment; large σ mainly injects phase disorder, depressing $\mathcal{G}$. In the intermediate band, Ω exceeds the typical linear relaxation rate of a homogeneous population, so identical units respond weakly and with a large lag. Moderate diversity populates a subset with $\lambda_i \gtrsim \Omega$ that the drive can entrain nearly in phase; through coupling, these leader units lower neighbors' effective activation barrier each cycle, compressing the phase distribution and sustaining a high coherent response. Excessive diversity, however, broadens lags and amplitudes to the point that leader influence is diluted, yielding the uniformly small $\mathcal{G}$ observed for $\sigma = 1.5$.

Overall, these results show that diversity can provide a broader and more robust response across varying operating conditions, even when the maximal gain is slightly below the single-point optimum, like in the low-frequency regime.

5. Drive-amplitude dependence

In Secs. III B 2–III B 4, we study the dependence of $\mathcal{G}$ on coupling strength, diversity, and driving frequency, while keeping the drive amplitude fixed at $\rho = 0.20$. This corresponds to the situation in which the drive span covers the entire bistable window, i.e., $\bar{\alpha}(1 - \rho) \leq \alpha_+$ and $\bar{\alpha}(1 + \rho) \geq \alpha_-$ (see Sec. III A 2). We now repeat the σ sweeps using two smaller drive amplitudes, i.e., $\rho = 0.10$ and $\rho = 0.05$, which are both unable to straddle the interval $[\alpha_+, \alpha_-]$. In line with the analysis developed in Sec. III A 2, the results reveal a different behavior with respect to the case $\rho = 0.20$, as illustrated in Fig. 6.

At $\Omega = 0.006$ [adiabatic regime, Fig. 6(a)], reducing ρ shifts the DIR maximum to slightly larger σ (from 0.3 to 0.4–0.6) and significantly increases the peak gain from $\mathcal{G}^* \approx 2.3$ to 3.3–4.2. This occurs because, when the mean drive no longer covers the entire distance between the two stable branches of the isotherm, diversity can still provide network elements with α_i values corresponding to the vicinity of a turning point of the isotherm, where the derivative $\frac{d \ln \alpha}{d \theta}$ tends to zero [Eqs. (20) and (21)] and the gain diverges. This yields a large first-harmonic response even in the absence of full switching between the stable branches. At $\Omega = 0.06$ [finite-rate entrainment, Fig. 6(b)], the peak weakens to $\mathcal{G}^* \approx 1.0$ –1.2 while shifting to larger σ (~ 0.7). This is consistent with the fact that, at smaller ρ values, the fraction of sites satisfying $\lambda_i \gtrsim \Omega$ is progressively reduced. At high frequency, $\Omega = 0.6$ [Fig. 6(c)], all curves remain low and only weakly dependent on ρ .

These additional results confirm the qualitative trends predicted by the analytic discussion in Sec. III A 2 and reinforce the general observation that diversity enables a significantly increased and more robust network response compared to a homogeneous system, across a wide range of network configurations and operating conditions.

6. Experimental validation

In a dynamic interfacial tension experiment the observable is the time series of interfacial tension, $\gamma(t)$, under a sinusoidal drive at frequency f . Therefore, these types of measurements lend themselves well to a comparison of experimental data with the model presented in this study. For instance, one may compute the first harmonic of γ , which we

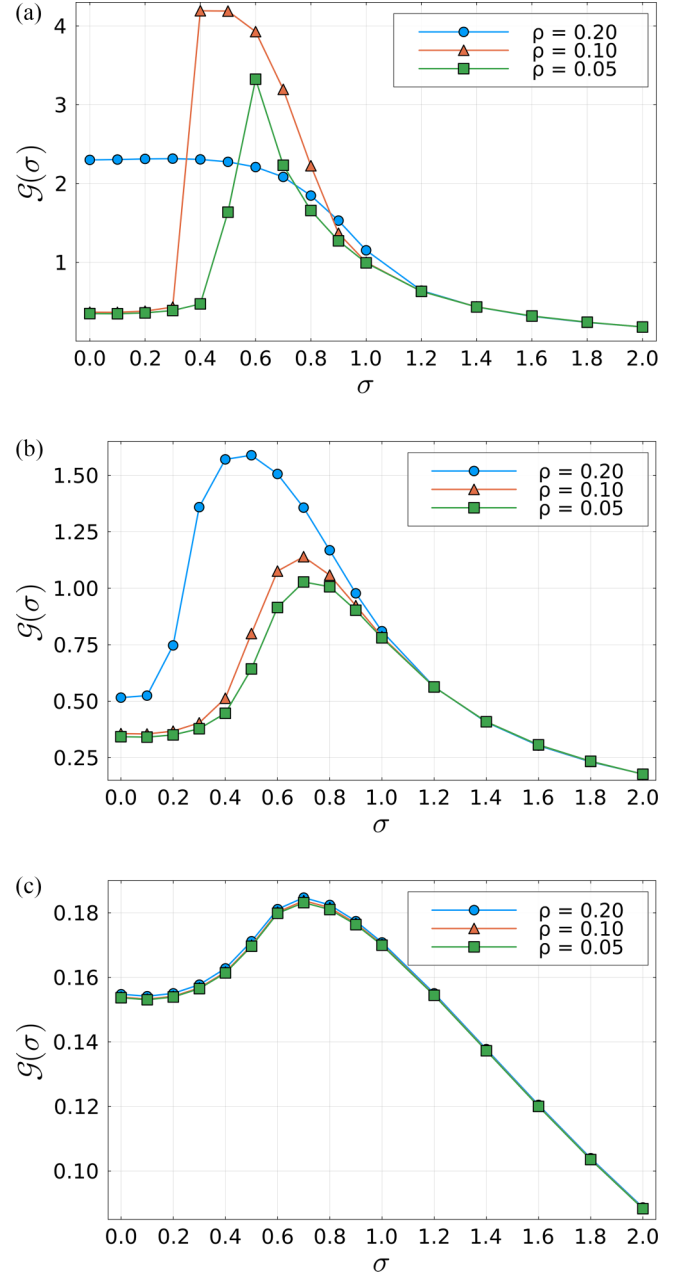


FIG. 6. Gain $\mathcal{G}$ as a function of diversity σ for a series of ρ (drive amplitude) values, at different frequencies Ω : (a) $\Omega = 0.006$; (b) $\Omega = 0.06$; (c) $\Omega = 0.6$. The lines are guides for the eye.

name A_1^γ , exactly as defined for $\bar{\theta}$ in Eqs. (8)–(10), replacing $\bar{\theta}(\tau)$ with $\gamma(t)$ and projecting at the drive frequency, and then normalize by the drive amplitude ρ :

$$A_1^\gamma \equiv \frac{2}{T} \int_{t_0}^{t_0+T} (\gamma(t) - \langle \gamma \rangle_T) e^{-i\omega t} dt,$$

$$\mathcal{G}_\gamma \equiv \frac{|A_1^\gamma|}{\rho}, \quad T = \frac{2\pi}{\omega}. \quad (31)$$

$\mathcal{G}_\gamma$ is the first-harmonic small-signal gain of the interfacial tension—i.e., the magnitude of the linear transfer from the imposed oscillatory drive to γ at the drive frequency. For small signals, a local linearization

$\gamma(\theta) \approx \gamma(\theta^*) + (\partial\gamma/\partial\theta)_{\theta^*}(\theta - \theta^*)$ gives

$$\mathcal{G}_\gamma \approx \left(\frac{\partial\gamma}{\partial\theta} \right)_{\theta^*} \mathcal{G}(\sigma; \Omega), \quad \Omega = \frac{\omega}{k_d} = \frac{2\pi f}{k_d}, \quad (32)$$

so the shape of $\mathcal{G}_\gamma$ versus σ or Ω should mirror the model's $\mathcal{G}(\sigma; \Omega)$.

In oscillating-drop or oscillating-bubble measurements, the external drive is the logarithmic area strain $\varepsilon(t) \equiv \Delta \ln S(t)$, where $S(t)$ is the surface area. Let A_1^ε be the complex first harmonic of $\varepsilon(t)$ obtained by the same projection used above for A_1^γ , and set the input amplitude $\rho := |A_1^\varepsilon|$. Then the first-harmonic gain of the interfacial tension is

$$\mathcal{G}_\gamma = \frac{|A_1^\gamma|}{|A_1^\varepsilon|} = |E^*|, \quad (33)$$

where $|E^*|$ is by definition the magnitude of the complex dilational modulus $E^* \equiv \frac{A_1^\gamma}{A_1^\varepsilon} = E' + iE''$, with E' and E'' being the elastic and viscous dilational modulus, respectively.

In a tailored experiment, diversity could be introduced into the system and varied by using a mixture of two surfactants with different adsorption kinetics (corresponding to different α_i values in the model) and changing their relative mole fraction. For instance, one may use a combination of two homologous surfactants with different chain lengths. By coadjusting the total monomer concentration c_0 [recall $\alpha \equiv \frac{k_a c_0}{k_d}$, Eq. (4)], it is possible to vary σ while keeping $\bar{\alpha}$ fixed.

There is experimental evidence in the literature that DIR can occur at surfactant-laden interfaces. For instance, Loglio *et al.* [56] studied the dynamic interfacial response of mixed adsorption layers of TTAB (tetradecyl trimethyl ammonium bromide) and C₁₃DMPO (tridecyl dimethyl phosphine oxide) at the water/hexane interface using oscillating-drop measurements (20 °C, strain amplitude $\approx 10\%$). At fixed TTAB concentration (45 μM), they varied C₁₃DMPO from 0.4 to 22 μM and reported the magnitude of the complex dilational modulus $|E^*|$ and the phase shift versus frequency. Across all compositions, $|E^*|$ increases with frequency; moreover, at any given frequency $|E^*|$ grows as C₁₃DMPO is increased from 0.4 to 8.0 μM , but then decreases at 22 μM , revealing a clear nonmonotonic dependence on composition. The phase shift likewise rises with frequency and shows composition-dependent crossovers: mixtures with intermediate C₁₃DMPO generally exhibit larger phase lags than very low or very high C₁₃DMPO, especially at higher frequencies. This interior maximum of $|E^*|$ with composition—together with the phase behavior—is consistent with a resonancelike effect due to optimal heterogeneity, as expected under the DIR mechanism.

Hsieh *et al.* [57] studied a binary system in which oil-soluble palmitic acid (PA) accesses the interface from the oil phase while the cationic surfactant TTAB (tetradecyl trimethyl ammonium bromide) arrives from water. They defined “dilational modulus synergism” and showed that at pH 7—when PA is mostly deprotonated—mixtures display simultaneous synergistic lowering of interfacial tension and an increase of the storage modulus E' , whereas no synergism is observed at pH 3. They attributed this behavior to electrostatic attraction and interfacial catanionic complexation between PA[−] and TTAB⁺, which strengthens the interfacial layer [57]. In our framework, ionization and complexation shift the mean

and broaden the spread of the local drive $\alpha_i = k_{a,i}c_{0,i}/k_{d,i}$ and modify λ_i , yielding a subset of patches with $\lambda_i \gtrsim \Omega$ that entrains neighbors and amplifies the first harmonic—i.e., the DIR mechanism. The absence of synergism at low pH is then consistent with reduced charge-mediated heterogeneity and weaker coupling.

Interior maxima of $|E^*|$ as a function of composition can, in principle, emerge also in multimode mixture theories that include coupled relaxation pathways, as discussed by Loglio *et al.* [56]. Our square-lattice, microreversible Frumkin model introduces a unique mechanistic route to explain these patterns: diversity creates multiple coupled relaxations whose rates and weights vary smoothly with composition, so that at intermediate diversity their contributions overlap and amplify the first-harmonic response. Notably, in our model, this is achieved without resorting to empirical, composition-specific time constants or extra relaxation pathways.

IV. CONCLUSIONS

We showed the occurrence in interfacial adsorption phenomena of diversity-induced resonance (DIR) [1]: a signal amplification mechanism well known in complex network physics that we extend here to interface science. To do this, we introduced a square-lattice model of a surfactant-laden interface based on laterally coupled, heterogeneous patches obeying Frumkin kinetics and driven by a sinusoidal concentration modulation: at a suitable, intermediate level of heterogeneity, the model interface response in terms of surface coverage exhibits a maximum, indicating a coherent, in-phase behavior of the patch network. We demonstrated the existence of this maximum and studied its dependence on key system parameters both analytically and via numerical simulations. From a broader perspective, the present work provides a physically grounded soft-matter realization of the DIR mechanism: the network units and their nonlinearity are not phenomenological oscillators but adsorption patches obeying microreversible Frumkin kinetics, and the coupling is a coarse-grained representation of lateral exchange along the interface.

Specifically, two amplification routes emerged and were traced to interfacial adsorption physics: (i) coherent branch switching when the drive span straddles the bistable window of the Frumkin isotherm, and (ii) near-fold amplification when trajectories graze a turning point on a stable branch. In both situations, moderate diversity significantly enhances (up to a factor of 3–4) the first-harmonic gain of the spatially averaged surface coverage and broadens the high-gain band relative to homogeneous interfaces. These findings constitute an explicit formulation of DIR at fluid interfaces, extending the original DIR concept to interfacial adsorption systems grounded in Frumkin thermodynamics [32,33].

Our central hypothesis is that physicochemical diversity of the interface structure—originating, for example, from surfactant mixtures or local environment variability—can, via lateral coupling, enhance the coherent interfacial response to periodic actuation through the DIR mechanism. While the present study is theoretical, we showed evidence of the occurrence of DIR in published experimental data [56,57] and provided a metric to observe DIR through dynamic

interfacial tension measures. More broadly, we emphasize that surfactant-laden interfaces provide a concrete soft-matter platform in which DIR can, in principle, be realized and tested experimentally: the nonlinear element dynamics are standard interfacial adsorption kinetics, diversity corresponds to patchwise variability in well-defined physicochemical parameters (e.g., adsorption propensity), coupling arises from lateral exchange along the interface, and the macroscopic response is directly measurable in dynamic surface-tension and dilational-rheology experiments. In this sense, our model provides a physically grounded realization of DIR beyond idealized oscillator networks and complements the still relatively limited set of engineered physical systems where constructive disorder effects have been explored.

Relative to prior literature, classical dynamic adsorption and dilational-rheology models quantify the frequency-dependent response of multicomponent interfacial layers. These models map kinetic rate limitations and cross-species coupling onto relaxation time spectra within a lattice-gas free-energy-kinetic framework. In equilibrium, this framework reduces to the Frumkin–Fowler–Guggenheim isotherm and, for mixtures, incorporates cross-interaction terms [26,32–35]. Typically, these models are spatially lumped.

Conversely, spatially resolved descriptions capture heterogeneity, domain formation, and pattern selection under spatiotemporal forcing. These include lattice-gas, Ising-type models that generate mosaic domains, continuum phase-field (Cahn–Hilliard-type) formulations that embed lateral diffusion and line tension, and mesoscale lattice-Boltzmann and lattice-gas schemes with explicit interfacial structure [22,36–40].

However, across these approaches, an unaddressed question is whether the combination of optimal diversity and lateral coupling, under the action of an external forcing, can yield a coherent interface-wide response. In this context, our patch network model establishes DIR as a macroscopic signal amplification mechanism, previously unexplored in interfacial processes. Importantly, the model does not utilize composition-specific empirical time constants or *ad hoc* relaxation pathways.

Although our numerical results are presented for a square lattice with four nearest neighbors, we expect the emergence of a DIR peak to be independent of the specific topology used. This is because the coupling enters the model only through a Laplacian operator on the coverage field. Therefore, the existence of a DIR peak is expected to be robust to reasonable changes in local connectivity (e.g., inclusion of diagonal couplings or alternative tilings), which mainly renormalize the

effective coupling scale and modify the Laplacian spectrum. A systematic study of topology dependence is beyond the scope of the present work; nonetheless, the mechanism underlying DIR here, i.e., the interplay of heterogeneity in α_i and the coupling-induced collective response under periodic forcing, does not depend on topology in a fundamental way.

In terms of future improvements, our model uses diffusive nearest-neighbor coupling and a sinusoidal modulation of $c(t)$ (concentration forcing). While this minimal Laplacian coupling captures recruitment and dephasing, it neglects additional interaction channels that may be relevant in real surfactant-laden interfaces. In particular, spatial variations in coverage can induce surface-tension gradients that drive Marangoni flows, yielding advective and potentially longer-ranged hydrodynamic coupling between patches. Moreover, the interfacial layer may exhibit finite dilational viscoelasticity, providing an additional mechanical coupling channel, especially under area-oscillation conditions. Incorporating these mechanisms—for example, by augmenting the lateral-transport term with an advective or nonlocal operator, or by coupling the coverage field to an interfacial mechanical mode—and exploring heterogeneity in additional parameters (e.g., A or k_d) represent sensible future extensions of this work, besides a direct experimental validation.

Finally, it is worth noting that disorder-enabled amplification has also been widely studied in the presence of fluctuations in the form of stochastic resonance [58]; a few examples include bistable nanomechanical resonators [59], ferroelectric systems [60], and driven quantum systems [61]. Exploring whether controlled static heterogeneity can induce DIR-like effects in these platforms—and, conversely, how thermal noise and diversity interact in interfacial adsorption—is another interesting direction for future work.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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