

Implications of the electromagnetic scale anomaly to the QCD chiral phase transition in the smaller quark mass regime: T_{pc} does not drop with eB

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The decrease of the chiral pseudocritical temperature T_{pc} with an applied strong magnetic field has been extensively investigated by various QCD low-energy effective models and lattice QCD at the physical point. We find that this decreasing feature may not hold in the case with a weak magnetic field and still depends on quark masses: when the quark masses get smaller, T_{pc} turns to increase with the weak magnetic field. This happens due to the significant electromagnetic scale anomaly contribution in the thermomagnetic medium. We demonstrate this salient feature by employing the Nambu-Jona-Lasinio model with $2 + 1$ quark flavors including the electromagnetic scale anomaly contribution. We observe that at $(m_0, m_{sc}) \simeq (2, 20)$ MeV for the isospin symmetric mass for up and down quarks, m_0 , and the strange quark mass, m_s , T_{pc} decreases with the magnetic field if the quark masses exceed the critical values, and increases as the quark masses become smaller. Related cosmological implications, arising when the supercooled electroweak phase transition or dark QCD cosmological phase transition is considered along with a primordial magnetic field, are also briefly addressed.

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I. INTRODUCTION

The chiral phase transition in QCD with a possible presence of magnetic fields has extensively been investigated in light of deeply understanding heavy ion collision experiments. External strong magnetic fields make significant effects on the hot QCD plasma and the nature of the chiral phase transition, such as what are called the reduction of the chiral pseudocritical temperature and the inverse magnetic catalysis [1–6], and so forth.

Also in the early hot Universe, the strong magnetic field could be generated via cosmological first-order phase transitions [7–14], or inflationary scenarios [15–29], which would be redshifted to a later epoch when the Universe undergoes the QCD phase transition. The QCD phase transition epoch may thus be magnetized by the redshifted primordial magnetic field, and the chiral phase transition could take place in the thermal bath with a background magnetic field coupled to the quarks. The strength of such a cosmic magnetic field background might be as small as or smaller than the intrinsic scale of QCD $\Lambda_{\text{QCD}} \sim (200\text{--}400)$ MeV. In that case, the QCD phase transition cannot simply be dominated by physics of the lowest Landau levels longitudinally polarized along the direction parallel to the magnetic field, which leads to the dimensional reduction into the $2 + 2$ system and can support interpretations on the inverse magnetic catalysis (see, e.g., [30]).

Thermal QCD with such a weak magnetic field has not yet been well explored. Lattice QCD has observed the inverse magnetic catalysis and the decrease of T_{pc} as the

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magnetic field strength eB gets larger at high enough $T = \mathcal{O}(100 \text{ MeV})$ [1–6]. On lattices, the size of eB is bounded from below as $eB \gtrsim T^2$ due to the currently applied method to create the magnetic field [1], which works only for the physical pion mass simulations at finite temperatures. Thus, applying a weak eB at high T with much smaller pion masses, particularly close to the chiral limit, is challenging due to the higher numerical cost, and it is still uncertain whether the normal or inverse magnetic catalysis persists and T_{pc} gets increased or reduced with eB .

Recently it has been clarified, based on the linear sigma model and the Nambu-Jona-Lasinio (NJL) model with the mean field approximation (MFA), that the electromagnetic scale anomaly effect is not negligible in a weak magnetic field and is shown to have important implications for the chiral phase transition properties regarding the quark mass dependence [31,32]. Without applying an external magnetic field, in the massless-two flavor case those chiral effective models (in the MFA) predict the chiral phase transition of second order, and in the massless three-flavor case, it goes like the first-order type.¹ The trend drastically gets altered, however, in the presence of a weak magnetic field [with $\sqrt{eB} \lesssim \Lambda_{\text{QCD}} \sim (200\text{--}400) \text{ MeV}$]: when eB exceeds a critical strength (depending on those models), the chiral phase transition turns to be crossoverlike in the massless-two flavor case [31], and the first-order nature is gone to be of the second order or crossoverlike also in the massless-three flavor case [32].

The disappearance of the first-order feature is tied with the crucial presence of the electromagnetic scale anomaly contributing to the chiral symmetry breaking at around the criticality. As has been emphasized in the literature [32], this trend is irrespective to details of the parameter setting of the chiral effective model, and hence is fairly generic and can be applied to a wide class of QCD-like theories, which might still hold even beyond the mean-field approximation in the NJL description: for instance, QCD can be replaced with dark QCD and the electromagnetic field with a dark photon field which is typically the portal between the dark and the Standard Model sectors. The disappearance of the first-order nature then makes crucial impacts on modeling the dark QCD scenario in light of the gravitational wave production sourced from the supercooled cosmological phase transition, such as those discussed in the literature [36–46],

as well as helps deeply understanding the chiral phase structure in ordinary QCD. Furthermore, QCD in the Standard Model with (almost) massless quarks can be viable also in the supercooled electroweak phase transition scenarios, in which QCD first triggers the electroweak symmetry breaking via the chiral phase transition along with massless six quarks [47–50].

Given such cosmological interests also at hand, in this paper, we extend the analysis in [32] by investigating the weak eB dependence on T_{pc} with the quark masses varied. We find that T_{pc} may not decrease with the magnetic field in the smaller quark mass regime. This happens due to the presence of the magnetic catalysis with a weak eB and the chiral first-order phase transition in the smaller quark mass regime. As quarks get heavier to go off the first-order regime, T_{pc} turns to decrease with eB . We demonstrate this salient feature by employing a NJL model with the lightest three flavor quarks in the MFA with electromagnetic scale anomaly.

Conventionally, the reduction of T_{pc} with respect to the magnetic field is thought to be identical to the inverse magnetic catalysis. However, the literature [5,51] suggests that at a large enough pion mass the inverse magnetic catalysis, i.e., the decrease of the quark condensate with the increase of the magnetic field will disappear, whereas the reduction of T_{pc} with the increase of the magnetic field will still persist. This might imply the universal feature of the reduction of T_{pc} in the quark mass plane, when projected onto the conventional Columbia plot [52]. However, what we argue in the present paper is to propose a counter case to this seemingly universal reduction feature, in a sense that the reduction of T_{pc} does not drop with the increase of eB if quark masses get small enough. We observe that at $(m_{0c}, m_{sc}) \simeq (2, 20) \text{ MeV}$ for the isospin symmetric up and down quark mass m_0 and the strange quark mass m_s , T_{pc} decreases with the magnetic field if the quark masses are greater than those values, and increases with the magnetic field if the quark masses are less than them.

Our finding helps deeply understand the chiral phase transition in the wider parameter space in QCD, including quark masses, temperature, and also the magnetic field strength. The increase of T_{pc} could also help investigate the cosmological consequences from the thermomagnetized QCD or dark QCD in the thermal history of the Universe, as briefly noted in the Summary and Discussion section.

II. NJL MODEL WITH ELECTROMAGNETIC SCALE ANOMALY

We start with introducing the NJL model with the $2 + 1$ flavors, which is also coupled to a constant external magnetic field. The Lagrangian describing the model for the quark triplet $\psi = (u, d, s)^T$ is written as follows:

¹Regarding the current status on the lattice QCD study with $eB = 0$, in the case with very light three flavors, the first-order scenario has not yet been excluded [33,34] for highly improved staggered fermions. Another analysis with unimproved staggered fermions suggests second-order-like phase transitions for $N_f \leq 6$ [35]. Thus the first-order nature in the case with $N_f = 3$ has not been conclusive yet even using lattice QCD.

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - \mathbf{m})\psi + \sum_{a=0}^8 G\{(\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi}i\gamma^5\lambda^a\psi)^2\} - K[\det\bar{\psi}(1 + \gamma_5)\psi + \det\bar{\psi}(1 - \gamma_5)\psi], \quad (1)$$

where the current quark mass matrix $\mathbf{m}$ takes the form $\mathbf{m} = \text{diag}\{m_0, m_0, m_s\}$, and $\lambda^a (a = 0, \dots, 8)$ represents the Gell-Mann matrices in the flavor space with $\lambda^0 = \sqrt{2/3}\text{diag}(1, 1, 1)$. The covariant derivative is defined as $D_\mu = \partial_\mu - iqA_\mu^{\text{em}}$, which includes the external electromagnetic field A_μ^{em} , with the electromagnetic charge matrix $q = \text{diag}\{q_u, q_d, q_s\} = e \cdot \text{diag}\{2/3, -1/3, -1/3\}$. The constant magnetic field B is embedded in A_μ^{em} as $A_\mu^{\text{em}} = (0, 0, Bx, 0)$. In Eq. (1), the four-fermion interaction term with the coupling G keeps the full chiral $U(3)_L \times U(3)_R$ symmetry, while the Kobayashi-Maskawa-'t Hooft determinant term $\propto K$ [53–55], which arises from the QCD instanton effect coupled to quarks, is invariant under the $SU(3)_L \times SU(3)_R \times U(1)_V$ symmetry, but is anomalous for the $U(1)_A$ symmetry.

We introduce the three-dimensional momentum cutoff Λ to regularize the NJL model in the MFA. We refer to empirical hadron observables in the isospin symmetric limit at $eB = T = 0$ [56]: the pion mass $m_\pi \simeq 135$ MeV, the pion decay constant $f_\pi = 92.4$ MeV, the η' mass $m_{\eta'} \simeq 957.8$ MeV, and the kaon mass $m_K \simeq 497.7$ MeV. This together with $m_0 = 5.5$ MeV as inputs fixes the model parameters as $\Lambda = 0.6023$ GeV, $G\Lambda^2 = 1.835$, and $K\Lambda^5 = 12.36$. Then we have $m_s \simeq 140.7$ MeV and the dynamical mass for up and down quarks, $m_{\text{dyn}} \simeq 367.7$ MeV. In the present study, we shall call this parameter set with $(m_0 = 5.5 \text{ MeV}, m_s = 140.7 \text{ MeV})$ the physical point in QCD.

Specifically important to note is that the thermal chiral phase transition in the NJL can be controlled by the following set of the parameter ratios:

$$G\Lambda^2 \simeq 1.835, \quad K\Lambda^5 \simeq 12.36, \quad \Lambda/f_\pi \simeq 6.518. \quad (2)$$

This is what is called the QCD-monitored parameter setup, which forms a wide class of QCD-like theories in the NJL-like description given by the scaling up of ordinary QCD with respect to Λ or f_π [32]. A similar parameter setting obtained based on a different regularization scheme has been employed for dark QCD scenarios using scaling up [36–38, 40, 41, 43]. Note also that as has been emphasized in [32], the present analysis is actually irrespective to the size of e itself, as the strength of the external magnetic field contributes to the chiral phase transition while preserving the gauge-invariant form $F_{12} = eB$ without separating e and B .

To evaluate the thermal and magnetic corrections to the chiral phase transition, we employ the imaginary time formalism and the Landau level decomposition with the following replacements:

$$\begin{aligned} p_0 &\leftrightarrow i\omega_N = i(2N + 1)\pi T, \\ \int \frac{d^4 p}{(2\pi)^4} &\leftrightarrow iT \sum_{N=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3} \\ &\leftrightarrow iT \sum_{N=-\infty}^{\infty} \sum_{n=0}^{\infty} \alpha_n \frac{|q_f B|}{4\pi} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} f_\Lambda(p_z, n). \end{aligned} \quad (3)$$

ω_N (with N being integers) denotes the Matsubara frequency; n represents the Landau levels (which will be summed up to 500 in the numerical analysis); $\alpha_n = 2 - \delta_{n,0}$ corresponds to the spin degeneracy factor with respect to the Landau levels. With those replacements, it is straightforward to extend the chiral phase transition in the vacuum to the case with $T \neq 0$ and $eB \neq 0$. In the present study, as seen from Eq. (3), we simply apply a conventional soft cutoff scheme [57] to regularize the integration over p_z in Eq. (3), in such a way that

$$f_\Lambda(p_z, n) = \frac{\Lambda^{10}}{\Lambda^{10} + (p_z^2 + 2n|q_f B|)^5}. \quad (4)$$

In addition to the conventional NJL model terms in Eq. (1), we incorporate the electromagnetic scale anomaly into the NJL model as in the literature [31, 32], which takes the form

$$V_{\text{eff}}^{(\text{Tad})} = -\frac{\varphi}{f_\varphi} T_\mu^\mu. \quad (5)$$

Here φ denotes the chiral-singlet part of the scalar-meson fields (like a QCD dilaton), associated with the radial component of the nonet-quark bilinear $\bar{q}_f q_{f'}$; T_μ^μ is the trace of the energy-momentum tensor, which dictates the electromagnetic scale anomaly, to be more explicitly evaluated right below; f_φ stands for the vacuum expectation value of φ , which will be explicitized below.

In the weak field limit $eB \ll \Lambda_{\text{QCD}}^2$, the trace anomaly T_μ^μ in Eq. (5) takes the form

$$\begin{aligned} T_\mu^\mu &= \frac{\beta(e)}{e^3} |eB|^2 + \frac{1}{2} N_c \sum_{f=u,d,s} \sum_{n=0}^{\infty} \alpha_n \frac{q_f^2}{e^2} M_f^2 \frac{|q_f B|}{4\pi} \\ &\times \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} F(T, eB, M_f) |eB|^2. \end{aligned} \quad (6)$$

The first term $\propto \beta(e)$ in Eq. (6) corresponds to the electromagnetic scale anomaly at the vacuum ($T = eB = 0$), which comes in there along with the β function coefficient of the electromagnetic coupling e , as is well known. At the one-loop level, the β function is given as

$$\beta(e) = \frac{e}{(4\pi)^2} \frac{4N_c}{3} \sum_f q_f^2. \quad (7)$$

The second term in Eq. (6) is the thermomagnetic part including the Fermi-Dirac thermal distribution function as (for details, see the Appendix)²

$$F(T, eB, M_f) = \frac{1}{E_{f,n}^5} \cdot \frac{-2}{\exp\left(\frac{E_{f,n}}{T}\right) + 1},$$

with $E_{f,n}^2 = p_z^2 + 2n|q_f B| + M_f^2,$ (8)

and the constituent (full) quark masses $M_{u,d,s}$ are defined as

$$\begin{aligned} M_u &= m_u - 4G\phi_u + 2K\phi_d\phi_s \equiv m_0 + \sigma_u, \\ M_d &= m_d - 4G\phi_d + 2K\phi_u\phi_s \equiv m_0 + \sigma_d, \\ M_s &= m_s - 4G\phi_s + 2K\phi_u\phi_d \equiv m_s + \sigma_s, \end{aligned} \quad (9)$$

where σ_f corresponds to the dynamical quark mass of the q_f quark, which also acts as a σ (or σ_s) meson field in the effective potential at the level of the MFA. In Eq. (9) ϕ_f stands for the quark condensate for the q_f quark, which will be evaluated as the solution of the stationary condition for the thermomagnetic potential. In terms of σ_f , the chiral-singlet scalar-meson field φ and its vacuum expectation value f_φ in Eq. (5) are expressed as

$$\frac{\varphi}{f_\varphi} = \sqrt{\frac{\sigma_u^2 + \sigma_d^2 + \sigma_s^2}{\sigma_{u0}^2 + \sigma_{d0}^2 + \sigma_{s0}^2}} = \sqrt{\frac{\sigma_u^2 + \sigma_d^2 + \sigma_s^2}{2\sigma_0^2 + \sigma_{s0}^2}}, \quad (10)$$

where σ_{f0} represents the vacuum expectation value of σ_f at $T = eB = 0$, and $\sigma_{u0} = \sigma_{d0} = \sigma_0$.

When eB gets stronger than the threshold strength, $eB \lesssim \Lambda_{\text{QCD}}^2$, the electromagnetic scale anomaly tends to be washed out, due to the fact that the dynamics of quarks are dominated by the lowest Landau level states, which are longitudinally polarized along the magnetic field direction. Therefore, the electromagnetic scale anomaly is most remarkable in a weak magnetic regime, as first clarified in the literature [31].

The constituent quark mass M_f in the second term of Eq. (6) potentially includes higher-order contributions in eB , while other possible higher-order terms in eB have been disregarded in Eq. (6). This may be a crude truncation that we have currently taken. In [58] a nonperturbative evaluation of the photon polarization function coupled to the chiral-singlet meson field has been attempted. It could

be possible to make a straightforward computation of the photon polarization in the weak magnetic field. The reliability of the present truncation prescription could be evaluated in such a way, which is to be left to a future work.

Thus, the total thermomagnetic potential in the MFA takes the form

$$\begin{aligned} \Omega(\phi_u, \phi_d, \phi_s, T, eB) &= \sum_{f=u,d,s} \Omega_f(\phi_f, T, eB) \\ &+ 2G(\phi_u^2 + \phi_d^2 + \phi_s^2) - 4K\phi_u\phi_d\phi_s \\ &+ V_{\text{eff}}^{(\text{Tad})}(M_f, T, eB), \end{aligned} \quad (11)$$

where

$$\begin{aligned} \Omega_f(\phi_f, T, eB) &= -N_c \sum_{n=0}^{\infty} \alpha_n \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \frac{|q_f B|}{2\pi} f_{\Lambda}(p_z, n) \\ &\times \left[E_{f,n} + 2T \ln \left(1 + e^{-\frac{E_{f,n}}{T}} \right) \right]. \end{aligned} \quad (12)$$

The pseudocritical temperature T_{pc} can be determined by the second-order partial derivative of the constituent-up quark mass (M_u) with respect to T , which is given as

$$\left. \frac{\partial^2 M_u}{\partial T^2} \right|_{T=T_{pc}} = 0. \quad (13)$$

III. PSEUDOCRITICAL TEMPERATURE T_{pc}

In this section, we show that due to the presence of the tadpole potential in Eq. (5), T_{pc} does not drop with eB when quark masses get small enough compared to the physical point values.

A. Analysis at physical point: T_{pc} drops with eB

See Fig. 1, where we plot M_u and its first derivative with respect to T , $\frac{dM_u}{dT}$, as a function of T , with varying eB in the case with or without the tadpole contribution in Eq. (6). When the tadpole contribution in Eq. (6) is not taken into account, M_u increases slightly with eB in the weak eB range, and T_{pc} also grows with eB (see dotted curves in the left panel of the figure). When the tadpole term in Eq. (6) is turned on, M_u is significantly lifted up by increasing eB at high temperatures ($T \gtrsim T_{pc}$), while M_u gets almost unchanged at low temperatures (see the solid curves in the left panel). This is due to the additional chiral-explicit breaking effect induced and enhanced by the thermomagnetic tadpole term (the second term) in Eq. (6). Thus, this trend can lead to the reduction of T_{pc} with the increase of eB (solid curves in the right panel), which is in contrast to the case without the tadpole term in Eq. (6) (dotted curves in the right panel). We have checked that similar trends are observed also for the constituent mass of the down quark, M_d , at the physical point. The constituent mass of the

²In [31,32] the thermomagnetic part in the second term of Eq. (6) was evaluated by introducing the typical magnetic field strength $\sqrt{eB}$ as the infrared-loop momentum cutoff, so that the overall weak eB dependence was reduced by 1 order of magnitude. In the present study, we have refined the evaluation without introduction of such a cutoff, so that the overall eB dependence is of quadratic order as seen in Eq. (6) and the kernel function $F(T, eB, M_f)$ has simply been set by the form of the thermal distribution function as in Eq. (8).

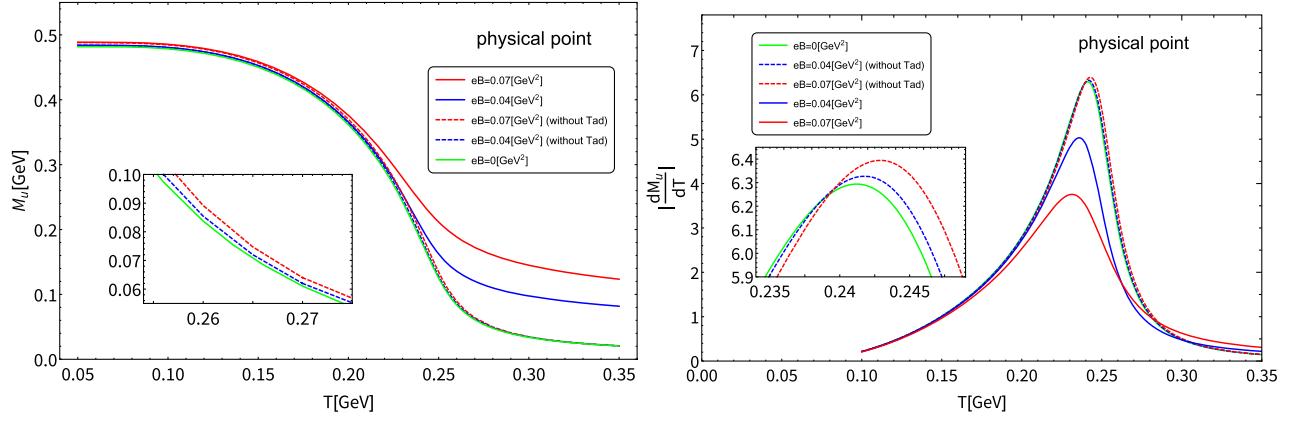


FIG. 1. Left panel: T dependence of M_u at the physical point with varying eB in the case with (solid curves) or without (dotted curves) the tadpole term in Eq. (6). Right panel: the same eB dependence on $\frac{dM_u}{dT}$, in which T_{pc} is defined at the extremum as in Eq. (13).

strange quark, M_s , behaves with T in a way different from M_u and M_d , and does not fast damp with T , simply because of its heaviness, and hence is not essential in discussing the (pseudo)criticality of the chiral symmetry restoration. Therefore, it is not the focus of the present paper and is to be disregarded in the present discussion.

B. T_{pc} does not drop with eB in the smaller quark mass range

In Fig. 2 we show $T_{pc}(eB)$, normalized to $T_{pc}(0)$ [$T_{pc}(eB)$ at $eB = 0$], as a function of eB with various quark masses. As clearly seen from the figure, T_{pc} does not drop with eB for smaller quark masses unlike the case at the physical point, but rather tends to increase with eB as quark masses get smaller. We have observed the critical point around $m_0 = 0.002$ GeV and $m_s = 0.02$ GeV. This implies that the expected universal drop trend of T_{pc} with the increase eB may not be seen in the small quark mass regime. Note that since the thermomagnetic potential in Eq. (11) is quark-flavor universal up to the electromagnetic charge difference, the currently observed trend is almost flavor universal.

To achieve a more quantitative agreement between the model and lattice data on the quark condensates, one may consider a smooth and simple extrapolation from lattice results into a strong eB regime to the weak eB regime. To this end, the present NJL model needs to be extended by including a running effect associated with an intrinsic eB and/or T dependence into the four-fermion coupling G . To this end, we may refer to the literature [59] and introduce the fitting function of G for the three-flavor model case as

$$G(B) = G \left(\frac{1 + a\xi^2 + b\xi^3}{1 + c\xi^2 + d\xi^4} \right), \quad (14)$$

with $\xi = eB/\Lambda_{\text{QCD}}^2$. The fitting parameters introduced above are fixed as follows [59]:

$$\begin{aligned} a &= 0.0108805, & b &= -1.0133 \times 10^{-4}, \\ c &= 0.02228, & d &= 1.84558 \times 10^{-4}, \end{aligned} \quad (15)$$

(for $\Lambda_{\text{QCD}} = 300$ MeV) so that the lattice data on the crossover of the averaged lightest-flavor quark condensate $(\langle \bar{u}u \rangle + \langle \bar{d}d \rangle)/2$ [60], including the inverse magnetic catalysis, can even quantitatively be produced well [59]. It has also been shown that thermodynamic quantities such as pressure, entropy density, and equation of state, can show eB and T dependence consistent with the lattice data [3].

When the fitting to the lattice data in the strong eB regime is simply extrapolated to a weaker eB regime ($eB \lesssim 0.07$ GeV^2 as in Fig. 2), the running G in Eq. (14) will almost become constant in eB . Hence, no sizable contribution to trigger the inverse phenomenon will be left in the weak eB regime. Also as to T_{pc} , the size of the correction is expected to be small enough not to alter the

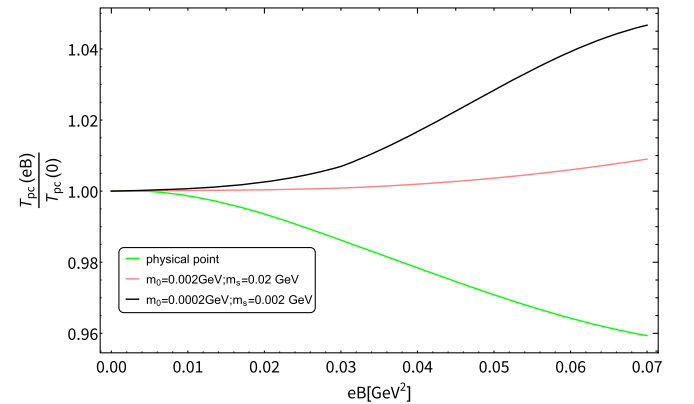


FIG. 2. The plot on the eB dependence of T_{pc} , normalized to T_{pc} at the vacuum ($eB = 0$), with varying quark masses. T_{pc} decreases with eB if the quark masses are greater than the critical values, and turns to increase when the quark masses get smaller than the critical ones.

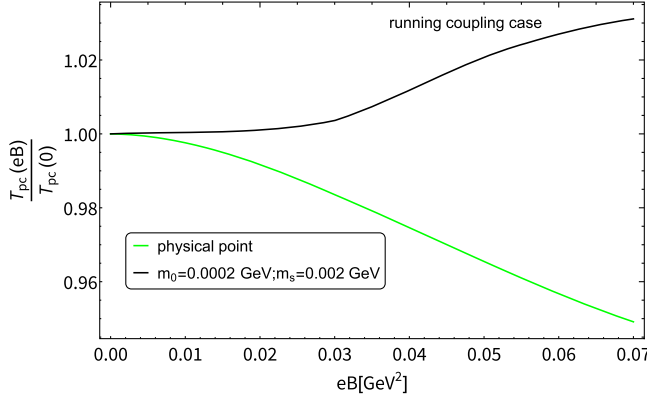


FIG. 3. The plots on T_{pc} versus eB varying the quark mass values, using the running G in Eq. (14) extrapolated from a strong eB to a weak eB regime based on the fitting parameter values listed in Eq. (15). The essentially same trend has been observed as predicted in the case without the running effect, in Fig. 2, in agreement with the analytic argument in the text: the electromagnetic scale anomaly is the crucial contribution to make T_{pc} highly dependent on m .

qualitative features as has been observed in Fig. 2, as long as the electromagnetic scale anomaly term is the dominant source to provide the eB dependence with M , as in the case with the vacuum place of M . In Fig. 3 we plot T_{pc} versus eB varying the quark mass values, for the cases with the running G extrapolated from a strong eB to a weak eB regime based on the fitting parameters in Eq. (15). The trend of nondropping T_{pc} still shows up for smaller quark masses in a way qualitatively similar to the case with the constant G in Fig. 2. At the physical point, the drop rate and grow rates of T_{pc} from $eB = 0$ to $eB = 0.07 \text{ GeV}^2$ get smaller by $\sim 1\%$.

IV. SUMMARY AND DISCUSSIONS

In summary, the drop trend of T_{pc} with respect to the increase of eB may not be universal for all the QCD parameter space. Employing the $2 + 1$ flavor NJL model in the MFA with the electromagnetic scale anomaly included [Eq. (6)], we have observed that T_{pc} with the weak magnetic field ($eB \lesssim \Lambda_{\text{QCD}}^2$) gets increased in a smaller quark mass regime, which is signaled by the significance of the thermomagnetic contribution to the electromagnetic scale anomaly (see Figs. 1–3).

We have also checked that this nondrop trend will actually be unseen in the case of the two-flavor NJL model: T_{pc} decreases with increasing eB even in the small quark mass regime. This is simply because of somewhat small contributions to the electromagnetic scale anomaly, compared to the three-flavor case.

Including the Polyakov loop variable into the NJL model, the tadpole potential term induced from the electromagnetic scale anomaly would get the Polyakov loop

dependence via the Fermi-Dirac distribution function part in F in the thermomagnetic term [see Eq. (8)]. Actually, this extension involves a theoretically nontrivial issue on the correlation between the scale anomaly and confinement, and hence would provide more phenomenological consequences other than the physics solely on T_{pc} . This issue is thus to be left to be pursued for another publication.

Lattice simulations have been systematically investigated for $2 + 1$ flavors chiral properties in thermal QCD [61] and in the strong magnetic field [4,62,63]. Currently, the lower bound of the magnetic field strength that lattice QCD can create at the physical point is $\sim 100 \text{ MeV}$ [1]. The functional renormalization group approach [64,65] has also developed methods to study the QCD phase diagram in the absence of the magnetic field close to the chiral limit. We expect that the enhanced trend of T_{pc} with respect to a weak magnetic field in the range of smaller quark masses can be cross-checked by those methods beyond the MFA in the future.

The present study reveals that depending on the size of the background field eB , the weakly magnetized QCD plasma with smaller quark masses may not keep the reduction trend on the chiral pseudocritical temperature T_{pc} with respect to eB . The phase transition becomes first order when $eB \lesssim 0.04 \text{ GeV}^2$, i.e., $T_{pc} = T_c$, where the electromagnetic scale anomaly magnifies T_c by about 2%, as has been observed in Figs. 2 and 3. This enhancement has not been found in the previous study [32]. We shall briefly address phenomenological impacts of this enhancement.

As has been noted in the discussion around Eq. (2), our present analysis and results can be applied also to a wide class of QCD-like theories, such as dark QCD and a nonstandard cosmological scenario having the supercooled electroweak phase transition [47–50]. In the latter case, the electroweak broken vacuum is to be created at temperatures much lower than the typical electroweak scale due to the supercooling triggered by some beyond the Standard Model, so that quarks would still remain massless until the dynamical chiral symmetry breaking in QCD takes place at around $T = \mathcal{O}(100 \text{ MeV})$. The vacuum expectation value of the Higgs field is temporarily governed by the dynamical quark condensate—hence the quarks get smaller current quark masses than the standard ones—and finally reaches the electroweak scale after the beyond the Standard Model sector finalizes the supercooling (when the associated potential barrier goes away). It has recently been argued, based on the ladder Schwinger-Dyson approach, that the QCD chiral phase transition in this type of scenario takes place along with keeping all six quarks light enough at the QCD scale [66]. In particular, the QCD-driven Higgs-vacuum expectation value yields $m_t \sim 100 \text{ MeV}$, and others $\lesssim \mathcal{O}(\text{MeV})$ depending on the hierarchical Yukawa couplings, and the QCD chiral phase transition is suggested to be first order in the framework of the

MFA [66]. Given this point, recall that the thermomagnetic part of the electromagnetic scale anomaly term is almost flavor universal and simply scales with the number of light quark flavors, N_f [see Eq. (6)]. Thus, when a redshifted primordial magnetic field is present in the QCD phase transition epoch, keep the eB strength weak enough that the electromagnetic scale anomaly is not washed out, the currently gained result implies an increased critical temperature for the supercooled electroweak phase transition in the scenarios of this class. This might also indicate a larger nucleation/percolation temperature for the supercooling cosmological phase transition when the created potential barrier does not substantially get affected by the weak eB compared to the case without eB . This might also affect the gravitational wave production in these types of scenarios. If the nucleation/percolation temperature gets enhanced as well as T_c by about 2% as noted above, the peak frequency of the produced gravitational wave would be shifted to higher by the same amount, as long as the strength of the magnetic field is small enough not to spoil the creation and nucleation of isotropic bubbles by the induced inhomogeneity. This shift may or may not be tiny or less sensitive to the predicted gravitational spectra, which highly depends on the modeling of the supercooling extension of the Standard Model. More on such cosmological implications would be noteworthy to pursue in another publication. Although the quantitative impact is still unclear at this point, at any rate, the present study has proposed a nontrivial indication arising from the electromagnetic scale anomaly to the prediction related to the gravitational wave production or the supercooling phase transition.

Regarding the quark mass and eB dependence in light of the impact on the cosmological dark QCD scenarios, note that the method currently employed is comparable in reliability to those discussed in Refs. [36–38,40,41,43]. As the Wilsonian renormalization group tells that the linear sigma model belongs to the same universality class as NJL-like models, the currently derived conclusion could also be directly extended to other dark QCD scenarios [39–42,44–46,67,68] based on the linear sigma model analysis like that done in the original Pisarski and Wilczek [69]. In this sense, we may say that the present study has an adequate impact as the first step. The analysis should anyhow be improved by going beyond the MFA, which is made possible, say, by using the functional renormalization group method. It would help get a more conclusive answer to whether the chiral T_{pc} in the thermomagnetic QCD drops with eB in the smaller quark mass regime.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: COMPUTATION OF THE ELECTROMAGNETIC SCALE ANOMALY AT FERMION-ONE-LOOP LEVEL

As was discussed in the literature [70], we apply the so-called “Higgs low-energy theorem” [71] to the chiral-singlet component of the quarkonic mesons in QCD, φ , so that the trace anomaly coupled to φ is evaluated in the *soft-dilaton limit* as³

$$\mathcal{L}_{\varphi-A^{\text{em}}-A^{\text{em}}} = \frac{\varphi}{f_\varphi} \left[\left(\lim_{q \rightarrow 0} q^\lambda \frac{\partial}{\partial q^\lambda} \Pi(q) \right) \frac{1}{4} F_{\mu\nu}^2 \right]. \quad (\text{A1})$$

Here $F_{\mu\nu}$ is the field strength of the photon field (A_μ^{em}), and $\Pi(q)$ denotes the photon polarization correlator associated with the photon polarization tensor $\Pi_{\mu\nu}(q)$,

$$\Pi^{\mu\nu}(q) = i(q^2 g^{\mu\nu} - q^\mu q^\nu) \Pi(q), \quad (\text{A2})$$

with the transfer momentum q (which is $\sim m_\varphi \sim 0$ in the soft-dilaton approximation). We evaluate $\lim_{q \rightarrow 0} q^\lambda \frac{\partial}{\partial q^\lambda} \Pi(q)$ at the quark-one loop level. Since φ acts as a quark mass m in the quark propagator and $\lim_{q \rightarrow 0} q^\mu \frac{\partial}{\partial q^\mu} \Pi(q)$ is nothing but the first-order term in the Taylor expansion coefficient with respect to φ for the photon-wave-function renormalization function, i.e., $\Pi(q^2)$, we have

$$\lim_{q \rightarrow 0} q^\mu \frac{\partial}{\partial q^\mu} \Pi(q) = \lim_{q \rightarrow 0} m \frac{\partial}{\partial m} \Pi(q). \quad (\text{A3})$$

Working in the imaginary time formalism, we thus compute $\lim_{q \rightarrow 0} m \frac{\partial}{\partial m} \Pi(q)$ to find

³See also the literature [72], for the discussion on the Ward-Takahashi identities for the scale symmetry breaking to get a composite dilaton coupling to gauge bosons.

$$\begin{aligned}
\lim_{q \rightarrow 0} m \frac{\partial}{\partial m} \Pi(q) &= 16e^2 N_c \frac{m^2}{3} T \sum_N \int \frac{d^3 l}{(2\pi)^3} \frac{1}{(\omega_N^2 + E_l^2)^3} \\
&= 16e^2 N_c \frac{m^2}{3} \int \frac{d^3 l}{(2\pi)^3} \frac{3}{16E_l^5} (1 - 2f(E_l)) \\
&= \frac{6e^2 N_c}{4(4\pi)^2} + e^2 m^2 N_c \int \frac{d^3 l}{(2\pi)^3} \frac{1}{E_l^5} (-2f(E_l)) \\
&= \frac{2\beta(e)}{e} + e^2 m^2 N_c \int \frac{d^3 l}{(2\pi)^3} \frac{1}{E_l^5} (-2f(E_l)),
\end{aligned} \tag{A4}$$

where $f(E_l)$ is the Fermi-Dirac distribution function, and $f(E_l) = 1/(1 + e^{E_l/T})$ with $E_l = \sqrt{\vec{l}^2 + m^2}$. For simplicity, we have taken the electromagnetic charge of the fermion $q_f = 1$. Passing through the replacement of the three-momentum integration with the sum of the Landau levels and the residual one-dimensional integration in the z direction, one can arrive at the expression in the right-hand side of Eq. (6) for each quark contribution with the charge of q_f .

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