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We analyze the mixing between the π^0 meson and the photon in the presence of a strong uniform magnetic field, in the context of a two-flavor Nambu-Jona-Lasinio model. It is shown that only one specific photon polarization state can get mixed with the π^0 state, a fact that can be understood on general grounds. For magnetic fields with values up to $1 \text{ GeV}^2/e$, it is found that the effect of the mixing on the pion mass and pion-quark couplings is relatively small (below a level of 15%), which is at variance with previous results reported in the literature.

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I. INTRODUCTION

The response of strongly interacting matter to the presence of extremely large magnetic fields has attracted considerable attention over the past few decades [1–4]. To a large extent, this interest is driven by the important role played by these fields in the physics of the early Universe [5,6], in noncentral high-energy heavy ion collisions [7,8], and in the properties of compact stellar objects such as magnetars [9,10]. In addition, strong magnetic fields have been shown to induce several remarkable phenomena in quantum chromodynamics (QCD), including the chiral magnetic effect [11–13], the enhancement of the quark-antiquark condensate (magnetic catalysis) [14], and the reduction of critical temperatures for chiral restoration and deconfinement transitions (inverse magnetic catalysis) [15].

Given this context, the understanding of how the properties of light hadrons get modified in the presence of an intense magnetic field becomes a highly relevant task. This is, however, a nontrivial problem, since first-principle calculations require one to deal with QCD in the low-energy, nonperturbative regime. Consequently, theoretical analyses have been carried out using a variety of approaches, including lattice QCD, chiral perturbation

theory, and effective quark models for strong interactions (see Ref. [4] and references therein).

An interesting aspect to consider is the mixing between hadronic states with different spin and/or isospin. Although these mixed states are typically forbidden by isospin and angular momentum conservation, they can exist in the presence of an external magnetic field. Effects of this kind have been recently studied by the authors of the present work within the framework of the Nambu-Jona-Lasinio (NJL) model. In particular, in Refs. [16–18] it was shown that magnetic-field-induced mixing between pseudoscalar and vector/axial-vector channels tends to improve the agreement between model predictions and available Lattice QCD results.

In this work we focus on another type of mixing induced by the external magnetic field, namely the π^0 - γ mixing. The mixing between photons and low-mass bosons has been extensively studied in the past, particularly in the context of scalar particles such as axions (see, e.g., Ref. [19]). Within strong interaction physics, magnetic-field-induced π^0 - γ mixing was previously analyzed in Ref. [20] using leading order chiral perturbation theory. In that study, nonanomalous one-loop corrections to the pion mass were neglected. Under those approximations, it was found that the mixing with the photon induces a significant increase of the neutral pion mass with the magnetic field.

Motivated by these results, the aim of the present work is to reexamine the impact of π^0 - γ mixing on the neutral pion mass within the NJL model. In this framework, mesons are described as quantum fluctuations in the random phase approximation (RPA) [21–23], i.e., they arise from the

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resummation of an infinite series of quark loop diagrams. In the presence of a magnetic field, these loops have to be evaluated using magnetized quark propagators. In the terminology of Ref. [20], this corresponds to the inclusion of the nonanomalous contribution to the π^0 mass. Thus, a consistent picture requires one also to consider the presence of π^0 - γ mixing, which also arises from loops of magnetized quarks. This approach has the important advantage that both contributions to the π^0 mass are treated on an equal footing.

Regarding the model setup, special care must be taken in the regularization of ultraviolet divergences, since the presence of an external magnetic field can lead to spurious effects. In this work we adopt the magnetic-field-independent regularization (MFIR) scheme [24–26], which has proven to be particularly suitable for magnetized systems [27].

The article is organized as follows. In Sec. II. A we introduce the main theoretical framework and obtain the quadratic terms of the effective action that are relevant for the π^0 - γ mixing. The quantities of interest (mixing matrix, pion mass, pion wave function renormalization, and pion-quark couplings) are analytically calculated in Secs. II. B, II. C, and II. D. In Sec. III we present the corresponding numerical results, while in Sec. IV we summarize our work and state our main conclusions. We also include an Appendix in which we collect some mathematical expressions used for the calculation of polarization functions.

II. THEORETICAL FRAMEWORK

A. Bosonized effective action in the NJL model

Let us start by considering an NJL Lagrangian density in which u and d quarks are coupled to an electromagnetic field A_μ . One has

$$\mathcal{L} = \bar{\psi}(iD - m_c)\psi + G[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad (1)$$

where $\psi = (ud)^T$, τ_i are the usual Pauli matrices, and m_c is the current quark mass, which is assumed to be equal for u and d quarks. The interaction between the fermions and the electromagnetic field is driven by the covariant derivative

$$D_\mu = \partial_\mu + i\hat{Q}A_\mu, \quad (2)$$

where $\hat{Q} = \text{diag}(Q_u, Q_d)$, with $Q_u = 2e/3$ and $Q_d = -e/3$, e being the proton electric charge. As usual, $F^{\mu\nu}$ is the electromagnetic field strength tensor, given by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

Since we consider the presence of an external magnetic field, we write the gauge field A^μ as the sum of an external (classical) field $\mathcal{A}_{cl}^\mu$ and a dynamical electromagnetic field a^μ , i.e., $A^\mu = \mathcal{A}_{cl}^\mu + a^\mu$. As stated, we consider here the

particular case in which we have an external homogenous and static magnetic field $\vec{B}$. Without loss of generality, it can be taken to be orientated along the axis 3, or z . Notice that to write down the explicit form of $\mathcal{A}_{cl}^\mu$ one has to choose a specific gauge. However, it will be seen that our calculations turn out to be manifestly gauge independent.

Since we are interested in studying the π^0 - γ mixing, it is convenient to bosonize the fermionic theory, introducing scalar and pseudoscalar fields $\sigma(x)$ and $\vec{\pi}(x)$, respectively. After integrating out the fermion fields one gets a bosonized action given by

$$S_{\text{bos}} = -i \ln \det(iD) - \frac{1}{4G} \int d^4x [\sigma(x)\sigma(x) + \vec{\pi}(x) \cdot \vec{\pi}(x)] + S_{\text{em}}, \quad (3)$$

with

$$iD_{x,x'} = [iD_x - m_0 - \sigma(x) - i\gamma_5\vec{\tau} \cdot \vec{\pi}(x)]\delta^{(4)}(x - x'), \quad (4)$$

where a direct product to an identity matrix in color space is understood. The pure electromagnetic field contribution S_{em} is given by

$$S_{\text{em}} = -\frac{1}{4} \int d^4x F_{\mu\nu}F^{\mu\nu} = -\frac{V^{(4)}B^2}{2} + \frac{1}{2} \int d^4x a^\mu(x) [\eta_{\mu\nu} \partial_\alpha \partial^\alpha - \partial_\mu \partial_\nu] a^\nu(x), \quad (5)$$

where the first term of the last expression corresponds to the classical piece and the second one includes the propagating degrees of freedom. Since we consider fixed values of the external magnetic field, in what follows the classical contribution to S_{em} will be dropped.

We proceed by expanding the bosonized action in powers of the fluctuations of the bosonic fields around the corresponding mean field (MF) values. We assume that the fields σ have nontrivial translational invariant MF values $\bar{\sigma}$, while vacuum expectation values of the pion fields are zero. Thus, we write

$$\mathcal{D}_{x,x'} = \mathcal{D}_{x,x'}^{\text{MF}} + \delta\mathcal{D}_{x,x'}. \quad (6)$$

The MF piece is diagonal in flavor space and includes the interaction of the underlying quark fields with the classical electromagnetic field $\mathcal{A}_{cl}^\mu$. One has

$$\mathcal{D}_{x,x'}^{\text{MF}} = \text{diag}(\mathcal{D}_{x,x'}^{\text{MF},u}, \mathcal{D}_{x,x'}^{\text{MF},d}), \quad (7)$$

where

$$\mathcal{D}_{x,x'}^{\text{MF},f} = -i(i\partial_x - Q_f \mathcal{A}_{cl}(x) - M)\delta^{(4)}(x - x'), \quad (8)$$

with $f = u, d$. Here $M = m_c + \bar{\sigma}$ stands for a quark effective mass. On the other hand, the correction $\delta\mathcal{D}_{x,x'}$ in Eq. (6) reads

$$\delta\mathcal{D}_{x,x'} = i[\hat{Q}\not{x} + \delta\sigma(x) + i\gamma_5 \vec{\tau} \cdot \delta\vec{\pi}(x)]\delta^{(4)}(x-x'). \quad (9)$$

The MF action per unit volume is given by

$$\frac{S_{\text{bos}}^{\text{MF}}}{V^{(4)}} = -\frac{\bar{\sigma}^2}{4G} - \frac{iN_c}{V^{(4)}} \sum_{f=u,d} \int d^4x d^4x' \text{tr}_D \ln(S_{x,x'}^{\text{MF},f})^{-1}, \quad (10)$$

where tr_D stands for the trace over Dirac indices, and $S_{x,x'}^{\text{MF},f} = (i\mathcal{D}_{x,x'}^{\text{MF},f})^{-1}$ is the MF quark propagator in the

presence of the magnetic field. Its explicit expression can be written as

$$S_{x,y}^{\text{MF},f} = e^{i\Phi_Q(x,y)} \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \bar{S}^f(p_{\parallel}, p_{\perp}), \quad (11)$$

where $\Phi_Q(x,y)$ is the so-called Schwinger phase associated to the classical field, and $\bar{S}^f(p_{\parallel}, p_{\perp})$ is given by

$$\begin{aligned} \bar{S}^f(p_{\parallel}, p_{\perp}) = & -i \int_0^\infty d\sigma \exp \left[-i\sigma \left(M^2 - p_{\parallel}^2 + \vec{p}_{\perp}^2 \frac{\tan(\sigma B_f)}{\sigma B_f} - i\epsilon \right) \right] \\ & \times \left[(p_{\parallel} \cdot \gamma_{\parallel} + M)(1 - s_f \gamma^1 \gamma^2 \tan(\sigma B_f)) - \frac{\vec{p}_{\perp} \cdot \vec{\gamma}_{\perp}}{\cos^2(\sigma B_f)} \right], \end{aligned} \quad (12)$$

with $B_f = |BQ_f|$ and $s_f = \text{sign}(BQ_f)$. Here we have defined the “parallel” and “perpendicular” four-vectors

$$p_{\parallel}^{\mu} = (p^0, 0, 0, p^3), \quad p_{\perp}^{\mu} = (0, p^1, p^2, 0), \quad (13)$$

and equivalent definitions have been used for $\gamma_{\parallel}, \gamma_{\perp}$. The Schwinger phase, which is a gauge-dependent quantity, reads

$$\Phi_Q(x,y) = Q_f \int_x^y d\xi_{\mu} \left[\mathcal{A}_{cl}^{\mu}(\xi) + \frac{1}{2} F_{cl}^{\mu\nu} (\xi_{\nu} - y_{\nu}) \right]. \quad (14)$$

In fact, since we deal here with a neutral pion, Schwinger phases cancel out in quark loops and will play no effective role in our calculations.

Let us consider the quark-antiquark condensates $\phi_f \equiv \langle \bar{\psi}_f \psi_f \rangle$. For each flavor $f = u, d$ we have

$$\phi_f = iN_c \int \frac{d^4p}{(2\pi)^4} \text{tr}_D \bar{S}^f(p_{\parallel}, p_{\perp}), \quad (15)$$

which is clearly gauge independent. The integral in this expression is divergent and has to be properly regularized. As stated in the Introduction, we use here the magnetic field independent regularization (MFIR) scheme: for a given unregularized quantity, the corresponding (divergent) $B \rightarrow 0$ limit is subtracted and then it is added in a regularized form. Thus, the quantities can be separated into a (finite) “ $B = 0$ ” part and a “magnetic” piece. Notice that, in general, the “ $B = 0$ ” part still depends implicitly on B (e.g., through the values of the dressed quark masses M); hence, it should not be confused with the value of the studied quantity at vanishing external field. To deal with the divergent “ $B = 0$ ” terms we use here a 3D cutoff regularization. Thus, we obtain

$$\phi_f^{\text{reg}} = \phi^{0,\text{reg}} + \phi_f^{\text{mag}}, \quad (16)$$

where

$$\phi^{0,\text{reg}} = -N_c M I_1^{\text{reg}}, \quad \phi_f^{\text{mag}} = -N_c M I_{1f}^{\text{mag}}, \quad (17)$$

with

$$I_1^{\text{reg}} = \frac{\Lambda^2}{2\pi^2} \left[\sqrt{1 + R_{\Lambda}^2} + R_{\Lambda}^2 \ln \left(\frac{R_{\Lambda}}{1 + \sqrt{1 + R_{\Lambda}^2}} \right) \right], \quad (18)$$

$$I_{1f}^{\text{mag}} = \frac{B_f}{2\pi^2} \left[\ln \Gamma(x_f) - \left(x_f - \frac{1}{2} \right) \ln x_f + x_f - \frac{\ln 2\pi}{2} \right]. \quad (19)$$

Here, Λ stands for the 3D momentum cutoff, and we have defined $R_{\Lambda} = M/\Lambda$ and $x_f = M^2/(2B_f)$. The corresponding gap equation for M , obtained from $\partial S_{\text{bos}}^{\text{MF}}/\partial \bar{\sigma} = 0$, can be written as

$$M = m_c - 2G \sum_f \phi_f^{\text{reg}}. \quad (20)$$

Next we consider quadratic terms in the expansion of S_{bos} in powers of $\delta\mathcal{D}_{x,x'}$. We just concentrate in neutral pion contributions, since charged pions do not mix with the photon. Since Schwinger phases cancel out in fermion loops, the corresponding polarization functions turn out to be both gauge and translational invariant. Moreover, momentum conservation implies that they have to be diagonal in the momentum basis. After a Fourier transformation the relevant terms of the bosonized action can be written as

$$\begin{aligned}
S_{\text{bos}}^{\text{neut}} = & -\frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} [\delta\pi^0(-q) \mathcal{G}_{\pi^0\pi^0}(q) \delta\pi^0(q) \\
& + a_\mu(-q) \mathcal{G}_{\gamma\gamma}^{\mu\nu}(q) a_\nu(q) \\
& + \delta\pi^0(-q) \mathcal{G}_{\pi^0\gamma}^\mu(q) a_\mu(q) + a_\mu(-q) \mathcal{G}_{\gamma\pi^0}^\mu(q) \delta\pi^0(q)],
\end{aligned} \quad (21)$$

where

$$\mathcal{G}_{\pi^0\pi^0}(q) = \frac{1}{2G} - \sum_f \Pi_{\pi^0\pi^0}^f(q), \quad (22)$$

$$\mathcal{G}_{\gamma\gamma}^{\mu\nu}(q) = q^2 \eta^{\mu\nu} - q^\mu q^\nu - \sum_f \Pi_{\gamma\gamma}^{f,\mu\nu}(q), \quad (23)$$

$$\mathcal{G}_{\pi^0\gamma}^\mu(q) = -\sum_f \Pi_{\pi^0\gamma}^{f,\mu}(q), \quad (24)$$

$$\mathcal{G}_{\gamma\pi^0}^\mu(q) = -\sum_f \Pi_{\gamma\pi^0}^{f,\mu}(q). \quad (25)$$

In the above expressions, the polarization functions $\Pi_{PP'}^f(q)$, where $P, P' = \pi^0$, or γ , are given by

$$\Pi_{PP'}^f(q) = -iN_c \int \frac{d^4 q}{(2\pi)^4} \text{tr}_D [i\tilde{S}^f(p_\parallel^+, p_\perp^+) \Gamma_P^f i\tilde{S}^f(p_\parallel^-, p_\perp^-) \Gamma_{P'}^f], \quad (26)$$

where $\Gamma_{\pi^0}^f = i\kappa_f \gamma_5$, $\Gamma_\gamma^{f,\mu} = Q_f \gamma^\mu$, with $\kappa_u = 1$, $\kappa_d = -1$ and $p^\pm = p \pm q/2$.

Let us proceed with the evaluation of the loop integrals in Eq. (26). The expression for the π^0 - π^0 polarization function using the MFIR scheme is given, e.g., in Refs. [16,17]. One has

$$\Pi_{\pi^0\pi^0}^f(q) = C_{\pi\pi}^{0,\text{reg}}(q^2) + C_{\pi\pi}^{f,\text{mag}}(q_\parallel^2, q_\perp^2), \quad (27)$$

where

$$C_{\pi\pi}^{0,\text{reg}}(q^2) = N_c [I_1^{\text{reg}} - q^2 I_2^{\text{reg}}(q^2)], \quad (28)$$

$$\begin{aligned}
C_{\pi\pi}^{f,\text{mag}}(q_\parallel^2, q_\perp^2) = & \frac{N_c}{8\pi^2} \int_0^\infty dz \int_{-1}^1 d\beta \left\{ e^{-z\omega^f(q_\parallel^2, q_\perp^2)} \left[\left(M_f^2 + \frac{1}{z} + \frac{1-\beta^2}{4} q_\parallel^2 \right) \frac{B_f}{\tanh(zB_f)} \right. \right. \\
& \left. \left. + \frac{B_f^2}{\sinh^2(zB_f)} \left(1 + \frac{\cosh(zB_f) - \cosh(\beta zB_f)}{2B_f \sinh(zB_f)} q_\perp^2 \right) \right] - \frac{e^{-z\omega_0(q^2)}}{z} \left(M^2 + \frac{2}{z} + \frac{1-\beta^2}{4} q^2 \right) \right\}.
\end{aligned} \quad (29)$$

Here I_1^{reg} is given by Eq. (18), and we have used the definitions

$$\begin{aligned}
I_2^{\text{reg}}(q^2) = & -\frac{1}{4\pi^2} \left[\text{arcsinh}(1/R_\Lambda) - \sqrt{1/r_q - 1} \arctan \left(\frac{1}{\sqrt{(1+R_\Lambda^2)(1/r_q - 1)}} \right) \right], \\
\omega^f(q_\parallel^2, q_\perp^2) = & M^2 - \frac{1-\beta^2}{4} q_\parallel^2 - \frac{\cosh(zB_f) - \cosh(\beta zB_f)}{2zB_f \sinh(zB_f)} q_\perp^2, \\
\omega_0(q^2) = & M^2 - \frac{1-\beta^2}{4} q^2,
\end{aligned} \quad (30)$$

with $r_q = q^2/(4M^2)$. It is assumed that the pion four-momentum squared is below the threshold $q^2 = 4M^2$, which corresponds to the formation of on-shell quark pairs.

The mixing contributions to the effective action are driven by the π^0 - γ and γ - π^0 polarization functions. These are found to be given by

$$\Pi_{\pi^0\gamma}^{f,\mu}(q) = -\Pi_{\gamma\pi^0}^{f,\mu}(q) = -iC_{\pi\gamma}^f(q_\parallel^2, q_\perp^2) v^\mu, \quad (31)$$

where we have introduced a four-vector v defined by $v^\mu = (q^3, 0, 0, q^0)$, and the function $C_{\pi\gamma}^f(q_\parallel^2, q_\perp^2)$ is

given by

$$C_{\pi\gamma}^f(q_\parallel^2, q_\perp^2) = \frac{1}{8\pi^2} N_c s_f M \kappa_f Q_f B_f \int_0^\infty dz \int_{-1}^1 d\beta e^{-z\omega^f(q_\parallel^2, q_\perp^2)}. \quad (32)$$

This result is consistent with the expressions obtained in Ref. [16] for the mixing between π^0 and ρ^0 mesons in a similar framework.

Finally, we consider the γ - γ polarization function. The expression for $\Pi_{\gamma\gamma}^{f,\mu\nu}$ can be written in terms of three transverse tensors, namely

$$\begin{aligned} P_0^{\mu\nu} &= q^2 \eta^{\mu\nu} - q^\mu q^\nu, & P_1^{\mu\nu} &= q_\parallel^2 \eta_\parallel^{\mu\nu} - q_\parallel^\mu q_\parallel^\nu, \\ P_2^{\mu\nu} &= q_\perp^2 \eta_\perp^{\mu\nu} - q_\perp^\mu q_\perp^\nu, \end{aligned} \quad (33)$$

where $\eta_\parallel^{\mu\nu} = \text{diag}(1, 0, 0, -1)$, $\eta_\perp^{\mu\nu} = \text{diag}(0, -1, -1, 0)$, $\eta^{\mu\nu} = \eta_\parallel^{\mu\nu} + \eta_\perp^{\mu\nu}$. One has

$$\begin{aligned} \Pi_{\gamma\gamma}^{f,\mu\nu}(q) &= C_{\gamma\gamma,0}^{f,\text{reg}}(q_\parallel^2, q_\perp^2) P_0^{\mu\nu} + C_{\gamma\gamma,1}^f(q_\parallel^2, q_\perp^2) P_1^{\mu\nu} \\ &\quad + C_{\gamma\gamma,2}^f(q_\parallel^2, q_\perp^2) P_2^{\mu\nu}, \end{aligned} \quad (34)$$

where the regularized function $C_{\gamma\gamma,0}^{f,\text{reg}}(q_\parallel^2, q_\perp^2)$ can be written as

$$C_{\gamma\gamma,0}^{f,\text{reg}}(q_\parallel^2, q_\perp^2) = C_{\gamma\gamma,0}^{f,\text{mag}}(q_\parallel^2, q_\perp^2) + C_{\gamma\gamma,0}^{f,0,\text{reg}}(q^2). \quad (35)$$

The above functions are given by [28,29]

$$C_{\gamma\gamma,i}^f(q_\parallel^2, q_\perp^2) = -\frac{N_c Q_f^2}{16\pi^2} B_f \int_0^\infty dz \int_{-1}^1 d\beta e^{-z\omega^f(q_\parallel^2, q_\perp^2)} f_i(z, \beta), \quad i=1,2, \quad (36)$$

$$C_{\gamma\gamma,0}^{f,\text{mag}}(q_\parallel^2, q_\perp^2) = -\frac{N_c Q_f^2}{16\pi^2} \int_0^\infty dz \int_{-1}^1 d\beta \left[B_f e^{-z\omega^f(q_\parallel^2, q_\perp^2)} f_0(z, \beta) - e^{-z\omega_0(q^2)} \frac{(1-\beta^2)}{z} \right], \quad (37)$$

$$C_{\gamma\gamma,0}^{f,0,\text{reg}}(q^2) = -\frac{N_c Q_f^2}{16\pi^2} \int_0^\infty dz \int_{-1}^1 d\beta [e^{-z\omega_0(q^2)} - e^{-z\omega_0(0)}] \frac{(1-\beta^2)}{z}, \quad (38)$$

with

$$\begin{aligned} f_0(z, \beta) &= \frac{\cosh(\beta z B_f) - \beta \coth(z B_f) \sinh(\beta z B_f)}{\sinh(z B_f)}, \\ f_1(z, \beta) &= (1 - \beta^2) \coth(z B_f) - f_0(z, \beta), \\ f_2(z, \beta) &= \frac{2[\cosh(z B_f) - \cosh(\beta z B_f)]}{\sinh^3(z B_f)} - f_0(z, \beta). \end{aligned} \quad (39)$$

Concerning the regularization of the coefficient of the $P_0^{\mu\nu}$ term ($C_{\gamma\gamma,1}^f$ and $C_{\gamma\gamma,2}^f$ are finite and do not have to be regularized), we have used once again the MFIR scheme, separating a “magnetic” contribution $C_{\gamma\gamma,0}^{f,\text{mag}}$ and a “ $B=0$ ” piece [see Eqs. (37) and (38)]. Then, as shown in Eq. (38), the (divergent) “ $B=0$ ” contribution $C_{\gamma\gamma,0}^{f,0}(q^2)$ has been regularized by subtracting the value of this function at $q^2=0$. This prescription has also been used in previous works, see, e.g., Refs. [28–30]. The integral over z in the regularized expression given by Eq. (38) can be performed analytically, leading to

$$C_{\gamma\gamma,0}^{f,0,\text{reg}}(q^2) = \frac{N_c Q_f^2}{16\pi^2} \int_{-1}^1 d\beta (1 - \beta^2) \ln \left[1 - (1 - \beta^2) \frac{q^2}{4M^2} \right]. \quad (40)$$

B. Inverse propagator matrix

To determine the π^0 mass in the presence of the external magnetic field it is necessary to diagonalize the inverse propagator matrix arising from the expansion of the bosonized action. Regarding the electromagnetic field a_μ , we have to consider the two relevant photon states that contribute to the quadratic action in Eq. (21). In the presence of the external field, the corresponding polarization vectors $\epsilon^\mu(\lambda)$ are given by (see, e.g., Ref [30])

$$\begin{aligned} \epsilon^\mu(1) &= (0, q^2, -q^1, 0) / \sqrt{-q_\perp^2}, \\ \epsilon^\mu(2) &= (q^3, 0, 0, q^0) / \sqrt{q_\parallel^2}, \end{aligned} \quad (41)$$

where q is the photon four-momentum. These vectors are manifestly gauge invariant, since they can be written in terms of the electromagnetic field strength tensor and its

dual as

$$\begin{aligned}\epsilon^\mu(1) &= \pm F^{\mu\nu} q_\nu / \sqrt{F^{\sigma\alpha} F^{\sigma\beta} q_\alpha q_\beta}, \\ \epsilon^\mu(2) &= \pm \tilde{F}^{\mu\nu} q_\nu / \sqrt{\tilde{F}^{\sigma\alpha} \tilde{F}^{\sigma\beta} q_\alpha q_\beta},\end{aligned}\quad (42)$$

where $+$ and $-$ signs correspond to $B > 0$ and $B < 0$, respectively, and the dual of $F^{\mu\nu}$ is given by

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}, \quad (43)$$

with the convention $\epsilon^{0123} = 1$.

Taking into account the above expressions for the polarization functions, it is seen that the photon with polarization $\lambda = 1$ decouples, while the $\lambda = 2$ state becomes mixed with the neutral pion. The inverse propagator matrix can be written as

$$\mathcal{G} = \begin{pmatrix} \mathcal{G}_{\gamma_1\gamma_1} & 0 & 0 \\ 0 & \mathcal{G}_{\gamma_2\gamma_2} & \mathcal{G}_{\gamma_2\pi^0} \\ 0 & \mathcal{G}_{\pi^0\gamma_2} & \mathcal{G}_{\pi^0\pi^0} \end{pmatrix}, \quad (44)$$

where the expression for $\mathcal{G}_{\pi^0\pi^0}$ can be read from Eqs. (22) and (27). The other entries of the matrix are given by

$$\mathcal{G}_{\gamma_1\gamma_1} = -\left(1 - \sum_f C_{\gamma\gamma,0}^{f,\text{reg}}\right) q^2 + \sum_f C_{\gamma\gamma,2}^f q_\perp^2, \quad (45)$$

$$\mathcal{G}_{\gamma_2\gamma_2} = -\left(1 - \sum_f C_{\gamma\gamma,0}^{f,\text{reg}}\right) q^2 + \sum_f C_{\gamma\gamma,1}^f q_\parallel^2, \quad (46)$$

$$\mathcal{G}_{\gamma_2\pi^0} = -\mathcal{G}_{\pi^0\gamma_2} = i \sum_f C_{\pi\gamma}^f \sqrt{q_\parallel^2}. \quad (47)$$

There is a symmetry reason that explains why the pion just mixes with one specific photon polarization state. It is important to note that when studying the π^0 - γ mixing in the presence of an external magnetic field, we are dealing with a massive particle. Therefore, observable quantities that are meaningless for the photon become fully relevant in this case. The whole physical system is invariant under rotations around the axis 3, and consequently the third component of total angular momentum, $J_3 = (\vec{x} \times \vec{q})_3 + S_3$, has to be a good quantum number. Thus, if we let $\vec{q}_\perp = 0$, the quantum number S_3 will be a good one to characterize the pion states. In Ref. [16], when dealing with π - ρ meson mixing in the presence of a magnetic field, it is discussed how S_3 becomes a good quantum number for the ρ meson. That analysis can also be applied to the π - γ mixing studied in the present work, showing that the polarization $\epsilon^\mu(2)$ corresponds to a state with $S_3 = 0$ (hence, it can get mixed with the neutral pion), whereas

$\epsilon^\mu(1)$ corresponds to a combination of $S_3 = +1$ and $S_3 = -1$ states.

It is worth noticing that up to now we have just taken into account light quark contributions to the effective action. However, for the evaluation of γ - γ terms, all charged fermion loops should be taken into account. Thus, the sums over f in the expressions for $\mathcal{G}_{\gamma_1\gamma_1}$ and $\mathcal{G}_{\gamma_2\gamma_2}$ have to be extended with the inclusion of electron and muon loops, the contributions of heavier quark and leptons being not at all significant. For lepton contributions we have to change $N_c \rightarrow 1$ and take electric charges $Q = -e$. In particular, in the case of the electrons, the loop contribution to $\mathcal{G}_{\gamma_2\gamma_2}$ will include some imaginary part, which can be evaluated by carrying out an analytical extension of the obtained expressions (see discussion in the Appendix). However, it is found that this absorptive piece is highly suppressed by a factor m_e^2/m_π^2 , and will be neglected in our numerical calculations.

C. Neutral pion mass and wave function renormalization

The π^0 mass can be obtained by diagonalizing the $\mathcal{G}$ matrix. We introduce the states

$$\begin{aligned}\tilde{\pi}^0 &= \cos \Theta \delta\pi^0 - i \sin \Theta a^{(2)}, \\ \tilde{a}^{(2)} &= -i \sin \Theta \delta\pi^0 + \cos \Theta a^{(2)},\end{aligned}\quad (48)$$

where $a^{(2)}$ is the electromagnetic field state associated to the polarization vector $\epsilon(2)$. Then we require that the off-diagonal entries of the rotated matrix $\tilde{\mathcal{G}}$ vanish. This leads to a Θ mixing angle given by

$$\tan 2\Theta = -\frac{2i\mathcal{G}_{\pi^0\gamma_2}}{\mathcal{G}_{\gamma_2\gamma_2} - \mathcal{G}_{\pi^0\pi^0}}. \quad (49)$$

Taking into account that Lorentz invariance is broken by the external magnetic field, we define the π^0 mass by considering the $\tilde{\pi}^0$ lowest energy state, which corresponds to the pion rest frame in which $\vec{q}_\perp = 0$. The pion mass, given by the condition

$$\tilde{\mathcal{G}}_{\tilde{\pi}^0\tilde{\pi}^0}|_{(q_\parallel^2, q_\perp^2)=(m_\pi^2, 0)} = 0, \quad (50)$$

can be obtained by solving the equation

$$[\mathcal{G}_{\pi^0\pi^0}\mathcal{G}_{\gamma_2\gamma_2} - |\mathcal{G}_{\pi^0\gamma_2}|^2]_{(q_\parallel^2, q_\perp^2)=(m_\pi^2, 0)} = 0. \quad (51)$$

In general, for $q_\perp^2 = 0$ the functions $C_{\pi\pi}^{f,\text{mag}}$, $C_{\pi\gamma}^f$, $C_{\gamma\gamma,i}^f$, and $C_{\gamma\gamma,0}^{f,\text{mag}}$ in Eqs. (29), (32), (36), and (37) become simplified. The corresponding expressions are given in the Appendix.

Now, as usual, the pion field wave function has to be renormalized. This is done by considering the pion mass term in the quadratic action, given by

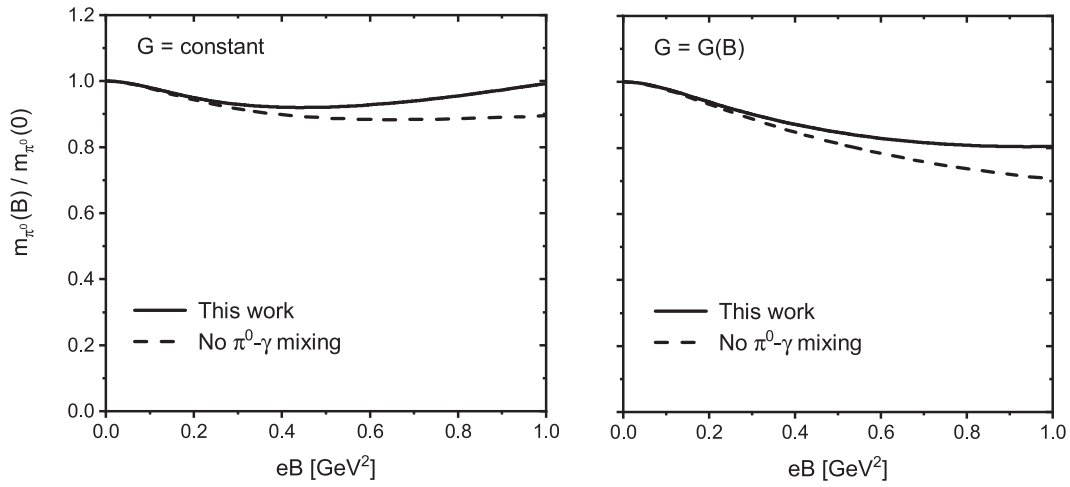


FIG. 1. π^0 mass as a function of the external magnetic field (solid lines), for the cases $G = \text{constant}$ and $G = G(B)$. Results for the models with no π^0 - γ mixing (dashed lines) are shown for comparison.

$$S_{\pi^0}^{\text{quad}} = -\frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} \tilde{\pi}^0(-q) \tilde{\mathcal{G}}_{\pi^0 \pi^0}(q_{\parallel}^2, q_{\perp}^2) \tilde{\pi}^0(q). \quad (52)$$

Expanding around the pion pole we have

$$S_{\pi^0}^{\text{quad}} = \frac{1}{2} \int \frac{d^4 q}{(2\pi)^4} \tilde{\pi}_R^0(-q) (q_{\parallel}^2 + u_{\pi^0}^2 q_{\perp}^2 - m_{\pi^0}^2) \tilde{\pi}_R^0(q), \quad (53)$$

where the pion field has been renormalized according to $\tilde{\pi}_R^0(q) = Z_{\parallel}^{-1/2} \tilde{\pi}^0(q)$, and we have used the definitions

$$Z_{\parallel, \pi^0}^{-1} = -\frac{\partial \tilde{\mathcal{G}}_{\pi^0 \pi^0}}{\partial q_{\parallel}^2} \Big|_{q_{\perp}^2=0, q_{\parallel}^2=m_{\pi^0}^2}, \quad Z_{\perp, \pi^0}^{-1} = -\frac{\partial \tilde{\mathcal{G}}_{\pi^0 \pi^0}}{\partial q_{\perp}^2} \Big|_{q_{\parallel}^2=0, q_{\perp}^2=m_{\pi^0}^2}, \quad (54)$$

$$u_{\pi^0}^2 = \frac{Z_{\parallel, \pi^0}}{Z_{\perp, \pi^0}}.$$

D. π^0 -fermion couplings

From Eq. (9) it is seen that one has two contributions to the effective $\tilde{\pi}^0$ -quark couplings. One of them arises from the standard vertex $i\gamma_5 \kappa_f$, where κ_f is associated to the light quark isospin ($\kappa_u = 1$, $\kappa_d = -1$). The second one, originated from the mixing between the neutral pion and the $a^{(2)}$ field, is given by $Q_f \gamma^\mu$, where Q_f is the quark electric charge. Notice that, as discussed in Sec. II B in relation to $\mathcal{G}_{\gamma\gamma}$ propagators, this second coupling can be extended to $Q_{\mathcal{F}} \gamma^\mu$, being $\mathcal{F}$ any charged fermion (all charged fermions will couple to the $a^{(2)}$ field). Thus, taking also into account the pion wave function renormalization, the vertex that couples the $\tilde{\pi}_R^0$ to a fermion $\mathcal{F}$ will be given by

$$iG_{\pi^0 q q} \gamma_5 \kappa_{\mathcal{F}} + iG_{\pi^0}^{\text{em}} \frac{Q_{\mathcal{F}}}{e} \epsilon_2, \quad (55)$$

where

$$G_{\pi^0 q q} = Z_{\parallel}^{1/2} \cos \Theta, \quad G_{\pi^0}^{\text{em}} = G_{\pi^0 q q} e \tan \Theta, \quad (56)$$

with $\kappa_{\mathcal{F}} = 0$ if $\mathcal{F} \neq u, d$. Notice that, from the equations in Secs. II B and II C, $\tan \Theta$ is of order e . Hence, as expected, the effective coupling $G_{\pi^0}^{\text{em}}$ is $\mathcal{O}(e^2)$.

It is clear that, in general, the existence of the π^0 - γ mixing implies that the pion mass state $\tilde{\pi}_R^0$ will interact with any charged particle through the coupling between the electromagnetic current associated to that particle and the $a^{(2)}$ component of the $\tilde{\pi}_R^0$ state.

III. NUMERICAL RESULTS

In this section we discuss the numerical results arising from our calculation of π^0 properties. For the model parameters we take the values $m_0 = 5.418$ MeV, $\Lambda = 613.4$ MeV, and $G\Lambda^2 = 2.252$, which—for a vanishing external magnetic field—correspond to a quark effective mass $M = 350$ MeV and a quark-antiquark condensate $\langle \bar{f}f \rangle = (-243.3 \text{ MeV})^3$. This parametrization, denoted as S350, reproduces adequately the empirical values of m_{π^0} and f_{π} in vacuum, namely $m_{\pi^0} = 135$ MeV and $f_{\pi} = 92.4$ MeV.

Besides the usual NJL model framework, we also consider the case in which the coupling constant G has not a fixed value but changes with the magnetic field. In fact, this can be understood as an effective way of incorporating the sea effect produced by the backreaction of gluons to magnetized quarks loops. Such a behavior has been proposed in order to achieve a good description of the inverse magnetic catalysis effect observed in lattice QCD at

finite temperature [15], which can be hardly reproduced within NJL-like models (an interesting exception is the case of models which include nonlocal interactions [31,32]). Moreover, as shown in Ref. [33], through the assumption of a magnetic-field-dependent coupling $G(B)$ one can obtain a good agreement with the behavior of the π^0 mass obtained from LQCD calculations (those calculations lead to a decrease of about 30% in m_{π^0} for $eB \simeq 1 \text{ GeV}^2$ [34,35], while a more flat behavior arises within the NJL model for a constant coupling G). In this way, we consider here a B -dependent coupling

$$G(B) = Gf(B), \quad (57)$$

where $f(0) = 1$. To set a definite form for $f(B)$ we follow Ref. [33], taking

$$f(B) = c_1 + (1 - c_1)e^{-c_2(eB)^2}, \quad (58)$$

with $c_1 = 0.321$ and $c_2 = 1.31 \text{ GeV}^{-4}$. For $B = 0$ we keep the parametrization stated above for the case of constant G , which is somewhat different from the one used in Ref. [33]. In fact, our values for the parameters are more adequate to reproduce the phenomenological values of m_π and f_π for $B = 0$, while the parametrization considered in Ref. [33] provides a better fit of LQCD data for quark condensates at nonzero B .

Our results for the π^0 mass, obtained from Eq. (51), are shown in Fig. 1. Left and right panels correspond to the models with constant and B -dependent couplings G , respectively. In the case of constant G , the behavior of the neutral pion mass with the magnetic field is non-monotonic: while a minimum is shown at $eB \simeq 0.5 \text{ GeV}^2$, the value $m_{\pi^0}(0)$ is approximately recovered for $eB \simeq 1 \text{ GeV}^2$. On the other hand, for $G = G(B)$ the value of m_{π^0} decreases steadily with B for the studied range of magnetic fields, reaching a value of about $0.8m_{\pi^0}(0)$ for $eB \simeq 1 \text{ GeV}^2$. By comparing with the results obtained for no π^0 - γ mixing (dashed lines in both panels of the figure), it is seen that in both cases the mixing leads to a relative enhancement of the π^0 mass. This enhancement gets increased with the magnetic field, reaching about 10–14% for $eB \simeq 1 \text{ GeV}^2$. It is worth noticing that the effect is much less significant than the one obtained in Ref. [20], where the mixing leads to a relative enhancement of $\sim 90\%$ in the π^0 mass for $eB \simeq 1 \text{ GeV}^2$. To have an estimation of the dependence on the NJL parameters, we have checked that our main results are qualitatively similar to those obtained considering the parametrization considered in Ref. [33].

In Fig. 2 we show the behavior of the effective π^0 -fermion couplings $G_{\pi^0 qq}$ and $G_{\pi^0}^{\text{em}}$ as functions of the magnetic field. We include in the graphs the results corresponding to the models with $G = \text{constant}$ and $G = G(B)$. In addition, for comparison we include the

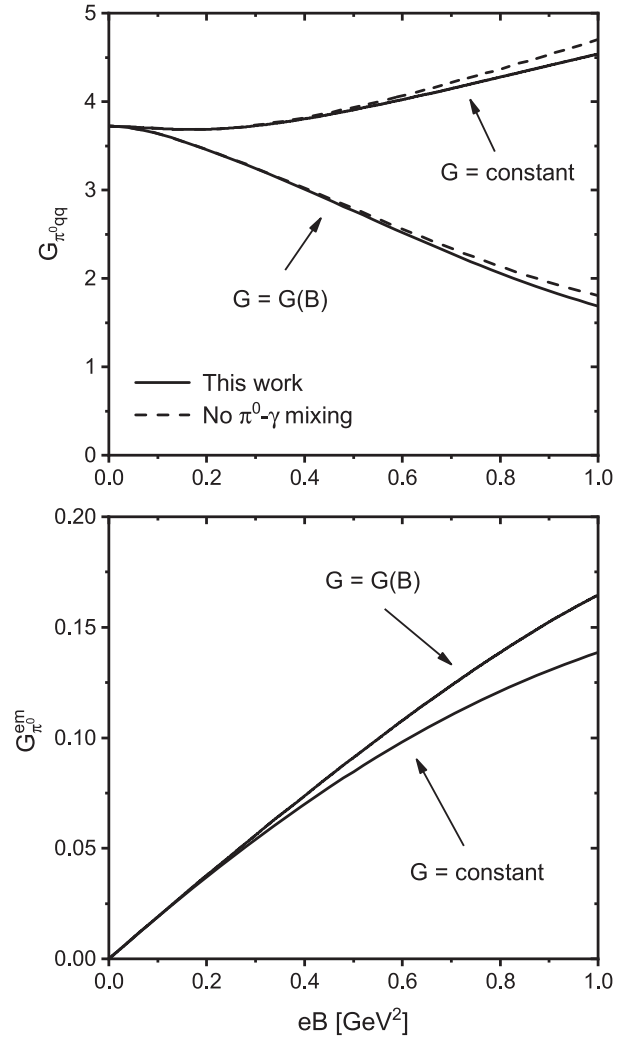


FIG. 2. $G_{\pi^0 qq}$ (top) and $G_{\pi^0}^{\text{em}}$ (bottom) effective couplings as functions of the external magnetic field, for the cases $G = \text{constant}$ and $G = G(B)$. Values of $G_{\pi^0 qq}$ for the models with no π^0 - γ mixing (dashed lines) are shown for comparison.

results for $G_{\pi^0 qq}$ in the case of no π^0 - γ mixing (dashed lines). The comparison between solid and dashed lines in the upper panel show that there is no significant effect on $G_{\pi^0 qq}$ arising from the mixing. On the other hand, it is seen that the behavior of $G_{\pi^0 qq}$ for the case of constant G is found to be opposite to the one for $G = G(B)$, in agreement with the results obtained in Ref. [33], where no π^0 - γ mixing was considered. Regarding the effective coupling $G_{\pi^0}^{\text{em}}$ (lower panel of Fig. 2), it is found that the behavior is similar for $G = \text{constant}$ and $G = G(B)$, reaching values of about 0.14 and 0.16, respectively, for $eB = 1 \text{ GeV}^2$. It is observed that the values of the coupling $G_{\pi^0}^{\text{em}}$ —which corresponds to the photon contribution to the π^0 -fermion vertex, and, therefore, is zero if there is no π^0 - γ mixing—are much smaller than those of $G_{\pi^0 qq}$ for the full considered range of values of the magnetic field.

IV. SUMMARY AND CONCLUSIONS

We have analyzed the mixing between the π_0 meson and photon fields in the presence of a strong uniform magnetic field. Our calculations have been carried out in the context of a two-flavor NJL model, considering both the cases of a constant and a B -dependent four-quark effective coupling G . The effect of the external magnetic field has been taken into account consistently in all involved polarization functions.

As a main conclusion, our numerical analysis indicates that, contrary to what has been stated in previous analyses, the effect of the π^0 - γ mixing on the mass of the π^0 meson turns out to be relatively small. For the considered range of values of eB (up to 1 GeV²), the correction with respect to the model with no mixing reaches a maximum value of about 14% for the case $G = G(B)$, and it is somewhat lower for the case of constant G . Regarding the π^0 -quark coupling $G_{\pi qq}$, the effect is even smaller, reaching maximum values of about 6% for the case $G = G(B)$. We find that these results are not expected to change qualitatively with the model parameters.

It is worth mentioning that the π^0 field mixes only with one specific photon polarization state. This can be understood from the fact that the system is invariant under rotations around the axis defined by the orientation of the

magnetic field. For a pion at rest, the total spin projection on that axis (say, axis 3) has to be a conserved quantity, and the quantum number S_3 turns out to be a good one to characterize the photon and pion states. As a consequence, it is seen that the pion can only mix with the photon polarization state $\epsilon^\mu(2)$, defined by Eq. (41).

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: INVERSE PROPAGATOR MATRIX IN THE PION REST FRAME

The integrals $C_{\pi\pi}^{f,\text{mag}}$, $C_{\pi\gamma}^f$, $C_{\gamma\gamma,i}^f$, and $C_{\gamma\gamma,0}^{f,\text{mag}}$ appearing in the inverse propagator matrix turn out to be simplified in the frame where $q_\parallel^2 = m_\pi^2$, $q_\perp^2 = 0$. From Eq. (29) we find

$$C_{\pi\pi}^{f,\text{mag}}(m_\pi^2, 0) = \frac{N_c}{8\pi^2} \int_0^\infty dz \int_{-1}^1 d\beta e^{-z\omega_0(m_\pi^2)} \times \left[\left(M^2 + \frac{1}{z} + \frac{1-\beta^2}{4} m_\pi^2 \right) \left(\frac{B_f}{\tanh(zB_f)} - \frac{1}{z} \right) + \frac{B_f^2}{\sinh^2(zB_f)} - \frac{1}{z^2} \right]. \quad (\text{A1})$$

Upon integration by parts, this quantity can alternatively be written as

$$C_{\pi\pi}^{f,\text{mag}}(m_\pi^2, 0) = N_c [I_{1f}^{\text{mag}} - m_\pi^2 I_{2f}^{\text{mag}}(m_\pi^2)], \quad (\text{A2})$$

where I_{1f}^{mag} is given by Eq. (19), while the function $I_{2f}^{\text{mag}}(q^2)$ reads

$$I_{2f}^{\text{mag}}(q^2) = -\frac{1}{16\pi^2} \int_{-1}^1 d\beta \int_0^\infty dz e^{-z\omega_0(q^2)/B_f} \left(\coth z - \frac{1}{z} \right). \quad (\text{A3})$$

If, as expected, quark effective masses M are large enough to satisfy $m_\pi < 2M$, the above integral is convergent. The integration over z can be carried out analytically, leading to

$$I_{2f}^{\text{mag}}(m_\pi^2) = \frac{1}{16\pi^2} \int_{-1}^1 d\beta \left[\psi(r_{\pi f}) - \ln r_{\pi f} + \frac{1}{2r_{\pi f}} \right], \quad (\text{A4})$$

where $\psi(x)$ is the digamma function, and we have defined

$$r_{\pi f} = \frac{\omega_0(m_\pi^2)}{2B_f} = \frac{M^2 - (1-\beta^2)m_\pi^2/4}{2B_f}. \quad (\text{A5})$$

In turn, for the π - γ mixing entry we have from Eq. (32)

$$C_{\pi\gamma}^f(m_\pi^2, 0) = \frac{1}{8\pi^2} N_c s_f M \kappa_f Q_f B_f \int_{-1}^1 d\beta \int_0^\infty dz e^{-z\omega_0(m_\pi^2)}. \quad (\text{A6})$$

These integrals can be performed analytically. Assuming $m_\pi < 2M$ we get

$$C_{\pi\gamma}^f(m_\pi^2, 0) = \frac{N_c s_f M \kappa_f Q_f B_f}{\pi^2 m_\pi^2 \sqrt{4M^2/m_\pi^2 - 1}} \arctan \left(\frac{1}{\sqrt{4M^2/m_\pi^2 - 1}} \right). \quad (\text{A7})$$

Finally, for the γ - γ polarization functions we have to consider the integrals in Eqs. (36)–(38). Changing the integration variable z to $\tau = zB_f$ and using the above definition for $r_{\pi f}$ we have

$$C_{\gamma\gamma,0}^{f,\text{mag}}(m_\pi^2, 0) = -\frac{N_c Q_f^2}{16\pi^2} \int_{-1}^1 d\beta \int_0^\infty d\tau e^{-2\tau r_{\pi f}} \gamma_0(\tau, \beta), \quad \gamma_0(\tau, \beta) = \frac{\cosh(\tau\beta) - \beta \coth \tau \sinh(\tau\beta)}{\sinh \tau} - \frac{1 - \beta^2}{\tau},$$

$$(A8) \quad \gamma_1(\tau, \beta) = (1 - \beta^2) \left(\coth \tau - \frac{1}{\tau} \right),$$

$$C_{\gamma\gamma,i}^f(m_\pi^2, 0) = -\frac{N_c Q_f^2}{16\pi^2} \int_{-1}^1 d\beta \int_0^\infty d\tau e^{-2\tau r_{\pi f}} [\gamma_i(\tau, \beta) - \gamma_0(\tau, \beta)], \quad \gamma_2(\tau, \beta) = \frac{2[\cosh \tau - \cosh(\tau\beta)]}{\sinh^3 \tau} - \frac{1 - \beta^2}{\tau}.$$

$$i=1,2, \quad (A9)$$

where

The integrals over τ can be carried out taking into account the relations

$$\int_0^\infty d\tau e^{-2\lambda\tau} \left(\frac{\cosh(\tau\beta)}{\sinh \tau} - \frac{1}{\tau} \right) = \ln \lambda - \frac{1}{2} \left[\psi \left(\frac{2\lambda + \beta + 1}{2} \right) + \psi \left(\frac{2\lambda - \beta + 1}{2} \right) \right],$$

$$\int_0^\infty d\tau e^{-2\lambda\tau} \frac{\sinh(\tau\beta)}{\sinh \tau} = \frac{1}{2} \left[\psi \left(\frac{2\lambda + \beta + 1}{2} \right) + \psi \left(\frac{2\lambda - \beta + 1}{2} \right) \right], \quad (A11)$$

where we have used the recurrence relation $\psi(x+1) = \psi(x) + 1/x$. We obtain

$$\int_{-1}^1 d\beta \int_0^\infty d\tau e^{-2\tau r_{\pi f}} \gamma_0(\tau, \beta) = \int_{-1}^1 d\beta \left[-\beta^2 + (1 - \beta^2) \ln r_{\pi f} + (\beta^2 + 2\beta r_{\pi f} - 1) \psi \left(r_{\pi f} + \frac{\beta + 1}{2} \right) \right],$$

$$\int_{-1}^1 d\beta \int_0^\infty d\tau e^{-2\tau r_{\pi f}} \gamma_1(\tau, \beta) = \int_{-1}^1 d\beta (1 - \beta^2) \left[\ln r_{\pi f} - \psi(r_{\pi f}) - \frac{1}{2r_{\pi f}} \right],$$

$$\int_{-1}^1 d\beta \int_0^\infty d\tau e^{-2\tau r_{\pi f}} \gamma_2(\tau, \beta) = \int_{-1}^1 d\beta \left[-2 + (1 - \beta^2) \left(\frac{3}{2} + \ln r_{\pi f} \right) - 2r_{\pi f} - 4r_{\pi f}^2 \psi(r_{\pi f}) \right. \\ \left. + [(\beta + 2r_{\pi f})^2 - 1] \psi \left(r_{\pi f} + \frac{\beta + 1}{2} \right) \right]. \quad (A12)$$

Now, from Eqs. (35) and (46) it is seen that the matrix element $\mathcal{G}_{\gamma_2\gamma_2}$ needed to determine the π^0 mass is given by

$$\mathcal{G}_{\gamma_2\gamma_2} = -m_\pi^2 + \sum_f [C_{\gamma\gamma,0}^{f,0,\text{reg}}(m_\pi^2) + C_{\gamma\gamma,0}^{f,\text{mag}}(m_\pi^2, 0) + C_{\gamma\gamma,1}^f(m_\pi^2, 0)] m_\pi^2. \quad (A13)$$

According to the above equations, the sum of the “magnetic” contributions is given by

$$C_{\gamma\gamma,0}^{f,\text{mag}}(m_\pi^2, 0) + C_{\gamma\gamma,1}^f(m_\pi^2, 0) = -\frac{N_c Q_f^2}{16\pi^2} \int_0^\infty d\tau e^{-2\tau r_{\pi f}} \gamma_1(\tau, \beta)$$

$$= -\frac{N_c Q_f^2}{16\pi^2} \int_{-1}^1 d\beta (1 - \beta^2) \left[\ln r_{\pi f} - \psi(r_{\pi f}) - \frac{1}{2r_{\pi f}} \right]. \quad (A14)$$

Adding now the regularized “ $B = 0$ ” contribution $C_{\gamma\gamma,0}^{f,0,\text{reg}}(m_\pi^2)$ given by Eq. (40), the log terms can be trivially integrated, and we end up with

$$\mathcal{G}_{\gamma_2\gamma_2} = -m_\pi^2 + \frac{N_c m_\pi^2}{16\pi^2} \sum_f Q_f^2 \left\{ -\frac{4}{3} \ln \left(\frac{M^2}{2B_f} \right) + \int_{-1}^1 d\beta (1 - \beta^2) \left[\psi(r_{\pi f}) + \frac{1}{2r_{\pi f}} \right] \right\}. \quad (A15)$$

The above expression is valid as long as the condition $m_\pi < 2M$ is fulfilled, which is expected to hold for quark loops. Now, as stated in the main text, to determine the pion mass one has to take into account all fermion loop contributions to $\mathcal{G}_{\gamma_2\gamma_2}$, which means to add QED lepton loops. These are similar to those included in the sum in Eq. (A15): one has just to replace $N_c \rightarrow 1$, $Q_f^2 \rightarrow e^2$ and $M \rightarrow m_l$, where $l = e, \mu$ (the contribution of the τ lepton can be safely neglected). In the case of the electron loop, the π^0 mass lies above the threshold of e^+e^- production. Therefore, the contribution of the loop to $\mathcal{G}_{\gamma_2\gamma_2}$ will have an absorptive part. As is usually done, we addressed this problem by carrying out an analytical extension of the integral in Eq. (A15). We define the function

$$F(q^2) = \int_{-1}^1 d\beta(1-\beta^2) \left[\psi(r(q^2)) + \frac{1}{2r(q^2)} \right], \quad (\text{A16})$$

where $r(q^2) = [4m_e^2 - (1-\beta^2)q^2]/(8B_e)$, with $B_e = |eB|$. Performing the analytic extension to the region $q^2 > 4m_e^2$ one has (see, e.g., Ref. [17])

$$F(q^2) = \int_{-1}^1 d\beta(1-\beta^2) \psi(r(q^2)) + N + 1 + \frac{8B_e}{q^2} \sum_{n=0}^N \frac{2-\delta_{n0}}{\rho_n} [\rho_n + (1-\rho_n^2) \text{arctanh} \rho_n] - i \frac{4\pi B_e}{q^2} \sum_{n=0}^N \frac{2-\delta_{n0}}{\rho_n} (1-\rho_n^2), \quad (\text{A17})$$

where we have introduced the definitions

$$\rho_n = \sqrt{1 - \frac{4(m_e^2 + 2nB_e)}{q^2}}, \quad N = \text{Floor} \left(\frac{q^2 - 4m_e^2}{8B_e} \right). \quad (\text{A18})$$

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