

Canonical differential equations for the elliptic two-loop five-point integral family relevant to $t\bar{t}$ + jet production at leading color

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(Received 13 March 2025; accepted 31 July 2025; published 21 August 2025)

We complete the construction of canonical differential equations (DEs) for all families of Feynman integrals appearing in the two-loop leading-color QCD amplitude for top-pair production in association with a jet at hadron colliders. This allows us to obtain analytic results for all required Feynman integrals in terms of iterated integrals with closed-form kernels. To achieve this, we study the two-loop five-point integral family for which canonical DEs were previously unavailable due to the appearance of elliptic functions and nested square roots. In addition to marking a significant step toward next-to-next-to-leading-order QCD predictions for a high-priority LHC process, this is the first time that canonical DEs are obtained for Feynman integrals where elliptic functions appear in conjunction with the high algebraic complexity of a process with more than four particles. Our results also reveal a number of new interesting analytic features, such as a “duplet” structure that generalizes the even/odd parity of square roots to nested square roots.

DOI: [10.1103/zt4w-c1jk](https://doi.org/10.1103/zt4w-c1jk)

Feynman integrals are central in the study of quantum field theory (QFT). Not only are they essential to obtain precise predictions for particle phenomenology, but they also drive formal studies and offer increasingly many links to mathematics and other areas of physics. Progress in their computation is crucial for developments across these fields. However, this progress is challenged by both algebraic and analytic complexity. The former grows rapidly with the number of variables and thus of scattered particles, whereas the latter escalates as new classes of special functions emerge. In this Letter, we present a computation that lies at the cutting edge of both types of complexity by obtaining—for the first time—analytic results for a family of two-loop five-particle Feynman integrals involving elliptic functions and nested square roots.

One of the most powerful frameworks for computing Feynman integrals consists in viewing them as the solution to certain differential equations (DEs) [1–6]. The insight that a “canonical” form of the DEs [6]—one which dramatically simplifies the solution—exists has made possible increasingly many results in high-energy physics. In the best understood case, canonical means that the

dependence on the regulator of dimensional regularization is factorized, and the DEs contain solely logarithmic differential 1-forms ($d \log$'s). In such a case, we have a well-developed mathematical technology to handle the solution (see Refs. [7–10] for reviews), and the state of the art has reached integrals with five [11] and six particles [12–14] at three- and two-loop order, respectively.

However, starting at two-loop order, Feynman integrals may involve transcendental functions linked to higher-genus geometries [15,16], such as periods of elliptic curves, whose DEs cannot be expressed in the form above. Such functions are typically linked to the presence of massive virtual particles. Despite recent progress in extending the construction of canonical DEs [17–24], the most advanced applications are confined to processes with comparatively simpler four-particle kinematics [19,25–30]. Advancing our methodology to handle high-multiplicity processes with massive virtual particles is imperative. This is one of the main bottlenecks to keep the theoretical uncertainties in line with the experimental ones for many processes in the Large Hadron Collider (LHC) program [31]. Moreover, thanks to their rich kinematics, high-multiplicity processes offer an extraordinary laboratory for uncovering new properties of QFT.

We break through the wall of elliptic integrals with five-point kinematics and obtain canonical DEs for a family of Feynman integrals relevant for top-pair production with a jet at hadron colliders that involves elliptic functions [32–34]. Our result completes the construction of canonical

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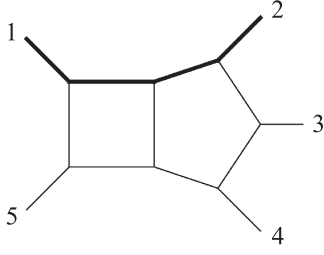


FIG. 1. Graph of the integral family studied in this Letter. Thin (thick) lines are massless (massive).

DEs for all families appearing in the leading-color two-loop QCD amplitude for this process, allowing us to express all required Feynman integrals analytically in terms of iterated integrals [35], thus marking a key step toward obtaining next-to-next-to-leading-order (NNLO) QCD predictions. This achievement is of immediate importance for the LHC, where top-pair production with a jet is a high-priority process [36–39]. Its high sensitivity to the top-quark mass [40–43] and the increasing experimental precision call for the NNLO QCD corrections to be computed. Our results advance the theoretical understanding of elliptic Feynman integrals to a new, high-multiplicity regime, provide analytical data to fuel ongoing formal studies, and open new research avenues.

Setting the stage. We consider the “pentagon-box” integral family shown in Fig. 1. We use the notation of Ref. [33]: the external momenta p_i are outgoing and on shell, i.e., $p_1^2 = p_2^2 = m_t^2$ and $p_3^2 = p_4^2 = p_5^2 = 0$, and the kinematics are described by six independent Lorentz invariants,

$$\vec{x} = (d_{12}, d_{23}, d_{34}, d_{45}, d_{15}, m_t^2), \quad (1)$$

where $d_{ij} = p_i \cdot p_j$. The family is the set of all scalar integrals of the form

$$I_{a_9, a_{10}, a_{11}}^{a_9, a_{10}, a_{11}}(\vec{x}, \epsilon) = e^{2\epsilon\gamma_E} \int \frac{d^d k_1}{i\pi^{\frac{d}{2}}} \frac{d^d k_2}{i\pi^{\frac{d}{2}}} \frac{D_9^{a_9} D_{10}^{a_{10}} D_{11}^{a_{11}}}{D_1^{a_1} \dots D_8^{a_8}}, \quad (2)$$

where $a_9, a_{10}, a_{11} \geq 0$, $d = 4 - 2\epsilon$, and the inverse propagators D_i are given in Table 1 of Ref. [33] (family PB_B). Modulo integration-by-parts [44–46] and symmetry relations, the family has a basis of 121 master integrals (MIs). The analytic structure of the integrals involves six square roots: $\sqrt{\beta_{12}}$, $\sqrt{\Delta_{31}}$, $\sqrt{\Delta_{32}}$, $\sqrt{\Delta_{33}}$, $\sqrt{\Lambda_4}$, and $\sqrt{\Delta_5}$.¹

In general, a set of MIs $\vec{\mathcal{I}}$ satisfies a system of DEs of the form [1–5]

¹With respect to Ref. [33], we make the square-root sign explicit and use $\sqrt{\Delta_5}$ rather than tr_5 to avoid confusion with the homonymous pseudoscalar invariant.

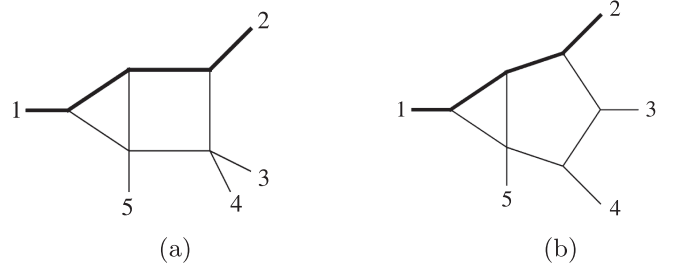


FIG. 2. Sectors whose maximal cuts introduce the elliptic curve (a), the nested square roots (b), and labels of the corresponding MIs. (a) MIs Nos. 35–37. (b) MIs Nos. 18–20.

$$d\vec{\mathcal{I}}(\vec{x}, \epsilon) = dA(\vec{x}, \epsilon) \cdot \vec{\mathcal{I}}(\vec{x}, \epsilon), \quad (3)$$

where d is the total differential with respect to $\vec{x}$. The connection matrix $dA(\vec{x}, \epsilon)$ for the basis $\vec{\mathcal{I}}$ of Ref. [33] depends quadratically on ϵ and contains nonlogarithmic 1-forms. These noncanonical features are, however, localized in the two sectors shown in Fig. 2, i.e., in the subsets of integrals having the propagators of the graphs in the figure. More concretely, the “homogeneous” blocks—i.e., the blocks on the diagonal of the connection matrix that couple the MIs of the same sector—were cast into “ $\epsilon \times d \log$ ” form for all sectors but the two in Fig. 2.

As noticed in Ref. [33], the homogeneous block of the sector in Fig. 2(b) is ϵ factorized by rotating $\mathcal{I}_{19}$ and $\mathcal{I}_{20}$ as

$$\mathcal{J}_{19} = \frac{\sqrt{n_+}}{d_{45} r_2} (\mathcal{I}_{20} + \mathcal{I}_{19}), \quad \mathcal{J}_{20} = \frac{\sqrt{n_-}}{d_{45} r_2} (\mathcal{I}_{20} - \mathcal{I}_{19}). \quad (4)$$

This transformation contains the nested square roots

$$\sqrt{n_{\pm}} = \sqrt{d_{23}^2 \Delta_5 - 8r_2 r_4 \pm 4d_{23} r_3 \sqrt{\Delta_5}}, \quad (5)$$

where r_k denotes a degree- k polynomial in $\vec{x}$. The resulting ϵ -factorized block was not written in terms of $d \log$'s, and it is unclear whether this is generally possible in the presence of nested roots [47]. We will show that this is indeed possible in our case and discuss the symmetries associated with the nested square roots. Note that the interior square root, $\sqrt{\Delta_5}$ in Eq. (5), can be rationalized, e.g., by using momentum-twistor variables [48], at the cost of increasing the algebraic complexity of the expressions and obscuring the symmetry properties discussed in the Analytic Structure section. We therefore refrain from doing such a change of variables.

The sector in Fig. 2(a) was shown to involve elliptic functions associated with the elliptic curve $y^2 = \mathcal{P}(z)$,

$$\mathcal{P}(z) = (z - e_1)(z - e_2)(z - e_3)(z - e_4), \quad (6)$$

where the roots e_i are functions of $\vec{x}$. In particular, the maximal cut of the scalar integral of this sector ($\mathcal{I}_{35}$, with a

suitable normalization) at $\epsilon = 0$ is given, depending on the integration cycle, by the periods of the elliptic curve [33]. We choose them as

$$\begin{aligned}\psi_1 &= \frac{2}{\pi} \int_{e_2}^{e_3} \frac{dz}{\sqrt{\mathcal{P}(z)}} = \frac{4K(\kappa^2)}{\pi \sqrt{(e_3 - e_1)(e_4 - e_2)}}, \\ \psi_2 &= 4i \int_{e_1}^{e_2} \frac{dz}{\sqrt{\mathcal{P}(z)}} = \frac{-8K(1 - \kappa^2)}{\sqrt{(e_3 - e_1)(e_4 - e_2)}},\end{aligned}\quad (7)$$

where K is the complete elliptic integral of the first kind,

$$K(\kappa^2) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\kappa^2 t^2)}}, \quad (8)$$

$\kappa^2 = e_{32}e_{41}/(e_{31}e_{42})$, with $e_{ij} = e_i - e_j$, is the elliptic modulus, and the normalizations in Eq. (7) were chosen to simplify the m_i^2 expansion. The presence of the elliptic curve is the main obstacle toward achieving canonical DEs, which we overcome in the next section.

Construction of a canonical basis. There is no consensus yet on what “canonical” means in addition to the factorization of ϵ beyond the $d \log$ case [24,49,50]. We adopt a local definition: we require that, for each singular point, it is possible to choose the appearing period of the elliptic curve such that all 1-forms have at most simple poles. In this section, we will be concerned with achieving the factorization of ϵ . We will verify that the resulting DEs fulfil this notion of canonical in the next section. We denote by $T(\vec{x}, \epsilon)$ the transformation matrix from the basis $\vec{\mathcal{I}}$ of Ref. [33] to the new, canonical basis $\vec{\mathcal{J}}$, i.e., $\vec{\mathcal{J}} = T \cdot \vec{\mathcal{I}}$. We include in T the transformation in Eq. (4).

We construct T in two steps. First, we transform the homogeneous block of the DEs related to the elliptic sector into ϵ -factorized form. Second, we perform a similar rotation to factorize ϵ in the inhomogeneous blocks that couple other sectors to the ones shown in Fig. 2. In both cases, we follow the approach of Ref. [19]. An important part of this method is to find a suitable starting basis that simplifies the DEs as much as is possible without introducing transcendental functions. We find the basis of Ref. [33] to be a suitable starting point. In particular, the maximal cut of $\mathcal{I}_{35}$ is related to the periods of the elliptic curve (see Appendix A of Ref. [33] and Supplemental Material [51]).

We begin by focusing on the derivative with respect to m_i^2 . The homogeneous block of the connection matrix of the elliptic sector takes the following form at $\epsilon = 0$:

$$dA_{\text{hom,ell}}^{\epsilon=0} = A_{m_i^2, \text{hom,ell}}^{\epsilon=0} dm_i^2 + \dots, \quad (9)$$

where we omit the differentials in the other variables, and

$$A_{m_i^2, \text{hom,ell}}^{\epsilon=0} = \begin{pmatrix} & 0 \\ B & \\ \star & \star & 0 \end{pmatrix}, \quad (10)$$

with a 2×2 block B coupling the first two MIs. The fundamental solution W for this block, i.e., the solution to

$$\partial_{m_i^2} W = B \cdot W, \quad (11)$$

can be found by converting the 2×2 system of first-order DEs into a single second-order DE,

$$L_{m_i^2}^{(2)} \psi_i \equiv [\partial_{m_i^2}^2 + a_1(\vec{x}) \partial_{m_i^2} + a_0(\vec{x})] \psi_i = 0, \quad (12)$$

by standard methods. The solutions to this equation are the two periods in Eq. (7). We then can convert back to the 2×2 system to write W in terms of the periods.

Following Ref. [19], we then split W into a lower-triangular semisimple part and an upper-triangular unipotent part, i.e., $W = W^{\text{ss}} \cdot W^{\text{u}}$, with

$$W^{\text{ss}} = \begin{pmatrix} \psi_1 & 0 \\ \psi_1' & \frac{\det(W)}{\psi_1} \end{pmatrix} \quad \text{and} \quad W^{\text{u}} = \begin{pmatrix} 1 & \frac{\psi_2}{\psi_1} \\ 0 & 1 \end{pmatrix}, \quad (13)$$

and rotate the first two MIs by the inverse of the semisimple part. Although W depends on both ψ_1 and ψ_2 (and their derivatives), its semisimple part can, in general, be written in terms of a single solution, say ψ_1 (and its m_i^2 derivative, ψ_1'), thanks to the fact that $\det(W) = \psi_1 \psi_2' - \psi_2 \psi_1'$ is a rational function of $\vec{x}$ due to the Legendre relation for elliptic integrals. The freedom of choosing which solution of $L_{m_i^2}^{(2)}$ appears in W^{ss} and thus in the canonical DEs is unphysical and the result for the original basis $\vec{\mathcal{I}}$ does not depend on it.

After this basis change and a rescaling by suitable factors of ϵ , the homogeneous DEs in m_i^2 are now linear in ϵ , with the leading term being lower triangular,

$$A_{m_i^2, \text{hom,ell}}^{\text{ss}} = \begin{pmatrix} 0 & 0 & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{pmatrix} + \epsilon \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix}. \quad (14)$$

Achieving ϵ factorization therefore amounts to integrating out the remaining elements at $\epsilon = 0$ [52]. However, as was the case for several examples in Ref. [19], not all elements are total derivatives in m_i^2 . As a result, we need to introduce a new function, G_1 , defined through its m_i^2 derivative, as $\partial_{m_i^2} G_1 = \left(A_{m_i^2, \text{hom,ell}}^{\text{ss}, \epsilon=0} \right)_{2,1}$, which is linear in ψ_1 and ψ_1' (the same as in W^{ss}) and rational in $\vec{x}$.

Having worked out the intricate dependence of the resulting transformation on all kinematic variables (see Supplemental Material [51]), we can successively check the

ϵ factorization of the homogeneous DEs not only in m_i^2 , but also in the other variables. We find that, thanks to our careful choice for the solutions of the appearing differential operators, these are automatically ϵ factorized, so that no further (m_i^2 -independent) transformations are required.

To achieve the same for the inhomogeneous blocks of the connection matrix, we again integrate out the elements at $\epsilon = 0$, introducing three additional functions, dubbed G_2 , $G_{3,\pm}$. Their partial derivatives are linear in ψ_1 and ψ'_1 , but now additionally contain various square roots: they are odd with respect to $\sqrt{\Delta_{32}}$, and $G_{3,\pm}$ originate from the coupling of the two sectors in Fig. 2 to each other and contain $\sqrt{\Delta_5}$ as well as $\sqrt{n_{\pm}}$, respectively.

Similar to the first step, these transformations suffice to ϵ factorize the DEs in all variables, even though they were derived by only considering the derivative with respect to m_i^2 . Moreover, while from the point of view of the DEs the new functions G_i are defined through their partial derivatives, we were also able to express them explicitly as complete elliptic integrals of the third kind by considering different Baikov representations [53,54] and performing the integrals appearing in them, see Supplemental Material [51]. This result will be useful in view of the numerical evaluation of the solution, which we leave to future work.

Analytic structure. The DEs for the basis $\vec{\mathcal{J}}$ constructed in the previous section take the form

$$d\vec{\mathcal{J}}(\vec{x}, \epsilon) = \epsilon \sum_{i=1}^{112} M_i \omega_i(\vec{x}) \cdot \vec{\mathcal{J}}(\vec{x}, \epsilon), \quad (15)$$

where M_i are constant rational matrices, and $\omega_i(\vec{x})$ are $\mathbb{Q}$ linearly independent differential 1-forms. We verified the linear independence either by treating the elliptic functions (ψ_1 , G_i) as independent variables—i.e., we view the entries of the connection matrix as rational functions of $\vec{x}$, ψ_1 , and G_i —or by evaluating them numerically. We expressed the 83 $\omega_i(\vec{x})$'s free of elliptic functions as $d \log W$ for some algebraic function W . This required constructing 13 new $d \log$'s involving the nested square roots, besides the ones of Ref. [33]. We chose the 29 one-forms containing elliptic functions from the entries of the connection matrix and wrote them in terms of differentials of $\vec{x}$. The definition of the MIs $\vec{\mathcal{J}}$ contains overall (nested) square roots and transcendental functions. This induces two gradings on the 1-forms.

First, it is well known that 1-forms and MIs can be chosen to be either even or odd upon flipping the sign of each square root. While this holds also for $\sqrt{n_{\pm}}$, flipping the interior root ($\sqrt{\Delta_5}$) exchanges $\sqrt{n_+}$ and $\sqrt{n_-}$, leading to a more complicated behavior. One way to generalize the even/odd parity to a picture that includes nested square roots is to view it as the trivial/sign representation of $\mathbb{Z}_2$, the

group formed by the operations of flipping each square root. The relevant group for the nested square roots $\sqrt{n_{\pm}}$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$: one $\mathbb{Z}_2$ flips the sign of the exterior root, the other of the interior one.² For the exterior root, we have the usual even/odd representation. For the interior root, instead, we have a two-dimensional representation of $\mathbb{Z}_2$ generated by the Pauli matrix σ_1 , acting on duplets. In other words, we arrange all objects into duplets whose elements transform into each other upon flipping $\sqrt{\Delta_5}$. Examples of such duplets are $(\sqrt{n_+}, \sqrt{n_-})$, $(\mathcal{J}_{19}, \mathcal{J}_{20})$, $(G_{3,+}, G_{3,-})$, and (ω_4, ω_5) . This duplet structure constrains the form of the DEs, as

$$M_a = \Sigma \cdot M_b \cdot \Sigma^{-1}, \quad (16)$$

for every duplet (ω_a, ω_b) , where Σ is the matrix representing the $\sqrt{\Delta_5}$ flip on the MIs.

The grading of 1-forms also informs the construction of a $d \log$ representation, when that is possible. While for $d \log$'s containing square roots there is a well-understood procedure [57,58], results involving nested square roots are scarce, and it is, in general, unclear whether a $d \log$ representation is possible [47]. We succeeded for all 1-forms free of elliptic functions, by generalizing the methods applied to square roots. For example, for a $\sqrt{\Delta_5}$ duplet of 1-forms $\omega_{\pm}$ odd with respect to $\sqrt{n_{\pm}}$ and one of the square roots, say, $\sqrt{\beta}$, we make the ansatz

$$\omega_{\pm} = d \log \left(\frac{c_1 \pm c_2 \sqrt{\Delta_5} + c_3 \sqrt{\beta} \sqrt{n_{\pm}}}{c_1 \pm c_2 \sqrt{\Delta_5} - c_3 \sqrt{\beta} \sqrt{n_{\pm}}} \right), \quad (17)$$

which manifests the grading. We then constrain the polynomials c_i by imposing that the singular locus is that of the target 1-form. This amounts to solving quartic equations, for which we adopt *ad hoc* methods leveraging the sparsity of the system and of the solution. In some cases, it is easier to integrate the 1-form in one variable, setting all others to numbers and rationalizing one of the roots, and use the result to constrain the ansatz. Interestingly, the polynomial $C := n_+ n_-$ does not appear as the argument of a $d \log$. This observation is relevant in view of the program aimed at determining the logarithmic 1-forms from the Landau equations [59–72]: C appears in the Landau discriminant, but only factoring it as $C = (a + b\sqrt{\Delta_5})(a - b\sqrt{\Delta_5})$, for polynomials a and b , reveals $n_{\pm} = a \pm b\sqrt{\Delta_5}$ as possible arguments of square roots.

²The mathematical framework is that of Galois theory. The Galois group of a polynomial is, roughly speaking, the group of unique permutations of its roots. Square roots are roots of quadratic polynomials whose Galois group is $\mathbb{Z}_2$. The nested square roots $\pm\sqrt{n_{\pm}}$ are instead roots of $P(z) = z^4 + az^2 + b$ (for a, b functions of $\vec{x}$), whose Galois group is $D_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. See also Refs. [55,56] for similar applications of Galois theory to Feynman integrals.

The second grading, which we dub “elliptic grading,” is associated with the elliptic functions. By grading +1 the periods and thus the G_i ’s, all MIs and 1-forms gain a uniform grade ranging from -2 to $+2$, as already observed in other cases. Note that a function having elliptic grade 0 may still contain elliptic functions, as ratios with ψ_1 in the denominator also appear, e.g., G_1/ψ_1 . We stress that the emergence of this grading is here a mere observation that we use to classify MIs and 1-forms; its implications are left for future study. Additionally, we note that elliptic 1-forms appear also in the entries of the connection matrix coupling MIs of sectors that are not elliptic on their maximal cut, and that ψ'_1 is absent from the DEs, although present in the transformation T .

Finally, we verified that the DEs in Eq. (15) are canonical according to the local definition given above. They contain transcendental functions, in particular, ψ_1 . What matters for the ϵ factorization is that ψ_1 satisfies the DE (12); any linear combination of ψ_1 and ψ_2 would work, and this freedom extends to the G_i ’s. However, in the neighborhood of certain singular points, one solution to Eq. (12) is holomorphic, while the other has logarithmic singularities; which one depends on the point. We verified that, for each singular point, there is a solution to Eq. (12) such that all 1-forms have at most simple poles. In practice, we pull the 1-forms back to paths parametrized by one variable and expand the elliptic functions around each singular point via the Frobenius method [73,74], choosing the holomorphic solution.

Our analytic results can be found at Ref. [75].

Conclusion and outlook. We derived canonical DEs for the two-loop, five-point integral family involving elliptic functions that contributes to the leading-color NNLO QCD corrections to top-pair production with a jet at hadron colliders. Together with the boundary values of Ref. [33], this makes it straightforward to write the solution analytically in terms of iterated integrals with closed-form kernels [35]. This result finalizes the construction of canonical DEs for all integral families needed for this prediction [32–34] and opens several interesting research avenues.

From a theoretical standpoint, this achievement opens the door to investigating the analytic structure of amplitudes involving higher-genus geometries in a high-multiplicity regime. In particular, recent studies have uncovered striking patterns of cancellations among combinations of iterated integrals involving elliptic functions and even functions defined over Calabi-Yau varieties [27,28,30,76–78], leading to remarkable simplifications at the amplitude level. It would be interesting to establish whether similar simplifications occur in the present context, determine their potential implications for phenomenology, and explore whether they can be linked to the elliptic grading analyzed in this Letter.

The generalization of the even/odd parity to incorporate nested square roots and the constraints this imposes on Feynman integrals through their DEs are other important insights raised in this Letter. This aspect is closely tied to the possibility of expressing DEs containing nested square roots in $d \log$ form. Our results lay a strong foundation to extend to nested square roots the active study of symbol letters [57,58,62–72,79–89], which underpins many bootstrap programs (see Ref. [90] for a review).

Another primary challenge is to obtain an analytic representation of the MIs that not only reveals simplifications in the amplitude but also enables efficient numerical evaluation. As of now, the best available approach to evaluate the MIs considered here is that of generalized series expansions [91] applied to noncanonical DEs [33,92]; while flexible, this method suffers from slow evaluation times with respect to what is typically needed for Monte Carlo integration. Recent progress in expressing MIs in terms of independent special functions [93–97] together with a specialized approach to their evaluation [94,98] have led to a surge of NNLO QCD predictions for $2 \rightarrow 3$ LHC processes with massless virtual particles [99–115]. Our results set the necessary ground to study the extension of these techniques to elliptic integrals in a context where this is essential to obtain theoretical predictions. In fact, the complexity of the Monte Carlo phase-space integration sets substantially higher efficiency demands for $2 \rightarrow 3$ processes as compared to lower multiplicities. The extension of the techniques above would thus have a transformative impact and remove one of the main bottlenecks to improve the predictions for a wide class of processes. More general strategies based on series expansions are also under study for iterated integrals with higher-genus geometries [27,28,30,48,76], and their potential merits further exploration.

Beyond being a critical step toward achieving NNLO QCD corrections for an important LHC process, our results prove that the methodology under development by the community has reached the maturity to tackle the computation of elliptic Feynman integrals with five-particle kinematics. This milestone is important also in view of other massive $2 \rightarrow 3$ amplitudes, which are the object of active research [47,116–119] and play a crucial role in LHC phenomenology. Furthermore, the mathematical challenges addressed in this Letter are relevant in other research areas, including gravitational waves [23,120–122], string theory [123], and pure mathematics, underscoring the broad impact of our findings.

Acknowledgments. We thank S. Badger, C. Nega, V. Sotnikov, L. Tancredi, and F. Wagner for useful discussions, the organizers of “MathemAmplitudes 2023,” and CERN for hospitality. We are grateful to S. Badger, J. Henn, and V. Sotnikov for useful comments

on the draft. This project received funding from the European Union’s Horizon Europe research and innovation program under the Marie Skłodowska-Curie Grant Agreement No. 101105486, ERC Starting Grant No. 101040760 *FFHiggsTop*, and from the Swiss

National Science Foundation under the Ambizione Grant No. 215960.

Data availability. The data that support the findings of this article are openly available [75], embargo periods may apply.

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