

$\xi R\phi^2$ nonminimal coupling, and the long range gravitational potential for different spin fields from 2-2 scattering amplitudes

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In this paper, we investigate the long range gravitational effect of curvature-scalar field nonminimal coupling in the form of $\xi R\phi^2$, in the perturbative quantum gravity framework. Such coupling is most naturally motivated from the renormalization of a scalar field theory with a quartic self interaction in a curved spacetime background. This coupling results in two scalar- n graviton vertices which contain no explicit momenta of the scalar, qualitatively different from the usual, e.g., $\kappa h^{\mu\nu} T_{\mu\nu}$ -type minimal matter-graviton vertices. Assuming the dimensionless coupling parameter ξ to be small, we compute the 2-2 scattering Feynman amplitudes between such scalars up to $\mathcal{O}(G^2\xi)$. From the nonrelativistic limit of these amplitudes, we compute the corresponding long range gravitational potential. There exists no tree level contribution ($\mathcal{O}(\xi G)$) here, and hence the one loop $\mathcal{O}(G^2\xi)$ result is leading. Recently, the effect of a cosmological constant in such nonminimal interaction and the subsequent gravitational potential was computed. In this work, we take the cosmological constant to be vanishing. The resulting potential is found to have r^{-4} leading behavior. We further extend these results for scalar-massive spin-1 and massive spin-1/2 scattering. Spin and polarization dependence of the two body potential have been explicitly demonstrated. We discuss some possible physical implications of these results.

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I. INTRODUCTION

Among the four fundamental interactions of nature, i.e., strong, weak, electromagnetic, and gravity, the first three can be quantized, and the resulting quantum field theories make predictions that are in excellent agreement with observation. A traditional approach to quantizing general relativity, however, shows that it is not perturbatively renormalizable. At each order of expansion of the space-time inverse metric with respect to a classical background, we generate new terms in the action, and accordingly we need new counterterms at every order of perturbation theory for renormalization. This was established in a series of pioneering works such as [1–18]. The fact that gravity is not a renormalizable quantum field theory has led to many alternative ideas for a consistent quantum theory of gravity. However, it will be fair enough to admit that none of them

has given us any complete or satisfactory answer so far, see, e.g., [19–21], and references therein.

Despite this shortcoming or incompleteness of gravity while treated as a quantum field theory, it is believed that at energy scales much below the Planck mass, at least the first couple of orders of computation in this framework might be physically meaningful. The predictions thus made can be hoped to be tested in the not too far away future, especially keeping in mind the improving observational capacity to probe strong gravity regime. Such observations can tell us whether there is any deviation from the classical theory of gravity. In particular, they can tell us whether the notion of graviton as the quantum of gravity makes any sense. Second, one also hopes that these low energy computations will one day be successfully embedded in a more complete theory of quantum gravity. With this motivation, there has really been a huge amount of effort by the community over past few decades to understand the physical predictions of perturbative quantum gravity, see, e.g., [22–48] and also references therein. They chiefly investigate the quantum corrections of the long range, two body gravitational potentials, renormalization of matter-graviton interaction processes, and running of various couplings. See also [49–51] for computation of gravitational light bending at leading and subleading orders using

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graviton exchanges between massive and massless fields. We refer our reader to [52–57] for inclusion of a cosmological constant in the perturbative quantum gravity calculations, for processes occurring much inside the cosmological event horizon. We further refer our reader to, e.g., [58–68] (also references therein), for an effective field theory description of quantum gravity. These computations also do not concern with the ultraviolet completion of gravity, and chiefly focus on the two body long range gravitational potential at various post Minkowski orders, as well as on gravitational radiation. Finally, see, e.g., [69–80] and references therein for discussion on perturbative quantum gravity in curved, especially the primordial inflationary de Sitter, backgrounds.

When one tries to renormalize a scalar field theory with a quartic self interaction in the presence of gravity, inclusion of a matter-gravity nonminimal term like $\xi R\phi^2/2$ in the action becomes mandatory, see [81] and the original references therein. Note also that $\xi = 1/6$ in particular leads to the trace or conformal anomaly in the massless limit. With a conformal scalar field, stationary hairy back hole solutions can be seen in [82–86]. What will be the contribution of such nonminimal interaction to the two body long range gravitational potential? While the minimal case has been extensively investigated in the literature, e.g., [29,33,34], (also, e.g., [63] and references therein for effective field theory computations), the case $\xi \neq 0$ seems to be relatively less addressed. This coupling generates two scalar- n graviton vertices, each containing explicitly the momenta carried by the graviton lines, but not of the two scalar lines, somehow complementing the usual matter-graviton vertices (like $\kappa h_{\mu\nu}T^{\mu\nu}$), where only the momenta carried by the matter field appear explicitly.

In this paper we wish to compute the effect of such nonminimal coupling in the two body gravitational potential, via the computation of the 2-2 scattering amplitudes between massive scalar-massive scalar, massive scalar-massive spin-1, and massive scalar-massive spin-1/2 fields. To the best of our knowledge, this question was first addressed in [57], where the effect of the three graviton vertex generated by the cosmological constant (via the $-2\Lambda\sqrt{-g}/\kappa^2$ term in the action) was studied for scalar-scalar scattering. In this paper, we will set $\Lambda = 0$. The rest of the paper is organized as follows. In the next section, we briefly discuss the basic ingredients we will be needing for our computations. In Sec. III, we compute the nonminimal gravitational scattering between two massive scalar fields. In Secs. IV and V, we respectively compute the same for massive scalar-massive spin-1 and massive scalar-massive spin-1/2 fields. The leading behavior of the potentials turns out to be

$\sim r^{-4}$. We compare our result with the well-known $\xi = 0$ case in Sec. VI. The nonminimal interaction will be taken only for the scalar field. We will assume in the following that the coupling parameter ξ is small, so that we will restrict our computations to linear order in ξ only.

We will work with the mostly positive signature of the metric in four spacetime dimensions. The incoming (outgoing) momenta in the 2-2 scattering process will always be denoted by k_1, k_2 (k'_1, k'_2) respectively, so that $k_1 + k_2 = k'_1 + k'_2$. k_1 and k'_1 will be associated with the scalar, whereas k_2, k'_2 will stand for various massive spin fields (0, 1, 1/2). All these external momenta will be taken to be nonrelativistic. The transfer momentum ($k_1 - k'_1$) = ($k'_2 - k_2$) will be denoted by q . For symmetrization, we will use the notation in this paper: $X_{(\alpha\beta)} = X_\alpha Y_\beta + X_\beta Y_\alpha$.

II. THE BASIC INGREDIENTS

Let us first briefly discuss the basic ingredients we will be needing for our main computations. Much of the discussion appearing below can be seen in, e.g., [38] (see also references therein). We begin with the action of the theory,

$$\begin{aligned} S = & \frac{2}{\kappa^2} \int d^4x \sqrt{-g} R - \frac{1}{2} \int d^4x \sqrt{-g} \left[g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) \right. \\ & + (M^2 + \xi R) \phi^2 \left. \right] - \frac{1}{2} \int d^4x \sqrt{-g} \left[g^{\mu\nu} (\nabla_\mu \varphi) (\nabla_\nu \varphi) \right. \\ & + (m^2 + \xi R) \varphi^2 \left. \right] + \int d^4x \sqrt{-g} (i \bar{\Psi} \not{\nabla} \Psi - m_f \bar{\Psi} \Psi) \\ & - \int d^4x \sqrt{-g} \left(\frac{1}{4} g^{\mu\rho} g^{\nu\lambda} F_{\mu\nu} F_{\rho\lambda} + \frac{1}{2} m_v^2 g^{\mu\nu} A_\mu A_\nu \right), \quad (1) \end{aligned}$$

where ξ is the dimensionless nonminimal coupling parameter, $\kappa^2 = 32\pi G$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. $\not{\nabla} \equiv \tilde{\gamma}^\mu \nabla_\mu$, where ∇ acting on the spinor is the spin covariant derivative. Also, since we are working with the mostly positive signature of the metric, the anticommutation relation for the curved space γ -matrices reads

$$[\tilde{\gamma}^\mu, \tilde{\gamma}^\nu]_+ = -2g^{\mu\nu} \mathbf{I}.$$

We wish to compute the scattering of φ, A_μ and Ψ off the scalar ϕ , one by one. We will assume ξ to be small, and will compute the amplitudes up to $\mathcal{O}(\xi G^2)$. Note also that we have kept the same value for the nonminimal coupling for the second scalar, φ . Extension with two different values of them is straightforward, although a little bit more tedious.

We will work in the weak gravity regime where the background can be taken to be Minkowski, so that we have the series expansion

$$\begin{aligned}
 g_{\mu\nu} &= \eta_{\mu\nu} + \kappa h_{\mu\nu}, & g^{\mu\nu} &= \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha} h_\alpha^\nu - \kappa^3 h^\mu_\alpha h^\alpha_\beta h^{\beta\nu} + \dots, \\
 \sqrt{-g} &= 1 + \frac{\kappa h}{2} + \frac{\kappa^2}{4} \left(\frac{h^2}{2} - h_{\mu\nu} h^{\mu\nu} \right) + \frac{\kappa^3}{48} (h^3 - 6h h_{\mu\nu} h^{\mu\nu} + 8h^\mu_\nu h^\nu_\rho h^\rho_\mu) + \dots \\
 \Gamma^\mu_{\nu\rho} &= \frac{\kappa}{2} \eta^{\mu\alpha} (\partial_\nu h_{\rho\alpha} + \partial_\rho h_{\nu\alpha} - \partial_\alpha h_{\nu\rho}) - \frac{\kappa^2}{2} h^{\mu\alpha} (\partial_\nu h_{\rho\alpha} + \partial_\rho h_{\nu\alpha} - \partial_\alpha h_{\nu\rho}) + \frac{\kappa^3}{2} h^{\mu\beta} h_\beta^\alpha (\partial_\nu h_{\rho\alpha} + \partial_\rho h_{\nu\alpha} - \partial_\alpha h_{\nu\rho}) + \dots
 \end{aligned} \quad (2)$$

Only up to $\mathcal{O}(\kappa^2)$ expansion of various quantities appearing above will effectively be necessary for our present tree [$\mathcal{O}(\kappa^2)$] and one loop [$\mathcal{O}(\kappa^4)$] level computations, as follows. We need 2-matter-1-graviton, 2-matter-2-graviton, and also the 3-graviton vertices. The first and the second require the expansion of the inverse metric and $\sqrt{-g}$ up to $\mathcal{O}(\kappa^2)$, for the minimal case. For the third, on the other hand, one needs to expand $\sqrt{-g}R = \sqrt{-g}g^{\mu\nu}R_{\mu\nu}$ at $\mathcal{O}(\kappa^3)$. However, the κ^2 sitting in the denominator of this term in the Einstein-Hilbert action [Eq. (1)] effectively reduces it to $\mathcal{O}(\kappa)$. Now, since the lowest order expansion of the metric and its determinant is $\mathcal{O}(\kappa^0)$, we must expand $R_{\mu\nu}$ up to $\mathcal{O}(\kappa^3)$. This means that for the terms in the Ricci tensor containing a single Γ , we must take Γ up to $\mathcal{O}(\kappa^3)$. However, the $\Gamma(\kappa^3)$ terms yield total divergences, and hence they do not contribute to the three-graviton interaction. Note also that for the nonminimal term $\xi\sqrt{-g}R\phi^2$, there is no κ^2 sitting in the denominator, Eq. (1), and hence this term requires an expansion of $\sqrt{-g}R$ only up to $\mathcal{O}(\kappa^2)$. This will be more explicit from the following calculations.

In the de Donder gauge,

$$\partial_\mu \left(h^\mu_\nu - \frac{1}{2} \delta^\mu_\nu h \right) = 0,$$

the graviton propagator reads

$$\Delta_{\mu\nu\alpha\beta}(k) = -\frac{i\mathcal{P}_{\mu\nu\alpha\beta}}{k^2}, \quad (3)$$

where

$$\mathcal{P}_{\mu\nu\alpha\beta} = \frac{1}{2} (\eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\nu\alpha} - \eta^{\mu\nu} \eta^{\alpha\beta}). \quad (4)$$

The tree level scalar propagator with mass m reads

$$\Delta(k) = -\frac{i}{k^2 + m^2}. \quad (5)$$

Let us now come to various vertex functions we will require for our purpose. By setting $\xi = 0$ in Eq. (1), the two scalar-one graviton minimal vertex reads, e.g., [34],

$$V_{\text{spin-0}}^{(1)\mu\nu}(k, k', m) = -\frac{i\kappa}{2} [k^\mu k'^\nu + k'^\mu k^\nu - \eta^{\mu\nu} (k \cdot k' + m^2)], \quad (6)$$

whereas the two scalar-two graviton minimal vertex reads

$$\begin{aligned}
 V_{\text{spin-0}}^{(2)\mu\lambda\rho\sigma}(k, k', m) &= i\kappa^2 \left[\left\{ I^{\mu\lambda\alpha\nu} I^{\rho\sigma\beta}_\nu - \frac{1}{4} (\eta^{\mu\lambda} I^{\rho\sigma\alpha\beta} + \eta^{\rho\sigma} I^{\mu\lambda\alpha\beta}) \right\} (k_\alpha k'_\beta + k_\beta k'_\alpha) \right. \\
 &\quad \left. - \frac{1}{2} \left(I^{\mu\lambda\rho\sigma} - \frac{1}{2} \eta^{\mu\lambda} \eta^{\rho\sigma} \right) (k \cdot k' + m^2) \right],
 \end{aligned} \quad (7)$$

where

$$I_{\mu\nu\lambda\rho} = \frac{1}{2} (\eta_{\mu\lambda} \eta_{\nu\rho} + \eta_{\mu\rho} \eta_{\nu\lambda}).$$

Note that the momenta appearing in Eqs. (6) and (7) are carried by the two scalar lines, originating from the derivative acting on the scalar. The nonminimal vertex, however, will just be the opposite of this, as will be demonstrated toward the end of this section.

For the spin-1 field with mass m_v , the propagator reads (e.g., [87])

$$D_{\mu\nu}(k) = -\frac{i}{k^2 + m_v^2} \left(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{m_v^2} \right). \quad (8)$$

The two massive spin-1-one graviton and the two massive spin-1-two graviton vertices respectively read [38]

$$V_{\beta,\alpha;\mu\nu}^{\text{spin-1}(1)}(k, k', m_v) = -\frac{i\kappa}{2}\eta_{\mu\nu}\left[(k \cdot k' + m_v^2)\eta_{\alpha\beta} - k_{(\alpha}k'_{\beta)}\right] + i\kappa I_{\mu\nu\kappa\lambda}\left[(k \cdot k' + m_v^2)I_{\alpha\beta}{}^{\kappa\lambda} + \frac{1}{2}k^{(\kappa}k'^{\lambda)}\eta_{\alpha\beta} - (k^\kappa k'_\alpha\delta_\beta^\lambda + k'^\kappa k_\beta\delta_\alpha^\lambda)\right], \quad (9)$$

and

$$V_{\beta,\alpha;\mu\nu\rho\sigma}^{\text{spin-1}(2)}(k, k', m_v) = \frac{i\kappa^2}{2}\mathcal{P}_{\mu\nu\rho\sigma}\left[(k \cdot k' + m_v^2)\eta_{\alpha\beta} - k_\beta k'_\alpha\right] - i\kappa^2\left\{I_{\mu\nu}{}^{\kappa\delta}I_{\rho\sigma\delta}{}^\lambda + I_{\rho\sigma}{}^{\kappa\delta}I_{\mu\nu\delta}{}^\lambda - \frac{1}{2}\left(\eta_{\mu\nu}I_{\rho\sigma}{}^{\kappa\lambda} + \eta_{\rho\sigma}I_{\mu\nu}{}^{\kappa\lambda}\right)\right\}\left[(k \cdot k' + m_v^2)I_{\alpha\beta\kappa\lambda} + \frac{1}{2}(k_\kappa k'_\lambda + k_\lambda k'_\kappa)\eta_{\alpha\beta} - (k_\kappa k'_\alpha\delta_{\beta\lambda} + k'_\kappa k_\beta\delta_{\alpha\lambda})\right] - \frac{i\kappa^2}{2}\left(I_{\mu\nu}{}^{\eta\theta}I_{\rho\sigma}{}^{\kappa\lambda} + I_{\rho\sigma}{}^{\eta\theta}I_{\mu\nu}{}^{\kappa\lambda}\right)\left[k_\eta\eta_{\alpha\kappa}(k'_\theta\eta_{\beta\lambda} - k'_\lambda\eta_{\beta\theta}) + k'_\eta\eta_{\beta\kappa}(k_\theta\eta_{\alpha\lambda} - k_\lambda\eta_{\alpha\theta})\right], \quad (10)$$

where $I_{\mu\nu\kappa\lambda}$ can be seen below Eq. (7). Let us now clarify the issue of the polarization, say $\epsilon^\mu(k)$, of the massive spin-1 field. Since the field has three independent degrees of freedom, we have the constraint $k \cdot \epsilon = 0$. Since the transfer momentum is given by $k_1 - k'_1 = k'_2 - k_2 = q$, we first write

$$\begin{aligned} \vec{k}_1 &= \vec{k} + \frac{\vec{q}}{2}, & \vec{k}'_1 &= \vec{k} - \frac{\vec{q}}{2}; \\ \vec{k}_2 &= \vec{k} - \frac{\vec{q}}{2}, & \vec{k}'_2 &= \vec{k} + \frac{\vec{q}}{2}, \end{aligned} \quad (11)$$

where $\vec{k}$ is the momentum of the centre of mass frame. We next write in the nonrelativistic limit for the polarization vector [38],

$$\epsilon^\mu(k)|_{\text{NR}} \simeq \left(\frac{\vec{k} \cdot \vec{\epsilon}}{m_v}, \vec{\epsilon}\right) \quad (12)$$

which trivially satisfies $k \cdot \epsilon = 0$ for $k_0 \approx -m_v$. We have

$$\begin{aligned} \epsilon_\mu(k_1)\epsilon^{\star\mu}(k'_1)|_{\text{NR}} &= \vec{\epsilon}(\vec{k}_1) \cdot \vec{\epsilon}^*(\vec{k}'_1) - \frac{i}{2m_v^2}\vec{S} \cdot (\vec{k} \times \vec{q}) \\ &\quad - \frac{1}{m_v^2}\vec{k} \cdot \vec{\epsilon}(\vec{k}_1)\vec{k} \cdot \vec{\epsilon}^*(\vec{k}'_1) \\ &\quad + \frac{1}{4m_v^2}\vec{q} \cdot \vec{\epsilon}(\vec{k}_1)\vec{q} \cdot \vec{\epsilon}^*(\vec{k}'_1), \end{aligned} \quad (13)$$

where we have written for the spin vector,

$$-i\vec{S} = \vec{\epsilon}(\vec{k}_1) \times \vec{\epsilon}^*(\vec{k}'_1), \quad (14)$$

and have used the trivial identity,

$$\vec{q} \cdot \vec{\epsilon} \vec{k} \cdot \vec{\epsilon}' = \vec{k} \cdot \vec{\epsilon} \vec{q} \cdot \vec{\epsilon}' + i\vec{S} \cdot (\vec{k} \times \vec{q}).$$

We use the abbreviation in the above and below: $\vec{\epsilon}^*(k') \equiv \vec{\epsilon}'$, for the sake of convenience.

Let us next come to the case of the massive spin-1/2 field. The propagator reads

$$S(k) = -\frac{i(k + m_f)}{k^2 + m_f^2}. \quad (15)$$

The relevant three and four point vertices respectively read [38]

$$V_{\mu\nu}^{\text{spin-1/2}(1)}(k, k', m_f) = -\frac{i\kappa}{2}\left[\frac{1}{8}\gamma_\mu(k + k')_\nu - \eta_{\mu\nu}\left(\frac{1}{2}(k + k') - m_f\right)\right], \quad (16)$$

and

$$\begin{aligned} V_{\mu\nu\rho\sigma}^{\text{spin-1/2}(2)}(k, k', m_f) &= i\kappa^2\left[-\frac{1}{2}\left\{\frac{1}{2}(k + k') - m_f\right\}\mathcal{P}_{\mu\nu\rho\sigma} - \frac{1}{32}\{\eta_{\mu\nu}\gamma_{(\rho}(k + k')_{\sigma)} + \eta_{\rho\sigma}\gamma_{(\mu}(k + k')_{\nu)}\}\right. \\ &\quad \left.+ \frac{3}{16}(k + k')^\epsilon\gamma^\xi(I_{\xi\phi\mu\nu}I_{\epsilon\rho\sigma}^\phi + I_{\xi\phi\rho\sigma}I_{\epsilon\mu\nu}^\phi) + \frac{i}{16}\epsilon^{\epsilon\phi\eta\lambda}\gamma_\lambda\gamma_5(I_{\rho\sigma\phi\xi}I_{\mu\nu\eta}{}^\xi k_\epsilon - I_{\mu\nu\phi\xi}I_{\rho\sigma\eta}{}^\xi(k + q)_\epsilon)\right], \end{aligned} \quad (17)$$

where in both the expressions above, the momenta are carried by the spin-1/2 field. We will also need the Gordon identity satisfied by the spinors,

$$\bar{u}_{s'}(k'_2)\gamma^\mu u_s(k_2) = \frac{1}{2m_f}\bar{u}_{s'}(k'_2)\left[(k_2 + k'_2)^\mu - \frac{1}{2}[\gamma^\mu, \gamma^\nu]q_\nu\right]u_s(k_2). \quad (18)$$

We take

$$u_s(\vec{k}) = \sqrt{E_{\vec{k}} + m_f} \begin{pmatrix} 1 \\ \frac{\vec{k} \cdot \vec{\sigma}}{E_{\vec{k}} + m_f} \end{pmatrix} \chi_s \quad (19)$$

where

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (20)$$

We next compute using Eqs. (19) and (20), in the non-relativistic limit,

$$\bar{u}_{s'}(\vec{k}_2) u_s(\vec{k}_2) = 2m_f \left[\delta_{ss'} - \frac{i}{2m_f^2} (\vec{k} \times \vec{q}) \cdot \vec{S}_{1/2} \right] + \text{subleading terms} \quad (21)$$

where we have defined the matrix elements of the spin vector as ($\hbar = 1$),

$$\vec{S}_{1/2}^{ss'} = \frac{1}{2} \chi_{s'}^\dagger \vec{\sigma} \chi_s.$$

The three graviton vertex reads, e.g., [38] and also references therein,

$$\begin{aligned} V_{\alpha\beta\gamma\delta}^{(3)\mu\nu}(k, q) = & -\frac{i\kappa}{2} \left[\mathcal{P}_{\alpha\beta\gamma\delta} \left(k^\mu k^\nu + (k-q)^\mu (k-q)^\nu + q^\mu q^\nu - \frac{3}{2} \eta^{\mu\nu} q^2 \right) + 2q_\lambda q_\sigma \left(I_{\alpha\beta}^{\sigma\lambda} I_{\gamma\delta}^{\mu\nu} + I_{\gamma\delta}^{\sigma\lambda} I_{\alpha\beta}^{\mu\nu} \right. \right. \\ & - I_{\alpha\beta}^{\mu\sigma} I_{\gamma\delta}^{\nu\lambda} - I_{\gamma\delta}^{\mu\sigma} I_{\alpha\beta}^{\nu\lambda} \left. \right) + \left\{ q_\lambda q^\mu \left(\eta_{\alpha\beta} I_{\gamma\delta}^{\nu\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}^{\nu\lambda} \right) + q_\lambda q^\nu \left(\eta_{\alpha\beta} I_{\gamma\delta}^{\mu\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}^{\mu\lambda} \right) \right. \\ & - q^2 \left(\eta_{\alpha\beta} I_{\gamma\delta}^{\mu\nu} + \eta_{\gamma\delta} I_{\alpha\beta}^{\mu\nu} \right) - \eta^{\mu\nu} q_\sigma q_\lambda \left(\eta_{\alpha\beta} I_{\gamma\delta}^{\sigma\lambda} + \eta_{\gamma\delta} I_{\alpha\beta}^{\sigma\lambda} \right) \left. \right\} + \left\{ -2q_\lambda \left(I_{\alpha\beta}^{\lambda\sigma} I_{\gamma\delta}^{\mu\nu} (k-q)^\mu \right. \right. \\ & + I_{\alpha\beta}^{\lambda\sigma} I_{\gamma\delta}^{\mu\nu} (k-q)^\nu + I_{\gamma\delta}^{\lambda\sigma} I_{\alpha\beta}^{\mu\nu} k^\mu + I_{\gamma\delta}^{\lambda\sigma} I_{\alpha\beta}^{\mu\nu} k^\nu \left. \right) + q^2 \left(I_{\alpha\beta}^{\mu\sigma} I_{\gamma\delta}^{\nu\sigma} + I_{\gamma\delta}^{\mu\sigma} I_{\alpha\beta}^{\nu\sigma} \right) \\ & + \eta^{\mu\nu} q_\sigma q_\lambda \left(I_{\alpha\beta}^{\lambda\rho} I_{\gamma\delta}^{\sigma\rho} + I_{\gamma\delta}^{\lambda\rho} I_{\alpha\beta}^{\sigma\rho} \right) \left. \right\} + \left\{ \left(k^2 + (k-q)^2 \right) \left(I_{\alpha\beta}^{\mu\sigma} I_{\gamma\delta}^{\nu\sigma} + I_{\gamma\delta}^{\mu\sigma} I_{\alpha\beta}^{\nu\sigma} \right. \right. \\ & \left. \left. - \frac{1}{2} \eta^{\mu\nu} \mathcal{P}_{\alpha\beta\gamma\delta} \right) - \left(I_{\gamma\delta}^{\mu\nu} \eta_{\alpha\beta} k^2 + I_{\alpha\beta}^{\mu\nu} \eta_{\gamma\delta} (k-q)^2 \right) \right\} \left. \right] \quad (22) \end{aligned}$$

where $\mathcal{P}_{\alpha\beta\gamma\delta}$ is given by Eq. (4).

Finally, we come to the issue of the nonminimal vertices ($\xi \neq 0$), Fig. 1. For one graviton-two scalar interaction, the relevant part of the action reads

$$S_\xi^{(1)} = \frac{\xi\kappa}{2} \int d^4x \eta_{\mu\nu} \phi^2 \partial^2 h^{\mu\nu} \quad (23)$$

The corresponding vertex function reads

$$V_{(\xi)}^{\alpha\beta}(q) = -i\xi\kappa\eta^{\alpha\beta}q^2. \quad (24)$$

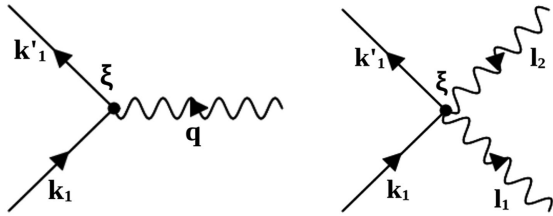


FIG. 1. The one and two graviton-two scalar nonminimal vertices. The dark circle on the junction represents that these vertices are nonminimal, compared to the minimal ones.

Note that the momentum appearing above is carried by the graviton, owing to the derivative acting on the same, Eq. (23). As we mentioned earlier, this is opposite to the minimal vertex, as the latter contains derivatives of only the scalar field and not of the graviton. The part of the action relevant for the two scalar-two graviton interaction reads

$$\begin{aligned} S_\xi^{(2)} = & \frac{\xi\kappa^2}{2} \int d^4x \phi^2 \left[\frac{1}{4} h \partial^2 h + \frac{1}{2} (\partial^\lambda h^{\mu\alpha}) (\partial_\alpha h_{\mu\lambda}) \right. \\ & \left. - h^{\mu\nu} \partial^2 h_{\mu\nu} - \frac{3}{4} (\partial^\lambda h^{\mu\nu}) (\partial_\lambda h_{\mu\nu}) \right] \quad (25) \end{aligned}$$

The two graviton-two scalar nonminimal vertex function reads,

$$\begin{aligned} V_{(\xi)}^{\mu\nu;\rho\sigma}(l_1, l_2) = & -\frac{i\xi\kappa^2}{4} \left[(l_1^2 + l_2^2) (\eta^{\mu\nu} \eta^{\rho\sigma} - 4\eta^{\mu\rho} \eta^{\nu\sigma}) \right. \\ & \left. - 2(l_1^\sigma l_2^\nu + l_1^\nu l_2^\sigma) \eta^{\mu\rho} + 6l_1 \cdot l_2 \eta^{\mu\rho} \eta^{\nu\sigma} \right]. \quad (26) \end{aligned}$$

As we have already mentioned earlier in Sec. I, there is a strong physical motivation to include a nonminimal interaction for a scalar owing to the issue of renormalisability [81]. However, to the best of our knowledge, there is no

sufficient purpose to consider such interactions for vector bosons or fermions. Hence, in order to maintain the simplest possible scenario, we will ignore any nonminimal interaction for nonspin-0 fields. In other words, ξ will affect the scattering of scalar and nonspin-0 fields via the exchange of gravitons only. With these equipments, we are ready for our scattering computations.

III. THE MASSIVE SPIN-0-MASSIVE SPIN-0 NONMINIMAL SCATTERING AND GRAVITATIONAL POTENTIAL

The scattering between two massive nonminimal scalars in the presence of a three point vertex due to a cosmological constant (originating from the $-2\Lambda\sqrt{-g}/\kappa^2$ term in the action) in a scale much smaller compared to the Hubble horizon was computed recently in [57]. For $\Lambda = 0$, it was also argued that the nonminimal coupling will result in a subleading long range gravitational potential compared to that of $\xi = 0$, but no explicit results were presented. We wish to find out in this section these explicit results for $\Lambda = 0$.

A. The tree diagrams

Let us begin with the tree diagram, given by the first of Fig. 2. Using Eqs. (3), (6), (24), the amplitude reads

$$i\mathcal{M}_{\text{ladder},1}^{\text{spin-0-spin-0}} = \frac{\xi\kappa^4}{2^3} \eta_{\mu\nu} \mathcal{P}^{\mu\nu\alpha\beta} \left\{ k'_{2(\alpha}(k_2 - l)_{\beta)} + \eta_{\alpha\beta} k'_2 \cdot l \right\} \frac{\left\{ k'_{1(\lambda}(l + k_1)_{\rho)} - \eta_{\rho\lambda} k_1 \cdot l \right\} \mathcal{P}^{\lambda\rho\gamma\delta} \left\{ k_{2(\gamma}(k_2 - l)_{\delta)} + \eta_{\gamma\delta} k_2 \cdot l \right\}}{(l + q)^2 [(l - k_2)^2 + m^2] [(l + k_1)^2 + M^2]}. \quad (28)$$

It is easy to see that the above amplitude also has no terms nonanalytic in the transfer momentum q^2 , and hence it has no contribution to long range gravitational potential. The same conclusion also holds for the three other ladder diagrams.

Let us now come to the cross-ladder diagrams, the third of Fig. 2, which also has four subcategories as of the ladder diagram. For the ξ -vertex placed upon the k'_1 line, we have

$$i\mathcal{M}_{\text{cross-ladder},1}^{\text{spin-0-spin-0}} = \frac{\xi\kappa^4}{2^3} \eta_{\mu\nu} \mathcal{P}^{\mu\nu\alpha\beta} \left\{ k'_{2(\alpha}(k'_2 + l)_{\beta)} + \eta_{\alpha\beta} k'_2 \cdot l \right\} \times \frac{\left\{ k'_{1(\lambda}(l + k_1)_{\rho)} - \eta_{\rho\lambda} k'_1 \cdot l \right\} \mathcal{P}^{\lambda\rho\gamma\delta} \left\{ k_{2(\gamma}(k'_2 + l)_{\delta)} + \eta_{\gamma\delta} k_2 \cdot l \right\}}{(l + q)^2 [(l + k'_2)^2 + m^2] [(l + k_1)^2 + M^2]}, \quad (29)$$

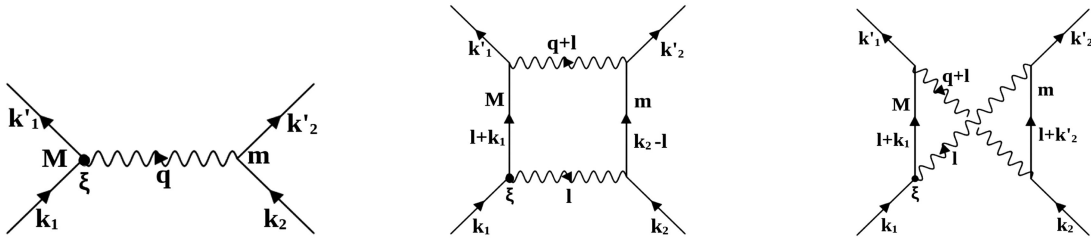


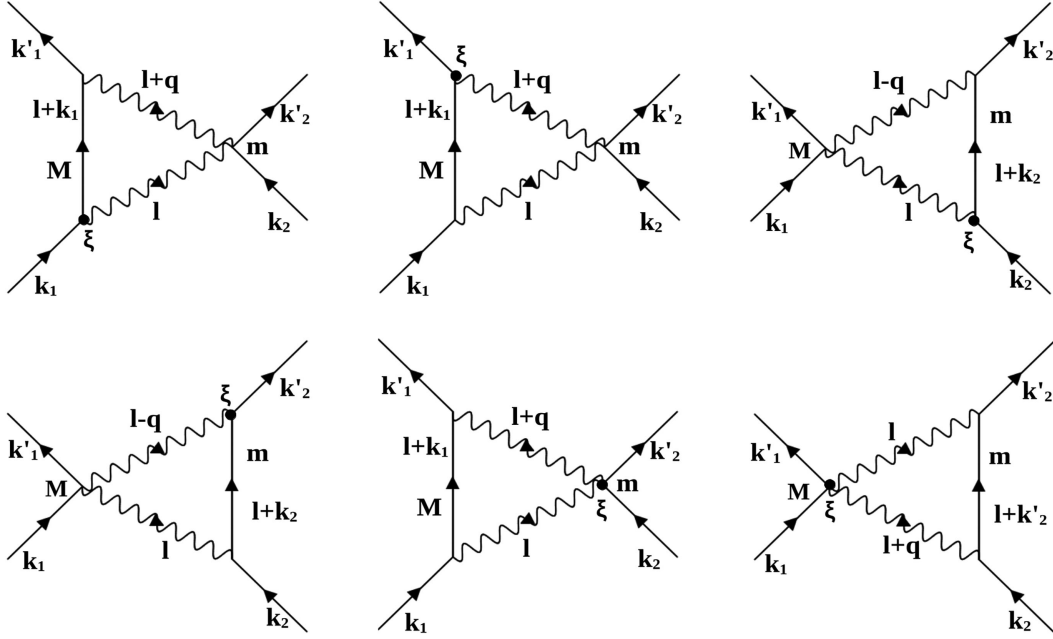
FIG. 2. The tree, ladder and cross-ladder diagrams at linear order in the nonminimal coupling parameter ξ . Hence the ladder and cross-ladder diagrams have four subcategories each, depending on the placement of the ξ -vertex, denoted by the thick circle. The tree diagram has two subcategories.

$$\begin{aligned} i\mathcal{M}_{\text{tree}}^{\text{spin-0-spin-0}} &= -i(-i\kappa\xi) \left(-\frac{i\kappa}{2} \right) \\ &\times \frac{q^2 \eta^{\alpha\beta} \mathcal{P}_{\mu\nu\alpha\beta} \left(k_2^\mu k_2^\nu + k_2^\nu k_2^\mu - \eta^{\mu\nu} (k_2 \cdot k'_2 + m^2) \right)}{q^2} \\ &= i\kappa^2 \xi m^2, \end{aligned} \quad (27)$$

where we have used the nonrelativistic limit, $k_2 \cdot k'_2 \approx -m^2$. Since this amplitude is independent of the transfer momentum q^2 , its Fourier transform is proportional to $\delta^3(\vec{r})$, and hence it does *not* contribute to any long range gravitational potential. Similar conclusion holds if we instead take the $k_2 - k'_2$ vertex to be nonminimal.

B. The ladder and cross-ladder diagrams

The ladder diagrams are given by the second of Fig. 2, having four subcategories depending upon whether the ξ -vertex is placed on the k_1 or k'_1 or k_2 or k'_2 lines. For the first, using $k_1 \cdot k'_1 \approx -M^2$, $k_2 \cdot k'_2 \approx -m^2$, we have


 FIG. 3. The triangle diagrams for massive spin-0-spin-0 fields nonminimal scattering at $\mathcal{O}(G^2\xi)$.

which also does not contribute to the long range potential. Likewise the other cross-ladder subcategories do not contribute to the same. This was first argued in [57].

C. The triangle diagrams

There are total six triangle diagrams given in Fig. 3. Note that for the first four of them the nonminimal vertices are three point, whereas for the last two, they are four point. The Feynman amplitude for the first diagram reads

$$\begin{aligned}
 i\mathcal{M}_{\text{Triangle-1}}^{\text{spin-0-spin-0}} &= \int \frac{d^4l}{(2\pi)^4} V_{\mu\nu}^{\text{spin-0}(1)}(l) V_{\alpha\beta}^{\text{spin-0}(1)}(l+k_1, k'_1, M) \\
 &\times \frac{-i\mathcal{P}^{\mu\nu\phi\lambda} - i\mathcal{P}^{\alpha\beta\rho\sigma}}{l^2} V_{\phi\lambda\rho\sigma}^{\text{spin-0}(2)}(k_2, k'_2, m) \\
 &\times \frac{-i}{[(l+k_1)^2 + M^2]} \quad (30)
 \end{aligned}$$

It is easy to check that the above reduces to an integral like

$$\sim \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2 [(l+k_1)^2 + M^2]},$$

which makes no contributions nonanalytic in the transfer momentum squared, q^2 . A similar conclusion holds for the second, third, and fourth triangle diagrams.

The last two diagrams containing four point nonminimal vertices [Eq. (25)] contribute to our present purpose as follows. Using the integrals given in Appendix, the Feynman amplitudes respectively read

$$\begin{aligned}
 i\mathcal{M}_{\text{Triangle-5}}^{\text{spin-0-spin-0}} &= \int \frac{d^4l}{(2\pi)^4} V_{\mu\nu}^{\text{spin-0}(1)}(k_1, l+k_1, M) V_{\alpha\beta}^{\text{spin-0}(1)}(l+k_1, k'_1, M) \frac{-i\mathcal{P}^{\mu\nu\phi\lambda} - i\mathcal{P}^{\alpha\beta\rho\sigma}}{l^2} V_{\phi\lambda\rho\sigma}^{\text{spin-0}(2)}(l+q, l) \frac{-i}{[(l+k_1)^2 + M^2]} \\
 &= iG^2\xi \left(\frac{2q^6 \ln q^2}{M^2} + 18q^4 \ln q^2 - \frac{140}{3} M^2 q^2 \ln q^2 + \frac{\pi^2 q^5}{M} + \frac{27}{2} \pi^2 M q^3 + \frac{16}{3} M^2 q^2 - 32\pi^2 M^3 q \right), \quad (31)
 \end{aligned}$$

and

$$\begin{aligned}
 i\mathcal{M}_{\text{Triangle-6}}^{\text{spin-0-spin-0}} &= \int \frac{d^4l}{(2\pi)^4} V_{\phi\lambda\rho\sigma}^{\text{spin-0}(2)}(l, l+q) \frac{-i\mathcal{P}^{\phi\lambda\mu\nu} - i\mathcal{P}^{\rho\sigma\alpha\beta}}{l^2} V_{\mu\nu}^{\text{spin-0}(1)}(l+k_2, k'_2, m) V_{\alpha\beta}^{\text{spin-0}(1)}(k_2, l+k_2, m) \frac{-i}{[(l+k'_2)^2 + m^2]} \\
 &= iG^2\xi \left(\frac{2q^6 \ln q^2}{m^2} + 18q^4 \ln q^2 - \frac{140}{3} m^2 q^2 \ln q^2 + \frac{\pi^2 q^5}{m} + \frac{27}{2} \pi^2 m q^3 + \frac{16}{3} m^2 q^2 - 32\pi^2 m^3 q \right), \quad (32)
 \end{aligned}$$

where $q^{2n+1} = (q^2)^{(2n+1)/2}$ is understood. We need to retain only the pieces that are nonanalytic in q^2 . In the nonrelativistic limit we take $q \approx \{0, \vec{q}\}$. The gravitational potential is then defined as the Fourier transform

$$V(\vec{r}) = -\frac{1}{4Mm} \int \frac{d^3\vec{q}}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \mathcal{M}(\vec{q}^2) \Big|_{\text{nonanalytic}}.$$

Using then Table I written in Appendix, the long range gravitational potentials corresponding to Eqs. (31) and (32), respectively reads

$$\begin{aligned} V_{\text{Triangle-5}}^{\text{spin-0-spin-0}}(G^2, r, \xi) &= \frac{G^2 \xi}{mr^4} \left(-8M^2 + \frac{35M}{\pi r} - \frac{81}{2r^2} + \frac{270}{\pi M r^3} - \frac{90}{M^2 r^4} + \frac{1260}{\pi M^3 r^5} \right), \\ V_{\text{Triangle-6}}^{\text{spin-0-spin-0}}(G^2, r, \xi) &= \frac{G^2 \xi}{Mr^4} \left(-8m^2 + \frac{35m}{\pi r} - \frac{81}{2r^2} + \frac{270}{\pi m r^3} - \frac{90}{m^2 r^4} + \frac{1260}{\pi m^3 r^5} \right). \end{aligned} \quad (33)$$

Note that the positions of m and M are flipped in the two above potentials, as one expects from the topology of the respective diagrams. Note also that the terms originating from the $\ln \vec{q}^2$ should be interpreted as quantum.

D. The seagull diagrams

There are total six seagull diagrams as depicted in Fig. 4. Note the two groups of different topologies here, i.e., diagrams one to four and five to six. The Feynman amplitude for the first diagram is,

$$\begin{aligned} i\mathcal{M}_{\text{seagull-1}}^{\text{spin-0-spin-0}} &= \int \frac{d^4 l}{(2\pi)^4} V_{\text{spin-0}(1)}^{\mu\nu}(k_1, k'_1, M) \frac{-i\mathcal{P}_{\mu\nu\alpha\beta}}{q^2} V_{\gamma\delta\rho\sigma}^{\alpha\beta(3)}(l-q, -q) \frac{-i\mathcal{P}^{\gamma\delta\lambda\phi}}{(l-q)^2} \frac{-i\mathcal{P}^{\rho\sigma\psi\theta}}{l^2} V_{\lambda\phi(\xi)}^{\text{spin-0}(1)}(l-q) \\ &\quad \times V_{\psi\theta}^{\text{spin-0}(1)}(k_2, l+k_2, m) \frac{-i}{[(l+k_2)^2 + m^2]}, \end{aligned} \quad (34)$$

which reduces to an integral like

$$\sim \frac{1}{q^2} \int \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 [(l+k_2)^2 + m^2]} = \frac{i}{16\pi^2 q^2} \left[\frac{2}{\epsilon} - \ln \frac{m^2}{4\pi\mu^2} + 2(\ln 2 - 1) \right].$$

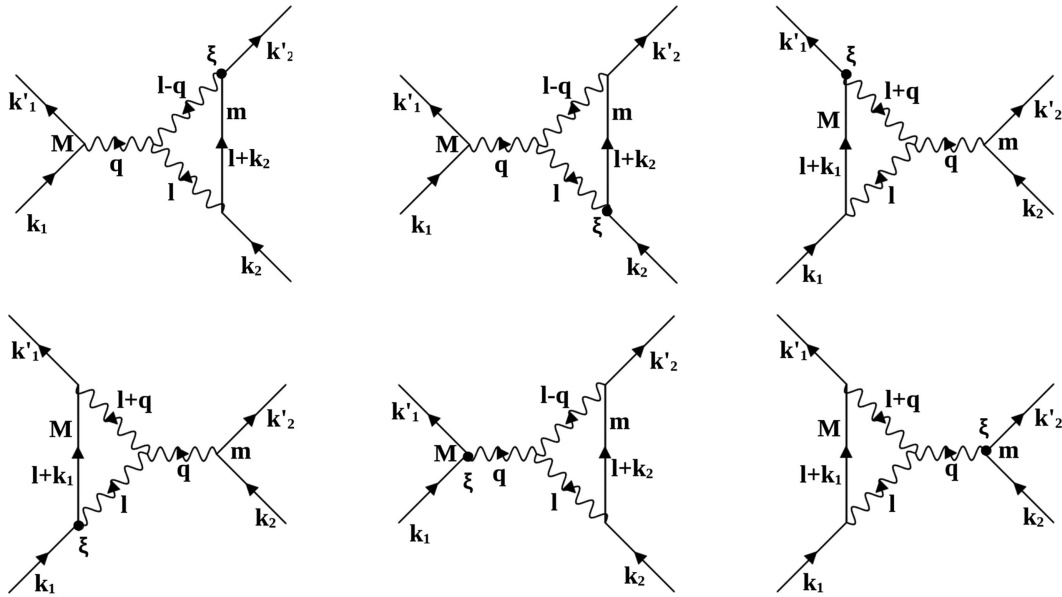


FIG. 4. The seagull diagrams for massive spin-0-spin-0 fields scattering at $\mathcal{O}(G^2 \xi)$.

Note that the q^2 appearing above comes from the graviton propagator, and it gets factorized with the rest of the amplitude, which is basically the 1PI one loop correction [$\mathcal{O}(\kappa^3\xi)$] of the three point nonminimal vertex. The above contribution coming from this vertex function is just a constant, and hence it can be absorbed

in a vertex counterterm. Thus Eq. (34) does not contribute to the gravitational potential. A similar conclusion holds for the seagull diagrams 2, 3 and 4. The situation is, however, different for the fifth and sixth seagull diagrams. The corresponding amplitudes respectively read

$$\begin{aligned} i\mathcal{M}_{\text{seagull-5}}^{\text{spin-0-spin-0}} &= \int \frac{d^4l}{(2\pi)^4} V_{\text{spin-0}(1)}^{\rho\sigma(\xi)}(q) \frac{-i\mathcal{P}_{\rho\sigma\mu\nu}}{q^2} V_{\alpha\beta\gamma\delta}^{\mu\nu(3)}(l-q, -q) \frac{-i\mathcal{P}_{\gamma\delta\psi\theta}}{l^2} \frac{-i\mathcal{P}^{\alpha\beta\lambda\phi}}{(l-q)^2} V_{\psi\theta}^{\text{spin-0}(1)}(k_2, l+k_2, m) \\ &\quad \times V_{\lambda\phi}^{\text{spin-0}(1)}(l+k_2, k'_2, m) \frac{-i}{[(l+k_2)^2 + m^2]} \\ &= iG^2\xi \left[q^2 \log q^2 \left(\frac{16q^4}{m^2} + 48q^2 + 72m^2 \right) + \frac{6\pi^2 q^5}{m} + 14\pi^2 m q^3 + 48\pi^2 m^3 q \right], \end{aligned} \quad (35)$$

and

$$\begin{aligned} i\mathcal{M}_{\text{seagull-6}}^{\text{spin-0-spin-0}} &= \int \frac{d^4l}{(2\pi)^4} V_{\lambda\phi}^{\text{spin-0}(1)}(l+k_1, k'_1, M) V_{\psi\theta}^{\text{spin-0}(1)}(k_1, l+k_1, M) \frac{-i\mathcal{P}^{\psi\theta\gamma\delta}}{l^2} \frac{-i\mathcal{P}^{\lambda\phi\alpha\beta}}{(l+q)^2} V_{\alpha\beta\gamma\delta}^{\mu\nu(3)}(l+q, q) \\ &\quad \times \frac{-i\mathcal{P}_{\mu\nu\rho\sigma}}{q^2} V_{\text{spin-0}(1)}^{\rho\sigma(\xi)}(q) \frac{-i}{[(l+k_1)^2 + M^2]} \\ &= iG^2\xi \left[q^2 \ln q^2 \left(\frac{16q^4}{M^2} + 48q^2 + 72M^2 \right) + \frac{6\pi^2 q^5}{M} + 14\pi^2 M q^3 + 48\pi^2 M^3 q \right]. \end{aligned} \quad (36)$$

The corresponding potentials are

$$\begin{aligned} V_{\text{seagull-5}}^{\text{spin-0-spin-0}}(G^2, r, \xi) &= \frac{6G^2\xi}{Mr^4} \left(2m^2 - \frac{9m}{\pi r} - \frac{7}{r^2} + \frac{120}{\pi m r^3} - \frac{90}{m^2 r^4} + \frac{1680}{\pi m^3 r^5} \right), \\ V_{\text{seagull-6}}^{\text{spin-0-spin-0}}(G^2, r, \xi) &= \frac{6G^2\xi}{mr^4} \left(2M^2 - \frac{9M}{\pi r} - \frac{7}{r^2} + \frac{120}{\pi M r^3} - \frac{90}{M^2 r^4} + \frac{1680}{\pi M^3 r^5} \right). \end{aligned} \quad (37)$$

Note once again as a consistency check the symmetry under the interchange of m and M in the above potentials.

E. The double seagull diagrams

Let us now come to the two double seagull diagrams given by Fig. 5. Their Feynman amplitudes read

$$\begin{aligned} i\mathcal{M}_{\text{double seagull-1}}^{\text{spin-0-spin-0}} &= \frac{1}{2!} \int \frac{d^4l}{(2\pi)^4} V_{\eta\lambda\rho\sigma(\xi)}^{\text{spin-0}(2)}(l, l+q) \frac{-i\mathcal{P}^{\rho\sigma\mu\nu}}{(l+q)^2} \frac{-i\mathcal{P}^{\eta\lambda\alpha\beta}}{l^2} V_{\alpha\beta\mu\nu}^{\text{spin-0}(2)}(k_2, k'_2, m) \\ &= i\frac{40}{3}G^2\xi q^4 \ln q^2 + i\frac{584}{3}G^2m^2\xi q^2 \ln q^2, \end{aligned} \quad (38)$$

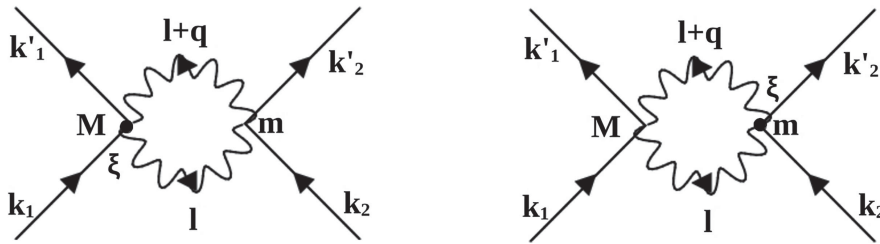


FIG. 5. The double seagull diagrams for massive spin-0-spin-0 fields scattering.

and

$$\begin{aligned}
 i\mathcal{M}_{\text{double seagull-2}}^{\text{spin-0-spin-0}} &= \frac{1}{2!} \int \frac{d^4 l}{(2\pi)^4} V_{\eta\lambda\rho\sigma}^{\text{spin-0}(2)}(k_1, k'_1, M) \frac{-i\mathcal{P}^{\eta\lambda\alpha\beta} - i\mathcal{P}^{\rho\sigma\mu\nu}}{l^2} \frac{-i\mathcal{P}^{\alpha\beta\mu\nu}}{(l+q)^2} V_{\alpha\beta\mu\nu}^{\text{spin-0}(2)}(l+q, l) \\
 &= i\frac{40}{3} G^2 \xi q^4 \ln q^2 + i\frac{584}{3} G^2 M^2 \xi q^2 \ln q^2.
 \end{aligned} \quad (39)$$

Accordingly, the corresponding contributions to the gravitational potential are evaluated to be

$$V_{\text{double seagull-1}}^{\text{spin-0-spin-0}}(G^2, r, \xi) = \frac{2G^2 \xi}{\pi M r^5} \left(-73m + \frac{100}{mr^2} \right), \quad V_{\text{double seagull-2}}^{\text{spin-0-spin-0}}(G^2, r, \xi) = \frac{2G^2 \xi}{\pi m r^5} \left(-73M + \frac{100}{Mr^2} \right). \quad (40)$$

F. The fish diagrams

There are total four fish diagrams, Fig. 6, all of which contribute to the long range gravitational potential. The corresponding Feynman amplitudes read,

$$\begin{aligned}
 i\mathcal{M}_{\text{fish-1}}^{\text{spin-0-spin-0}} &= \frac{1}{2!} \int \frac{d^4 l}{(2\pi)^4} V_{\text{spin-0}(1)}^{\lambda\phi(\xi)}(q) \frac{-i\mathcal{P}_{\lambda\phi\mu\nu}}{q^2} V_{\alpha\beta\gamma\delta}^{\mu\nu(3)}(l, -q) \frac{-i\mathcal{P}^{\alpha\beta\psi\theta} - i\mathcal{P}^{\gamma\delta\rho\sigma}}{l^2} \frac{-i\mathcal{P}^{\rho\sigma\mu\nu}}{(l+q)^2} V_{\psi\theta\rho\sigma}^{\text{spin-0}(2)}(k_2, k'_2, m) \\
 &= -i64G^2 \xi q^4 \ln q^2 - i1136G^2 m^2 \xi q^2 \ln q^2,
 \end{aligned} \quad (41)$$

$$\begin{aligned}
 i\mathcal{M}_{\text{fish-2}}^{\text{spin-0-spin-0}} &= \frac{1}{2!} \int \frac{d^4 l}{(2\pi)^4} V_{\rho\sigma\psi\theta}(k_1, k'_1, M) \frac{-i\mathcal{P}^{\rho\sigma\gamma\delta} - i\mathcal{P}^{\psi\theta\alpha\beta}}{l^2} \frac{-i\mathcal{P}^{\mu\nu(3)}}{(l+q)^2} V_{\alpha\beta\gamma\delta}^{\mu\nu(3)}(l+q, q) \frac{-i\mathcal{P}_{\mu\nu\lambda\phi}}{q^2} V_{\text{spin-0}(1)}^{\lambda\phi(\xi)}(q) \\
 &= -i64G^2 \xi q^4 \ln q^2 - i1136G^2 M^2 \xi q^2 \ln q^2,
 \end{aligned} \quad (42)$$

$$\begin{aligned}
 i\mathcal{M}_{\text{fish-3}}^{\text{spin-0-spin-0}} &= \frac{1}{2!} \int \frac{d^4 l}{(2\pi)^4} V_{\text{spin-0}(1)}^{\lambda\phi}(k_1, k'_1, m) \frac{-i\mathcal{P}_{\lambda\phi\mu\nu}}{q^2} V_{\alpha\beta\gamma\delta}^{\mu\nu(3)}(l, -q) \frac{-i\mathcal{P}^{\alpha\beta\psi\theta} - i\mathcal{P}^{\gamma\delta\rho\sigma}}{l^2} \frac{-i\mathcal{P}^{\rho\sigma\mu\nu}}{(l+q)^2} V_{\rho\sigma\psi\theta}^{\text{spin-0}(2)}(l+q, l) \\
 &= -\frac{310i}{3} G^2 \xi q^4 \ln q^2 - \frac{2060i}{3} G^2 M^2 \xi q^2 \ln q^2,
 \end{aligned} \quad (43)$$

and

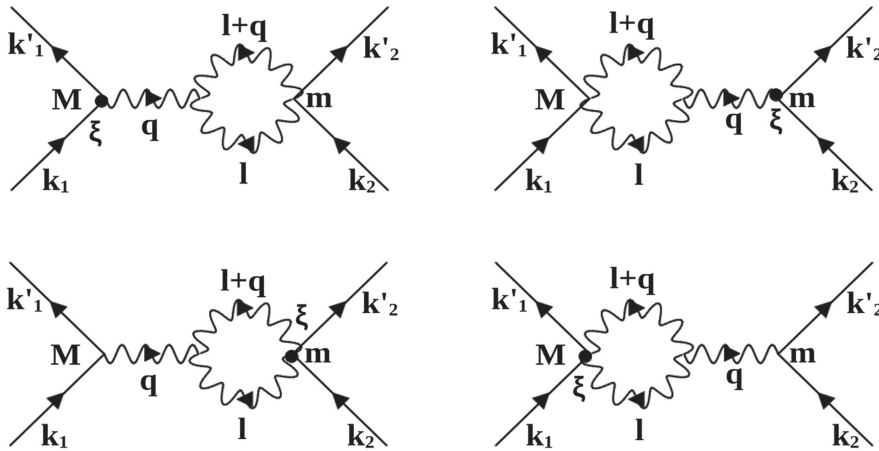


FIG. 6. The fish diagrams for massive spin-0-spin-0 fields scattering.

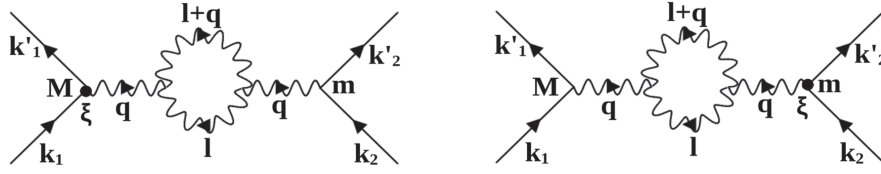


FIG. 7. The vacuum polarization diagrams for massive spin-0-spin-0 fields scattering. The contributions from the ghost loop needs also to be added.

$$\begin{aligned}
 i\mathcal{M}_{\text{fish-4}}^{\text{spin-0-spin-0}} &= \frac{1}{2!} \int \frac{d^4 l}{(2\pi)^4} V_{\rho\sigma\psi\theta(\xi)}^{\text{spin-0}(2)}(l, l+q) \frac{-i\mathcal{P}^{\rho\sigma\gamma\delta}}{l^2} \frac{-i\mathcal{P}^{\psi\theta\alpha\beta}}{(l+q)^2} V_{\alpha\beta\gamma\delta}^{\mu\nu(3)}(l+q, q) \frac{-i\mathcal{P}_{\mu\nu\lambda\phi}}{q^2} V_{\text{spin-0}(1)}^{\lambda\phi}(k_2, k'_2, m) \\
 &= -\frac{310i}{3} G^2 \xi q^4 \ln q^2 - \frac{2060i}{3} G^2 m^2 \xi q^2 \ln q^2.
 \end{aligned} \tag{44}$$

Their respective contributions to the gravitational potential are given by

$$\begin{aligned}
 V_{\text{fish-1}}^{\text{spin-0-spin-0}}(G^2, r, \xi) &= \frac{4G^2 \xi}{\pi M r^5} \left(213m - \frac{240}{mr^2} \right), & V_{\text{fish-2}}^{\text{spin-0-spin-0}}(G^2, r, \xi) &= \frac{4G^2 \xi}{\pi m r^5} \left(213M - \frac{240}{Mr^2} \right), \\
 V_{\text{fish-3}}^{\text{spin-0-spin-0}}(G^2, r, \xi) &= \frac{5G^2 \xi}{\pi m r^5} \left(103M - \frac{310}{Mr^2} \right), & V_{\text{fish-4}}^{\text{spin-0-spin-0}}(G^2, r, \xi) &= \frac{5G^2 \xi}{\pi M r^5} \left(103m - \frac{310}{mr^2} \right).
 \end{aligned} \tag{45}$$

G. The vacuum polarization diagrams

Finally, we come to the two vacuum polarization diagrams at $\mathcal{O}(G^2 \xi)$ as shown in Fig. 7. The Feynman amplitudes are given by

$$\begin{aligned}
 \mathcal{M}_{\text{vac pol-1}}^{\text{spin-0-spin-0}} &= V_{\mu\nu(\xi)}^{\text{spin-0}(1)}(q) \frac{-i\mathcal{P}^{\mu\nu\rho\sigma}}{q^2} \Pi_{\rho\sigma\lambda\phi}(q) \frac{-i\mathcal{P}^{\lambda\phi\gamma\delta}}{q^2} V_{\gamma\delta}^{\text{spin-0}(1)}(k_2, k'_2, m) \\
 &= 12G^2 \xi q^4 \ln q^2 + 24G^2 m^2 \xi q^2 \ln q^2,
 \end{aligned} \tag{46}$$

and

$$\begin{aligned}
 \mathcal{M}_{\text{vac pol-2}}^{\text{spin-0-spin-0}} &= V_{\mu\nu}^{\text{spin-0}(1)}(k_1, k'_1, M) \frac{-i\mathcal{P}^{\mu\nu\rho\sigma}}{q^2} \Pi_{\rho\sigma\lambda\phi}(q) \frac{-i\mathcal{P}^{\lambda\phi\gamma\delta}}{q^2} V_{\gamma\delta(\xi)}^{\text{spin-0}(1)}(q) \\
 &= 12G^2 \xi q^4 \ln q^2 + 24G^2 M^2 \xi q^2 \ln q^2,
 \end{aligned} \tag{47}$$

where $\Pi_{\alpha\beta\gamma\delta}$ is the one loop graviton self-energy due to itself after adding the ghost contribution, reading [8],

$$\begin{aligned}
 \Pi_{\alpha\beta\gamma\delta} &= -\frac{2G}{\pi} \ln q^2 \left[\frac{21}{120} q^4 I_{\alpha\beta\gamma\delta} + \frac{23}{120} q^4 \eta_{\alpha\beta} \eta_{\gamma\delta} - \frac{23}{120} q^2 (\eta_{\alpha\beta} q_\gamma q_\delta + \eta_{\gamma\delta} q_\alpha q_\beta) \right. \\
 &\quad \left. - \frac{21}{240} q^2 (q_\alpha q_\delta \eta_{\beta\gamma} + q_\beta q_\delta \eta_{\alpha\gamma} + q_\alpha q_\gamma \eta_{\beta\delta} + q_\beta q_\gamma \eta_{\alpha\delta}) + \frac{11}{30} q_\alpha q_\beta q_\gamma q_\delta \right].
 \end{aligned} \tag{48}$$

The potentials for these two diagrams read

$$V_{\text{vac pol-1}}^{\text{spin-0-spin-0}}(G^2, r, \xi) = \frac{18G^2 \xi}{\pi M r^5} \left(-m + \frac{10}{mr^2} \right), \quad V_{\text{vac pol-2}}^{\text{spin-0-spin-0}}(G^2, r, \xi) = \frac{18G^2 \xi}{\pi m r^5} \left(-M + \frac{10}{Mr^2} \right). \tag{49}$$

H. The full long range gravitational potential at $\mathcal{O}(G^2 \xi)$

Combining now the different contributions from Eqs. (33), (37), (40), (45), (63), we finally obtain the total long range gravitational potential at $\mathcal{O}(G^2 \xi)$,

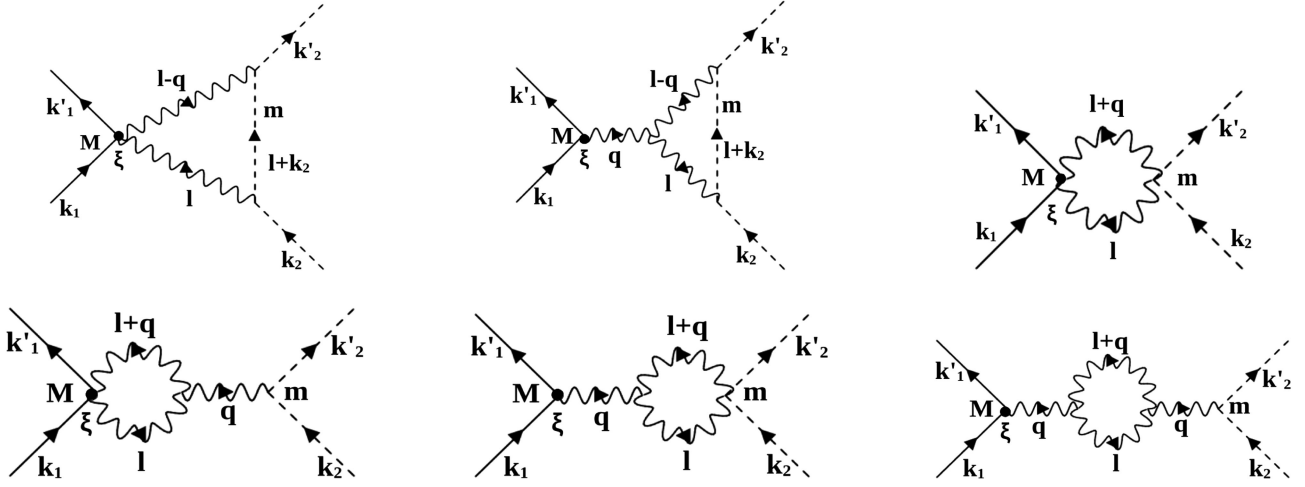


FIG. 8. The diagrams for massive spin-0-spin-1 and massive spin-0-spin-1/2 fields scattering. The broken lines will consecutively represent the massive spin-1 field, and in the next section, a massive spin-1/2 field. We have fixed (k_1, k'_1) for the scalar. Since there are no nonminimal interactions for the spin fields, we have much less number of diagrams compared to the scalar-scalar scattering discussed in the preceding section.

$$\begin{aligned}
 V^{\text{spin-0-spin-0}}(G^2, r, \xi) &= \frac{4G^2\xi}{r^4} \left[\left(\frac{M^2}{m} + \frac{m^2}{M} \right) + \left(\frac{M}{m} + \frac{m}{M} \right) \frac{296}{\pi r} - \left(\frac{1}{m} + \frac{1}{M} \right) \frac{165}{8r^2} \right. \\
 &\quad \left. - \frac{570}{\pi m M r^3} - \left(\frac{1}{m} + \frac{1}{M} \right) \frac{315}{2m M r^4} + \left(\frac{1}{M^2} + \frac{1}{m^2} \right) \frac{2835}{\pi m M r^5} \right]. \quad (50)
 \end{aligned}$$

Since there is no tree level contribution in this case, the above is the *leading* result in the perturbative expansion. Note that the result is symmetric under the interchange of the two masses, as is expected.

We now wish to extend Eq. (50) for scattering of massive spin-1 and spin-1/2 fields. For $\xi = 0$, relevant computations can be seen in [38]. The Feynman diagrams that contribute to this purpose are shown in Fig. 8. The solid lines and the thick circles, as earlier, represent the scalar and the nonminimal vertex. The broken lines will

respectively stand for the massive spin-1 (Sec. IV) and massive spin-1/2 (Sec. V) fields. As in the scalar-scalar scattering, the tree and (cross-)ladder diagrams do not make any contributions in these cases as well. Note in particular that there are no nonminimal interactions for these spin fields. Hence there are many fewer subcategories of diagrams here, as compared to the scalar-scalar scattering discussed above.

IV. MASSIVE SPIN-0-SPIN-1 INTERACTION

Let us compute the long range gravitational potential between a massive spin-0 and a massive spin-1 field at the leading order $\mathcal{O}(G^2\xi)$ in this section.

A. The triangle diagram

There is only one triangle diagram that contributes to our present purpose in this case, given by the first of Fig. 8. The Feynman amplitude in the nonrelativistic limit reads

$$\begin{aligned}
 i\mathcal{M}_{\text{Triangle}}^{\text{spin-0-spin-1}}|_{\text{NR}} &= \int \frac{d^4l}{(2\pi)^4} V_{\rho\sigma\mu\theta(\xi)}^{\text{spin-0}(2)}(l, l-q) \frac{-i\mathcal{P}^{\psi\theta\mu\nu} - i\mathcal{P}^{\rho\sigma\alpha\beta}}{l^2} V_{\tau\chi\mu\nu}^{\text{spin-1}(1)}(k_2, l+k_2, m_v) \epsilon^\tau(k_2) \\
 &\quad \times V_{\eta,\xi,\alpha\beta}^{\text{spin-1}(1)}(l+k_2, k'_2, m_v) \epsilon^{\star\eta}(k'_2) \frac{-i\mathcal{P}^{\chi\eta}}{[(l+k_2)^2 + m_v^2]} \Big|_{\text{NR}} \\
 &= iG^2\xi \left\{ \vec{q}^2 \ln \vec{q}^2 \left(\frac{7\vec{q}^6}{2m_v^4} + \frac{\vec{q}^4}{m_v^2} - \frac{215\vec{q}^2}{3} - 116m_v^2 \right) + \frac{35\pi^2\vec{q}^7}{32m_v^3} + \frac{21\pi^2\vec{q}^5}{16m_v} + 6\vec{q}^4 - \frac{315}{4}m_v\pi^2\vec{q}^3 \right. \\
 &\quad \left. + \frac{8}{3}m_v^2\vec{q}^2 - 26m_v^3\pi^2\vec{q} \right\} \vec{\epsilon} \cdot \vec{\epsilon}' + \left\{ \vec{q}^2 \ln \vec{q}^2 \left(-\frac{7\vec{q}^6}{2m_v^4} - \frac{\vec{q}^4}{m_v^2} + \frac{215\vec{q}^2}{3m_v^2} \vec{q} + 116 \right) - \frac{35\pi^2\vec{q}^7}{32m_v^3} \right. \\
 &\quad \left. - \frac{21\pi^2\vec{q}^5}{16m_v^3} - \frac{6\vec{q}^4}{m_v^2} + \frac{315\pi^2}{4m_v} \vec{q}^3 - \frac{8\vec{q}^2}{3} + 26m_v\pi^2\vec{q} \right\} \vec{k} \cdot \vec{\epsilon} \vec{k}' \cdot \vec{\epsilon}' + \left\{ \vec{q}^2 \ln \vec{q}^2 \left(\frac{7\vec{q}^6}{8m_v^6} - \frac{23\vec{q}^4}{6m_v^4} - \right. \right.
 \end{aligned}$$

$$\begin{aligned} & \times \frac{217\vec{q}^2}{12m_v^2} - 120 \Big) - 20m_v^2 \ln \vec{q}^2 + \frac{35\pi^2\vec{q}^7}{128m_v^5} - \frac{5\pi^2}{4m_v^3} - 2\vec{q}^5 + \frac{41\vec{q}^4}{6m_v^2} - \frac{137\pi^2\vec{q}^3}{8m_v} + \frac{4\vec{q}^2}{3} \\ & + \frac{43}{8}m_v\pi^2\vec{q} - \frac{13}{2}m_v\pi^2\vec{q} + \frac{10m_v^3\pi^2}{\vec{q}} \Big\} \vec{q} \cdot \vec{\epsilon} \vec{q} \cdot \vec{\epsilon}' + \left\{ \vec{q}^2 \ln \vec{q}^2 \left(-\frac{7\vec{q}^6}{4m_v^6} - \frac{\vec{q}^4}{2m_v^4} + \frac{215\vec{q}^2}{6m_v^2} + 58 \right) \right. \\ & \left. - \frac{35\pi^2\vec{q}^7}{64m_v^5} - \frac{21\pi^2\vec{q}^5}{32m_v^3} - \frac{3\vec{q}^4}{m_v^2} + \frac{315\pi^2\vec{q}^3}{8m_v} - \frac{4\vec{q}^2}{3} + 13m_v\pi^2\vec{q} \right\} i(\vec{k} \times \vec{q}) \cdot \vec{S} \Big], \end{aligned} \quad (51)$$

where ϵ and $\vec{S}$ stand respectively for the polarization and spin vector of the massive spin-1 field, introduced and discussed in the nonrelativistic limit in the centre of mass frame in Eqs. (11)–(14). Also, $\vec{q}^n = |\vec{q}|^n$ in the above expression and in the following is understood. We also have abbreviated for convenience

$$\mathcal{P}^{\chi\eta} = \eta^{\chi\eta} + \frac{(l+k_2)^\chi(l+k_2)^\eta}{m_v^2}. \quad (52)$$

The corresponding potential reads

$$\begin{aligned} V_{\text{triangle}}^{\text{spin-0-spin-1}}(G^2, r, \xi) = & \frac{G^2\xi}{r^4} \left[\frac{1}{M} \left(-9m_v^2 + \frac{92m_v}{\pi r} + \frac{1881}{8r^2} - \frac{625}{\pi m_v r^3} + \frac{5877}{8m_v^2 r^4} - \frac{5145}{4\pi m_v^3 r^5} - \frac{26325}{4m_v^4 r^6} - \frac{227745}{2\pi m_v^5 r^7} \right. \right. \\ & + \frac{55125}{4m_v^6 r^8} - \frac{436590}{\pi m_v^7 r^9} \Big) \vec{\epsilon} \cdot \vec{\epsilon}' + \frac{1}{M} \left(\frac{13}{2} - \frac{87}{\pi m_v r} - \frac{945}{4m_v^2 r^2} + \frac{1075}{\pi m_v^3 r^3} - \frac{945}{8m_v^4 r^4} - \frac{630}{\pi m_v^5 r^5} \right. \\ & + \frac{11025}{2m_v^6 r^6} + \frac{158760}{\pi m_v^7 r^7} \Big) \vec{k} \cdot \vec{\epsilon} \vec{k} \cdot \vec{\epsilon}' + \frac{1}{Mr} \left(-13 + \frac{435}{2\pi m_v r} + \frac{2835}{4m_v^2 r^2} - \frac{7525}{2\pi m_v^3 r^3} + \frac{945}{2m_v^4 r^4} \right. \\ & + \frac{2835}{\pi m_v^5 r^5} - \frac{55125}{2m_v^6 r^6} - \frac{873180}{\pi m_v^7 r^7} \Big) (\vec{k} \times \hat{r}) \cdot \vec{S} + \frac{1}{Mr^2} \left(10m_v^2 r^2 + \frac{27}{4} - \frac{3150}{\pi m_v r} - \frac{4932}{m_v^2 r^2} \right. \\ & \left. \left. + \frac{68985}{4\pi m_v^3 r^3} + \frac{21375}{2m_v^4 r^4} - \frac{987525}{2\pi m_v^5 r^5} - \frac{165375}{m_v^6 r^6} + \frac{5675670}{\pi m_v^7 r^7} \right) \hat{r} \cdot \vec{\epsilon} \hat{r} \cdot \vec{\epsilon}' \right]. \end{aligned} \quad (53)$$

B. The seagull diagram

There is one seagull diagram here given by the second of Fig. 8, the Feynman amplitude for which in the nonrelativistic limit reads

$$\begin{aligned} i\mathcal{M}_{\text{seagull}}^{\text{spin-0-spin-1}}|_{\text{NR}} = & \int \frac{d^4l}{(2\pi)^4} V_{\text{spin-0}(1)}^{\rho\sigma(\xi)}(q) \frac{-i\mathcal{P}_{\rho\sigma\mu\nu}}{q^2} V_{\alpha\beta\gamma\delta}^{\mu\nu(3)}(l-q, -q) \frac{-i\mathcal{P}^{\gamma\delta\psi\theta}}{l^2} \frac{-i\mathcal{P}^{\alpha\beta\lambda\phi}}{(l-q)^2} V_{\tau,\chi,\psi\theta}^{\text{spin-1}(1)}(k_2, l+k_2, m_v) \epsilon^\tau(k_2) \\ & \times V_{\eta,\zeta,\lambda\phi}^{\text{spin-1}(1)}(l+k_2, k'_2, m_v) \epsilon^{\star\zeta}(k'_2) \frac{-i\mathcal{P}^{\chi\eta}}{[(l+k_2)^2 + m_v^2]} \Big|_{\text{NR}} \\ = & iG^2\xi \left[\left\{ \vec{q}^2 \ln \vec{q}^2 \left(-\frac{62\vec{q}^6}{3m_v^4} + \frac{64\vec{q}^4}{m_v^2} + \frac{1828\vec{q}^2}{3} + 696m_v^2 \right) - \frac{13\pi^2\vec{q}^7}{2m_v^3} - \frac{8\vec{q}^6}{3m_v^2} + \frac{209\pi^2\vec{q}^5}{8} - \frac{68\vec{q}^4}{3} \right. \right. \\ & + 508m_v\pi^2\vec{q}^3 + 112m_v^3\pi^2\vec{q} \Big\} \vec{\epsilon} \cdot \vec{\epsilon}' + \left\{ \vec{q}^2 \ln \vec{q}^2 \left(\frac{62\vec{q}^6}{3m_v^6} - \frac{64\vec{q}^4}{m_v^4} - \frac{1828\vec{q}^2}{3m_v^2} - 696 \right) + \frac{13\pi^2\vec{q}^7}{2m_v^5} \right. \\ & + \frac{8\vec{q}^6}{3m_v^4} - \frac{209\pi^2\vec{q}^5}{8m_v^3} + \frac{68\vec{q}^4}{3m_v^2} - \frac{508\pi^2\vec{q}^3}{m_v} - 112m_v\pi^2\vec{q} \Big\} \vec{k} \cdot \vec{\epsilon} \vec{k} \cdot \vec{\epsilon}' + \left\{ \ln \vec{q}^2 \left(-\frac{59\vec{q}^8}{6m_v^6} + \frac{80\vec{q}^6}{m_v^4} \right. \right. \\ & \left. \left. + \frac{1091\vec{q}^4}{3m_v^2} + \frac{1882\vec{q}^2}{3} + 16m_v^2 \right) - \frac{27\pi^2\vec{q}^7}{8m_v^5} - \frac{2\vec{q}^6}{m_v^4} + \frac{993\pi^2\vec{q}^5}{32m_v^3} - \frac{133\vec{q}^4}{3m_v^2} + \frac{1125\pi^2\vec{q}^3}{4m_v} - \frac{56\vec{q}^2}{3} \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{169m_v\pi^2\vec{q}}{2} + \frac{64m_v^3\pi^2}{\vec{q}} \Big\} \vec{q} \cdot \vec{\epsilon} \vec{q} \cdot \vec{\epsilon}' + \left\{ \ln q^2 \left(\frac{31\vec{q}^8}{3m_v^6} - \frac{32\vec{q}^6}{m_v^4} - \frac{914\vec{q}^4}{3m_v^2} - 348\vec{q}^2 \right) + \frac{13\pi^2\vec{q}^7}{4m_v^5} \right. \\
& \left. + \frac{4\vec{q}^6}{3m_v^4} - \frac{209\pi^2\vec{q}^5}{16m_v^3} + \frac{34\vec{q}^4}{3m_v^2} - \frac{254\pi^2\vec{q}^3}{m_v} - 56G^2m_v\pi^2\xi\vec{q} \right\} i(\vec{k} \times \vec{q}) \cdot \vec{S} \Big]. \quad (54)
\end{aligned}$$

The potential is given by

$$\begin{aligned}
V_{\text{seagull}}^{\text{spin-0-spin-1}}(G^2, r, \xi) = & \frac{G^2\xi}{Mr^4} \left[\left(12m_v^2 - \frac{526m}{r} - \frac{3217}{2r^2} + \frac{9140}{\pi m_v r^3} - \frac{29655}{4m_v^2 r^4} + \frac{38640}{\pi m_v^3 r^5} + \frac{110205}{2m_v^4 r^6} + \frac{483840}{\pi m_v^5 r^7} \right. \right. \\
& - \frac{170100}{m_v^6 r^8} + \frac{4906440}{\pi m_v^7 r^9} \Big) \vec{\epsilon} \cdot \vec{\epsilon}' + \left(-28 + \frac{522}{\pi m_v r} + \frac{1524}{m_v^2 r^2} - \frac{9140}{\pi m_v^3 r^3} + \frac{9405}{4m_v^4 r^4} \right. \\
& - \frac{38640}{\pi m_v^5 r^5} - \frac{32760}{m_v^6 r^6} - \frac{937440}{\pi m_v^7 r^7} \Big) \vec{k} \cdot \vec{\epsilon} \vec{k} \cdot \vec{\epsilon}' + \frac{1}{r} \left(56 - \frac{1305}{\pi m_v r} - \frac{4572}{m_v^2 r^2} + \frac{31990}{\pi m_v^3 r^3} \right. \\
& + \frac{9405}{m_v^4 r^4} + \frac{173880}{\pi m_v^5 r^5} + \frac{163800}{m_v^6 r^6} + \frac{5155920}{\pi m_v^7 r^7} \Big) (\vec{k} \times \hat{r}) \cdot \vec{S} + \left(64m_v^2 + \frac{507}{r^2} + \frac{40500}{m_v^2 r^4} \right. \\
& \left. \left. - \frac{223425}{m_v^4 r^6} + \frac{4989600}{\pi m_v^5 r^7} + \frac{2041200}{m_v^6 r^8} - \frac{63783720}{\pi m_v^7 r^9} \right) \hat{r} \cdot \vec{\epsilon} \hat{r} \cdot \vec{\epsilon}' \right]. \quad (55)
\end{aligned}$$

C. The double seagull diagram

The Feynman amplitude for the third diagram of Fig. 8 is

$$\begin{aligned}
i\mathcal{M}_{\text{double seagull}}^{\text{spin-0-spin-1}}|_{\text{NR}} = & \frac{1}{2!} \int \frac{d^4l}{(2\pi)^4} V_{\eta\lambda\rho\sigma(\xi)}^{\text{spin-0}(2)}(l+q, l) \frac{-i\mathcal{P}^{\rho\sigma\mu\nu} - i\mathcal{P}^{\eta\lambda\rho\sigma}}{(l+q)^2 l^2} V_{\beta,\alpha,\mu\nu\rho\sigma}^{\text{spin-1}(2)}(k_2, k'_2, m_v) \epsilon^\beta \epsilon^{\star\alpha} \Big|_{\text{NR}} \\
= & iG^2\xi \ln \vec{q}^2 \left[\left(-\frac{5\vec{q}^4}{3} - \frac{208m_v^2\vec{q}^2}{3} \right) \vec{\epsilon} \cdot \vec{\epsilon}' + \left(\frac{5\vec{q}^4}{3m_v^2} + \frac{208\vec{q}^2}{3} \right) \vec{k} \cdot \vec{\epsilon} \vec{k} \cdot \vec{\epsilon}' \right. \\
& \left. + \left(-\frac{5\vec{q}^4}{12m_v^2} - \frac{52\vec{q}^2}{3} + 20m_v^2 \right) \vec{q} \cdot \vec{\epsilon} \vec{q} \cdot \vec{\epsilon}' + \left(\frac{5\vec{q}^4}{6m_v^2} + \frac{104\vec{q}^2}{3} \right) i(\vec{k} \times \vec{q}) \cdot \vec{S} \right] \quad (56)
\end{aligned}$$

The corresponding potential reads

$$\begin{aligned}
V_{\text{double seagull}}^{\text{spin-0-spin-1}}(G^2, r, \xi) = & \frac{G^2\xi}{2\pi Mr^5} \left[\left(188m_v + \frac{160}{m_v r^2} - \frac{175}{m_v^3 r^4} \right) \vec{\epsilon} \cdot \vec{\epsilon}' + \frac{4}{m_v} \left(-52 + \frac{25}{m_v^2 r^2} \right) \vec{k} \cdot \vec{\epsilon} \vec{k} \cdot \vec{\epsilon}' \right. \\
& \left. + \frac{10}{m_v r} \left(52 - \frac{35}{m_v^2 r^2} \right) (\vec{k} \times \hat{r}) \cdot \vec{S} + \frac{35}{m_v r^2} \left(-52 + \frac{45}{m_v^2 r^2} \right) \hat{r} \cdot \vec{\epsilon} \hat{r} \cdot \vec{\epsilon}' \right]. \quad (57)
\end{aligned}$$

D. The fish diagrams

There are two fish diagrams for this scattering process given by the fourth and fifth of Fig. 8. The Feynman amplitudes for them respectively read

$$\begin{aligned}
 i\mathcal{M}_{\text{fish-1}}^{\text{spin-0-spin-1}} \Big|_{\text{NR}} &= \frac{1}{2!} \int \frac{d^4 l}{(2\pi)^4} V_{\rho\sigma\psi\theta(\xi)}^{\text{spin-0}(2)}(l, l+q) \frac{-i\mathcal{P}^{\rho\sigma\gamma\delta}}{l^2} \frac{-i\mathcal{P}^{\psi\theta\alpha\beta}}{(l+q)^2} V_{\alpha\beta\gamma\delta}^{\mu\nu(3)}(l+q, q) \frac{-i\mathcal{P}_{\mu\nu\lambda\phi}}{q^2} \\
 &\quad \times V_{\text{spin-1}(1)}^{\tau, \chi, \lambda\phi}(k_2, k'_2, m_v) \epsilon_\tau(k_2) \epsilon_\chi^*(k'_2) \Big|_{\text{NR}} \\
 &= iG^2 \xi \ln \vec{q}^2 \left[\left(\frac{310\vec{q}^4}{3} + \frac{2060m_v^2\vec{q}^2}{3} \right) \vec{\epsilon} \cdot \vec{\epsilon}' + \left(-\frac{310\vec{q}^4}{3m_v^2} - \frac{2060\vec{q}^2}{3} \right) \vec{k} \cdot \vec{\epsilon} \vec{k} \cdot \vec{\epsilon}' \right. \\
 &\quad \left. + \left(\frac{155\vec{q}^4}{6m_v^2} + \frac{515\vec{q}^2}{3} \right) \vec{q} \cdot \vec{\epsilon} \vec{q} \cdot \vec{\epsilon}' + \left(-\frac{155\vec{q}^4}{3m_v^2} - \frac{1030\vec{q}^2}{3} \right) i(\vec{k} \times \vec{q}) \cdot \vec{S} \right], \tag{58}
 \end{aligned}$$

and

$$\begin{aligned}
 i\mathcal{M}_{\text{fish-2}}^{\text{spin-0-spin-1}} \Big|_{\text{NR}} &= \frac{1}{2!} \int \frac{d^4 l}{(2\pi)^4} V_{\text{spin-0}(1)}^{\lambda\phi(\xi)}(q) \frac{-i\mathcal{P}_{\lambda\phi\mu\nu}}{q^2} \\
 &\quad \times V_{\alpha\beta\gamma\delta}^{\mu\nu(3)}(l, -q) \frac{-i\mathcal{P}^{\gamma\delta\rho\sigma}}{(l+q)^2} \frac{-i\mathcal{P}^{\alpha\beta\psi\theta}}{l^2} V_{\tau, \chi, \rho\sigma\psi\theta}^{\text{spin-1}(2)}(k_2, k'_2, m_v) \epsilon^\tau(k_2) \epsilon^{\star\chi}(k'_2) \Big|_{\text{NR}} \\
 &= iG^2 \xi \ln \vec{q}^2 \left[(16\vec{q}^4 + 752m^2\vec{q}^2) \vec{\epsilon} \cdot \vec{\epsilon}' + \left(-\frac{16\vec{q}^4}{m^2} - 752\vec{q}^2 \right) \vec{k} \cdot \vec{\epsilon} \vec{k} \cdot \vec{\epsilon}' \right. \\
 &\quad \left. + \left(\frac{4\vec{q}^4}{m^2} + 188\vec{q}^2 - 192m^2 \right) \vec{q} \cdot \vec{\epsilon} \vec{q} \cdot \vec{\epsilon}' + \left(-\frac{8}{m^2} \vec{q}^4 - 376\vec{q}^2 \right) i(\vec{k} \times \vec{q}) \cdot \vec{S} \right]. \tag{59}
 \end{aligned}$$

Their contribution to the gravitational potential, respectively reads

$$\begin{aligned}
 V_{\text{fish-1}}^{\text{spin-0-spin-1}}(G^2, r, \xi) &= \frac{5G^2\xi}{2\pi Mr^5} \left[\left(-206m_v + \frac{725}{2m_v r^2} + \frac{1085}{m_v^3 r^4} \right) \vec{\epsilon} \cdot \vec{\epsilon}' + \frac{2}{m_v} \left(103 - \frac{310}{m_v^2 r^2} \right) \vec{k} \cdot \vec{\epsilon} \vec{k} \cdot \vec{\epsilon}' \right. \\
 &\quad \left. + \frac{5}{m_v r} \left(-105 + \frac{434}{m_v^2 r^2} \right) (\vec{k} \times \hat{r}) \cdot \vec{S} \right] + \frac{5}{m_v r^2} \left(\frac{721}{2} - \frac{1953}{m_v^2 r^2} \right) \hat{r} \cdot \vec{\epsilon} \hat{r} \cdot \vec{\epsilon}', \\
 V_{\text{fish-2}}^{\text{spin-0-spin-1}}(G^2, r, \xi) &= \frac{G^2\xi}{\pi Mr^5} \left[\left(-516m_v - \frac{465n}{m_v r^2} + \frac{420}{m_v^3 r^4} \right) \vec{\epsilon} \cdot \vec{\epsilon}' + \frac{12}{m_v} \left(47 - \frac{20}{m_v^2 r^2} \right) \vec{k} \cdot \vec{\epsilon} \vec{k} \cdot \vec{\epsilon}' \right. \\
 &\quad \left. + \frac{30}{m_v r} \left(-47 + \frac{28}{m_v^2 r^2} \right) (\vec{k} \times \hat{r}) \cdot \vec{S} + \frac{105}{m_v r^2} \left(47 - \frac{36}{m_v^2 r^2} \right) \hat{r} \cdot \vec{\epsilon} \hat{r} \cdot \vec{\epsilon}' \right]. \tag{60}
 \end{aligned}$$

E. The vacuum polarization diagram

The Feynman amplitude for the last of Fig. 8 is given by,

$$\begin{aligned}
 \mathcal{M}_{\text{vac-pol}}^{\text{spin-0-spin-1}} \Big|_{\text{NR}} &= V_{\mu\nu(\xi)}^{\text{spin-0}(1)}(q) \frac{-i\mathcal{P}^{\mu\nu\rho\sigma}}{q^2} \Pi_{\rho\sigma\lambda\phi}(q) \frac{-i\mathcal{P}^{\lambda\phi\gamma\delta}}{q^2} V_{\beta, \alpha, \gamma\delta}^{\text{spin-1}(1)}(k_2, k'_2, m_v) \epsilon^\beta(k_2) \epsilon^{\star\alpha}(k'_2) \Big|_{\text{NR}} \\
 &= 3G^2 \xi \vec{q}^2 \ln \vec{q}^2 \left[-4(2m_v^2 + \vec{q}^2) \vec{\epsilon} \cdot \vec{\epsilon}' + 4 \left(2 + \frac{\vec{q}^2}{m_v^2} \right) \vec{k} \cdot \vec{\epsilon} \vec{k} \cdot \vec{\epsilon}' - \left(2 + \frac{\vec{q}^2}{m_v^2} \right) \vec{q} \cdot \vec{\epsilon} \vec{q} \cdot \vec{\epsilon}' \right. \\
 &\quad \left. + 2 \left(2 + \frac{\vec{q}^2}{m^2} \right) i(\vec{k} \times \vec{q}) \cdot \vec{S} \right], \tag{61}
 \end{aligned}$$

where the expression of the gauge invariant one loop graviton self-energy due to itself, $\Pi_{\rho\sigma\lambda\phi}(q)$, can be seen in Eq. (48) [8]. The potential reads,

$$V_{\text{vac-pol}}^{\text{spin-0-spin-1}}(G^2, r, \xi) = \frac{9G^2\xi}{\pi Mr^5} \left[\left(2m_v - \frac{35}{2m_v r^2} - \frac{35}{m_v^3 r^4} \right) \vec{\epsilon} \cdot \vec{\epsilon}' + \frac{2}{m_v} \left(-1 + \frac{10}{m_v^2 r^2} \right) \vec{k} \cdot \vec{\epsilon} \vec{k} \cdot \vec{\epsilon}' \right. \\ \left. + \frac{5}{m_v r} \left(1 - \frac{14}{m_v^2 r^2} \right) (\vec{k} \times \hat{r}) \cdot \vec{S} + \frac{35}{m_v r^2} \left(-\frac{1}{2} + \frac{9}{m_v^2 r^2} \right) \hat{r} \cdot \vec{\epsilon} \hat{r} \cdot \vec{\epsilon}' \right]. \quad (62)$$

F. The full result

Combining now the individual contributions from Eqs. (53), (55), (57), (60), and (62), we obtain the long range nonminimal gravitational potential between a massive spin-0 and massive spin-1 field at leading $\mathcal{O}(G^2\xi)$,

$$V^{\text{spin-0-spin-1}}(G^2, r, \xi) = \frac{G^2\xi}{Mr^4} \left[\left(3m_v^2 - \frac{1353m_v}{\pi r} - \frac{10987}{8r^2} \right) \vec{\epsilon} \cdot \vec{\epsilon}' + \left(-\frac{43}{2} + \frac{1392}{\pi m_v r} + \frac{5151}{4m_v^2 r^2} \right) \vec{k} \cdot \vec{\epsilon} \vec{k} \cdot \vec{\epsilon}' \right. \\ \left. + \left(\frac{43}{r} - \frac{3505}{\pi m_v r^2} \right) (\vec{k} \times \hat{r}) \cdot \vec{S} + \left(74m_v^2 + \frac{2055}{4r^2} \right) \hat{r} \cdot \vec{\epsilon} \hat{r} \cdot \vec{\epsilon}' \right] + \mathcal{O}(r^{-7}) + \dots \quad (63)$$

V. MASSIVE SPIN-0-SPIN-1/2 SCATTERING

We will next recompute the diagrams of Fig. 8, assuming the broken lines to represent the fermions. Using the expressions given in Eqs. (15)–(21), we find out the following results.

A. The triangle diagram

The Feynman amplitude and its nonrelativistic limit for the triangle diagram reads

$$i\mathcal{M}_{\text{Triangle}}^{\text{spin-0-spin-1/2}}|_{\text{NR}} = \int \frac{d^4l}{(2\pi)^4} V_{\rho\sigma\psi\theta(\xi)}^{\text{spin-0}(2)}(l, l-q) \frac{-i\mathcal{P}^{\psi\theta\mu\nu}}{l^2} \frac{-i\mathcal{P}^{\rho\sigma\alpha\beta}}{(l-q)^2} V_{\mu\nu}^{\text{spin-1/2}(1)}(k_2, l+k_2, m_f) \\ \times V_{\alpha\beta}^{\text{spin-1/2}(1)}(l+k_2, k'_2, m_f) \bar{u}_{s'}(k_2) \frac{-i(\gamma^\lambda(l+k_2)_\lambda + m_f)}{[(l+k_2)^2 + m_f^2]} u_s(k_2) \Big|_{\text{NR}} \\ = iG^2\xi \left[\left\{ \vec{q}^2 \ln \vec{q}^2 \left(-\frac{105\vec{q}^4}{8m_f^2} + \frac{519\vec{q}^2}{4} + 22m_f^2 \right) - \frac{315\pi^2\vec{q}^5}{64m_f} + \frac{885\pi^2m_f\vec{q}^3}{16} \right. \right. \\ \left. \left. - \frac{313\pi^2m_f^3\vec{q}}{2} \right\} \delta_{ss'} + \left\{ \vec{q}^2 \ln \vec{q}^2 \left(\frac{105\vec{q}^4}{16m_f^4} - \frac{2349\vec{q}^2}{16m_f^2} - \frac{3985m_f^2}{12} \right) \right. \right. \\ \left. \left. + \frac{315\pi^2\vec{q}^5}{128m_f^3} - \frac{7283\pi^2\vec{q}^3}{128m_f} - \frac{179\pi^2m_f\vec{q}}{4} \right\} i(\vec{k} \times \vec{q}) \cdot \vec{S}_{1/2} \right]. \quad (64)$$

The corresponding potential is given by

$$V_{\text{Triangle}}^{\text{spin-0-spin-1/2}}(G^2, r, \xi) = \frac{G^2\xi}{Mr^4} \left[\left(-\frac{313m_f^2}{8} - \frac{33m_f}{2\pi r} - \frac{2655}{16r^2} + \frac{7785}{4\pi m_f r^3} - \frac{14175}{32m_f^2 r^4} - \frac{33075}{4\pi m_f^3 r^5} \right) \delta_{ss'} \right. \\ \left. + \frac{1}{r} \left(45 - \frac{19925}{16\pi m_f r} - \frac{65547}{64m_f^2 r^2} + \frac{246645}{16\pi m_f^3 r^3} - \frac{14175}{8m_f^4 r^4} - \frac{297675}{8\pi m_f^5 r^5} \right) (\vec{k} \times \hat{r}) \cdot \vec{S}_{1/2} \right]. \quad (65)$$

B. The seagull diagram

The Feynman amplitude for the second diagram of 8 reads in the nonrelativistic limit,

$$\begin{aligned}
 i\mathcal{M}_{\text{seagull}}^{\text{spin-0-spin-1/2}}|_{\text{NR}} &= \int \frac{d^4l}{(2\pi)^4} V_{\text{spin-0}(1)}^{\rho\sigma(\xi)}(q) \frac{-i\mathcal{P}_{\rho\sigma\mu\nu}}{q^2} V_{\alpha\beta\gamma\delta}^{\mu\nu(3)}(l-q, -q) \frac{-i\mathcal{P}_{\gamma\delta\psi\theta}}{l^2} \frac{-i\mathcal{P}^{\alpha\beta\lambda\phi}}{(l-q)^2} V_{\tau,\chi,\psi\theta}^{\text{spin-1/2}(1)}(k_2, l+k_2, m_f) \\
 &\quad \times V_{\lambda\phi}^{\text{spin-1/2}(1)}(l+k_2, k'_2, m_f) \bar{u}_{s'}(k'_2) \frac{-i(\gamma^\lambda(l+k_2)_\lambda + m_f)}{[(l+k_2)^2 + m_f^2]} u_s(k_2) \Big|_{\text{NR}} \\
 &= iG^2\xi \left[\left\{ \vec{q}^2 \ln \vec{q}^2 \left(\frac{8\vec{q}^4}{m_f^2} - 1085\vec{q}^2 - 1770m_f^2 \right) + \frac{89\pi^2\vec{q}^5}{32m_f} - \frac{2515\pi^2m_f\vec{q}^3}{4} \right. \right. \\
 &\quad \left. \left. - 406\pi^2m_f^3\vec{q} \right\} \delta_{ss'} + \left\{ \vec{q}^2 \ln \vec{q}^2 \left(-\frac{4\vec{q}^4}{m_f^4} + \frac{3475\vec{q}^2}{4m_f^2} + 230 \right) - \frac{89\pi^2\vec{q}^5}{64m_f^3} + \frac{7163\pi^2\vec{q}^3}{16m_f} \right. \right. \\
 &\quad \left. \left. - 1215\pi^2m_f\vec{q} \right\} i(\vec{k} \times \vec{q}) \cdot \vec{S}_{1/2} \right]. \tag{66}
 \end{aligned}$$

The corresponding potential is found to be

$$\begin{aligned}
 V_{\text{seagull}}^{\text{spin-0-spin-1/2}}(G^2, r, \xi) &= \frac{G^2\xi}{Mr^4} \left[\left(-\frac{203m_f^2}{2} + \frac{2655m_f}{2\pi r} + \frac{7545}{4r^2} - \frac{16275}{\pi m_f r^3} + \frac{4005}{16m_f^2 r^4} + \frac{5040}{\pi m_f^3 r^5} \right) \delta_{ss'} \right. \\
 &\quad \left. + \frac{1}{r} \left(1215 + \frac{1725}{2\pi m_f r} + \frac{64467}{8m_f^2 r^2} - \frac{364875}{4\pi m_f^3 r^3} + \frac{4005}{4m_f^4 r^4} + \frac{22680}{\pi m_f^5 r^5} \right) (\vec{k} \times \hat{r}) \cdot \vec{S}_{1/2} \right]. \tag{67}
 \end{aligned}$$

C. The double seagull diagram

The Feynman amplitude and its nonrelativistic limit for the third diagram of 8 reads,

$$\begin{aligned}
 i\mathcal{M}_{\text{double seagull}}^{\text{spin-0-spin-1/2}}|_{\text{NR}} &= \frac{1}{2!} \int \frac{d^4l}{(2\pi)^4} V_{\eta\lambda\rho\sigma(\xi)}^{\text{spin-0}(2)}(l+q, l) \frac{-i\mathcal{P}^{\rho\sigma\mu\nu}}{(l+q)^2} \frac{-i\mathcal{P}^{\eta\lambda\rho\sigma}}{l^2} \bar{u}_{s'}(k'_2) V_{\mu\nu\rho\sigma}^{\text{spin-1/2}(2)}(k_2, k'_2, m_f) u_s(k_2) \Big|_{\text{NR}} \\
 &= \frac{2924i}{3} G^2 m_f^2 \xi \vec{q}^2 \ln \vec{q}^2 \delta_{ss'} + \frac{214}{3} G^2 i \xi \vec{q}^2 \ln \vec{q}^2 (\vec{k} \times \vec{q}) \cdot \vec{S}_{1/2}. \tag{68}
 \end{aligned}$$

Accordingly, the potential reads

$$V_{\text{double seagull}}^{\text{spin-0-spin-1/2}}(G^2, r, \xi) = -\frac{731G^2m_f\xi\delta_{ss'}}{\pi Mr^5} + \frac{535G^2\xi(\vec{k} \times \hat{r}) \cdot \vec{S}_{1/2}}{2\pi mMr^6}. \tag{69}$$

D. The fish diagrams

The Feynman amplitudes for the fourth and fifth diagrams of 8 in the nonrelativistic limit are respectively given by

$$\begin{aligned}
 i\mathcal{M}_{\text{fish-1}}^{\text{spin-0-spin-1/2}}|_{\text{NR}} &= \frac{1}{2!} \int \frac{d^4l}{(2\pi)^4} V_{\rho\sigma\psi\theta(\xi)}^{\text{spin-0}(2)}(l, l+q) \frac{-i\mathcal{P}^{\rho\sigma\gamma\delta}}{l^2} \frac{-i\mathcal{P}^{\psi\theta\alpha\beta}}{(l+q)^2} V_{\alpha\beta\gamma\delta}^{\mu\nu(3)}(l+q, q) \frac{-i\mathcal{P}_{\mu\nu\lambda\phi}}{q^2} \\
 &\quad \times \bar{u}_{s'}(k'_2) V_{\text{spin-1/2}(1)}^{\lambda\phi}(k_2, k'_2, m_f) u_s(k_2) \Big|_{\text{NR}} \\
 &= \frac{17930i}{3} G^2 m_f^2 \xi \vec{q}^2 \ln \vec{q}^2 \delta_{ss'} - \frac{515i}{3} G^2 i \xi \vec{q}^2 \ln \vec{q}^2 (\vec{k} \times \vec{q}) \cdot \vec{S}_{1/2}, \tag{70}
 \end{aligned}$$

and

$$\begin{aligned}
i\mathcal{M}_{\text{fish-2}}^{\text{spin-0-spin-1/2}}|_{\text{NR}} &= \frac{1}{2!} \int \frac{d^4 l}{(2\pi)^4} V_{\text{spin-0}(1)}^{\lambda\phi(\xi)}(q) \frac{-i\mathcal{P}_{\lambda\phi\mu\nu}}{q^2} V_{\alpha\beta\gamma\delta}^{\mu\nu(3)}(l, -q) \frac{-i\mathcal{P}^{\delta\rho\sigma} - i\mathcal{P}^{\alpha\beta\psi\theta}}{(l+q)^2 l^2} \\
&\quad \times \bar{u}_{s'}(k'_2) V_{\rho\sigma\psi\theta}^{\text{spin-1/2}(2)}(k_2, k'_2, m_f) u_s(k_2) \Big|_{\text{NR}} \\
&= -i5600G^2 m_f^2 \xi \vec{q}^2 \ln \vec{q}^2 \delta_{ss'} - i400G^2 i\xi \vec{q}^2 \ln \vec{q}^2 (\vec{k} \times \vec{q}) \cdot \vec{S}_{1/2}.
\end{aligned} \tag{71}$$

The corresponding two body potentials read

$$\begin{aligned}
V_{\text{fish-1}}^{\text{spin-0-spin-1/2}}(G^2, r, \xi) &= -\frac{8965G^2 m_f \xi \delta_{ss'}}{2\pi M r^5} - \frac{2575G^2 \xi (\vec{k} \times \hat{r}) \cdot \vec{S}_{1/2}}{4\pi m_f M r^6}, \\
V_{\text{fish-2}}^{\text{spin-0-spin-1/2}}(G^2, r, \xi) &= \frac{4200G^2 m_f \xi \delta_{ss'}}{\pi M r^5} - \frac{1500G^2 \xi (\vec{k} \times \hat{r}) \cdot \vec{S}_{1/2}}{\pi m_f M r^6}.
\end{aligned} \tag{72}$$

E. The vacuum polarization diagram

The Feynman amplitude for the last diagram of 8 reads in the nonrelativistic limit

$$\begin{aligned}
i\mathcal{M}_{\text{vac-pol}}^{1/2}|_{\text{NR}} &= V_{\mu\nu(\xi)}^{\text{spin-0}(1)}(q) \frac{-i\mathcal{P}^{\mu\nu\rho\sigma}}{q^2} \Pi_{\rho\sigma\lambda\phi}(q) \frac{-i\mathcal{P}^{\lambda\phi\gamma\delta}}{q^2} \bar{u}_{s'}(k'_2) V_{\gamma\delta}^{\text{spin-1/2}(1)}(k_2, k'_2, m_f) u_s(k_2) \Big|_{\text{NR}} \\
&= -276iG^2 m^2 \xi \vec{q}^2 \ln \vec{q}^2 \delta_{ss'} + 6G^2 \xi i\vec{q}^2 \ln \vec{q}^2 (\vec{k} \times \vec{q}) \cdot \vec{S}_{1/2},
\end{aligned} \tag{73}$$

where the expression of $\Pi_{\rho\sigma\lambda\phi}(q)$ can be seen in Eq. (48). The corresponding potential reads

$$V_{\text{vac-pol}}^{\text{spin-0-spin-1/2}}(G^2, r, \xi) = \frac{207G^2 m_f \xi \delta_{ss'}}{\pi M r^5} + \frac{45G^2 \xi (\vec{k} \times \hat{r}) \cdot \vec{S}_{1/2}}{2\pi m_f M r^6}. \tag{74}$$

F. The full result

Combining now the individual contributions of Eqs. (65), (67), (69), (72), (74), we obtain the long range nonminimal gravitational potential between a massive scalar and fermion at the leading order $\mathcal{O}(G^2\xi)$,

$$\begin{aligned}
V^{\text{spin-0-spin-1/2}}(G^2, r, \xi, ss') &= \frac{G^2 \xi}{M r^4} \left[\left(-\frac{1125m_f^2}{8} + \frac{1009m_f}{2\pi r} + \frac{27525}{16r^2} \right) \delta_{ss'} \right. \\
&\quad \left. + \left(\frac{1260}{r} - \frac{35785}{16\pi m_f r^2} \right) (\vec{k} \times \hat{r}) \cdot \vec{S}_{ss'} \right] + \mathcal{O}(r^{-7}) + \dots.
\end{aligned} \tag{75}$$

VI. DISCUSSION

We have computed the effect of gravity-scalar $\xi R\phi^2$ nonminimal coupling in the two body long range gravitational potential for massive fields. We have considered the 2-2 scattering between scalars, scalar-spin-1, and scalar-spin-1/2 fields. From the nonrelativistic limit of the scattering Feynman amplitudes, we have computed the gravitational potentials in Eqs. (50), (63) and (75). These are the main results of this paper. Note that there is no tree level contribution here, and hence the $\mathcal{O}(G^2\xi)$ results found here are leading. Also, the leading behavior of the potential

is $\sim r^{-4}$. Let us now compare our result with the well-known $\xi = 0$ case [35],

$$V(r, \xi = 0) = -\frac{GMm}{r} \left(1 + \frac{3G(M+m)}{r} + \frac{41G\hbar}{10\pi r^2} \right). \tag{76}$$

The higher radial powers of our case compared to the above result certainly originate from the explicit appearance of the transfer momentum carried by the graviton lines in the nonminimal vertices of various scattering amplitudes [cf., discussion below Eq. (24)].

Imagine now that $M \gg m$ and we compare the leading r^{-4} part of Eq. (50) with that of $V(r, \xi = 0)$. The ratio of this leading part and the most subleading (quantum) part of $V(r, \xi = 0)$ reads ($\hbar = 1$),

$$3.06\xi \times \frac{M}{m} \times \frac{1}{mr}.$$

The masses appearing above are to be understood as their inverse Compton wavelengths. Let us now imagine a particle of electron mass on the surface of the Earth. We take $M \sim 10^{24}$ kg, $m \sim 10^{-31}$ kg, $r \sim 10^6$ m (the radius of the earth), and taking the Compton wavelength of m to be $\sim 10^{-12}$ m, we see that the above ratio is about $10^{37}\xi$. Thus, for any reasonable value less than unity of the nonminimal coupling parameter, Eq. (50) will dominate over the r^{-3} part of the one loop minimal gravitational potential. The second term of $V(r, \xi = 0)$ is, however, much more dominant. Nevertheless, it is clear that the effect of this r^{-4} term on the test particles near a massive object like a black hole can be interesting, for maybe in future observation it can distinguish the nonminimal interaction, if it really exists.

For a massive vector or spinor field, however, the ratio becomes

$$\sim \frac{m}{M} \times \frac{1}{Mr}$$

which is much subleading if we take the earlier values of the parameters.

Note also that apart from having different coefficients, the scalar-scalar potential Eq. (50) is symmetric under the interchange of the two masses, in contrast to the result for scalar-vector and scalar-fermion interactions, Eqs. (63) and (75). This is expected, as for the scalar-scalar interaction the particles are of the same species (with the same taken value of the nonminimal coupling, ξ). Hence, interchanging the masses leaves the potential unchanged. For the two other interactions, however, first, the particle species are different with different kinds of propagators, and second, there is no nonminimal interaction for the nonspin-0 fields. This (a) symmetry feature is also evident from the various Feynman diagrams.

It seems to be an important task to understand the effect of motion or macroscopic spin of the massive scalar body on the gravitational potential, in the context of $\xi R\phi^2$ interaction. It will be further important to understand the effect of ξ on gravitational light bending. Investigation of various other scalar tensor theories [88] seems also to be an important task. We hope to come back to these issues in our future publications.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: SOME USEFUL FORMULAS

1. List of Fourier transforms

The 3-D Fourier transform of a function $f(\vec{q})$ is defined as

$$\int \frac{d^3\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} f(\vec{q}) = g(\vec{r})$$

where n is any positive integer, and k is any odd positive integer. $A(n)$ and $B(k)$ are defined as

$$A(n) = \frac{(-1)^{[n/2+1]}\Gamma(n)}{2\pi^{1/2(3-(-1)^n)}}, \quad B(k) = \frac{2^k\Gamma(\frac{3+k}{2})}{\pi^{3/2}\Gamma(-\frac{k}{2})},$$

where the $[.]$ appearing in $A(n)$ is the greatest integer function, i.e., it returns the largest integer less than or equal to the given number.

TABLE I. Table for required Fourier transforms.

Sl.no.	$f(\vec{q})$	$g(\vec{r})$	Sl.no.	$f(\vec{q})$	$g(\vec{r})$
1.	$(\vec{q})^n \ln \vec{q}^2$	$\frac{A(n)}{r^{n+3}}$	2.	$(\vec{q})^{2n}$	$(-\vec{\partial}^2)^n \delta^{(3)}(\vec{r})$
3.	$q_i(\vec{q})^n \ln \vec{q}^2$	$A(n) \frac{i(n+3)r_i}{r^{n+5}}$	4.	$q_i(\vec{q})^{2n}$	$-i\partial_i(-\vec{\partial}^2)^n \delta^{(3)}(\vec{r})$
5.	$q_i q_j(\vec{q})^n \ln \vec{q}^2$	$A(n)(n+3) \left(\frac{\delta_{ij}}{r^{n+5}} - (n+5) \frac{r_i r_j}{r^{n+7}} \right)$	6.	$(\vec{q})^k$	$\frac{B(k)}{r^{k+3}}$
7.	$q_i q_j(\vec{q})^k$	$B(k)(k+3) \left(\frac{\delta_{ij}}{r^{k+5}} - (k+5) \frac{r_i r_j}{r^{k+7}} \right)$	8.	$q_i(\vec{q})^k$	$B(k) \frac{i(k+3)r_i}{r^{k+5}}$

2. List of some essential integrals

$$\begin{aligned} \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2(l+q)^2} &= -\frac{i}{16\pi^2} \ln q^2, & \int \frac{d^d l}{(2\pi)^d} \frac{l_\mu}{l^2(l+q)^2} &= \frac{i q_\mu}{32\pi^2} \ln q^2 \\ \int \frac{d^d l}{(2\pi)^d} \frac{l_\mu l_\nu}{l^2(l+q)^2} &= -\frac{i \ln q^2}{64\pi^2} \left[\frac{4}{3} q_\mu q_\nu - \frac{1}{3} q^2 \eta_{\mu\nu} \right] \\ \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2(l+q)^2((l+k)^2-m^2)} &= -\frac{i}{32\pi^2 m^2} \left[\ln q^2 + \frac{\pi^2 m}{q} \right]. \end{aligned} \quad (\text{A1})$$

$$\int \frac{d^d l}{(2\pi)^d} \frac{l_\mu}{l^2(l+q)^2((l+k)^2-m^2)} = \frac{i}{32\pi^2 m^2} \left[\left\{ \left(-1 - \frac{q^2}{2m^2} \right) \ln q^2 - \frac{1}{4} \frac{q^2}{m^2} \frac{\pi^2 m}{q} \right\} k_\mu + \left\{ \ln q^2 + \frac{\pi^2 m}{2q} \right\} q_\mu \right]. \quad (\text{A2})$$

$$\begin{aligned} \int \frac{d^d l}{(2\pi)^d} \frac{l_\mu l_\nu}{l^2(l+q)^2((l+k)^2-m^2)} &= \frac{i}{32\pi^2 m^2} \left[q_\mu q_\nu \left(-\ln q^2 - \frac{3}{8} \frac{\pi^2 m}{q} \right) - \frac{k_\mu k_\nu q^2}{2m^2} \left(\ln q^2 + \frac{\pi^2 q^2}{4mq} \right) \right. \\ &\quad \left. + \frac{1}{2} (q_\mu k_\nu + q_\nu k_\mu) \left\{ \left(1 + \frac{q^2}{m^2} \right) \ln q^2 + \frac{3\pi^2 mq}{8} \right\} + \frac{q^2}{4} \eta_{\mu\nu} \left(\ln q^2 + \frac{\pi^2 m}{2q} \right) \right]. \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} \int \frac{d^4 l}{(2\pi)^4} \frac{l_\mu l_\nu l_\alpha}{l^2(l+q)^2((l+k)^2-m^2)} &= \frac{i}{32\pi^2 m^2} \left[q_\mu q_\nu q_\alpha \left(\ln q^2 + \frac{5}{16} \frac{\pi^2 m}{q} \right) + k_\mu k_\nu k_\alpha \left(-\frac{1}{6} \frac{q^2}{m^2} \right) \right. \\ &\quad + (q_\mu k_\nu k_\alpha + q_\nu k_\mu k_\alpha + q_\alpha k_\mu k_\nu) \left(\frac{1}{3} \frac{q^2}{m^2} \ln q^2 + \frac{1}{16} \frac{q^2}{m^2} \frac{\pi^2 m}{q} \right) \\ &\quad + (q_\mu q_\nu k_\alpha + q_\mu q_\alpha k_\nu + q_\nu q_\alpha k_\mu) \left(\left(-\frac{1}{3} - \frac{1}{2} \frac{q^2}{m^2} \right) \ln q^2 - \frac{5}{32} \frac{q^2}{m^2} \frac{\pi^2 m}{q} \right) \\ &\quad + (\eta_{\mu\nu} k_\alpha + \eta_{\mu\alpha} k_\nu + \eta_{\nu\alpha} k_\mu) \times \frac{q^2}{12} \ln q^2 \\ &\quad \left. + (\eta_{\mu\nu} q_\alpha + \eta_{\mu\alpha} q_\nu + \eta_{\nu\alpha} q_\mu) \left(-\frac{1}{6} q^2 \ln q^2 - \frac{1}{16} q^2 \frac{\pi^2 m}{q} \right) \right]. \end{aligned} \quad (\text{A4})$$

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