

Undecidability in spacetime geometry via the AdS/CFT correspondence

Sameer Ahmad Mir^{1,2,*} Francesco Marino^{3,4,†} Arshid Shabir^{5,‡} Lawrence M. Krauss^{6,§} and Mir Faizal^{5,7,8,9,||}

¹Department of Physics, *Jamia Millia Islamia*, New Delhi, 110025, India

²Department of Computer Sciences, *Asian School of Business*, Noida, Uttar Pradesh, 201303, India

³*CNR-Istituto Nazionale di Ottica*, Via Sansone 1, I-50019 Sesto Fiorentino (FI), Italy

⁴*INFN, Sezione di Firenze*, Via Sansone 1, I-50019 Sesto Fiorentino (FI), Italy

⁵*Canadian Quantum Research Center*, 204-3002 32 Avenue, Vernon, Columbia V1T 2L7, Canada

⁶*The Origins Project Foundation*, Phoenix, Arizona 85028, USA

⁷*Irving K. Barber School of Arts and Sciences*, University of British Columbia Okanagan, Kelowna, British Columbia V1V 1V7, Canada

⁸Department of Mathematical Sciences, *Durham University*,

Upper Mountjoy, Stockton Road, Durham DH1 3LE, United Kingdom

⁹Faculty of Sciences, *Hasselt University*, Agoralaan Gebouw D, Diepenbeek, 3590, Belgium



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Undecidability, a hallmark of Gödel incompleteness theorems, has recently emerged in quantum many-body physics through the spectral gap problem. We demonstrate how this logical limitation can be holographically transmitted to a class of gravitational theories via the AdS/CFT correspondence. By embedding a translationally invariant spin Hamiltonian with undecidable gap status into a large- N gauge theory, we generate an anti-de Sitter (AdS) dual in which the selection of dominant bulk saddle (Poincaré AdS or AdS soliton) is itself undecidable. Consequently, under standard semiclassical holographic assumptions, even determining which smooth spacetime geometry emerges from quantum gravity can be beyond the limits of computability.

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Introduction. The AdS/CFT correspondence provides a concrete realization of the holographic principle [1,2], relating a quantum gravitational theory (string theory or M theory) in anti-de Sitter (AdS) spacetime to a conformal field theory (CFT) on its boundary [3–5]. A powerful feature of this correspondence is its strong-weak nature: when the boundary CFT is strongly coupled, the bulk gravity theory is weakly coupled and perturbatively tractable. This makes it possible to compute correlation functions [4] and transport properties in strongly interacting systems. Beyond these applications, the correspondence represents a major advance, providing a nonperturbative formulation of string theory with specific boundary conditions. In principle, AdS/CFT allows one to extract all bulk physics from the boundary CFT [6–12], provided that a

complete mapping between the observables of the two theories is established [13].

An important problem when exploiting this correspondence is to identify which bulk saddle dominates the gravitational path integral for a given boundary CFT or, equivalently, which semiclassical spacetime emerges, since multiple bulk saddles may be consistent with the same boundary data. For instance, both thermal AdS and AdS black holes satisfy the same boundary conditions [14], with the dominant saddle determining whether the boundary CFT is in a confined or deconfined phase [15]. Similarly, an AdS soliton can energetically dominate Poincaré AdS under certain conditions [16], indicating a different preferred vacuum state. Saddle-point selection underlies bulk reconstruction [17,18], sets the semiclassical background for Hawking radiation and the black hole information problem [19], and influences locality in emergent spacetime [20].

The selection of the dominant bulk geometry is therefore central to holography and to the ability to use an underlying quantum gravitational theory to derive semiclassical results. Here, we show that there exists a class of AdS/CFT scenarios in which this selection is undecidable.

Our starting point is the work by Cubitt-Pérez-García-Wolf (CPW) [21], who demonstrated that given a quantum many-body Hamiltonian it is, in general, computationally

*Contact author: sameerphst@gmail.com

†Contact author: francesco.marino@ino.cnr.it

‡Contact author: aslone186@gmail.com

§Contact author: lawrence@originsproject.org

||Contact author: mirfaizalmir@googlemail.com

impossible to determine whether the system it describes is gapped or gapless. The meaning of this statement is twofold. First, no algorithm can decide in finite time whether such a system is gapped or gapless, in the same sense that the halting problem is algorithmically undecidable [22]. Second, for any consistent axiomatization of mathematics, there exist specific Hamiltonians for which the presence or absence of a spectral gap cannot be decided within those axioms. This is the form of undecidability established by Gödel's incompleteness theorems [23].

We demonstrate here that the problem of the selection of dominant bulk geometry (Poincaré AdS vs AdS soliton) is dual to the spectral-gap problem of the boundary quantum many-body system. We map the local, translationally invariant Hamiltonian of Ref. [21] into a Euclidean path-integral formulation using a Trotter decomposition, and decouple its interactions by introducing an auxiliary scalar field through a Hubbard-Stratonovich transformation. The scalar field encodes the halting bit of the Turing machine through a coupling that persists in the continuum limit. We then recast this scalar field theory as an adjoint matrix model with a non-Abelian gauge structure. This reformulation allows us to properly define the large- N limit, in which the matrix theory admits a holographic dual described by gravity in asymptotically AdS spacetime with a bulk scalar mode sourced by the boundary scalar operator.

The key observation is that the sign of the scalar effective mass term, fixed by the same undecidable halting predicate as in the boundary spectral-gap problem, determines which smooth geometry dominates: Poincaré AdS (gapless phase) or the AdS soliton (gapped phase).

Preliminaries. We briefly recall the CPW construction [21], which established the undecidability of the spectral-gap problem (see also Sec. I of the Supplemental Material [24]). Consider translation-invariant, spin Hamiltonians H_L with nearest-neighbor, finite-range interactions on a 2D lattice of size $L \times L$, with each site carrying a local Hilbert space dimension d . The spectral gap is the difference between the first excited state and the ground state, $\gamma_L = E_1(L) - E_0(L)$. A uniform spectral gap, i.e., the existence of a lower bound $\Delta > 0$ such that $\inf_L \gamma_L \geq \Delta$, implies exponential decay of correlations (clustering) and stability of the quantum phase [37]. On the other hand, if $\liminf_{L \rightarrow \infty} \gamma_L = 0$, the system is gapless and supports arbitrarily low-energy excitations [38].

Using the Feynman-Kitaev history state [39–42], a universal Turing machine on input u is mapped to a frustration free k -local Hamiltonian. If the computation halts after runtime T , the history state carries weight $1/(T+1)$ on the halting configuration. In this case an additional output term, $H_{\text{halt}} = \delta(T)\Pi_{\text{halt}}$, then shifts the ground state energy by $\delta(T)/(T+1)$, and by zero otherwise.

The CPW construction embeds this computational layer into the above lattice and couples it to a spectral scaling

mechanism so that, if halting occurs, the effect produces a finite shift in the energy density as $L \rightarrow \infty$. This defines a computable map $u \mapsto H(u)$ that assigns to each input u a set of nearest-neighbor interactions, namely a translation-invariant nearest-neighbor Hamiltonian on $\mathbb{Z}^2$ (here $\mathbb{Z}^2$ denotes the two-dimensional integer lattice), with the one of two alternatives: if the machine halts, there exists $\Delta(u) > 0$ with $\gamma_L(u) \geq \Delta(u) \forall L$ (gapped phase); if it does not, $\liminf_{L \rightarrow \infty} \gamma_L(u) = 0$ (gapless phase). Any algorithm that would allow one to decide, from the finite description of $h(u)$, which case holds, would solve the halting problem. Because no such algorithm therefore exists, the spectral-gap promise problem is Turing undecidable. This obstruction persists even in one-dimensional quantum spin chains [21].

Gödel incompleteness and Turing's halting problem are two faces of the same coin: a consistent, recursively axiomatizable theory cannot be complete (or it would decide halting), and a halting decider would make such a theory complete [43,44]. The obstruction for quantum chains is therefore Gödelian: the Turing undecidability of the spectral-gap predicate means that, for certain Hamiltonians, the statement of whether they are gapped or gapless is formally independent of any consistent computably axiomatizable theory, i.e., it can be neither proved nor refuted within that theory.

Euclidean path-integral representation. We denote by $\vartheta(u)$ the CPW halting predicate, equal to 1 if the machine halts on input u and 0 otherwise. As we outline here, in the continuum, $\vartheta(u)$ survives as a relevant scalar-mass operator, making the infrared (IR) choice between gapped and gapless phases algorithmically undecidable. We refer the reader to Supplemental Material [24], Sec. II, for details.

Starting from the partition function $Z_L(\beta, u) = \text{Tr}_{\mathcal{H}_L} [e^{-\beta H_L(u)}]$, a Suzuki-Trotter decomposition [45–47] can map the system to a $(2+1)$ D classical statistical model, accurate up to $\mathcal{O}(a_\tau^2)$ with $a_\tau = \beta/N_\tau$. Here, N_τ denotes the number of discrete time slices in the Euclidean-time direction, so that the continuum limit corresponds to $N_\tau \rightarrow \infty$. The Lieb-Robinson bound [48] ensures a finite signal velocity (compatible with relativity) but does not by itself fix the dynamical exponent z that sets the anisotropic scaling between space and time. We restrict to microscopic models that flow to a Lorentz-invariant ($z = 1$) fixed point [e.g., the $O(N)$ Wilson-Fisher class], so that our Hubbard-Stratonovich (HS) continuum description applies.

A Hubbard-Stratonovich (HS) decoupling yields, after integrating out spins, an $O(N_v)$ scalar theory (with N_v the number of spin components) with bare Euclidean Lagrangian,

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \varphi^A)(\partial^\mu \varphi^A) + \frac{m^2}{2} \varphi^A \varphi^A + \frac{\lambda}{4!} (\varphi^A \varphi^A)^2 + g_{\text{halt}} \varphi^A \varphi^A, \quad (1)$$

where $g_{\text{halt}} \equiv \vartheta(u)\mu_h^2$ includes the halting bit, μ_h is a scale factor with the dimensions of a mass, and $A = 1, \dots, N_v$. A one-loop renormalization group analysis shows that g_{halt} is a relevant deformation (its coupling grows under coarse graining). The effective mass of the scalar field is $m_{\text{eff}}^2(u) = m^2 + g_{\text{halt}} + \Sigma_{\text{loop}}(\lambda)$. The $\Sigma_{\text{loop}}(\lambda)$ encodes loop corrections to the scalar mass. Renormalizing the system by enforcing $m^2 + \Sigma_{\text{loop}}(\lambda) = 0$ gives $m_{\text{eff}}^2(u) = \vartheta(u)\mu_h^2$ to leading order. In this case, the effective mass is controlled entirely by the halting deformation: if $\vartheta(u) = 0$ then $m_{\text{eff}}^2 = 0$, placing the IR theory at the Wilson-Fisher critical point (gapless, with long-range/power-law correlations) [49,50]. If $\vartheta(u) = 1$ then $m_{\text{eff}}^2 > 0$, yielding a uniformly gapped phase with correlation length $\xi = \Delta^{-1}$ and exponential clustering. Since $m_{\text{eff}}^2(u)$ depends on the uncomputable predicate $\vartheta(u)$, the distinction between gapless and gapped outcomes is algorithmically undecidable.

Matrix theory. We promote the $O(N_v)$ vector theory to an $SU(N_c)$ matrix theory by choosing $N_v = N_c^2 - 1$ and repackaging the vectors as an adjoint matrix $M(x) = \sum_{A=1}^{N_v} \phi_A(x) T^A$, where $T^A \in \mathfrak{su}(N_c)$ (with $\text{Tr}[T^A T^B] = \frac{1}{2} \delta^{AB}$). Gauging this global symmetry to $SU(N_c)$ introduces a non-Abelian gauge field A_μ with covariant derivative $D_\mu M = \partial_\mu M - i[A_\mu, M]$ and field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ (see Supplemental Material [24], Sec. III, for details). The adjoint action is

$$S_{\text{adj}} = N_c \int d^3x \text{Tr} \left[(D_\mu M)^2 + m^2 M^2 + \lambda M^4 + \frac{1}{4g^2} F_{\mu\nu}^2 \right] + N_c \int d^3x g_{\text{halt}} \text{Tr}(M^2), \quad (2)$$

with fixed 't Hooft coupling $\lambda = g^2 N_c$ [51].

This choice ensures that the planar (large- N_c) limit of the matrix model reproduces the large- N_v behavior of the HS vector theory introduced above. At large N_c , vacuum diagrams are organized by genus h , with contributions scaling as N_c^{2-2h} . As a consequence, connected correlators factorize at leading order $\langle \text{Tr} M^k \text{Tr} M^\ell \rangle = \langle \text{Tr} M^k \rangle \langle \text{Tr} M^\ell \rangle + O(N_c^{-2})$, so that a single planar saddle controls the dynamics [52]. Standard $d = 3$ renormalization gives a finite planar one-loop shift $\Sigma_{\text{loop}}(\lambda, \Lambda) = O(1)$, (where Λ is the UV cutoff) giving the classical quadratic action

$$S_{\text{cl}} = N_c \int d^3x \text{Tr} \left[(\partial_\mu M)^2 + m_{\text{eff}}^2 M^2 \right]. \quad (3)$$

Here, Σ_{loop} is the finite planar one-loop self-energy, and imposing $m^2 + \Sigma_{\text{loop}} = 0$ gives $m_{\text{eff}}^2 = g_{\text{halt}}$ to leading order.

The self-energy has a genus expansion,

$$\Pi(k) = \sum_{h=0}^{\infty} N_c^{-2h} \Pi_h(k, \lambda, \Lambda), \quad (4)$$

with planar part $\Pi_0 = O(1)$ and $\Pi_h = O(N_c^{-2h})$ for $h \geq 1$ [53]. Therefore, nonplanar terms cannot significantly affect m_{eff}^2 , and in the planar limit the IR phase is still determined by the undecidable predicate $\vartheta(u)$.

Equivalently, examining the problem in terms of the spectral gap, for a finite cubic box of side L with periodic boundary conditions, the propagator and dispersion relation are, respectively (for details, see Supplemental Material [24])

$$G(p) = \frac{1}{p^2 + m_{\text{eff}}^2}, \quad E(p) = \sqrt{p^2 + m_{\text{eff}}^2}. \quad (5)$$

The spectral gap for nonzero momentum modes is $\gamma_L^{(p \neq 0)}(u) = \sqrt{(2\pi/L)^2 + m_{\text{eff}}^2(u)}$. In the thermodynamic limit, $\Delta(u) = \lim_{L \rightarrow \infty} \gamma_L(u) = m_{\text{eff}}(u)$. A positive $\Delta(u)$ implies exponentially decaying correlations with correlation length $\xi = \Delta^{-1}$, and a stable gapped phase. When $m_{\text{eff}}^2(u) = 0$, correlations display a power-law behavior corresponding to an IR CFT with dynamical exponent $z = 1$.

The halting term is again crucial in determining which regime is realized: $\vartheta(u) = 0$ yields criticality; if $\vartheta(u) = 1$, then $m_{\text{eff}}^2(u) \geq c > 0$ uniformly in L , leading to a gapped phase (see Supplemental Material [24]). Since no Turing machine can infer the behavior of $\vartheta(u)$ from microscopic data [21], undecidability holds even for the large- N_c continuum matrix theory. As we will see, the question of which phase ultimately arises corresponds holographically to the selection of the dominant bulk geometry.

Ultraviolet completion. Having shown that the halting deformation survives at large N_c and governs the IR behavior, we now justify its microscopic origin. By performing a real-space coarse-graining and introducing a HS field to linearize the interblock couplings, one obtains an adjoint $SU(N_c)$ gauge-scalar completion with an emergent non-Abelian connection and Yang-Mills kinetic term $\text{Tr} F_{\mu\nu} F^{\mu\nu}$. As shown in Sec. IV of [24], the Yang-Mills sector arises naturally under this construction, while the halting deformation contributes only through the scalar mass shift $m^2 \rightarrow m_{\text{eff}}^2$. To leading order, the halting deformation enters only through the mass shift and leaves the gauge kinetic term unchanged. Hence, the only relevant quantities are m_{eff}^2 and the large- N_c scaling, where m_{eff}^2 is determined by the undecidable predicate $\vartheta(u)$.

Confinement is probed by the Wilson loop $W(C) = \frac{1}{N_c} \text{Tr} \mathcal{P} \exp(i \oint_C A_\mu dx^\mu)$, whose area law encodes the string tension. At fixed 't Hooft coupling $\lambda = g^2 N_c$, connected vacuum diagrams of genus h scale as N_c^{2-2h} ; hence, nonplanar terms are $O(N_c^{-2})$ and do not affect the leading behavior of m_{eff}^2 (see ribbon-graph analysis in Sec. III of [24]). The partition function admits the large- N_c expansion $\ln Z(N_c, \lambda) = \sum_{h \geq 0} N_c^{2-2h} F_h(\lambda)$ which shows that the

planar sector $h = 0$ dominates. Interpreted holographically, this is a string-loop expansion with coupling $g_s \sim 1/N_c$, while λ sets the effective string tension. Hence, the planar limit corresponds to classical bulk gravity in AdS/CFT and $1/N_c$ terms encode string loop corrections [51,52]. Crucially, any N_c -independent deformation, in particular the halting term g_{halt} in m_{eff}^2 , survives at planar order and determines the classical bulk geometry, while $1/N_c$ effects are parametrically subleading and cannot overturn this shift.

Holographic duality and competing geometries. In the holographic planar limit, the stress tensor of adjoint matter scales as $\langle T_{\mu\nu} \rangle \sim O(N_c^2)$, while the AdS/CFT matching fixes the gravitational coupling through $L^{d-1}/G_{d+1} \sim N_c^2$. Their product $G_4 \langle T_{\mu\nu} \rangle$ is therefore $O(1)$, indicating that the halting-bit deformation produces a classical backreaction on the geometry. Accordingly, our analysis is performed at leading classical order in supergravity, with higher-derivative and loop effects subleading.

We now examine how undecidability manifests in the competition between semiclassical bulk geometries. In the $N_c \rightarrow \infty$ limit, the source for the operator $\mathcal{O} = \text{Tr } M^2$ in the boundary theory is implemented by Gubser-Klebanov-Polyakov-Witten (GKPW) prescription,

$$Z_{\text{QFT}}[J] = \exp \left[-S_{\text{grav}}^{\text{on-shell}}(\Phi|_{z=0} = J) \right], \quad (6)$$

where the bulk action includes the Einstein-Hilbert term, the Gibbons-Hawking-York boundary term, and counterterms:

$$S_{\text{grav}} = \frac{1}{16\pi G_4} \int_{\mathcal{M}} d^4x \sqrt{-g} \left(R + \frac{6}{L^2} \right) + \frac{1}{8\pi G_4} \int_{\partial\mathcal{M}} d^3x \sqrt{-\gamma} K + S_{\text{ct}}[\gamma]. \quad (7)$$

Holographic renormalization ensures that the boundary stress tensor is traceless for a flat 3D boundary ($d = 3$) [54,55], as expected for a CFT. Validity of the semiclassical approximation for the bulk geometry requires that the AdS radius is much larger $L \gg \ell_s$ ($\lambda_{\text{th}} \gg 1$, $\ell_s^2/L^2 \sim \lambda_{\text{th}}^{-1/2}$) to suppress $1/N_c^2$ loops (where ℓ_s is the fundamental string length) [56–58].

The source J couples to a bulk scalar Φ satisfying $(\square_g - m_\Phi^2)\Phi = 0$ with near-boundary expansion $\Phi = z^{3-\Delta}\phi_{(0)}(x) + z^\Delta\phi_{(2\Delta-3)}(x) + \dots$, and $m_\Phi^2 L^2 = \Delta(\Delta - 3)$, where $\phi_{(0)} \equiv J$ and $\phi_{(2\Delta-3)} \propto \langle \text{Tr } M^2 \rangle$ [59]. Tuning $\phi_{(0)} = 0$ gives Poincaré AdS₄,

$$ds^2 = \frac{L^2}{z^2} (dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu), \quad (8)$$

which represents the gapless critical phase. Turning on the halting deformation, $\vartheta(u) = 1$ ($\phi_{(0)} \neq 0$) drives Φ away from the critical point and can give rise, within the (T, L_θ) window discussed below, to the Euclidean AdS₄ soliton geometry [16]:

$$ds^2 = \frac{r^2}{L^2} (d\tau^2 + dx^2 + f(r)d\theta^2) + \frac{L^2}{r^2 f(r)} dr^2, \\ f(r) = 1 - \left(\frac{r_0}{r} \right)^3, \quad \theta \sim \theta + \frac{4\pi L^2}{3r_0}. \quad (9)$$

Here, the boundary topology is $S_\tau^1(\beta) \times \mathbb{R}_x \times S_\theta^1(L_\theta)$, with the soliton caps S_θ^1 (contractible) and the black brane caps S_τ^1 (thermal).

The corresponding free-energy densities per unit x length are $F_{\text{sol}} = -c_s/L_\theta^2$ and $F_{\text{bb}} = -c_T T^3 L_\theta$, which coincide at the critical temperature $T_c(L_\theta) = (\frac{c_s}{c_T})^{1/3} \frac{1}{L_\theta}$. A relevant boundary (multitrace) deformation proportional to $\vartheta(u)$ modifies the UV coefficients (c_s, c_T) and the mixed (Robin) boundary condition of the dual scalar, shifting the critical temperature T_c while preserving large- N_c scalings and the high- T dominance of the AdS₄ black brane (see [24], Secs. V and VI). We focus in a low-temperature window with $\beta/L_\theta \gg 1$, where the only two smooth, static, translation-invariant saddles are thermal Poincaré AdS₄ and the AdS₄ soliton. Their renormalized on-shell actions satisfy $I[g_P] = 0$ and $I[g_{\text{sol}}] < 0$. In the undeformed theory, the transition occurs at $T_c \sim 1/L_\theta$; within this window $\vartheta(u) = 0$ selects the deconfined (Poincaré) branch ($T > T_c$) and $\vartheta(u) = 1$ selects the confining (soliton) branch ($T < T_c$). In the planar limit, large- N_c factorization implies that the deformation affects only one-point data, e.g., sources and expectation values, without affecting the saddle competition.

We implement the deformation through mixed (Robin) boundary conditions for the bulk scalar, following standard GKPW holography [55,60]. Near the boundary $z \rightarrow 0$, $\Phi(z, x) = z^{d-\Delta}\phi_{(0)}(x) + z^\Delta \frac{\langle \mathcal{O}(x) \rangle}{2\Delta-d} + \dots$, and $m_{\text{bulk}}^2 L^2 = \Delta(\Delta - d)$, so $\phi_{(0)}$ sources J and Δ obeying the Breitenlohner-Freedman relation [61]. The mixed boundary condition, $\phi_{(2\Delta-d)}(x) = \kappa_{\text{bdy}} \phi_{(0)}(x)$, implements the double-trace deformation. Matching the stress tensor fixes $L^{d-1}/G_{d+1} \sim N_c^2$ [54], so that $c_T \propto (L^{d-1}/G_{d+1}) \sim N_c^2$ and $\langle TT \rangle$ scales accordingly [62].

The classical supergravity regime requires both large $\lambda = g^2 N_c \gg 1$, ensuring that higher-derivative corrections are small, and large N_c . In this regime, the halting deformation enters only through the boundary coupling $\partial_z \Phi|_{z=0} = \kappa_{\text{bdy}} \varphi_{(0)}$, $\kappa_{\text{bdy}} \propto g_{\text{halt}} L$ that eventually determines which bulk phase is realized. Since the halting predicate is uncomputable, the resulting choice of dominant bulk geometry is itself undecidable within the holographic picture (see [24], Sec. V).

Vacuum selection on $S_\beta^1 \times \mathbb{R}^2$ (Casimir test). As an explicit example, consider the boundary metric $\gamma_{ij}^{(0)} dx^i dx^j = d\tau^2 + dx^2 + dy^2$, $0 \leq \tau < \beta$, which describes a spatially infinite system compactified on a Euclidean time circle of circumference β . If the theory is gapless, the vacuum is scale and translation invariant with vanishing expectation

value of the stress tensor $\langle T_{ij} \rangle = 0$. The corresponding bulk saddle is Poincaré AdS₄, where the Brown-York tensor also vanishes [54,63]. If the spectrum is gapped, compactification on S^1_β gives a negative Casimir energy. For a free massive scalar of mass m_* with periodic boundary conditions for $\gamma_{ij}^{(0)}$, zeta-function methods [64,65] give

$$\varepsilon_{\text{Cas}}(\beta, m_*) = -\frac{m_*^2}{2\pi^2\beta} \sum_{n=1}^{\infty} \frac{K_1(m_* n \beta)}{n} < 0 \quad (10)$$

with K_1 the modified Bessel function. As expected, the Casimir energy vanishes as $m_*\beta \rightarrow \infty$. The dual saddle is in this case the AdS₄ soliton [24], whose Brown-York tensor matches (10) at leading order in $G_4 \langle T_{\tau\tau} \rangle_{\text{sol}} = -\frac{r_0^3}{16\pi G_4 L^4}$, $\langle T_{xx} \rangle = \langle T_{yy} \rangle = -\frac{1}{2} \langle T_{\tau\tau} \rangle$, with $r_0 = \frac{4\pi L^2}{3L_0}$, ensuring the expected tracelessness [16]. Putting these results together leads to the following cases:

$$\langle T^i_i \rangle = \begin{cases} 0, & m_{\text{eff}}^2 = 0 \\ -\frac{3r_0^3}{16\pi G_4 L^4} < 0, & m_{\text{eff}}^2 > 0. \end{cases} \quad (11)$$

Therefore, $\vartheta(u)$ determines whether the vacuum stress tensor vanishes (gapless CFT/Poincaré AdS₄) or carries a negative Casimir energy (gapped CFT/AdS₄ soliton). This stress-tensor makes the holographic undecidability concrete: the boundary signature of the vacuum, just like the bulk geometry itself, is fixed by an uncomputable predicate (see Sec. V of the Supplemental Material [24]). Under standard holographic assumptions (large N_c , large λ , classical gravity, and static translation-invariant bulk geometries), deciding the dominant Euclidean AdS₄ saddle on $S^1_\beta \times \mathbb{R}^2$ is therefore algorithmically undecidable (see [24], Sec. VI). So, here Tarski's external truth predicate [66,67] would be needed to fix which semiclassical saddle (Poincaré AdS₄ or the AdS₄ soliton) emerges as the dominant spacetime.

String theoretic interpretation. A natural concern is that semiclassical holography is highly nongeneric. Large N and large 't Hooft coupling are necessary but not, by themselves, sufficient to ensure an Einstein gravity regime. In known examples, semiclassical bulk locality is tied to additional CFT structure, in particular a sparse set of low-dimension single-trace operators together with a parametrically large gap to higher-spin and stringy excitations [68–70]. We therefore phrase our conclusions in the explicitly conditional form adopted throughout the paper. We construct a computable family of large N theories in which an undecidable microscopic input selects a relevant deformation, and we analyze the resulting competition of smooth Euclidean bulk fillings under the standard GKPW hypothesis that the theories under consideration admit a semiclassical holographic description [3–5].

The simplified model is chosen only to keep the reduction and the saddle comparison completely explicit.

The underlying computational logic is not model dependent. In any holographic CFT, one follows the same sequence of steps: specify the deformation, identify the dual bulk field and the corresponding boundary data, construct the candidate saddles compatible with the prescribed boundary geometry and sources, and determine dominance by comparing the renormalized on-shell actions using standard holographic renormalization [55,71]. In this precise sense, our construction provides a prototype for analogous computations in any holographic setting where the dual is established and the semiclassical gravitational path integral is under quantitative control, including benchmark dual pairs such as planar $\mathcal{N} = 4$ super-Yang-Mills theory and Aharony–Bergman–Jafferis–Maldacena (ABJM) theory [3,72].

From a string-theoretic viewpoint, the extra ingredient required to export the mechanism to such top-down settings is not a new dynamical sector, but a choice of boundary coupling. In the AdS/CFT dictionary, switching on a relevant deformation corresponds to prescribing boundary data for the dual bulk scalar, usually by fixing the non-normalizable mode in standard quantization and, more generally, by imposing mixed boundary conditions when multitrace deformations are included [60,73]. Equivalently, one chooses a point in the space of couplings of the brane worldvolume theory. In the bulk this is implemented by turning on the corresponding supergravity mode with specified asymptotics, often interpretable in terms of moduli in the underlying brane construction. The role of the undecidable predicate $\vartheta(u)$ is then minimal, as it only selects which value of a chosen relevant coupling is realized.

To make the export to top-down settings concrete while staying within the same semiclassical logic, one may place the boundary theory on $S^{d-1} \times S^1$ (or on a spatial circle), and work in regimes where more than one smooth Euclidean bulk filling is admissible for the same prescribed boundary geometry and sources. A protected relevant deformation then shifts the renormalized on-shell action difference between the competing saddles. In planar $\mathcal{N} = 4$ super-Yang-Mills theory, a canonical example is the $\mathcal{N} = 2^*$ mass deformation [74–76], while in ABJM one can consider supersymmetry-preserving mass deformations with a well-developed large- N and holographic description [77–79]. In suitable deformation families of this type, varying the relevant coupling can, in favorable cases, lead to distinct semiclassical saddles with qualitatively different infrared behavior, in direct analogy with the branch structure exhibited by our explicit computable family. Once the boundary choice is made, the remainder of the analysis reduces to the standard semiclassical comparison of admissible Euclidean fillings via their renormalized on-shell actions.

Conclusions. We have shown that undecidability in the spectral-gap problem for quantum many-body systems can be faithfully transmitted, via the AdS/CFT correspondence,

into the bulk geometry selection problem of semiclassical AdS gravity. To our knowledge, this provides the first concrete realization of a Gödel-Turing type obstruction in a holographic setting. The validity of this conclusion is conditioned on standard assumptions in holography: suppression of higher-derivative and string corrections, uniqueness of smooth saddles under fixed boundary conditions, and the validity of the large- N planar limit.

While our result relies on a specific holographic construction, it suggests that undecidability can be a robust property of holographic duality and can constrain the very emergence of spacetime itself. A natural question is whether similar undecidable structures might arise in other holographic settings, such as black hole microstates, string compactifications, or cosmological duals. If so,

undecidability could represent a pervasive feature of quantum gravity. This would imply that many aspects of spacetime structure could be beyond algorithmic reach [80], and would require a form of nonalgorithmic understanding that transcends the Gödel-Turing barrier [67,81]. Just like algorithmic computation, such non-algorithmic understanding is intrinsic to nature.

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Data availability. No data were created or analyzed in this study.

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