

Classical Hamiltonian of Reissner-Nordström black holes at second post-Minkowskian order from scattering amplitudes

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We employ scattering amplitudes in Einstein-Maxwell theory to compute the classical Hamiltonian of a binary system of two charged, nonspinning compact objects. The effective Hamiltonian is valid to all orders in velocity and up to second post-Minkowskian (2PM) order, i.e., $\mathcal{O}(G^2)$. The classical interaction potential is extracted via matching the four-point tree and one-loop amplitudes to those of a suitably chosen classical effective field theory. We derive the 2PM scattering angle from the Hamiltonian and find complete agreement with known results. Using also the third-order result for the scattering angle computed by Wilson-Gerow, we expand the scattering angle at second post-Newtonian (2PN) order. Additionally, we provide expressions for the binding energy and periastron advance at 2PN order via the boundary-to-bound dictionary introduced by Kälin and Porto and find agreement with previous findings.

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I. INTRODUCTION

Understanding the relativistic two-body dynamics of compact binary mergers remains a central challenge in gravitational wave physics. Since the initial direct detection of the first gravitational waves [1–5] by LIGO and Virgo and later joined by KAGRA, the field has transformed tremendously toward precision measurements, which allows us to study the properties of black holes and perform consistency tests of general relativity [6]. Future upgrades of the LIGO-Virgo-KAGRA detectors [7], together with upcoming observatories, such as Cosmic Explorer [8], Einstein Telescope [9], LISA [10], and TianQin [11], will significantly enhance detector sensitivity and thereby expand the potential to uncover hints of new physics [12]. To exploit the complete physics potential of these observatories, an improved theoretical modeling of gravitational waveforms is necessary [13].

Even though most studies to date have focused on electrically neutral black holes, there are compelling theoretical motivations to investigate compact objects carrying a conserved charge. The presence of a charge has a profound impact on the dynamics of the two-body system, as in addition to the attractive gravitational pull

there can be another repulsive or attractive force at play. This will modify the observed gravitational wave signals in all three stages, the inspiral [14], the merger [15], and the ringdown [16–22].

Our motivation to study the two-body dynamics of charged black holes is twofold. First, within general relativity the no-hair theorem [23–26] guarantees that stationary black holes are characterized solely by three fundamental properties: mass, spin, and charge. However, in astrophysical environments it is not expected that black holes can sustain any macroscopic electrical charge for a long time [27–29]. Nonetheless, even if the presence of electrically charged black holes is not expected it would make future consistency tests of general relativity more robust if the charge degree of freedom would be inferred from data in the analysis of gravitational waves.

Second, many extensions of the standard model contain so-called dark sectors, which modify dynamics through the presence of additional long-range interactions that are associated with hidden gauge symmetries. These models often give candidates for dark matter particles [30–33] or include dark photon interactions [34,35]. In these models, compact objects may carry a dark U(1) charge, and the same astrophysical restrictions that would lead to the rapid neutralization of the electric charge do not apply in this case. Furthermore, if compact objects beyond the conventional black holes and neutron stars exist, such as boson stars, then those boson stars can carry charge [36–40], and mergers involving those objects, therefore, can showcase the dynamics of a charged two-body system.

We will focus on the systematic improvement of the description of the inspiral phase of charged black hole

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mergers. In comparison to Schwarzschild or Kerr black holes their charged partners received only very little attention so far. Most of the existing literature concerns the post-Newtonian (PN) expansion, in which one simultaneously expands around weak gravitational fields, i.e., $Gm/(rc^2) \ll 1$, and small velocities, i.e., $v^2/c^2 \ll 1$, of the celestial objects. The first PN corrections to the Lagrangian governing the dynamics of a charged binary black hole system have been computed in Refs. [41–43], as well as the dynamics in an external magnetic field [44]. The accuracy of the Lagrangian has been extended to the second PN order in Refs. [45,46]. Furthermore, the radiation reaction has been recently computed at 2PN accuracy [47]. Focusing only on binary systems that exhibit extreme mass ratios, tidal effects have been studied in Refs. [48,49].

An alternative approach is the so-called post-Minkowskian (PM) expansion in which only weak fields are assumed, i.e., $Gm/(rc^2) \ll 1$, but fully special relativistic effects are taken into account. For instance, in Ref. [50] the dynamics of charged black holes have been investigated in the probe limit to all orders in the PM expansion. Furthermore, Ref. [51] explored the Hamiltonian formulation of many-body interactions. The article derives the three-body Hamiltonian at second order in the PM expansion. We want to stress that the latter results depend on the two-body Hamiltonian, which, however, was not shown explicitly in the article. In Ref. [52] the spherically symmetric solutions for the metric and gauge field in Einstein-Maxwell theory were derived from scattering amplitudes at 3PM order. Finally, the scattering angle for the charged binary black hole system was computed at the 3PM order in Ref. [53] using an effective field theory approach to binary systems with extreme mass ratios.

The goal of this article is to present the calculation of the classical Hamiltonian of charged binary black hole systems at second order in the PM expansion using scattering amplitudes. We follow the methodology that has been established in Refs. [54–58] by computing directly the necessary gravitational scattering amplitudes in Einstein-Maxwell theory. Our results show consistency with the 2PN results of Ref. [46]. It should be noted that to obtain full 2PN accuracy we rely on the 3PM result for the scattering angle of Ref. [53]. Furthermore, for observables of bound systems we use the boundary-to-bound dictionary of Refs. [59,60].

The article is organized as follows: in Sec. II we discuss the Einstein-Maxwell theory in the weak-field expansion as well as the traditional black hole solution in unperturbed general relativity. In Sec. III we discuss how classical terms can be extracted from quantum scattering amplitudes via soft expansions. Afterwards, in Sec. IV we elaborate how the classical two-body Hamiltonian can be computed in terms of scattering amplitudes, followed by Sec. V, where we discuss classical observables and cross-checks. Finally, in Sec. VI we present our conclusions and outlook.

II. EINSTEIN-MAXWELL FIELD THEORY

In this section we will first discuss some properties of the induced spacetime metric by the presence of a stationary charged black hole. Afterwards, we develop the weak-field expansion of the Einstein-Maxwell theory that we will employ to compute perturbative scattering amplitudes in quantum field theory.

A. The static Reissner-Nordström black hole solution

The solution to the Einstein equations for a spherically symmetric, statically charged, stationary black hole in Einstein-Maxwell theory [61–64] is given by

$$ds^2 = \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) dt^2 - \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1} dr^2 - r^2 d\Omega^2, \quad (1)$$

where $r_s = 2GM$ is the Schwarzschild radius, $r_Q^2 = Ge^2 Q^2 / (4\pi)$ is a characteristic length scale associated with the electric charge Q of the black hole (in units of e), and $d\Omega^2$ is the volume element of the two-sphere. An alternative representation of the Reissner-Nordström metric can be obtained by factorizing the radial function as below:

$$1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right), \quad (2)$$

where

$$r_{\pm} = \frac{1}{2} \left(r_s \pm \sqrt{r_s^2 - 4r_Q^2} \right) = GM \pm \sqrt{G^2 M^2 - \frac{Ge^2 Q^2}{4\pi}}. \quad (3)$$

Consequently, Reissner-Nordström black holes exhibit two horizons if $2|r_Q| < r_s$. Here, r_+ corresponds to the outer horizon and r_- to the inner one. In the case where $2|r_Q| = r_s$, the outer and inner horizons coincide and the electric charge of the black hole, called an extremal black hole, reaches its maximum, which is given by

$$eQ_{\max} = \sqrt{4\pi GM}. \quad (4)$$

For later convenience, we also present the form of the Reissner-Nordström metric in isotropic coordinates. We use the following transformation given in Ref. [65] to bring the metric in Eq. (1) to isotropic coordinates:

$$r \mapsto r_{\text{iso}} \left(\left(1 + \frac{r_s}{4r_{\text{iso}}}\right)^2 - \frac{r_Q^2}{4r_{\text{iso}}^2} \right). \quad (5)$$

The Reissner-Nordström metric then reads

$$ds^2 = \left(\frac{(1 - \frac{r_s}{4r})(1 + \frac{r_s}{4r}) + \frac{r_Q^2}{4r^2}}{(1 + \frac{r_s}{4r})^2 - \frac{r_Q^2}{4r^2}} \right)^2 dt^2 - \left(\left(1 + \frac{r_s}{4r}\right)^2 - \frac{r_Q^2}{4r^2} \right)^2 (dr^2 + r^2 d\Omega), \quad (6)$$

where we have dropped the subscript in r_{iso} for notational simplicity. Finally, the electromagnetic four-potential $A_\mu(x)$ for a nonrotating charged black hole is given by

$$A_0(r) = \frac{eQ}{4\pi r}, \quad A_i(r) = 0. \quad (7)$$

Inserting Eq. (5) in the equation above, we obtain the expression for the four-potential in isotropic coordinates,

$$A_0(r) = \frac{eQ}{4\pi r \left((1 + \frac{r_s}{4r})^2 - \frac{r_Q^2}{4r^2} \right)}, \quad A_i(r) = 0. \quad (8)$$

We will make use of the metric and four-potential in isotropic coordinates later to perform a cross-check of our results in the probe limit.

B. Weak-field expansion of Einstein-Maxwell theory

We consider a two-body system of electrically charged, nonspinning black holes with masses m_1 and m_2 , and charges eQ_1 and eQ_2 . The dynamics of the system is governed by the Einstein-Maxwell Lagrangian minimally coupled to matter,

$$\mathcal{L} = \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{EM}} + \mathcal{L}_m, \quad (9)$$

where $\mathcal{L}_{\text{EH}}$ is the Einstein-Hilbert Lagrangian, $\mathcal{L}_{\text{EM}}$ is the electromagnetic Lagrangian, and $\mathcal{L}_m$ represents the matter Lagrangian. Here, we are considering the point-particle approximation by neglecting finite-size effects [66]. This effective description is valid under the assumption that the characteristic length scale R of each black hole and the separation r between them satisfy the hierarchy of scales $R \ll r$.

The Einstein-Hilbert Lagrangian, describing pure gravitational interactions, reads

$$\mathcal{L}_{\text{EH}} = -\frac{2}{\kappa^2} \sqrt{-g} R, \quad (10)$$

where $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor $g_{\mu\nu}$, R is the Ricci scalar, and $\kappa = \sqrt{32\pi G}$ is a coupling related to Newton's constant G . We will consider an expansion of the metric around a flat background spacetime metric. The choice

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \quad (11)$$

defines the *weak-field expansion* and $h_{\mu\nu}$ is identified with quantum fluctuations in the gravitational field, i.e., the dynamic graviton quantum field. The expansion generates infinite many terms through

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\lambda} h^\nu{}_\lambda + \dots, \quad \sqrt{-g} = 1 + \frac{\kappa}{2} h + \frac{\kappa^2}{8} (h^2 - 2h_{\mu\nu} h^{\mu\nu}) + \dots, \quad (12)$$

where we have defined $h \equiv h^\lambda{}_\lambda$. The quantization procedure of the graviton field $h_{\mu\nu}$ requires to supplement the Lagrangian with a gauge-fixing term, where we have chosen the commonly used *linearized harmonic gauge*. The corresponding Lagrangian is explicitly given by

$$\mathcal{L}_{\text{GF-GR}} = \eta^{\mu\nu} \left(\partial^\lambda h_{\mu\lambda} - \frac{1}{2} \partial_\mu h \right) \left(\partial^\lambda h_{\nu\lambda} - \frac{1}{2} \partial_\nu h \right), \quad (13)$$

where indices have been contracted using the flat spacetime metric $\eta_{\mu\nu}$. In this gauge the D -dimensional graviton propagator takes the form [67]

$$P_{\mu\nu,\alpha\beta} = \frac{i}{p^2 + i\epsilon} \frac{1}{2} \left[\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \frac{2}{D-2} \eta_{\mu\nu} \eta_{\alpha\beta} \right]. \quad (14)$$

The electromagnetic Lagrangian is given by

$$\frac{\mathcal{L}_{\text{EM}}}{\sqrt{-g}} = -\frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} - \frac{1}{2\xi} (g^{\mu\nu} \nabla_\mu A_\nu)^2, \quad (15)$$

where ξ is the gauge-fixing parameter and $\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda{}_{\mu\nu} A_\lambda$ is the covariant derivative of general relativity. Furthermore, the electromagnetic field strength tensor is given by $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$. Lastly, the matter Lagrangian includes massive, complex scalar fields ϕ_i , which couple minimally to $g_{\mu\nu}$ and to A_μ via their electric charge,

$$\frac{\mathcal{L}_m}{\sqrt{-g}} = \sum_{i=1}^2 [g^{\mu\nu} (D_\mu \phi_i)^\dagger (D_\nu \phi_i) - m_i^2 \phi_i^\dagger \phi_i], \quad (16)$$

where $D_\mu = \partial_\mu - ieQ_i A_\mu$ is the covariant derivative of quantum electrodynamics.

III. SCATTERING AMPLITUDES AND THEIR CLASSICAL LIMIT

The classical dynamics of a binary system of nonrotating charged black holes are encapsulated in the scattering process of two massive complex scalar fields $\phi_{1,2}$ with masses $m_{1,2}$ and charges $eQ_{1,2}$, respectively,

$$\phi_1(-p_1) + \phi_2(-p_2) \rightarrow \phi_1(p_4) + \phi_2(p_3), \quad (17)$$

where the signs for the external momenta follow an all-outgoing convention. Following Ref. [68] we parametrize the momenta by special kinematic variables that facilitate the extraction of classical physics. The explicit parametrization is given by

$$\begin{aligned} p_1 &= -\bar{p}_1 + q/2, & p_4 &= \bar{p}_1 + q/2, \\ p_2 &= -\bar{p}_2 - q/2, & p_3 &= \bar{p}_2 - q/2, \end{aligned} \quad (18)$$

for which $\bar{p}_i \cdot q = 0$ by construction, and where $q = p_1 + p_4$ is the momentum transfer. We further define normalized velocity vectors u_i through the relations $\bar{p}_i = \bar{m}_i u_i$ and $\bar{m}_i^2 = m_i^2 - q^2/4$, and the dimensionless parameter $y = u_1 \cdot u_2$, which is a measure of the relative velocity between the two scattered particles.

In accordance with the correspondence principle, the classical limit can be defined as the limit of large conserved charges. In the context of black hole scattering, we have large orbital angular momentum, masses, and center-of-mass energy. This implies the hierarchy of scales [68]

$$m_\star^2 \sim s, |u|, m_1^2, m_2^2 \sim J^2 |t| \gg |t| = |q|^2, \quad (19)$$

where $m_\star$ is some characteristic mass scale and s, t , and u are Mandelstam variables. Thus, the classical dynamics are defined by an expansion for small $|q|$. From the hierarchy above, the phase space can be divided into two momentum regions defined based on the possible scalings of the momentum components $k = (\omega, \mathbf{k})$ of a given (external or loop) momentum,

$$\begin{aligned} \text{hard: } (\omega, \mathbf{k}) &\sim (m_\star, m_\star), \\ \text{soft: } (\omega, \mathbf{k}) &\sim (|q|, |q|). \end{aligned} \quad (20)$$

Following Ref. [69], we can further subdivide the soft region through an additional small parameter $v \ll 1$, related to $y \approx 1/\sqrt{1-v^2}$. We then have the following subregions inside the soft region:

$$\begin{aligned} \text{potential: } (\omega, \mathbf{k}) &\sim (|q|v, |q|), \\ \text{radiation: } (\omega, \mathbf{k}) &\sim (|q|v, |q|v). \end{aligned} \quad (21)$$

For our purposes the classical conservative dynamics arise from interactions mediated by photons and gravitons in the potential region. The reason is because potential modes are off-shell and hence can be integrated out and its effects captured through an instantaneous contact interaction potential in an effective field theory description. It is known that the strict separation of potential and radiation regions breaks down at higher post-Minkowskian orders [70].

On dimensional grounds, let us consider the relevant length scales in the scattering process,

$$\text{Compton wavelength: } \ell_c \sim \frac{\hbar}{m},$$

$$\text{Outer horizon radius: } r_+ \sim Gm + \sqrt{G^2 m^2 - Ge^2 Q^2},$$

$$\text{Impact parameter: } |b| \sim \frac{\hbar}{|q|}, \quad (22)$$

where $m \sim m_1 + m_2$ is some common mass scale, and the impact parameter $|b|$ is the Fourier conjugate to the momentum transfer $|q|$. The distance r_+ is bounded as $r_s/2 \leq r_+ \leq r_s$, the lower bound is given by the extremal limit $eQ \rightarrow \sqrt{4\pi G}m$, and the upper bound is given by the limit of vanishing charge $Q \rightarrow 0$. Hence, we can make the replacement $r_+ \rightarrow r_s$ when considering the characteristic size of the charged black holes without any loss of generality. With this in mind, the PM expansion is established from the following hierarchy of scales [71]:

$$\ell_c \ll r_s \ll |b| \Leftrightarrow \frac{\hbar}{m} \ll Gm \ll \frac{\hbar}{|q|}, \quad (23)$$

which implies the dimensionless ratios

$$\frac{\ell_c}{|b|} \ll \frac{r_s}{|b|} \ll 1 \Leftrightarrow \frac{|q|}{m} \ll \frac{Gm|q|}{\hbar} \ll 1. \quad (24)$$

The PM expansion is defined as an expansion in $r_s/|b| \sim Gm/|b| \sim Gm|q|$ (in natural units). Therefore, at each order in G , we need to expand the amplitude to an additional order in $|q|$ in the small- $|q|$ expansion in order to extract the classical dynamics. Specifically, at $\mathcal{O}(G^n)$ the scattering amplitude only needs to be expanded up to $\mathcal{O}(|q|^{n-3})$.

There is an additional length scale that we can construct, which is given by the classical charge radius $r_{\text{charge}} = e^2 Q^2 / (4\pi m)$. Multiplying this scale by 1 through the ratio G/G we can establish the relation

$$r_{\text{charge}} = \frac{Ge^2 Q^2}{4\pi} \frac{1}{Gm} \leq G^2 m^2 \frac{1}{Gm} \sim r_s, \quad (25)$$

where the inequality above follows from the relation $2|r_Q| \leq r_s$ required for physically relevant charged black hole solutions [see Eq. (3)]. If we define the dimensionless charge-to-mass ratio $\eta = eQ/(\sqrt{4\pi G}m)$, then we see that $r_{\text{charge}} = Gm\eta^2 = r_s\eta^2$. The value of $|\eta|$ ranges from 0 in the vanishing charge limit to 1 in the extremal limit. It follows that

$$\frac{r_{\text{charge}}}{|b|} \leq \frac{r_s}{|b|} \ll 1 \Leftrightarrow \frac{Gm\eta^2|q|}{\hbar} \leq \frac{Gm|q|}{\hbar} \ll 1. \quad (26)$$

An expansion in orders of the small ratio $r_{\text{charge}}/|b| \ll 1$ defines the so-called post-Lorentzian (PL) regime [72,73]. The above implies that the classical PL regime is automatically captured when enforcing the hierarchy of scales

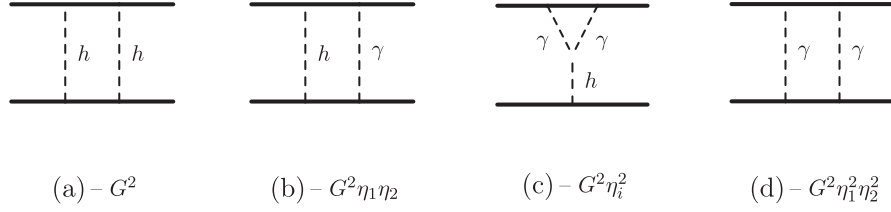


FIG. 1. Representative diagrams for the various gauge-invariant building block of the one-loop scattering amplitude in Einstein-Maxwell theory.

in Eq. (23) of the classical PM regime. Therefore, we only need to concern ourselves with the PM expansion in orders of the small ratio $r_s/|b| \ll 1$.

The necessary scattering amplitude can be decomposed into four classes of gauge-invariant building blocks parametrized by powers of η_i , as shown in Fig. 1. For charged black holes the scattering amplitude at $\mathcal{O}(G^2)$, i.e., at 2PM, involves pure gravitational and pure electromagnetic interactions, as shown in Fig. 1(a) and (d). Note that, due to the definition of $\eta_i = eQ_i/(\sqrt{4\pi G}m_i)$ the pure electromagnetic contribution in (d) does not depend on Newton's coupling, since $G^2\eta_1^2\eta_2^2 \sim G^2e^2Q_1^2Q_2^2/G^2 = e^2Q_1^2Q_2^2$. Furthermore, at this order we find, as depicted in Fig. 1(b), the first mixed contributions with graviton and photon exchanges. Finally, in Fig. 1(c) we find corrections that do not have an underlying tree-level contribution and are due

to the gravitational field interacting directly with the energy density of the electromagnetic field.

To compute the scattering amplitudes we obtain the necessary Feynman rules for Einstein-Maxwell theory using the weak-field expansion of the Lagrangian of Sec. II using the `xAct` packages [74,75]. We implemented the obtained rules in `Caravel` [76], which implements a numerical variant of the generalized unitarity method [77–81]. We evaluated the one-loop scattering amplitudes numerically using finite modular arithmetics [82,83], which allows the reconstruction of the complete analytical expression of the quantum scattering amplitudes. Afterwards, we expand the amplitudes in the potential region to extract the classical contributions.

The final expressions for the 1PM and 2PM classical scattering amplitudes in the center-of-mass frame are given by

$$\mathcal{M}_1 = -\frac{4\pi G\nu^2 m^2}{\gamma^2 \xi q^2} (1 - 2\sigma^2 + \eta_1\eta_2\sigma), \quad (27)$$

$$\begin{aligned} \mathcal{M}_2 = & \frac{\pi^2 G^2 \nu^2 m^3}{2\gamma^2 \xi |q|} \left[-3(1 - 5\sigma^2) + 2\eta_1^2\eta_2^2 - 12\eta_1\eta_2\sigma + \left(\frac{\eta_1^2 + \eta_2^2}{2} + \frac{\eta_1^2 - \eta_2^2}{2} \sqrt{1 - 4\nu} \right) (1 - 3\sigma^2) \right] \\ & + \frac{1}{\sqrt{\sigma^2 - 1}} \frac{2\pi i \nu^3 m^4 G^2}{\gamma^2 \xi q^2} (1 - 2\sigma^2 + \eta_1\eta_2\sigma)^2 g(\epsilon). \end{aligned} \quad (28)$$

In writing the above expressions, we have included the nonrelativistic normalization factor $1/(4E_1E_2)$, where $E_i = \sqrt{\mathbf{p}^2 + m_i^2}$. For convenience we also introduced the total energy E , the symmetric energy ratio ξ , the total mass m , the symmetric mass ratio ν , the energy-mass ratio γ , and the relativistic invariant σ as

$$E = E_1 + E_2, \quad \xi = \frac{E_1 E_2}{(E_1 + E_2)^2}, \quad \sigma = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{m_1 m_2}, \quad (29)$$

$$m = m_1 + m_2, \quad \nu = \frac{m_1 m_2}{(m_1 + m_2)^2}, \quad \gamma = \frac{E}{m}. \quad (30)$$

Without loss of generality, we choose $m_1 > m_2$ such that

$$m_1 = \frac{m}{2} [1 + \sqrt{1 - 4\nu}], \quad m_2 = \frac{m}{2} [1 - \sqrt{1 - 4\nu}]. \quad (31)$$

We also define

$$g(\epsilon) \equiv \left(\frac{-q^2}{\bar{\mu}^2} \right)^{-\epsilon} e^{\gamma_E \epsilon} \frac{\Gamma(-\epsilon)^2 \Gamma(1 + \epsilon)}{\Gamma(-2\epsilon)}, \quad (32)$$

where $\bar{\mu}^2 = 4\pi e^{-\gamma_E} \mu^2$, and μ is the dimensional regularization scale. The term in $\mathcal{M}_2$ proportional to $g(\epsilon)$ is divergent, which reflects the infrared (IR) divergence of the original integral that originated the term. It is also proportional to $(1 - 2\sigma^2 + \eta_1\eta_2\sigma)^2$ and represents the double iteration of the tree-level diagram. In the next section, we will see that this IR divergence will cancel with a similar contribution originating from the effective theory, that we employ to extract the classical interaction potential.

IV. EFFECTIVE FIELD THEORY

Our goal is to extract from the 1PM and 2PM scattering amplitudes the classical Hamiltonian for interacting spinless particles. As mentioned earlier, the amplitudes contain infrared divergent and superclassical contributions that need to be explicitly cancelled. To do so we construct an effective field theory (EFT) that includes only the relevant dynamical degrees of freedom and captures the classical physics of the considered $2 \rightarrow 2$ scattering process. Then, the Hamiltonian can simply be extracted by matching the EFT to the full theory results.

The physics we want to capture is the *elastic* scattering of *positive* energy states. Since we are interested in conservative dynamics and want to describe interactions via some classical potential, we also impose that interactions be instantaneous. The EFT that describes the physics of two nonrelativistic (i.e., no antiparticles) massive scalar fields interacting through a long-range potential was introduced in Ref. [84]. In the center-of-mass frame, the EFT Lagrangian in momentum space is given by

$$\mathcal{L} = \sum_{i=1}^2 \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \phi_i^\dagger(-\mathbf{k}) \left(i\partial_t - \sqrt{\mathbf{k}^2 + m_i^2} \right) \phi_i(\mathbf{k}) - \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \frac{d^{D-1}\mathbf{k}'}{(2\pi)^{D-1}} V(\mathbf{k}, \mathbf{k}') \phi_1^\dagger(\mathbf{k}') \phi_1(\mathbf{k}) \phi_2^\dagger(-\mathbf{k}') \phi_2(-\mathbf{k}). \quad (33)$$

This effective field theory can be obtained from the Einstein-Maxwell theory minimally coupled to matter by integrating out the matter negative energy states and the potential modes of photons and gravitons. As a result, the potential, which appears as some contact interaction, is nonlocal in space and can be thought of as a Wilson coefficient. Following Ref. [72], the form of the potential in momentum space is written in a PM expansion as

$$V(\mathbf{k}, \mathbf{k}') = \sum_{n=1}^{\infty} \frac{(G/2)^n (4\pi)^{(D-1)/2} \Gamma[(D-1-n)/2]}{|\mathbf{k} - \mathbf{k}'|^{D-1-n} \Gamma(n/2)} c_n \left(\frac{\mathbf{k}^2 + \mathbf{k}'^2}{2} \right) = \frac{4\pi G}{|\mathbf{k} - \mathbf{k}'|^2} c_1 \left(\frac{\mathbf{k}^2 + \mathbf{k}'^2}{2} \right) + \frac{2\pi^2 G^2}{|\mathbf{k} - \mathbf{k}'|} c_2 \left(\frac{\mathbf{k}^2 + \mathbf{k}'^2}{2} \right) + \dots, \quad (34)$$

where the c_n are free coefficients that are fixed by matching to the full theory amplitudes and encode the conservative dynamics. The various normalization factors in Eq. (34) are chosen such that the expression of the potential in position space is simplified. The resulting Feynman rules for the nonrelativistic propagator and interaction vertex can be obtained from Eq. (33) and are given by

$$\begin{array}{c} (k_0, \mathbf{k}) \\ \longrightarrow \end{array} = \frac{i}{k_0 - \sqrt{\mathbf{k}^2 + m_i^2} + i\varepsilon}, \quad \begin{array}{c} -\mathbf{k} \quad -\mathbf{k}' \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \mathbf{k} \quad \mathbf{k}' \end{array} = -iV(\mathbf{k}, \mathbf{k}'). \quad (35)$$

The amplitude in the effective field theory consists of iterated bubble diagrams, and up to the one-loop order all relevant diagrams are shown in Fig. 2. Here, we assume $|\mathbf{p}| = |\mathbf{p}'|$ for elastic scattering and energies are given by $E_i = \sqrt{\mathbf{p}^2 + m_i^2}$. In the center-of-mass frame, the momentum transfer $q = p_1 + p_4$ is purely spatial and is given by $\mathbf{q} = \mathbf{p} - \mathbf{p}'$.

To perform the EFT amplitude computations it is convenient to integrate the two nonrelativistic propagators in a bubble over the energy component of the loop momentum. This yields an effective “two-body” propagator,

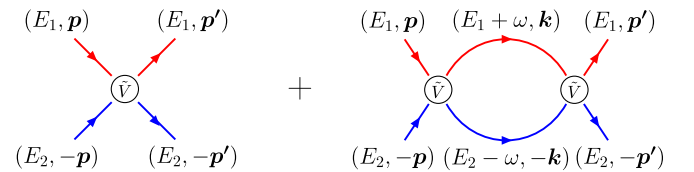


FIG. 2. The tree-level and one-loop EFT diagrams necessary for the matching. The loop momentum is given by $\ell^\mu = (\omega, \boldsymbol{\ell})$ and, therefore, $\mathbf{k} = \mathbf{p} + \boldsymbol{\ell}$.

$$i\Delta(k) \equiv \int \frac{d\omega}{2\pi} \frac{i}{E_1 + \omega - \sqrt{k^2 + m_1^2}} \frac{i}{E_2 - \omega - \sqrt{k^2 + m_2^2}} = \frac{i}{E_1 + E_2 - \sqrt{k^2 + m_1^2} - \sqrt{k^2 + m_2^2}}, \quad (36)$$

where the Feynman $i\epsilon$ -prescription is left implicit. The 1PM amplitude in the effective theory is given by the tree-level diagram, which involves only the $\mathcal{O}(G)$ term of the potential in Eq. (34) and, therefore,

$$\mathcal{M}_1^{\text{EFT}} = -V(-\mathbf{p}, -\mathbf{p}') = -\frac{4\pi G}{q^2} c_1(\mathbf{p}^2), \quad (37)$$

where we have set $\mathbf{k} - \mathbf{k}' = -\mathbf{p} + \mathbf{p}' = -\mathbf{q}$ and $k^2 = k'^2 = p^2$ in $c_1(\frac{k^2+k'^2}{2})$ for on-shell kinematics. In the above, the first argument of the potential corresponds to the three-momenta of p_1 , and we had to change from an outgoing to an incoming convention given how the Feynman interaction vertex is defined, and the second argument corresponds to the three-momenta of p_4 .

The 2PM amplitude receives contributions from two diagrams: the tree-level diagram given by the contact interaction vertex, keeping only the $\mathcal{O}(G^2)$ term of the potential in Eq. (34), and the one-loop bubble diagram with only the $\mathcal{O}(G)$ term of the potential inserted at each vertex,

$$\begin{aligned} \mathcal{M}_2^{\text{EFT}} &= -\frac{2\pi^2 G^2}{|q|} c_2(\mathbf{p}^2) - \int \frac{d^{D-1}\ell}{(2\pi)^{D-1}} V(-\mathbf{p}, \ell) \Delta(\ell) V(\ell, -\mathbf{p}') \\ &= -\frac{2\pi^2 G^2}{|q|} c_2(\mathbf{p}^2) - \int \frac{d^{D-1}\ell}{(2\pi)^{D-1}} \frac{4\pi G}{|-\mathbf{p} - \ell|^2} c_1\left(\frac{\mathbf{p}^2 + \ell^2}{2}\right) \Delta(\ell) \frac{4\pi G}{|\ell + \mathbf{p}'|^2} c_1\left(\frac{\ell^2 + \mathbf{p}'^2}{2}\right). \end{aligned} \quad (38)$$

By changing integration variables $\ell \rightarrow \ell - \mathbf{p}$, we arrive at the expression

$$\mathcal{M}_2^{\text{EFT}} = -\frac{2\pi^2 G^2}{|q|} c_2(\mathbf{p}^2) - 16\pi^2 G^2 \int \frac{d^{D-1}\ell}{(2\pi)^{D-1}} \frac{1}{\ell^2(\ell - \mathbf{q})^2} c_1^2\left(\frac{\mathbf{p}^2 + (\ell - \mathbf{p})^2}{2}\right) \Delta(\ell - \mathbf{p}). \quad (39)$$

For off-shell momenta, the potential $V(\mathbf{k}, \mathbf{k}')$ in the loop integral will introduce quantum corrections, which, of course, do not contribute to the classical dynamics. Therefore, we have to expand the integrand in the classical limit. For this we first define the variable

$$Y \equiv \ell^2 - 2\ell \cdot \mathbf{p}, \quad (40)$$

so that we obtain

$$c_1^2\left(\frac{\mathbf{p}^2 + (\ell - \mathbf{p})^2}{2}\right) = c_1^2\left(\mathbf{p}^2 + \frac{Y}{2}\right), \quad (41)$$

and the two-body propagator can be written as

$$\Delta(\ell - \mathbf{p}) = \frac{1}{E - \sqrt{E_1^2 + Y} - \sqrt{E_2^2 - Y}}. \quad (42)$$

In the classical limit $|\ell| \sim |q|$, and we can see that Y does not have a homogeneous $|q|$ -scaling. Nevertheless, the leading-order piece in Y is of order $|q|$ so in the classical limit enforced by the hierarchy of scales $|\ell| \sim |q| \ll |p_i|, m_i$, we can expand both the two-body propagator and the coefficient function c_1 around Y . We obtain the following expansions:

$$\begin{aligned} \Delta(\ell - \mathbf{p}) &= -\frac{2\xi E}{Y} - \frac{(1 - 3\xi)}{2\xi E} + \dots, \\ c_1^2\left(\mathbf{p}^2 + \frac{Y}{2}\right) &= c_1^2(\mathbf{p}^2) + Y c_1(\mathbf{p}^2) c_1'(\mathbf{p}^2) + \dots, \end{aligned} \quad (43)$$

where c_1' denotes the first derivative of c_1 with respect to $\mathbf{p}^2$. We plug the above expansion back in Eq. (39) and keep the terms that scale up to $\mathcal{O}(q)$, which is consistent with the potential in Eq. (34) up to $\mathcal{O}(G^2)$. The resulting $(D-1)$ -dimensional integrals have been computed in dimensional regularization and are in full agreement with those reported in Ref. [68]. The final expressions for the EFT amplitudes up to 2PM order are given by

$$\mathcal{M}_1^{\text{EFT}} = -\frac{4\pi G c_1}{q^2}, \quad (44)$$

$$\begin{aligned} \mathcal{M}_2^{\text{EFT}} &= -\frac{2\pi^2 G^2 c_2}{|q|} + \frac{\pi^2 G^2}{E\xi|q|} [(1 - 3\xi)c_1^2 + 4\xi^2 E^2 c_1 c_1'] \\ &\quad + \frac{1}{\sqrt{\sigma^2 - 1}} \frac{2\pi i \xi E^2 G^2 c_1^2}{m^2 \nu q^2} g(\epsilon). \end{aligned} \quad (45)$$

Since the EFT in Eq. (33) was constructed to capture the classical dynamics described by the full Einstein-Maxwell theory for the scattering of massive complex scalars interacting through the exchange of potential photons and gravitons, the scattering amplitudes of both theories must be equal order-by-order in the PM expansion

$$\mathcal{M}_n^{\text{EFT}} \stackrel{!}{=} \mathcal{M}_n. \quad (46)$$

This matching procedure yields the following coefficient functions for the potential:

$$\begin{aligned}
c_1(\mathbf{p}^2) &= \frac{\nu^2 m^2}{\gamma^2 \xi} (1 - 2\sigma^2 + \eta_1 \eta_2 \sigma), \\
c_2(\mathbf{p}^2) &= \frac{\nu^2 m^3}{\gamma^2 \xi} \left[\frac{3}{4} (1 - 5\sigma^2) - \frac{1}{2} \eta_1^2 \eta_2^2 + 3\eta_1 \eta_2 \sigma - \frac{1}{4} (1 - 3\sigma^2) \left(\frac{\eta_1^2 + \eta_2^2}{2} + \frac{\eta_1^2 - \eta_2^2}{2} \sqrt{1 - 4\nu} \right) \right] \\
&\quad + \frac{\nu^3 m^3}{\gamma^3 \xi^2} [-4\sigma(1 - 2\sigma^2) + \eta_1^2 \eta_2^2 \sigma + \eta_1 \eta_2 (1 - 6\sigma^2)] \\
&\quad + \frac{\nu^4 m^3}{\gamma^5 \xi^3} (1 - \xi) \left[-\frac{1}{2} (1 - 2\sigma^2)^2 - \frac{1}{2} \eta_1^2 \eta_2^2 \sigma^2 - \eta_1 \eta_2 \sigma (1 - 2\sigma^2) \right].
\end{aligned} \tag{47}$$

Let us note that the matching coefficients are, as expected, fully finite. The cancellation of infrared divergent and superclassical contributions is made apparent in the EFT amplitude. The superclassical and divergent piece at the 2PM order, the third term in Eq. (45), is completely determined by the lower-order coefficient and, therefore, shows that this term is indeed an iteration term and, therefore, does not contain any new classical information about the two-body system.

V. CLASSICAL HAMILTONIAN AND OBSERVABLES

From the matching procedure we obtained the coefficient functions of the effective two-body potential describing the classical conservative dynamics of charged, compact binary systems. The 2PM center-of-mass Hamiltonian in isotropic coordinates is given by

$$H(\mathbf{p}, \mathbf{r}) = \sqrt{\mathbf{p}^2 + m_1^2} + \sqrt{\mathbf{p}^2 + m_2^2} + \sum_{n=1}^2 c_n(\mathbf{p}^2) \left(\frac{G}{|\mathbf{r}|} \right)^n, \tag{48}$$

where $\mathbf{r}$ is the relative position vector between the two compact objects, which is the Fourier conjugate variable to the momentum transfer $\mathbf{k} - \mathbf{k}' = \mathbf{q}$ in Eq. (34), and the coefficients $c_n(\mathbf{p}^2)$ are given in Eq. (47). This Hamiltonian was determined by considering the scattering of two electrically charged, massive spinless particles. Nevertheless, it is valid not only for hyperbolic orbits but also for bound orbits. We proceed to perform consistency checks against known results in the literature.

A. Point charge in a Reissner-Nordström background

We start by checking that in the probe limit $m_2 \ll m_1$ our Hamiltonian coincides with the Hamiltonian for a point charge in a Reissner-Nordström background at 2PM order.

The motion of a massive charged relativistic particle of mass μ and charge eq in an electromagnetic field and in a

spacetime with metric $g_{\mu\nu}$ is determined by the Lagrangian (see, e.g., Ref. [85])

$$L(x^\mu, \dot{x}^\mu) = -\mu \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} - eq A_\mu(x^\mu) \dot{x}^\mu, \tag{49}$$

where we defined $\dot{x}^\mu \equiv dx^\mu/d\tau$ with τ referring to the proper time. We are interested in the case where $g_{\mu\nu}$ refers to the Reissner-Nordström metric of Eq. (1). The metric components g_{i0} explicitly vanish. Moreover, we choose a fixed reference frame with a worldline static gauge $x^0 = \tau$. The Lagrangian then reads

$$L(x^\mu, \dot{x}^\mu) = -\mu \sqrt{g_{00} + g_{ij} \dot{x}^i \dot{x}^j} - eq A_0 + eq A_i \dot{x}^i. \tag{50}$$

The Reissner-Nordström metric has a timelike Killing vector associated with time translation invariance since the metric does not explicitly depend on time. This implies the existence of a conserved quantity, namely, the energy of the particle along a geodesic $x^\mu(\tau)$. We can, therefore, set up the Hamiltonian formulation of our problem. The conjugate momenta are

$$p_i \equiv \frac{\partial L}{\partial \dot{x}^i} = -\mu \frac{g_{ij} \dot{x}^j}{\sqrt{g_{00} + g_{ij} \dot{x}^i \dot{x}^j}} + eq A_i. \tag{51}$$

The Hamiltonian is obtained in the usual fashion by taking the Legendre transform of the Lagrangian. The result is

$$H = p_i \dot{x}^i - L = \sqrt{g_{00}} \sqrt{\mu^2 - g^{ij} p_i p_j} + eq A_0. \tag{52}$$

If we now substitute for the Reissner-Nordström metric of Eq. (6) and the electromagnetic four-potential of Eq. (8), both in isotropic coordinates, we obtain what we call the *Reissner-Nordström Hamiltonian*,

$$H^{\text{RN}}(\mathbf{r}, \mathbf{p}) = \mu \frac{(1 - \frac{r_s}{4r})(1 + \frac{r_s}{4r}) + \frac{r_Q^2}{4r^2}}{(1 + \frac{r_s}{4r})^2 - \frac{r_Q^2}{4r^2}} \sqrt{1 + \left[\left(1 + \frac{r_s}{4r}\right)^2 - \frac{r_Q^2}{4r^2} \right]^{-2} \frac{\mathbf{p}^2}{\mu^2} + \frac{e^2 q Q}{4\pi r \left[(1 + \frac{r_s}{4r})^2 - \frac{r_Q^2}{4r^2} \right]}}. \quad (53)$$

In the neutral limit, this Hamiltonian reduces to the Hamiltonian for a test particle in a Schwarzschild background, as given in Ref. [86].

Let us now take the probe limit of this Hamiltonian. Our use of the symbols μ (the reduced mass) to represent the point-charge mass and m (the total mass) for the mass of the Reissner-Nordström background was so that in the probe limit, $m_2 \ll m_1$, we can make the replacements $\mu \rightarrow m_2$ and $m \rightarrow m_1$. If we then perform a series expansion of Eq. (53) up to $\mathcal{O}(G^2)$, take the probe limit, and perform a Fourier transform of the potential term, we will obtain a potential in the form of Eq. (34) with the coefficients

$$\begin{aligned} c_1^{\text{RN}}(\mathbf{p}^2) &= -\frac{m_1}{E_2} (2\mathbf{p}^2 + m_2^2) + m_1 m_2 \eta_1 \eta_2, \\ c_2^{\text{RN}}(\mathbf{p}^2) &= \frac{m_1^2}{4E_2^3} (9\mathbf{p}^2 + 13m_2^2 \mathbf{p}^2 + 2m_2^4) \\ &\quad + \frac{m_1^2 \eta_1^2}{4E_2} (3\mathbf{p}^2 + 2m_2^2) - m_1^2 m_2 \eta_1 \eta_2, \end{aligned} \quad (54)$$

where we identified $\eta_1 = eQ/(\sqrt{4\pi G}m_1)$ and $\eta_2 = eQ/(\sqrt{4\pi G}m_2)$. These coefficients agree with those in Eq. (47) after taking the probe limit of the latter.

B. Scattering angle

To obtain the conservative scattering angle for two charged black holes from the Hamiltonian in Eq. (48) we write down Hamilton's equations of motion. Since the Hamiltonian is isotropic and does not contain terms involving $\mathbf{p} \cdot \mathbf{r}$ but only $\mathbf{p}^2$, Hamilton's equations imply

$$\dot{\mathbf{r}} = \frac{\partial H(\mathbf{p}^2, \mathbf{r}^2)}{\partial \mathbf{p}} = \left[\frac{\partial H(\mathbf{p}^2, \mathbf{r}^2)}{\partial \mathbf{p}^2} \right] 2\mathbf{p}, \quad (55)$$

$$\dot{\mathbf{p}} = -\frac{\partial H(\mathbf{p}^2, \mathbf{r}^2)}{\partial \mathbf{r}} = \left[-\frac{\partial H(\mathbf{p}^2, \mathbf{r}^2)}{\partial \mathbf{r}^2} \right] 2\mathbf{r}. \quad (56)$$

The central-field potential $V(\mathbf{r}^2, \mathbf{p}^2)$ confines the dynamics to a plane, which we parametrize by polar coordinates (r, θ) . In these coordinates, vectors are decomposed into a sum of radial and tangential parts—the relative position vector is $\mathbf{r} = r\mathbf{e}_r$ and the momentum is $\mathbf{p} = p_r\mathbf{e}_r + p_\theta\mathbf{e}_\theta$, where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors in the radial and tangential directions, respectively.

Since the potential has spherical symmetry, the total energy E and angular momentum J are conserved throughout the scattering process,

$$E = H(r^2, p_r^2 + p_\theta^2), \quad J = |\mathbf{r} \times \mathbf{p}| = r p_\theta. \quad (57)$$

Since by assumption the potential V vanishes as $r \rightarrow \infty$, the total energy is the same as the kinetic energy of the system at infinity,

$$E = \sqrt{p_\infty^2 + m_1^2} + \sqrt{p_\infty^2 + m_2^2}, \quad (58)$$

and the angular momentum at infinity is given by

$$J = b p_\infty, \quad (59)$$

where $b = |\mathbf{b}|$ is the magnitude of the impact parameter. Equation (55) states that $\dot{\mathbf{r}}$ and $\mathbf{p}$ are parallel. Using the chain rule and the conservation of angular momentum, this determines the scattering trajectory in terms of the radial momentum

$$\frac{d\theta}{dr} = \frac{d\theta}{dt} \frac{dt}{dr} = \frac{p_\theta}{r p_r} = \frac{J}{r^2 p_r}. \quad (60)$$

The scattering angle is then obtained by integrating the above equation,

$$\chi = -\pi + 2J \int_{r_{\min}}^{\infty} \frac{dr}{r^2 p_r(r)}, \quad (61)$$

where p_r as a function of r is determined by the conservation of energy $E = H(r^2, p_r^2 + J^2/r^2)$, and $r_{\min}$ denotes the minimum distance between the two particles and is the point at which the radial momentum vanishes,

$$p_r(r_{\min}) = 0. \quad (62)$$

From the conservation of energy and Eq. (58) it follows that

$$\begin{aligned} &\sqrt{p_\infty^2 + m_1^2} + \sqrt{p_\infty^2 + m_2^2} \\ &= \sqrt{p_r^2 + J^2/r^2 + m_1^2} + \sqrt{p_r^2 + J^2/r^2 + m_2^2} \\ &\quad + \sum_{n=1}^2 c_n (p_r^2 + J^2/r^2) \left(\frac{G}{r} \right)^n, \end{aligned} \quad (63)$$

which we can solve for the radial momentum as a function of r perturbatively in G . To order G^2 , we obtain

$$p_r^2(r) = p_\infty^2 - \frac{J^2}{r^2} + \sum_{n=1}^2 P_n \left(\frac{G}{r} \right)^n, \quad (64)$$

with the coefficients in the series given by

$$P_1 = -2E\xi\bar{c}_1, \quad (65)$$

$$P_2 = -2E\xi\bar{c}_2 + (1 - 3\xi)\bar{c}_1^2 + 4E^2\xi^2\bar{c}_1\bar{c}_1', \quad (66)$$

where $\bar{c}_n = c_n(p_\infty^2)$, and the prime denotes a derivative with respect to the argument. The minimum distance is then obtained by solving Eq. (62) perturbatively in G , which to order G^2 gives

$$r_{\min}^2 = b^2 - b^3 P_1 \frac{G}{J^2} + \frac{b^4}{2} (P_1^2 - 2p_\infty^2 P_2) \frac{G^2}{J^4} + \mathcal{O}(G^3). \quad (67)$$

Using Eqs. (64) and (67) in Eq. (61), the scattering angle through second order in the PM expansion is given by [69]

$$\chi = \sum_{i \geq 1} \chi^{i\text{PM}} = \frac{P_1}{p_\infty} \left(\frac{G}{J} \right) + \frac{\pi}{2} P_2 \left(\frac{G}{J} \right)^2 + \mathcal{O}((G/J)^3). \quad (68)$$

Using Eqs. (65) and (66) along with the coefficient functions in (47), the conservative scattering angle up to $\mathcal{O}(G^2)$ is

$$\begin{aligned} \chi^{1\text{PM}} &= \frac{2(2\sigma^2 - 1 - \eta_1\eta_2\sigma)}{\sqrt{\sigma^2 - 1}} \frac{Gm^2\nu}{J}, \\ \chi^{2\text{PM}} &= \frac{\pi}{4\gamma} \left[3(5\sigma^2 - 1) + 2\eta_1^2\eta_2^2 - 12\eta_1\eta_2\sigma \right. \\ &\quad \left. + \left(\frac{\eta_1^2 + \eta_2^2}{2} + \frac{\eta_1^2 - \eta_2^2}{2} \sqrt{1 - 4\nu} \right) (1 - 3\sigma^2) \right] \frac{G^2 m^4 \nu^2}{J^2}. \end{aligned} \quad (69)$$

We find perfect agreement between our result and the 2PM scattering angle provided in Ref. [53], which was obtained using an effective theory for extreme mass ratios. In the neutral limit of vanishing electric charges, we correctly reproduce the 2PM scattering angle in Ref. [69], and in the limit $G \rightarrow 0$ the 2PL scattering angle in Ref. [72].

C. Post-Newtonian expansion and observables for bound systems

In this section we compute the post-Newtonian expansion of the scattering angle as well as the binding energy and periastron advance—two adiabatic invariants—for a charged black hole binary in *bound* orbits. Instead of calculating these observables using the Hamiltonian in Eq. (48), we follow Refs. [59,60], where an explicit dictionary is introduced relating scattering data and observables for bound states.

The dictionary is based on a relation between the relative momentum and the scattering amplitude that is presumed to

hold to all orders in the PM expansion, namely,¹

$$\mathbf{p}^2 = p_\infty^2 + \frac{1}{2E} \int d^3\mathbf{r} \mathcal{M}(\mathbf{p}, \mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}}, \quad (70)$$

where $\mathcal{M}(\mathbf{p}, \mathbf{q})$ represents the IR-finite part of the relativistically normalized scattering amplitude in the classical limit. The above equation is then expanded perturbatively as

$$\mathbf{p}^2 = p_\infty^2 + \sum_{n=1} \widetilde{\mathcal{M}}_n \frac{G^n}{r^n}, \quad (71)$$

which when compared with Eq. (64), extended to all PM orders, gives the relation

$$\widetilde{\mathcal{M}}_n = P_n \equiv p_\infty^2 m^n f_n. \quad (72)$$

The coefficients f_n in the expansion of the amplitude are the ones that appear more naturally in the PM expansion of the scattering angle (68),

$$\begin{aligned} \chi &= p_\infty f_1 \frac{Gm}{J} + \frac{\pi}{2} p_\infty^2 f_2 \left(\frac{Gm}{J} \right)^2 \\ &\quad + p_\infty^3 \left(2f_3 + f_2 f_1 - \frac{f_1^3}{12} \right) \left(\frac{Gm}{J} \right)^3 \\ &\quad + \frac{3\pi}{8} p_\infty^4 (f_2^2 + 2f_1 f_3 + 2f_4) \left(\frac{Gm}{J} \right)^4 + \dots, \end{aligned} \quad (73)$$

where we have explicitly written the expansion of the scattering angle to order G^4 for later use. If we now compare with our result in Eq. (69) and use the relation

$$p_\infty = \frac{m\nu}{\gamma} \sqrt{\sigma^2 - 1}, \quad (74)$$

we can extract the coefficients f_1 and f_2 for Einstein-Maxwell theory,

$$\begin{aligned} f_1 &= 2\gamma \left(\frac{2\sigma^2 - 1 - \eta_1\eta_2\sigma}{\sigma^2 - 1} \right), \\ f_2 &= \frac{\gamma}{2(\sigma^2 - 1)} \left[3(5\sigma^2 - 1) + 2\eta_1^2\eta_2^2 - 12\eta_1\eta_2\sigma \right. \\ &\quad \left. + \left(\frac{\eta_1^2 + \eta_2^2}{2} + \frac{\eta_1^2 - \eta_2^2}{2} \sqrt{1 - 4\nu} \right) (1 - 3\sigma^2) \right]. \end{aligned} \quad (75)$$

In order to obtain results at 2PN accuracy we need to go to 3PM, i.e., include the f_3 contribution. From the result of the scattering angle at 3PM [53], we obtain that

¹We are ignoring radiation reaction effects, which include tail effects, since they start appearing at 4PM order.

$$\begin{aligned}
 f_3 = \frac{\gamma}{6(\sigma^2 - 1)^4} & \left\{ 2\gamma^2(\sigma^2 - 1)(2\sigma^2 - 1 - \eta_1\eta_2\sigma)^3 - (1 - 2\nu)(\sigma^2 - 1)[10 - 3(\eta_1^2 + \eta_2^2) - 18\eta_1^2\eta_2^2 \right. \\
 & - 3\eta_1\eta_2\sigma(-30 + 3\eta_1^2 + 3\eta_2^2 + 2\eta_1^2\eta_2^2) + 3\sigma^2(-40 + 9\eta_1^2 + 9\eta_2^2 + 24\eta_1^2\eta_2^2) \\
 & + \eta_1\eta_2\sigma^3(-240 + 21\eta_1^2 + 21\eta_2^2 + 4\eta_1^2\eta_2^2) - 48\sigma^4(-5 + \eta_1^2 + \eta_2^2 + \eta_1^2\eta_2^2) - 12\eta_1\eta_2\sigma^5(-12 + \eta_1^2 + \eta_2^2) \\
 & + 8\sigma^6(-16 + 3\eta_1^2 + 3\eta_2^2)] - 3\gamma(\sigma^2 - 1)^2(2\sigma^2 - 1 - \eta_1\eta_2\sigma) \left[3(5\sigma^2 - 1) + 2\eta_1^2\eta_2^2 - 12\eta_1\eta_2\sigma \right. \\
 & + \left(\frac{\eta_1^2 + \eta_2^2}{2} + \frac{\eta_1^2 - \eta_2^2}{2}\sqrt{1 - 4\nu} \right)(1 - 3\sigma^2) \left. \right] - 3\sqrt{1 - 4\nu}(\sigma^2 - 1)^2[(\eta_1^2 - \eta_2^2)(1 - 8\sigma^2 + 8\sigma^4) \\
 & + \eta_1\eta_2\sigma(\eta_1^2 - \eta_2^2)(3 - 4\sigma^2)] + 2\nu(\sigma^2 - 1)[2\eta_1^3\eta_2^3(3 - 3\sigma^2 + \sigma^4) + 6\eta_1^2\eta_2^2\sigma(-1 - 4\sigma^2 + 4\sigma^4) \\
 & + \eta_1\eta_2(16 - 42\sigma^2 + 96\sigma^4 - 64\sigma^6) + \eta_1\eta_2(\eta_1^2 + \eta_2^2)(7 - 9\sigma^2 - 6\sigma^4 + 8\sigma^6) \\
 & + (\eta_1^2 + \eta_2^2)(23\sigma - 51\sigma^3 + 36\sigma^5 - 8\sigma^7) + 2\sigma(-55 + 132\sigma^2 - 114\sigma^4 + 36\sigma^6)] + 6\nu(\sigma^2 - 1)^{5/2}[12 \\
 & + 48\sigma^2 - 16\sigma^4 + 16\eta_1\eta_2\sigma(\sigma^2 - 2) - 4(\eta_1^2 + \eta_2^2)(2\sigma^2 + 1) + 4\eta_1^2\eta_2^2(1 - 2\sigma^2)]\text{arccosh}(\sigma) \left. \right\}. \quad (76)
 \end{aligned}$$

As expected, there is perfect agreement between Eqs. (5.60) and (5.65) of Ref. [59] for Einstein gravity and the f_n coefficients above upon taking the neutral limit.

In Ref. [60], a surprisingly simple relation between the scattering angle and the periastron advance was established, valid to all orders in the PM expansion. The relation is given by

$$\Delta\Phi = \chi(J) + \chi(-J), \quad (77)$$

and substituting Eq. (73) we find that

$$\begin{aligned}
 \Delta\Phi = \pi\hat{p}_\infty^2 f_2 \left(\frac{1}{j^2} \right) + \frac{3\pi}{4} \hat{p}_\infty^4 (f_2^2 + 2f_1 f_3 + 2f_4) \left(\frac{1}{j^4} \right) \\
 + \dots, \quad (78)
 \end{aligned}$$

where $\hat{p}_\infty = p_\infty/(m\nu)$, and we introduced the rescaled angular momentum $j = J/(Gm^2\nu)$. We define the binding energy as

$$\mathcal{E} = \frac{E - m}{m\nu}, \quad (79)$$

in terms of which it is easy to show that

$$\begin{aligned}
 \sigma & \equiv \frac{E^2 - m_1^2 - m_2^2}{2m_1 m_2} = 1 + \mathcal{E} + \frac{1}{2}\nu\mathcal{E}^2, \\
 \gamma & \equiv E/m = \sqrt{1 + 2\nu(\sigma - 1)} = 1 + \nu\mathcal{E}. \quad (80)
 \end{aligned}$$

Using Eq. (80) to write the f_n 's as a function of the binding energy, substituting the expressions thereof in Eq. (78), and performing a Taylor series expansion in powers of the binding energy, we obtain

$$\begin{aligned}
 K \equiv 1 + \frac{\Delta\Phi}{2\pi} = 1 + \frac{1}{j^2} & \left[3 - \frac{\eta_1^2}{4}(1 + X_{12}) - 3\eta_1\eta_2 - \frac{\eta_2^2}{4}(1 - X_{12}) + \frac{\eta_1^2\eta_2^2}{2} \right] \frac{1}{c^2} \\
 & + \left\{ \left[\frac{15}{2} - 3\nu + \frac{\eta_1^2}{4}(1 + X_{12})(-3 + \nu) + 3\eta_1\eta_2(-1 + \nu) + \frac{\eta_2^2}{4}(1 - X_{12})(-3 + \nu) - \frac{\nu}{2}\eta_1^2\eta_2^2 \right] \frac{\mathcal{E}}{j^2} \right. \\
 & + \left[\frac{105}{4} - \frac{15}{2}\nu + \frac{15}{2}\eta_1\eta_2(-7 + 3\nu) + \frac{3}{4}\eta_1^2\eta_2^2(45 - 36\nu) + \frac{15}{2}\eta_1^3\eta_2^3(-1 + 2\nu) + \frac{3}{8}\eta_1^4\eta_2^4(1 - 6\nu) \right. \\
 & + \frac{3\eta_1^4}{32}(1 + X_{12})^2 - \frac{45}{8}\eta_1^2(1 + X_{12}) - \frac{3\nu}{8}\eta_1^2(1 - X_{12}) + \frac{15}{2}\eta_1^3\eta_2(1 + X_{12}) + \frac{3\nu}{4}\eta_1^3\eta_2(3 - X_{12}) \\
 & + \frac{3}{8}\eta_1^4\eta_2^2X_{12}(-6 + \nu) - \frac{3}{8}\eta_1^4\eta_2^2(6 + 5\nu) + \frac{3\eta_2^4}{32}(1 - X_{12})^2 - \frac{45}{8}\eta_2^2(1 - X_{12}) - \frac{3\nu}{8}\eta_2^2(1 + X_{12}) \\
 & \left. \left. + \frac{15}{2}\eta_1\eta_2^3(1 - X_{12}) + \frac{3\nu}{4}\eta_1\eta_2^3(3 + X_{12}) - \frac{3}{8}\eta_1^2\eta_2^4X_{12}(-6 + \nu) - \frac{3}{8}\eta_1^2\eta_2^4(6 + 5\nu) \right] \frac{1}{j^4} \right\} \frac{1}{c^4}, \quad (81)
 \end{aligned}$$

where we define the (m -rescaled) mass difference $X_{12} = \sqrt{1 - 4\nu} = (m_1 - m_2)/m$, and we have restored factors of c to indicate the PN order by the formal PN expansion parameter $1/c^2$. The complete expression is symmetric under the interchange of charges $\eta_1 \leftrightarrow \eta_2$ and the replacement $X_{12} \rightarrow -X_{12}$. Equation (81) is the 2PN-accurate expression for the periastron advance and in the neutral limit reproduces the known 2PN result in general relativity, see, e.g., Eq. (C26) in Ref. [87]. Of course, the full expression in Eq. (78) contains additional PN corrections beyond 2PN order. In particular, the $\mathcal{O}(1/j^2)$ contribution is exact to all orders in the binding energy, though we would still need the f_4 coefficient (three-loop amplitude) to give the complete $1/j^4$ term valid to all orders in binding energy.

For the case of circular orbits, the following relation between the rescaled angular momentum and the scattering data holds at 3PM order [59]:

$$j^2 = \frac{1 - \sigma^2}{1 + 2\nu(\sigma - 1)} \left[\left(\frac{f_1}{2} \right)^2 - f_2 + \frac{2f_3}{f_1} \right] + \dots, \quad (82)$$

where we have truncated the full expression to only the terms we need for 2PN accuracy. Substituting the expressions for the f_n 's written as functions of the binding energy one can then compute the orbital frequency from the first law of binary black hole mechanics [88],

$$Gm\Omega = \left(\frac{dj(\mathcal{E})}{d\mathcal{E}} \right)^{-1}. \quad (83)$$

If we now introduce the so-called charge-flexed frequency parameter of Ref. [46],

$$x_q \equiv \left[\frac{Gm\Omega}{c^3} (1 - \eta_1\eta_2) \right]^{2/3}, \quad (84)$$

we see that Eq. (83) allows us to express this parameter as a function of the binding energy, $x_q(\mathcal{E})$. By performing a Taylor series expansion in powers of $\mathcal{E}$ and inverting the series, we then obtain the following expression for the binding energy at 2PN order:

$$\begin{aligned} \mathcal{E}(x_q) = & -\frac{x_q}{2} + \frac{x_q^2}{24} (1 - \eta_1\eta_2)^{-2} \{9 + \nu - 2(1 + X_{12})\eta_1^2 - 2(3 + \nu)\eta_1\eta_2 - 2(1 - X_{12})\eta_2^2 + (1 + \nu)\eta_1^2\eta_2^2\} \frac{1}{c^2} \\ & + \frac{x_q^3}{48} (1 - \eta_1\eta_2)^{-4} \{81 - 57\nu + \nu^2 - 18(1 + X_{12})\eta_1^2 - 2\nu(5 - X_{12})\eta_1^2 - 2(81 - 84\nu + 2\nu^2)\eta_1\eta_2 \\ & - 18(1 - X_{12})\eta_2^2 - 2\nu(5 + X_{12})\eta_2^2 - 2(1 + X_{12})^2\eta_1^4 + 4\eta_1^3\eta_2[9 + 11\nu + (9 - \nu)X_{12}] \\ & + (90 - 254\nu + 6\nu^2)\eta_1^2\eta_2^2 + 4\eta_1\eta_2^3[9 + 11\nu - (9 - \nu)X_{12}] - 2(1 - X_{12})^2\eta_2^4 \\ & + 2\eta_1^2\eta_2^4(2\nu - 19 - 9X_{12}) + 2\eta_1^2\eta_2^4[14(1 + X_{12}) - \nu(19 + X_{12})] + 2\eta_1^4\eta_2^2(2\nu - 19 + 9X_{12}) \\ & + 2\eta_1^4\eta_2^2[14(1 - X_{12}) - \nu(19 - X_{12})] - 2\eta_1^3\eta_2^3(9 - 76\nu + 2\nu^2) + (1 - 25\nu + \nu^2)\eta_1^4\eta_2^4\} \frac{1}{c^4}. \end{aligned} \quad (85)$$

Next, by taking Eq. (82) and plugging it in Eq. (81) we can obtain the periastron advance as a function of the binding energy. If we then substitute Eq. (85) we obtain the periastron advance as a function of the x_q . To 2PN order, we obtain

$$\begin{aligned} K = & 1 + \frac{x_q}{4} (1 - \eta_1\eta_2)^{-2} [12 - \eta_1^2 - 12\eta_1\eta_2 - \eta_2^2 + 2\eta_1^2\eta_2^2 + X_{12}(\eta_2^2 - \eta_1^2)] \frac{1}{c^2} \\ & + \frac{x_q^2}{48} (1 - \eta_1\eta_2)^{-4} \left\{ 648 - 120(1 + X_{12})\eta_1^2 - \frac{7}{2}(1 + X_{12})^2\eta_1^4 - 1368\eta_1\eta_2 + 204(1 + X_{12})\eta_1^3\eta_2 \right. \\ & - 120(1 - X_{12})\eta_2^2 + 4(222 - 7\nu)\eta_1^2\eta_2^2 - 70(1 + X_{12})\eta_1^4\eta_2^2 + 204(1 - X_{12})\eta_1\eta_2^3 - 192\eta_1^3\eta_2^3 \\ & - \frac{7}{2}(1 - X_{12})^2\eta_2^4 - 70(1 - X_{12})\eta_1^2\eta_2^4 + 10\eta_1^4\eta_2^4 + \nu[-336 - 20\eta_1^2 + 1008\eta_1\eta_2 + 112\eta_1^3\eta_2 - 20\eta_2^2 \\ & - 1256\eta_1^2\eta_2^2 - 92\eta_1^4\eta_2^2 + 112\eta_1\eta_2^3 + 688\eta_1^3\eta_2^3 - 92\eta_1^2\eta_2^4 - 104\eta_1^4\eta_2^4 + 16X_{12}(\eta_1^2 - 2\eta_1^3\eta_2 - \eta_2^2 + \eta_1^4\eta_2^2 \\ & \left. + 2\eta_1\eta_2^3 - \eta_1^2\eta_2^4)] \right\} \frac{1}{c^4}. \end{aligned} \quad (86)$$

Lastly, we give the PN expansion of the 3PM scattering angle. We take Eq. (73) truncated to 3PM order and substitute in the expressions for f_n 's. We also use Eqs. (74) and (80) so that the scattering angle is written as a function of the relativistic Lorentz factor, $\chi(\sigma)$. If we introduce the reduced angular momentum $p_\phi = J/(m\nu)$ and the variable $v_\infty = \sqrt{\sigma^2 - 1}$, which

measures the relative velocity at infinite separation (in units of c), we can Taylor expand the scattering angle in powers of v_∞ . The result up to 2PN order is

$$\begin{aligned}
 \chi = & \frac{2Gm}{p_\phi v_\infty} (1 - \eta_1 \eta_2) - \frac{2G^3 m^3}{3p_\phi^3 v_\infty^3} (1 - \eta_1 \eta_2)^3 \\
 & + \frac{1}{c^2} \left\{ \frac{Gm v_\infty}{p_\phi} (4 - \eta_1 \eta_2) + \frac{G^2 m^2 \pi}{p_\phi^2} \left[3 - \frac{\eta_1^2}{4} (1 + X_{12}) - 3\eta_1 \eta_2 - \frac{\eta_2^2}{4} (1 - X_{12}) + \frac{\eta_1^2 \eta_2^2}{2} \right] \right. \\
 & + \frac{G^3 m^3}{p_\phi^3 v_\infty} [8 - \eta_1^2 (1 + X_{12}) - 15\eta_1 \eta_2 + \eta_1^3 \eta_2 (1 + X_{12}) - \eta_2^2 (1 - X_{12}) + 8\eta_1^2 \eta_2^2 + \eta_1 \eta_2^3 (1 - X_{12}) - \eta_1^3 \eta_2^3] \Big\} \\
 & + \frac{1}{c^4} \left\{ \frac{G^2 m^2 \pi v_\infty^2}{p_\phi^2} \left[\frac{3}{4} (5 - 2\nu) + \frac{\eta_1^2}{8} (1 + X_{12}) (-3 + \nu) + \frac{3}{2} (-1 + \nu) \eta_1 \eta_2 + \frac{\eta_2^2}{8} (1 - X_{12}) (-3 + \nu) - \nu \frac{\eta_1^2 \eta_2^2}{4} \right] \right. \\
 & + \frac{G^3 m^3 v_\infty}{p_\phi^3} \left[-16(-3 + \nu) - \frac{17}{2} \eta_1^2 (1 + X_{12}) + \frac{\eta_1^2}{2} (1 - 2\nu + X_{12}) + \nu \eta_1^2 (1 + X_{12}) + \frac{\eta_1 \eta_2}{4} (128\nu - 225) \right. \\
 & + \eta_1^3 \eta_2 (1 + X_{12}) (6 - \nu) - \frac{3}{2} \eta_1^3 \eta_2 (1 - 2\nu + X_{12}) - \frac{17}{2} \eta_2^2 (1 - X_{12}) + \frac{\eta_2^2}{2} (1 - 2\nu - X_{12}) + \nu \eta_2^2 (1 - X_{12}) \\
 & + (16 - 24\nu) \eta_1^2 \eta_2^2 + \eta_1 \eta_2^3 (1 - X_{12}) (6 - \nu) - \frac{3}{2} \eta_1 \eta_2^3 (1 - 2\nu - X_{12}) + \frac{\eta_1^3 \eta_2^3}{4} (-3 + 16\nu) \Big] \\
 & \left. + \frac{Gm v_\infty^3}{4p_\phi} \eta_1 \eta_2 \right\}. \tag{87}
 \end{aligned}$$

We have compared our results for the binding energy (85), the periastron shift (86), and the scattering angle (87) at 2PN order with the results obtained in Ref. [46] and found agreement in each case.

VI. CONCLUSIONS

In this paper we have computed the conservative dynamics for classical scattering of two spinless charged black holes in Einstein-Maxwell theory at $\mathcal{O}(G^2)$. Specifically, using an EFT matching procedure, we extracted the 2PM-accurate conservative Hamiltonian that governs the orbital dynamics of charged compact binaries. We considered binary systems of arbitrary masses and charges so we can interpolate between Einstein gravity and electromagnetism by taking the appropriate limits. In both of these limits, our Hamiltonian reproduces known results [69,72].

In the probe limit, we have shown the equivalence of our Hamiltonian with the Hamiltonian for a point charge in a Reissner-Nordström background. This provides a check in the probe limit valid to all orders in velocity. Furthermore, by computing the scattering angle we found perfect agreement with the result previously reported in Ref. [53]. We also found agreement with observables for bound systems at 2PN with Ref. [46].

The results we have reported in this paper are limited to the first two orders in the PM expansion and obtaining higher orders is a natural continuation. Some additional avenues for future work are also immediately apparent, e.g.,

to consider the scattering of higher-spin fields in order to incorporate the effect of the initial angular momentum of the binary constituents on the dynamics, as well as taking into account finite-size effects. Another future direction involves incorporating the effects of radiation and computing radiative classical observables, such as the impulse, radiated momentum, and radiated energy.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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