

Can baryon acoustic oscillations illuminate the parity-violating galaxy four-point correlation function?

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Recent measurements of the galaxy four-point correlation function (4PCF) from the Sloan Digital Sky Survey Baryon Oscillation Spectroscopic Survey indicate potential statistical evidence for parity violation. If this signal is of genuine physical origin, it must stem from beyond-Standard Model physics, most likely during the very early Universe, prior to decoupling ($z \sim 1020$). Since the baryon acoustic oscillation (BAO) features imprint at decoupling, they are expected in the parity-odd galaxy 4PCF. Thus detecting them would be an additional piece of evidence that the signal is genuine. We demonstrate, using a toy parity-violating model, how the BAO imprint on the parity-odd 4PCF. We then show how to do a model-independent search for BAO in the odd 4PCF. This is desirable since, if the signal is real, we may not know for some time what model of e.g. inflation produces it. If BAO are detected in the parity-odd sector, they can be used as a standard ruler as is already done in the two-point correlation function and three-point correlation function. We derive a simple formula relating the expected precision on the BAO scale to the overall parity-odd detection significance. Searching for the BAO in the odd 4PCF of redshift surveys such as DESI, Euclid, Sphex, and Roman will be a valuable additional avenue to determine if parity violation in the distribution of galaxies is real.

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I. INTRODUCTION

In developing fundamental theories, symmetries such as translation and rotation invariance are often used as starting points. Until the 1950s, invariance under spatial inversion (parity) was also believed to be a symmetry of nature. However, [1] found that this is violated in the weak interaction. Shortly thereafter, [2] pointed out that CP (C for charge) violation is necessary to explain the observed excess of baryons over antibaryons in the Universe.

On cosmological scales, unlike translation and rotation symmetry, parity symmetry has not been systematically tested with observational data. Several theoretical proposals [3–16] and observational studies [17–21] have focused on two- and three-point correlations of vector and tensor quantities. Scalar quantities such as the galaxy density fluctuation field have been less associated with parity violation, as scalars are parity conserving by nature. Indeed, to probe parity violation using scalar quantities requires at minimum a four-point correlation function (4PCF).

In the context of galaxy surveys, [22] proposed using the galaxy 4PCF to test parity violation; [23] suggested this prospect for the cosmic microwave background (CMB). The approach of [22] exploits the isotropic basis functions of [24] and efficient N -point correlation function algorithms built on them [25,26]. [27] applied this idea to Sloan Digital Sky Survey Baryon Oscillation Spectroscopic Survey (SDSS BOSS) data [28], resulting in a 7σ detection of the parity-odd 4PCF for the larger, higher-redshift

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constant (stellar) mass (CMASS) sample and a 3σ detection for the smaller, lower-redshift low-redshift sample. [29] performed a similar study on CMASS only with the same covariance matrix [30] but different choices in galaxy weights and binning scheme, finding $\sim 3\sigma$.

A central challenge of these analyses is how well the covariance used matches the true covariance of the data, as discussed extensively in [22,27]. To deal with this challenge, [31] proposed a novel method that exploited cross-correlating independent patches of BOSS to try to separate out any true signal from misestimated covariance; a similar test using the north and south galactic caps of CMASS was already performed in [27], which showed that a true signal could be present yet lead to inconclusive cross-correlation. The setup of [31] was slightly different and found detection significances ranging from null to a maximum of 2.5σ if all the patches from the same hemisphere were combined.

A central aspect of parity-odd 4PCF analyses is their large number of degrees of freedom, which makes the covariance an even more difficult challenge. Compressed statistics, to reduce the number of degrees of freedom, are thus another route that has been pursued. Important early work by [32] suggested one such statistic. More recently, [33] introduced two types of parity-odd power spectra. These compressed statistics are complementary to the galaxy 4PCF, since they are sensitive to different parity-odd tetrahedral shapes (see footnote [34]).

Given the theoretical work, observational results, and algorithmic developments, it is worth exploring additional methods for determining whether the observed parity violation has a genuine cosmological origin. One approach is simply to use a larger survey volume or different, independent tracers. Another approach is to consider whether there is a parity-violating model that fits the data (e.g. [35–39] for recent examples). A third approach is to investigate whether there are distinctive characteristics of a true signal that might serve as a complementary test [27].

Here we propose to use the baryon acoustic oscillation (BAO) features in the distribution of galaxies as such a complementary test. BAO originate from a sharp feature in the baryon velocity at the moment of decoupling ($z \sim 1020$) produced by acoustic waves in the baryon-photon plasma before that time [40–44].

This velocity feature, at a scale of approximately $100h^{-1}$ Mpc, serves as the initial condition for the subsequent growth of the baryon density perturbations [45]. The dark matter, through gravitational attraction, then converges with the baryons, resulting in the observed BAO feature at late times, as an excess probability of finding galaxies separated by this characteristic scale [46]. These features have already been detected in the galaxy two-point correlation function (2PCF) and power spectrum [47,48] and the galaxy three-point correlation function (3PCF) and bispectrum [49,50], as well as pointed out in

theoretical modeling of the parity-even, gravitationally induced 4PCF [51].

Any pattern that existed in the density field prior to decoupling, such as parity-violating correlations produced during inflation, would carry the imprint of BAO. However, the absence of BAO in a potential parity-odd signal would also have important implications.

There are two critical “time stamps” to consider: the end of inflation, and the moment of decoupling. The first time stamp is important because current models for the parity-odd galaxy 4PCF primarily focus on inflationary mechanisms. The second, decoupling, is the time stamp associated with BAO. The absence of BAO in a parity-odd signal would raise significant doubts about its predecoupling origin, provided there is enough signal-to-noise ratio in the overall parity-odd detection that BAO would be expected to be evident.

The primary goal of the present work is to demonstrate the feasibility of searching for BAO imprints in the odd 4PCF. We illustrate the basic concept in Fig. 1. For demonstration purposes, in this work we will only consider parity-violating mechanisms of inflationary origin.

An important additional goal of this work is to propose a model-agnostic approach for the BAO search in the parity-odd sector; this offers the same flexibility that the original 4PCF test proposed in [22]. However, this approach also presents similar challenges, such as potential spurious detection due to underestimated covariance (e.g. as discussed in [22,27]). Moreover, a simultaneous detection of both a parity-odd signal and BAO features in it does not guarantee that systematics are fully excluded. To ensure this, a specific model for parity violation would still be required.

This work is structured as follows. In Sec. II, we outline the framework for understanding how BAO imprint on the density field and their effect on the trispectrum. We then briefly review the 4PCF and the isotropic basis functions. We next present a parity-violating simulation with a template whose signal peaks in the large-scale, “soft” (low wave-number) limit to show one example where we see an overall parity-odd signal and the BAO imprinted on it. In Sec. IV, we propose a model-agnostic approach for the BAO search and outline an efficient numerical method for performing it. This method remains model agnostic by essentially using the data as their own model. As we show, this dewiggling can actually be done at the estimator level rather than the field level, by modifying the radial binning function used in the 4PCF measurement. This makes the dewiggling highly efficient, with the same cost as measuring a 4PCF. In Sec. V, we discuss a possible spurious significance offset introduced by this dewiggling if applied in the absence of a real parity-odd signal. We also explore the impact of various systematics and their potential to degrade the BAO signal.

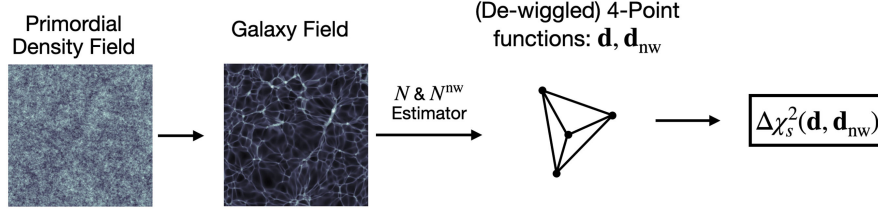


FIG. 1. The flowchart illustrates the main idea of this paper. From left to right: consider a parity-violating mechanism present in the primordial density field, which sources parity-odd configurations in the late-time, galaxy density field. Since BAO modulate density fields, we can “dewiggle” the BAO for each $N = D - R$ estimator. The 4PCFs are then constructed from these dewiggled fields (for more details, see Sec. IV). The significance of BAO detection in the parity-odd 4PCF can be assessed by comparing the dewiggled 4PCF with the original one.

Finally, we consider the prospect of using BAO from the parity-odd 4PCF as a new standard ruler. Section VI concludes.

II. BAO AND THE PARITY-ODD GALAXY 4PCF

In this work, we consider a scenario where a parity-violating mechanism during inflation generates initial curvature perturbations $\mathcal{R}(\mathbf{x})$ (but see further discussion in our footnote [52]). The curvature perturbations then seed the density perturbations at late times, $\delta(\mathbf{x}) \equiv (\rho(\mathbf{x}) - \bar{\rho})/\bar{\rho}$. $\rho(\mathbf{x})$ is the density at $\mathbf{x}$ and $\bar{\rho}$ is the Universe’s average density. In Fourier space, the late-time perturbation at redshift z , $\tilde{\delta}(\mathbf{k}, z)$, is related to the curvature perturbation via the matter transfer function $T_\delta(k)$ multiplied by the linear growth rate $D(z)$:

$$\tilde{\delta}(\mathbf{k}, z) = -\frac{2}{5} \left(\frac{k}{\mathcal{H}} \right)^2 \frac{D(z)}{D_0} T_\delta(k) \tilde{\mathcal{R}}(\mathbf{k}), \quad (1)$$

with $D_0 \equiv D(z=0)$, $\mathcal{H} = aH$ the comoving Hubble parameter, H the Hubble parameter, and a the scale factor (we discuss the relationship to the potential, ϕ , in footnote [53]). Tilde denotes a field in Fourier space. Importantly, the matter transfer function encodes the BAO.

As noted in Sec. I, the lowest-order statistic sensitive to parity in the density field is the 4PCF [22]; here, we begin with its Fourier space analog, the trispectrum. The primordial curvature trispectrum $T_{\mathcal{R}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ is defined via

$$\left\langle \prod_{i=1}^4 \tilde{\mathcal{R}}(\mathbf{k}_i) \right\rangle \equiv (2\pi)^3 \delta_{\text{D}}^{[3]}(\mathbf{k}_{1234}) T_{\mathcal{R}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4), \quad (2)$$

where $\tilde{\mathcal{R}}(\mathbf{k}_i)$ is the primordial curvature perturbation in Fourier space, $\delta_{\text{D}}^{[3]}(\mathbf{k}_{1234})$ is a 3D Dirac delta function and $\mathbf{k}_{1234} \equiv \sum_{i=1}^4 \mathbf{k}_i$.

The primordial trispectrum $T_{\mathcal{R}}$ can be split into a parity-even component $T_{\mathcal{R},+}$ and a parity-odd component $T_{\mathcal{R},-}$. We have

$$T_{\mathcal{R}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = T_{\mathcal{R},+}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) + iT_{\mathcal{R},-}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4). \quad (3)$$

The imaginary, parity-odd part will be nonzero on large scales only if there is beyond-Standard Model physics during inflation.

In linear theory, the primordial trispectrum $T_{\mathcal{R}}$ is evolved into the matter trispectrum T_{m} by applying the matter transfer functions, leading to

$$T_{\text{m}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \prod_{i=1}^4 \left[\frac{2}{5} \left(\frac{k_i}{\mathcal{H}} \right)^2 \frac{D(z)}{D_0} T_\delta(k_i) \right] T_{\mathcal{R}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4). \quad (4)$$

Everything inside the square brackets has a product taken over it; in other words even the unsubscripted prefactors should be repeated.

Now, since the right-hand side of Eq. (4) has real and imaginary parts, so will the left-hand side. In short, the late-time linear matter trispectrum will have even and odd pieces if the primordial trispectrum does. We note that both even and odd parts of the linear matter trispectrum will be subject to the standard nonlinear gravitational evolution e.g. [54].

Combining Eqs. (3) and (4), we see that BAO imprint on the parity-odd late-time trispectrum through the product of transfer functions T_δ . Thus, if there is a nonzero parity-odd primordial trispectrum, the BAO will imprint on it, and hence manifest in the parity-odd linear matter trispectrum and, eventually, in the parity-odd galaxy trispectrum.

Finally, the linear matter 4PCF, ζ , is the inverse Fourier transform (FT) of the linear matter trispectrum T_{m} :

$$\zeta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \mathcal{FT}^{-1} \left[(2\pi)^3 \delta_{\text{D}}^{[3]}(\mathbf{k}_{1234}) T_{\text{m}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \right], \quad (5)$$

where $\mathbf{r}_i \equiv \mathbf{x}_i - \mathbf{x}_0$ for $i = 1, 2, 3$. $\mathbf{x}_i$ and $\mathbf{x}_0$ are the absolute positions of four points, but due to the cosmological assumption of homogeneity, we have translation symmetry, so we may losslessly average over $\mathbf{x}_0$.

This averaging is equivalent by ergodicity to averaging over realizations of the density field, and enables eliminating $\mathbf{x}_0$ in favor of the relative coordinates $\mathbf{r}_i$. The Dirac delta $\delta_D^{[3]}(\mathbf{k}_{1234})$ in Fourier space can be understood as a consequence of this averaging; the delta function enforces momentum conservation, and momentum is the conserved quantity associated with translation invariance.

Here and throughout, our FT convention is that the forward FT is $\mathcal{FT}[f(\mathbf{x})](\mathbf{k}) \equiv \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x})$ and the inverse FT is $\mathcal{FT}^{-1}[\tilde{f}(\mathbf{k})](\mathbf{x}) \equiv \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{f}(\mathbf{k})$, where $\int_{\mathbf{k}} \equiv \int d^3\mathbf{k}/(2\pi)^3$.

III. ILLUSTRATION OF BAO IN THE 4PCF

In this section, we illustrate the BAO signal in the parity-odd 4PCF. We will first review the 4PCF in the isotropic basis of [24]. Then we will compute an example detection significance of the BAO signal using a suite of simulations with a toy model parity-violating 4PCF encoded in them.

A. 4PCF in the isotropic basis

A direct measurement of the 4PCF in Eq. (5) is computationally challenging. To accelerate the process, we decompose the 4PCF into a basis of isotropic basis functions $\mathcal{P}_{\ell_1\ell_2\ell_3}$ [24] that capture its angular behavior about one galaxy (the “primary”), times radial coefficients that capture its dependence on tetrahedron side lengths r_1, r_2, r_3 from that primary. The isotropic basis [24] has rotational invariance and can well capture the isotropic information in the large-scale distribution of galaxies. The radial coefficients are

$$\zeta_{\ell_1\ell_2\ell_3}(r_1, r_2, r_3) = \int d\hat{\mathbf{r}}_1 d\hat{\mathbf{r}}_2 d\hat{\mathbf{r}}_3 \zeta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \mathcal{P}_{\ell_1\ell_2\ell_3}^*(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3), \quad (6)$$

where ℓ_i , for $i = 1, 2, 3$ are the angular momenta associated with the three direction vectors $\mathbf{r}_i$, and star denotes a conjugate.

To avoid an overcomplete basis, we assign physically observable meaning to the subscripts 1, 2, 3 by ordering the tetrahedron sides so that $r_1 < r_2 < r_3$. The isotropic functions $\mathcal{P}_{\ell_1\ell_2\ell_3}(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3)$ of three arguments are

$$\mathcal{P}_{\ell_1\ell_2\ell_3}(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3) = \sum_{m_1 m_2 m_3} (-1)^{\ell_1+\ell_2+\ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \times Y_{\ell_1 m_1}(\hat{\mathbf{r}}_1) Y_{\ell_2 m_2}(\hat{\mathbf{r}}_2) Y_{\ell_3 m_3}(\hat{\mathbf{r}}_3), \quad (7)$$

where the $Y_{\ell_i m_i}(\hat{\mathbf{r}}_i)$ are spherical harmonics and the m_i are the z components of their angular momenta ℓ_i . The isotropic basis is fully separable in the $\mathbf{r}_i$, thus reducing the formal complexity of the 4PCF estimator to pairwise operations [25,55] (the actual scaling in practice is outlined in our footnote [56]).

While the full trispectrum in Eq. (3) encodes both even and odd parts, it is nontrivial to isolate only the odd (imaginary) part of the trispectrum in practice. From Eq. (7) we see that the basis functions behave under parity, $\mathbb{P}$: $(x, y, z) \rightarrow (-x, -y, -z)$, as

$$\mathbb{P}[\mathcal{P}_{\ell_1\ell_2\ell_3}(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3)] = \mathcal{P}_{\ell_1\ell_2\ell_3}(-\hat{\mathbf{r}}_1, -\hat{\mathbf{r}}_2, -\hat{\mathbf{r}}_3) = (-1)^{\ell_1+\ell_2+\ell_3} \mathcal{P}_{\ell_1\ell_2\ell_3}(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3). \quad (8)$$

An odd sum of ℓ_i therefore means a parity-odd basis function. The odd functions pick up any difference between the frequency of appearance of a given tetrahedron, and of its mirror image, in a galaxy sample. Thus, if they are nonzero at a statistically significant level, they reveal parity violation in the sample.

We note that, in the presence of realistic survey geometry, one must apply a mode decoupling matrix to correct for the mode mixing due to the geometry’s breaking the orthogonality of the basis functions [25,55]. We return to this point in Sec. IV.

B. Detection significance of BAO in parity-odd toy simulations

1. Setup and simulations

In this section, we discuss the setup of the simulations that we use to illustrate the BAO imprint on the odd 4PCF. As discussed in Sec. II, we consider the parity-violating mechanism to arise from some inflationary scenario. Accordingly, we simulate the curvature perturbation $\mathcal{R}(\mathbf{k})$, which includes both a Gaussian (subscript “G”) and a non-Gaussian (subscript “NG”) component:

$$\mathcal{R}(\mathbf{k}) = \mathcal{R}_G(\mathbf{k}) + \mathcal{R}_{NG}(\mathbf{k}). \quad (9)$$

A simple template for $\mathcal{R}_{NG}$ that produces an odd trispectrum is [33,57]

$$\mathcal{R}_{NG}(\mathbf{k}) = ig \int_{\mathbf{q}_1 \mathbf{q}_2 \mathbf{q}_3} \delta_D^{[3]}(\mathbf{k} - \mathbf{q}_{123}) \frac{\mathbf{q}_1 \cdot (\mathbf{q}_2 \times \mathbf{q}_3)}{q_1^\alpha q_2^\beta q_3^\gamma} \times \mathcal{R}_G(\mathbf{q}_1) \mathcal{R}_G(\mathbf{q}_2) \mathcal{R}_G(\mathbf{q}_3), \quad (10)$$

with $\mathbf{q}_{123} \equiv \mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3$, and g a coupling constant that controls the amplitude of the non-Gaussianity. α, β , and γ are free parameters that satisfy $\alpha + \beta + \gamma = -3$ to keep g dimensionless. In particular, we choose $\{\alpha, \beta, \gamma\} = \{-2, -1, 0\}$, following [33,57].

The leading-order parity-odd primordial curvature trispectrum from this template is

$$T_-(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = g \mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) (2\pi^2 A_s)^3 \times (k_1^{\alpha-4+n_s} k_2^{\beta-4+n_s} k_3^{\gamma-4+n_s} k_4^0 \pm 23 \text{ signed perms.}), \quad (11)$$

where “perms.” means to permute the four wave numbers $\{k_1, k_2, k_3, k_4\}$ relative to the powers $\alpha, \beta, \gamma, 0$. There are 23 permutations of the original ordering, and the sign is determined by whether it is cyclic (+1) or anticyclic (−1).

Although this template is not directly linked to any particular physical model, it is useful for gaining insight into how BAO might manifest within an odd 4PCF. We note that the template is fully separable in the $\mathbf{k}_i$. By isolating the scalar triple product $\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3)$, the remaining parts are parity even and have only radial dependence. A previous analysis [33] revealed that this template peaks in the “soft” (low- k) limit. Specifically, the template reaches its maximum when $k_1 \rightarrow 0$ with the wave numbers ordered as $k_1 < k_2 < k_3 < k_4$. However, the range of scales explored in the galaxy 4PCF thus far has been restricted to tetrahedra where the sides are more roughly equal in length [27,29], making the galaxy 4PCF (at least as used thus far) less sensitive to this “squeezed” trispectrum template. To increase sensitivity, we could

study the signal over a much larger range of scales (which might pose algorithmic challenges). Alternatively, we could consider other templates that have higher amplitude in roughly equilateral 4PCF configurations. We leave this for future work.

We construct simulations with both positive and negative coupling constants, $g = \pm 2 \times 10^7$ (see our footnote [58]), calibrated so that their power spectra match that of the simulation without a non-Gaussian field. We then apply transfer functions with and without BAO features to these density fields, and evolve to redshift $z = 0$ using CLASS [59]. The simulations with positive and negative coupling constants, as well as those with and without BAO features, all share the same initial conditions. In total, there are $100 \times 4 = 400$ simulation boxes, each with $N_{\text{mesh}} = 256$ mesh points per side. We interpret the box side length to be $L_{\text{box}} = 1 h^{-1}$ Gpc, which corresponds to a cell resolution of $\Delta L = 3.91 h^{-1}$ Mpc. Following the analysis of [27], we measure the 4PCF with a minimum tetrahedron side length

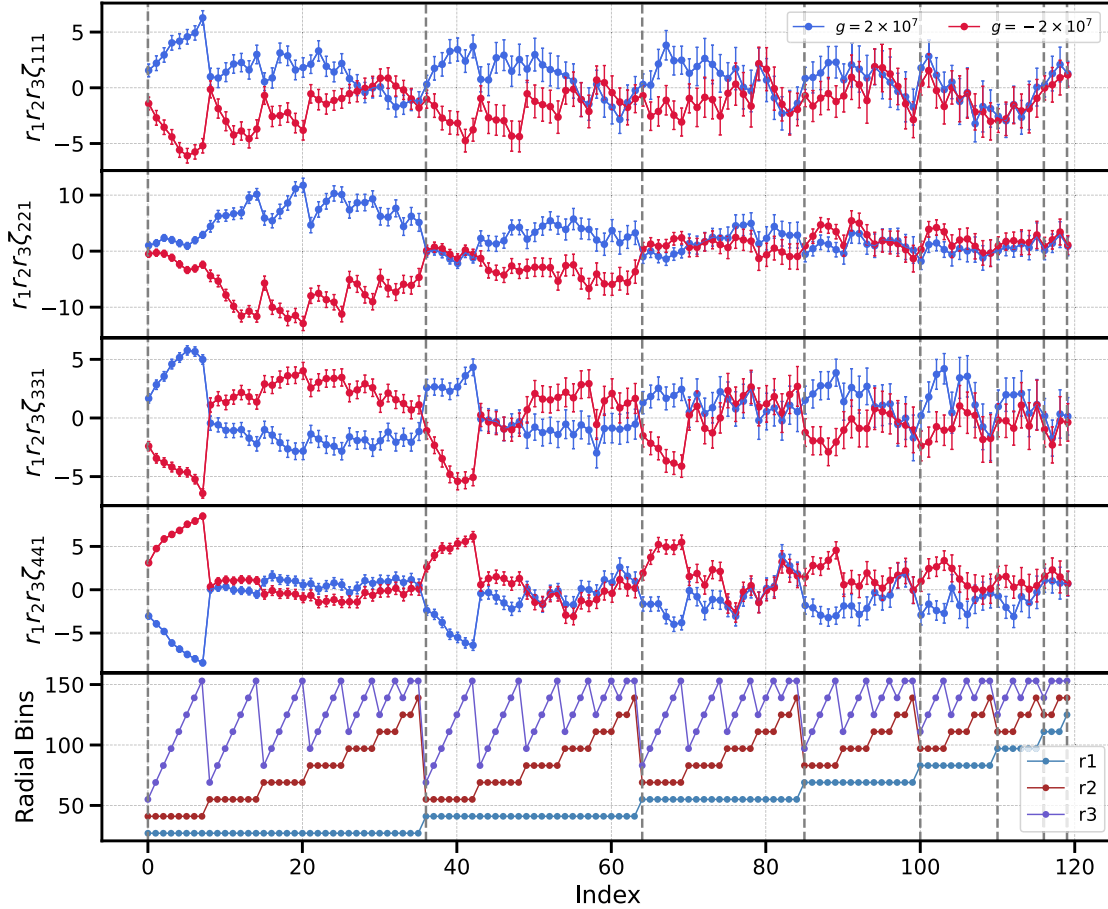


FIG. 2. Measurement of the lowest-lying few 4PCF coefficients on parity-violating simulations with coupling constants $g = \pm 2 \times 10^7$ [the template is in Eq. (11)]. For each g we have 100 averaged realizations. Here we show the angular channels for $\ell_1 = \ell_2 \leq 4$ at a fixed $\ell_3 = 1$. As explained in [24] one must order the side lengths such that $r_1 < r_2 < r_3$ of the tetrahedra to avoid redundancy of the basis. For plotting we mapped them to a 1D index, and this mapping is displayed in the bottom panel. Since the template peaks in the squeezed limit, the signal-to-noise ratio is higher when the three sides of the tetrahedron are far apart and lower when the sides are of similar length. To guide the eye, we emphasize the transition in the r_1 index.

of $r_{\min} = 20h^{-1}$ Mpc and a maximum value of the r_i of $r_{\max} = 160h^{-1}$ Mpc, with a radial bin width of $\Delta r = 14h^{-1}$ Mpc. We note that this maximum is on any side extending from the primary point; a side that does not attach to the primary can range up to twice this (from having an angle of 180° between two sides that do). Information about sides that do not attach to the primary is captured in the ℓ_i dependence of the 4PCF coefficients.

Figure 2 shows the measurements of the 4PCF coefficients for angular channels $\ell_1 = \ell_2 \leq 4$ at a fixed $\ell_3 = 1$ (both odd parity); the lengths and arrangement of the three sides $\{r_1, r_2, r_3\}$ is shown in the bottom panel. We notice that the signal is maximized when $\ell_3 = 1$ and when the separations between r_1 , r_2 , and r_3 are the largest. This behavior is consistent with our Fourier-space template's peaking in the soft limit, which in position space means when the tetrahedron side lengths are most disparate. As the three sides approach equal sizes (equilateral), the signal significantly drops. As shown in Fig. 2, starting from index ~ 80 , the signal-to-noise ratio declines, making it difficult to distinguish between the positive and negative coupling constants.

Figure 3 shows the parity-odd 4PCF measurements for two angular channels, $\{\ell_1, \ell_2, \ell_3\} = \{1, 1, 1\}$ and $\{\ell_1, \ell_2, \ell_3\} = \{2, 2, 1\}$, as functions of two of the side lengths (r_2 and r_3), with the shortest side (r_1) held fixed.

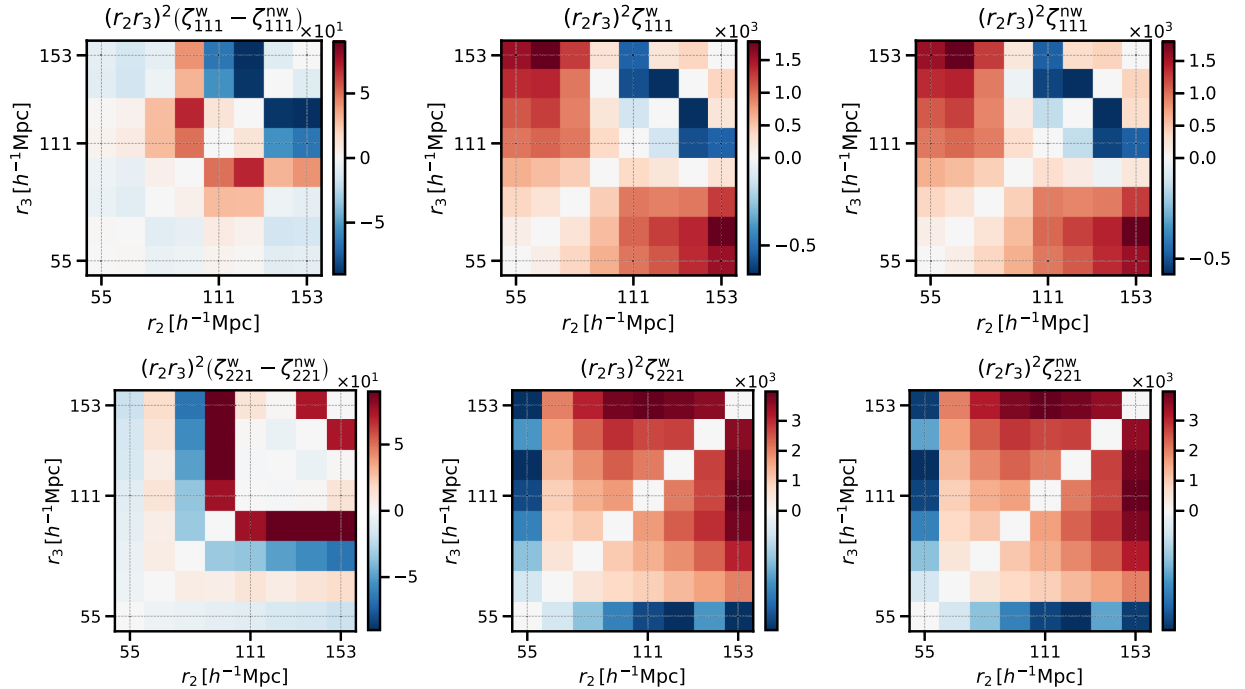


FIG. 3. Here we show the BAO features in a few low-lying angular channels of the parity-odd 4PCF, $\zeta_{\ell_1\ell_2\ell_3}(r_2, r_3)$, vs the middle and longest tetrahedron side lengths, r_2 and r_3 , fixing the shortest, r_1 , to be $41h^{-1}$ Mpc. The left-hand panel shows the difference between the 4PCF coefficients with BAO and without BAO. The middle panel shows the 4PCF without BAO, and the right panel shows it with BAO. Thus the leftmost panel is the difference of the middle and the rightmost. The top row shows the $\ell_1, \ell_2, \ell_3 = 1, 1, 1$, channel and the bottom row shows the $\ell_1, \ell_2, \ell_3 = 2, 2, 1$, channel. These plots are all weighted by $(r_2r_3)^2$ since the uniformly random distributed galaxy counts in each radial bin scale as r_i^2 .

The leftmost panels show the difference between the parity-odd 4PCF coefficients with (superscript “w”), and without (superscript “nw”) the BAO, at fixed $r_1 = 41h^{-1}$ Mpc. The middle and rightmost panels are the original 4PCF measurements respectively with and without BAO. Although the BAO is only a few percent effect, and the original 4PCF coefficients appear nearly identical, the difference plots reveal that the BAO manifest as transition from red to blue (excess of configurations over and above random, to decrement) or blue to red (vice versa) at scales of $\sim 100h^{-1}$ Mpc.

2. Overall parity-violation detection significance

We first compute χ^2 for the *overall* detection significance of the parity-odd signal:

$$\chi^2 = \sum (\zeta_d - \zeta_m) \mathbb{C}^{-1} (\zeta_d - \zeta_m)^T. \quad (12)$$

$\mathbb{C}$ is the covariance matrix, ζ_d is the measured 4PCF from the data (hence, subscript “d”), and ζ_m is a 4PCF model (hence subscript “m”). Following [27], we take our null hypothesis to be that $\zeta_m = 0$.

We consider two methods of computing the covariance matrix. The first uses the analytic template of [30], which takes it that the density fluctuations are a Gaussian random

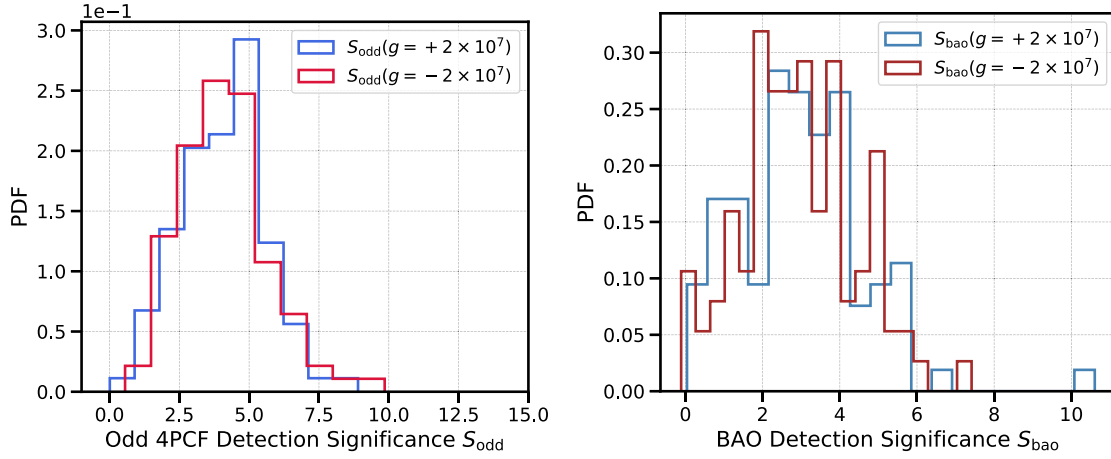


FIG. 4. Detection significance for both coupling constants $g = \pm 2 \times 10^7$. Left: overall parity-odd detection significance. Right: BAO detection significance in the parity-odd measurement. In both measurements, we treat each simulation as an independent realization of the Universe, and quantify the excess probability in units of the standard deviation of a null distribution given by our Gaussian simulations. The significance in both cases reflects how well we can reject the null hypothesis.

field (GRF). The second uses 5000 GRFs that we constructed to have matching power spectrum to that of the parity-violating simulations. In this second method, the χ^2 distribution is modified to be a multivariate T distribution [60] to account for noise in the covariance since this latter is estimated from a finite sample. We further explore the impact of covariance matrix method choice in Appendix B, and show it has only a marginal impact on the significances (though see footnote [61]).

Figure 4 shows the detection significance of the overall parity-odd signal in the left-hand panel with both the positive and negative coupling constants. Here, we treat each simulation as one realization of the Universe. We then quantify the detection significance for each realization in units of the standard deviation of the null distribution, given by our Gaussian simulations. The overall detection significance for parity violation in each realization is typically about 4σ .

3. BAO detection significance

Now, to assess the BAO detection significance, we need to develop an alternative to the standard approach used in BAO searches for 2- and 3-point statistics (e.g. [47–50]). The standard approach computes the difference between the χ^2 of the observable under the best-fit model with BAO, and the χ^2 of the observable under the best-fit model with no BAO. This difference is $\Delta\chi^2$, and $\sqrt{\Delta\chi^2}$ between the two models is the BAO detection significance. The presence in the χ^2 of the inverse covariance matrix, $\mathbb{C}^{-1} = \mathbb{S}^{-1}\mathbb{D}^{-1}\mathbb{S}$, both rotates the data vectors into an orthogonal basis (role of $\mathbb{S}$ and $\mathbb{S}^{-1}$), where each mode is independent, and also scales each such mode to have variance of unity (role of $\mathbb{D}^{-1}$). Thus the $\sqrt{\Delta\chi^2}$ is simply the distance between the two models in units suitable to be

compared with a normal distribution (mean zero, unit variance) when computing probabilities.

However, the standard approach requires a model for the expected signal, which is not known *a priori*. Instead, we propose a different method that uses the data as their own model. We can remove the BAO from the density field and use the dewiggled observable (4PCF measured on the dewiggled density field) in place of the “no-wiggle” model employed in the standard approach. We denote the χ^2 difference computed in this fashion with a tilde.

Our χ^2 difference is now

$$\Delta\tilde{\chi}_{\text{BAO}}^2 = (\zeta_{\text{d}} - \zeta_{\text{d}}^{\text{nw}})\mathbb{C}^{-1}(\zeta_{\text{d}} - \zeta_{\text{d}}^{\text{nw}})^{\text{T}}, \quad (13)$$

where ζ_{d} and $\zeta_{\text{d}}^{\text{nw}}$ are the 4PCF measured from the same data, but superscript “nw” means dewiggling has been applied first. In Appendix A, we show that in the high signal-to-noise ratio limit, our new statistic $\Delta\tilde{\chi}_{\text{BAO}}^2$ recovers the mean of the usual $\Delta\chi_{\text{BAO}}^2$. In the low signal-to-noise ratio limit, the mean of the new statistic is more sensitive to statistical noise, particularly for high-dimensional data vectors. Nevertheless, since both ζ are measured from the same data, they also have the comparable noise (up to the effect of dewiggling), the variance in the new statistic is also reduced.

It is important to notice that the dewiggling approach proposed here does result in an offset in the modified statistic $\Delta\tilde{\chi}_{\text{BAO}}^2$. In particular, even in the absence of a true parity-odd 4PCF, the expected BAO detection significance $\Delta\chi_{\text{s}}^2$ is nonzero. This occurs because the dewiggling process itself alters the spatial configuration of the density field, regardless of whether there is any true parity-odd component. When we compute the $\Delta\chi_{\text{BAO}}^2$, in the absence of a true parity-odd 4PCF, we are just comparing the variance of two data vectors (with BAO, and without BAO)

that should each be centered around zero. However, the presence or absence of BAO causes the variances to differ, leading to a nonvanishing $\Delta\chi_{\text{BAO}}^2$. This difference ultimately can be understood by examining the ratio of the matter transfer function with BAO to that without BAO. We return to this issue in Sec. VA.

To address this issue, we quantify the BAO significance by comparing $\Delta\chi_{\text{BAO}}^2$ measured from parity-violating simulations to a null distribution $\Delta\tilde{\chi}_{\text{BAO,null}}^2$ measured from purely Gaussian (no parity-violation) simulations. These latter already contain the variance effect outlined above, and so using them to define our null distribution removes any spurious $\Delta\chi^2$ so produced. We thus define our BAO significance as

$$S_{\text{BAO}} = \frac{\Delta\tilde{\chi}_{\text{BAO}}^2 - \Delta\tilde{\chi}_{\text{BAO,null}}^2}{\sigma(\Delta\tilde{\chi}_{\text{BAO,null}}^2)}, \quad (14)$$

where $\sigma(\Delta\tilde{\chi}_{\text{BAO,null}}^2)$ is the standard deviation of the null distribution.

The right-hand panel of Fig. 4 shows the detection significance of BAO in the odd 4PCF. We treat each simulation without BAO (“nw”) as the “model.” We then compare the $\Delta\tilde{\chi}_{\text{BAO}}^2 (g = \pm 2 \times 10^7)$ for parity-violating simulations to that for purely Gaussian simulations. We find the typical BAO significance computed in this way is about 3σ .

We pause to highlight a fundamental difference in the detection significance quoted here as compared to that used in the standard BAO search method. In the standard approach, the significance is based on the signal-to-noise ratio. Here, our null hypothesis is that we have BAO but in a parity-conserving scenario; our significance reflects how strongly we may reject this null hypothesis.

4. Dependence on template choice and coupling constant

The actual values of the significance found here are influenced both by the template choice and the coupling constant. Given that our template is strongest in the soft (low- k) limit, as noted in Sec. III B 1, the detection here may well be a conservative estimate of what could be found if parity violation is genuine. This is because we might expect that the BAO signal is strongest when one side is at the BAO scale, or two sides add up to the BAO scale, or add to twice the BAO scale; this is the behavior seen in the 3PCF (for instance, Fig. 8, $\ell = 1$ panel, in [62]). Thus, equilateral or semiequilateral configurations are likely preferred for BAO. Yet, these configurations do not appear in position space with very large amplitude given our template.

In the current work, we set our coupling constant g to be as large as possible without introducing beyond-leading-order terms in the parity-odd trispectrum, while also ensuring that the power spectrum of the non-Gaussian simulations remains consistent with observational constraints.

5. Caveats

In the real Universe, nonlinear gravitational evolution affects the sharpness of the BAO feature, as is well known for the two-point functions, and can also impact the detection significance in the odd 4PCF. We leave the impact of nonlinear gravitational evolution to future work, such as using N-body codes to evolve parity-violating simulations. Furthermore, our work here assumes linear bias only, and treats the continuous simulation as a proxy for the galaxy field, when in fact the latter is discrete, which leads to Poisson noise that also will impact the BAO significance.

IV. METHOD FOR EFFICIENT DEWIGGLING AND BAO SEARCH ON OBSERVATIONAL DATA

The 4PCF of the dewiggled density field, ζ^{nw} , is one key ingredient in our BAO search [cf. Eq. (13)]. In Sec. III B 1, we constructed simulations without BAO by evolving the primordial curvature perturbations using the no-wiggle transfer function.

One approach to remove the BAO feature would be to dewiggle the density field and then measure the 4PCF. For a continuous field without survey geometry, this can be done more efficiently by slightly altering the 4PCF estimator, so that dewiggle is done “on the fly” as we measure the 4PCF. Here we outline such an approach. We begin with a brief review of the 4PCF estimator.

A. 4PCF estimator

Estimators for the 4PCF fall into two broad classes: Fourier-space estimators that use FFTs and operate in k space [63], and position-space estimators based on direct galaxy counting [25]. For a continuous density field in the absence of survey geometry, the estimated density fluctuation is $\hat{\delta}(\mathbf{x}) = \rho(\mathbf{x})/\bar{\rho} - 1$, with $\bar{\rho}$ the mean of the density field; both the FFT-based and the direct-counting algorithms are applicable. For discrete data or in the presence of survey geometry, position-space estimators naturally incorporate the window via random catalogs. We start by defining the weighted density fluctuation field $N(\mathbf{x}) \equiv D(\mathbf{x}) - R(\mathbf{x})$, where $D(\mathbf{x})$ and $R(\mathbf{x})$ are the weighted data and randoms at a point $\mathbf{x}$. Here the random catalogs are required to convert the galaxy counts into a density fluctuation field and to correct for survey geometry, and are already normalized in our notation. Below, we briefly review the direct-count algorithm. We define the estimator

$$\begin{aligned} \hat{\zeta}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) &= \mathcal{I}^{-1} \int d^3\mathbf{x} N(\mathbf{x}) N(\mathbf{x} + \mathbf{r}_1) N(\mathbf{x} + \mathbf{r}_2) N(\mathbf{x} + \mathbf{r}_3) \\ &\equiv \mathcal{I}^{-1} \mathcal{N}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \end{aligned} \quad (15)$$

where the second line defines $\mathcal{N}$. Depending on the choice of the estimator, the normalization can either be a constant,

cf. [64], or a spatial integral over the random counts that retains the mode information associated with the separation vectors $\mathbf{r}_i$ (cf. extended Landy-Szalay estimator [27,65]). The $\mathcal{N}$ field can be expanded as

$$\mathcal{N}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \sum_{\ell_1 \ell_2 \ell_3} \mathcal{N}_{\ell_1 \ell_2 \ell_3}(r_1, r_2, r_3) \mathcal{P}_{\ell_1 \ell_2 \ell_3}(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3), \quad (16)$$

where $\mathcal{N}_{\lambda_1 \lambda_2 \lambda_3}$ is the radial expansion coefficients. In the following sections, we will focus on the $\mathcal{N}$ field, since it is the most relevant term for removing the BAO feature.

B. Field-level approach

As mentioned at the start of this section, to remove the BAO features, the most straightforward method is to operate at the field level: take the Fourier transform of the $N(\mathbf{x})$ field, multiply by the ratio of the matter to the no-wiggle transfer function, inverse-Fourier transform the field back into configuration space, and proceed by computing the 4PCFs as in [22,27]. While there are more efficient methods available, building a solid understanding of this basic procedure will pave the way for improvements.

In the field-level dewiggling approach in position space, we treat each late-time galaxy as a proxy for a primordial density perturbation. To understand the evolution of such a perturbation, consider an isolated primordial density perturbation modeled as a Dirac delta distribution. The resulting late-time matter distribution depends on both space and time. At late times during the matter-dominated era, the growth rate of perturbations becomes approximately scale independent and the matter distribution becomes separable in space and time [66]. Dividing the matter distribution by the time-dependent growth factor, the remaining spatial function is equivalent to the inverse-Fourier transform of the matter transfer function [46].

To remove the BAO feature from the density field, we can revert to the primordial density distribution by dividing the density perturbation by the matter transfer function. Then, to evolve the primordial field forward in a universe without BAO, we multiply the density field by a transfer function that excludes the BAO wiggles. As a result, we can

define a dewiggling kernel that captures both of these operations, as

$$\begin{aligned} G(\mathbf{r}) &\equiv \mathcal{F}\mathcal{T}^{-1}[\mathcal{T}_{\delta, \text{nw}}(k)/\mathcal{T}_{\delta}(k)](\mathbf{r}) \\ &\equiv \mathcal{F}\mathcal{T}^{-1}[\tilde{G}(k)](\mathbf{r}), \end{aligned} \quad (17)$$

where the second line defines $\tilde{G}$ as the transfer function ratio. Since this latter depends only on the magnitude k , the inverse FT can depend only on $|\mathbf{r}|$; in other words, $G(\mathbf{r}) = G(|\mathbf{r}|)$, and so is isotropic. The idea for this dewiggling kernel was motivated by works on the position-space approach to BAO [44,67] (also discussed in [46]).

C. Dewiggling at the estimator level

With the dewiggling kernel in hand, we may now compute the dewiggled field N^{nw} as

$$N^{\text{nw}}(\mathbf{u}) = \int d^3\mathbf{u}_1 G(\mathbf{u} - \mathbf{u}_1) N(\mathbf{u}_1). \quad (18)$$

We may now project a product of 4 dewiggled fields at appropriate points for the 4PCF onto our isotropic basis. We also include radial binning, which enables us to rewrite the angular integrals over the relative positions $\hat{\mathbf{r}}_i$ as full 3D integrals. We have

$$\begin{aligned} \mathcal{N}_{\ell_1 \ell_2 \ell_3}^{\text{nw}}(R_1, R_2, R_3) &= \int d^3\mathbf{x} N^{\text{nw}}(\mathbf{x}) \prod_{i=1}^3 \left[\int d^3\mathbf{r}_i N^{\text{nw}}(\mathbf{x} + \mathbf{r}_i) \Theta(r_i; R_i) \right] \\ &\quad \times \mathcal{P}_{\ell_1 \ell_2 \ell_3}^*(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3). \end{aligned} \quad (19)$$

Here Θ is a binning function ensuring that r_i is within a spherical shell of width Δ and center R_i , and R_i is used to identify the bin. Each bin is normalized to have unit volume.

We now expand the isotropic function in the last line in terms of spherical harmonics using its definition Eq. (7), and replace all the $\mathbf{r}_i$ -dependent fields $N^{\text{nw}}(\mathbf{x} + \mathbf{r}_i)$ by Eq. (18): $N^{\text{nw}}(\mathbf{x} + \mathbf{r}_i) \rightarrow \int d^3\mathbf{u}_i G(|\mathbf{x} + \mathbf{r}_i - \mathbf{u}_i|) N^{\text{nw}}(\mathbf{x} + \mathbf{u}_i)$. We obtain

$$\begin{aligned} \mathcal{N}_{\ell_1 \ell_2 \ell_3}^{\text{nw}}(R_1, R_2, R_3) &= \int d^3\mathbf{x} N^{\text{nw}}(\mathbf{x}) \sum_{m_1 m_2 m_3} (-1)^{\ell_1 + \ell_2 + \ell_3} \\ &\quad \times \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \prod_{i=1}^3 \left[\int d^3\mathbf{r}_i \int d^3\mathbf{u}_i G(|\mathbf{x} + \mathbf{r}_i - \mathbf{u}_i|) N(\mathbf{u}_i) \Theta(r_i; R_i) Y_{\ell_i m_i}^*(\hat{\mathbf{r}}_i) \right] \\ &\equiv \int d^3\mathbf{x} N^{\text{nw}}(\mathbf{x}) \sum_{m_1 m_2 m_3} (-1)^{\ell_1 + \ell_2 + \ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \int d\mathbf{u}_i N(\mathbf{u}_i) \prod_{i=1}^3 \mathcal{I}_{\ell_i m_i}(\mathbf{x}, \mathbf{u}_i, \mathbf{r}_i; R_i). \end{aligned} \quad (20)$$

We observe that our N fields no longer depend on the $\mathbf{r}_i$; thus, these integrals may be performed first, independently of the data. We also see that Eq. (20) contains three factors of the binned, dewiggled density fields. This motivates introducing

$$\begin{aligned} \mathcal{I}_{\ell_i m_i}(\mathbf{x}, \mathbf{u}_i, \mathbf{r}_i; R_i) & \\ & \equiv \int d^3 \mathbf{r}_i G(|(\mathbf{u}_i - \mathbf{x}) - \mathbf{r}_i|) \Theta(r_i; R_i) Y_{\ell_i m_i}^*(\hat{\mathbf{r}}_i) \\ & = \int d^3 \mathbf{r}_i [G(|\mathbf{r}_i|) \otimes (\Theta Y_{\ell_i m_i}^*)(\mathbf{r}_i)](\mathbf{u}_i - \mathbf{x}), \end{aligned} \quad (21)$$

where to make the convolutional structure more evident, in the second line above we rearranged the argument of the dewiggling function relative to how it appears in Eq. (20) using that $G(|\mathbf{r}_i + (\mathbf{x} - \mathbf{u}_i)|) = G(|(\mathbf{u}_i - \mathbf{x}) - \mathbf{r}_i|)$. By using

$$\begin{aligned} \mathcal{FT}[\Theta(r_i; R_i) Y_{\ell_i m_i}^*(\hat{\mathbf{r}}_i)](\mathbf{k}_i) & \\ = 4\pi \sum_{\ell_i m_i} (-i)^{\ell_i} Y_{\ell_i m_i}(\hat{\mathbf{k}}_i) \int r_i^2 dr_i j_{\ell_i}(k_i r_i) \Theta(r_i; R_i), \end{aligned} \quad (22)$$

the integrand of Eq. (21), as a convolution, can be written as a product in Fourier space:

$$\begin{aligned} [G(|\mathbf{r}_i|) \otimes (\Theta Y_{\ell_i m_i}^*)(\mathbf{r}_i)](\mathbf{u}_i - \mathbf{x}) & \\ = Y_{\ell_i m_i}(\widehat{\mathbf{u}_i - \mathbf{x}}) \int \frac{k_i^2 dk_i}{2\pi^2} j_{\ell_i}(k_i |\mathbf{u}_i - \mathbf{x}|) \tilde{G}(k_i) \tilde{\Theta}_{\ell_i}(k_i; R_i), & \\ \equiv Y_{\ell_i m_i}(\widehat{\mathbf{u}_i - \mathbf{x}}) g_{\ell_i}(|\mathbf{u}_i - \mathbf{x}|; R_i), \end{aligned} \quad (23)$$

where the Fourier transform of the binning function is

$$\tilde{\Theta}_{\ell_i}(k_i; R_i) \equiv 4\pi \int r^2 dr \Theta(r; R_i) j_{\ell_i}(k_i r). \quad (24)$$

The integral giving $\tilde{\Theta}_{\ell}$ can be performed analytically [68]. While there is no analytic solution for g_{ℓ_i} because it is a transform of $\tilde{G}$, which later involves a ratio of the transfer functions, g_{ℓ_i} can be obtained at each ℓ_i with a 1D FFTLog [69–72].

We may now write

$$\begin{aligned} \mathcal{N}_{\ell_1 \ell_2 \ell_3}^{\text{nw}}(R_1, R_2, R_3) & \\ = \int d^3 \mathbf{x} N^{\text{nw}}(\mathbf{x}) \sum_{m_1 m_2 m_3} (-1)^{\ell_1 + \ell_2 + \ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} & \\ \times \prod_{i=1}^3 \int d^3 \mathbf{u}_i N(\mathbf{u}_i) Y_{\ell_i m_i}^*(\widehat{\mathbf{u}_i - \mathbf{x}}) g_{\ell_i}(|\mathbf{u}_i - \mathbf{x}|) & \\ = \int d^3 \mathbf{x} N^{\text{nw}}(\mathbf{x}) \sum_{m_1 m_2 m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} & \\ \times \prod_{i=1}^3 [[N \otimes (Y_{\ell_i m_i}^* g_{\ell_i})](\mathbf{x})]. \end{aligned} \quad (25)$$

To obtain the last equality above, we used that the integrals over $\mathbf{u}_i$ are convolutions as well. Since $Y_{\ell m}(-\hat{\mathbf{u}}) = (-1)^\ell Y_{\ell m}(\hat{\mathbf{u}})$, the phase $(-1)^{\ell_1 + \ell_2 + \ell_3}$ ends up canceling out.

The algorithm for estimating the 4PCF of a field in the isotropic basis is exactly as Eq. (25) but with $g_{\ell_i}(|\mathbf{u}_i - \mathbf{x}|; R_i) \rightarrow \Theta(|\mathbf{u}_i - \mathbf{x}|; R_i)$, the usual spherical shell binning. In practice, we note that due to the dewiggling kernel, Eq. (23) results in a highly oscillatory function, and requires applying additional damping to regulate the oscillatory behavior. Implementing such damping is more straightforward in Fourier-space estimators; however, these latter require developments by extending the treatment of the survey window function in e.g. [73,74].

V. DISCUSSION

Here we further discuss what might result from application of our dewiggling procedure in the absence of a true parity-odd 4PCF. We then turn to the possible impact of systematics or other new physics on our approach, and finally, consider use of odd-4PCF BAO as a new standard ruler.

A. Offset in detection significance from dewiggling without genuine odd 4PCF

We notice that the proposed dewiggling procedure leads to a nonvanishing $\Delta \tilde{\chi}_{\text{BAO}}^2$ also in the absence of a parity-odd 4PCF signal, as already briefly discussed in Sec. III B 3. This occurs because dewiggling alters the spatial configuration of the density field, by convolving the original density field with the dewiggling function. This change to the density field will then alter the measured 4PCF relative to the original field.

In order to understand the change more quantitatively, we begin by considering the 4PCF coefficients, taking the inverse FT of the trispectrum, and projecting onto the isotropic basis functions:

$$\begin{aligned} \zeta_{\ell_1 \ell_2 \ell_3}^{\text{nw}}(r_1, r_2, r_3) & \\ = \int d\hat{\mathbf{r}}_1 d\hat{\mathbf{r}}_2 d\hat{\mathbf{r}}_3 \mathcal{P}_{\ell_1 \ell_2 \ell_3}^*(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3) & \\ \times \mathcal{FT}^{-1} \left[(2\pi)^3 \delta_{\text{D}}^{[3]}(\mathbf{k}_{1234}) T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \prod_{i=1}^4 \tilde{G}(k_i) \right], \end{aligned} \quad (26)$$

where $\mathbf{k}_{1234} = \sum_{i=1}^4 \mathbf{k}_i$ and $\tilde{G}(k_i)$ is the dewiggling function of Eq. (17) in Fourier space. The dewiggling function is the ratio between the wiggled- and nonwiggled matter transfer function. Since the BAO wiggles is an $\mathcal{O}(1\%)$ feature, we can split the ratio into a constant and a small modulation $\tilde{\epsilon}(k)$. As a result, the dewiggling function $\tilde{G}$ in Fourier space can be expressed as

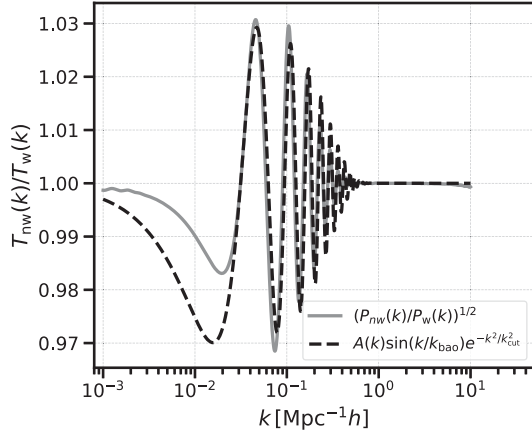


FIG. 5. Solid gray: ratio of the transfer function without BAO (no wiggle, subscript “nw”) from [75] to that with BAO (wiggle, subscript “w”) from CLASS. Dashed black: our approximation for this ratio, Eq. (27). We see that our approximation describes the ratio rather well; the parameters we use in it are $A = 0.03$, $k_{\text{BAO}} = 0.01 \text{ Mpc}^{-1}h$, and $k_{\text{cut}} = 0.3 \text{ Mpc}^{-1}h$.

$$\begin{aligned} \tilde{G}(k) &\equiv 1 - \tilde{\epsilon}(k) \\ &\approx 1 - A \sin\left(\frac{k}{k_{\text{BAO}}}\right) e^{-(k/k_{\text{cut}})^2}; \end{aligned} \quad (27)$$

in the second equation, we approximate the ratio with a sinusoidal function with exponential damping. We approximate amplitude $A = 0.03$, the BAO wave number $k_{\text{BAO}} = 0.01 \text{ Mpc}^{-1}h$, and we impose a cutoff scale $k_{\text{cut}} = 0.3 \text{ Mpc}^{-1}h$. As shown in Fig. 5, this formula gives a good empirical fit to the transfer function ratio.

In the absence of genuine parity violation, the set of odd 4PCF coefficients, which we denote with curly brackets, follows a multivariate Gaussian distribution, centered at zero and with covariance matrix $\mathbb{C}$:

$$\{\zeta_{\ell_1 \ell_2 \ell_3}(r_1, r_2, r_3)\} \sim \mathcal{N}(0, \mathbb{C}). \quad (28)$$

When applying the dewiggling function to the data vector, the density fields get modulated, therefore altering the variance of the data vector. In particular, the variance becomes modulated by the dewiggling function itself. As a result, the modified $\tilde{\chi}_{\text{BAO}}^2$ acquires a different expectation value compared to the original χ_{BAO}^2 (see Appendix A). The expectation value of the modified $\tilde{\chi}_{\text{BAO}}^2$, as defined in Eq. (13), is given as

$$\begin{aligned} \langle \Delta \tilde{\chi}_{\text{BAO}}^2 \rangle &= \langle (\Delta_\zeta - \Delta_\epsilon) \mathbb{C}^{-1} (\Delta_\zeta - \Delta_\epsilon)^T \rangle \\ &\approx \Sigma_{\text{BAO}}^2 \left(1 + \frac{\langle \Delta_\epsilon \mathbb{C}^{-1} \Delta_\epsilon \rangle}{\Sigma_{\text{BAO}}^2} \right), \end{aligned} \quad (29)$$

where $\Delta_\zeta \equiv \zeta_{\text{m}}^w - \zeta_{\text{m}}^{\text{nw}}$ denotes the deterministic difference between the model with and without the BAO wiggle, while $\Delta_\epsilon \equiv \epsilon^w - \epsilon^{\text{nw}}$ represents the change in statistical

fluctuations induced by the dewiggling function. In going from the first to the second line, we also assume that the expectation value of the statistical fluctuations vanishes, $\langle \epsilon \rangle = 0$. We define the significance of BAO detection in the presence of a parity-odd signal as

$$\Sigma_{\text{BAO}}^2 \equiv \Delta_\zeta \mathbb{C}^{-1} \Delta_\zeta^T. \quad (30)$$

The expectation values of the original and the modified statistics are equivalent when $\langle \Delta_\epsilon \mathbb{C}^{-1} \Delta_\epsilon \rangle \ll \Sigma_{\text{BAO}}^2$; however, in general, they can differ. Moreover, in the absence of parity-odd signal, the BAO significance naturally vanishes $\Sigma_{\text{BAO}} = 0$; the expectation value becomes only sensitive to the statistical fluctuations, which can be approximated as $\langle \Delta_\epsilon \mathbb{C}^{-1} \Delta_\epsilon \rangle \approx (4\epsilon)^2 N$, where ϵ is the modulation of BAO [see Eq. (27)] in the position space and N is the number of degrees of freedom of the data vector [76].

B. Systematics and additional physics with possible impact on detection significance

Here, we discuss systematics that could either contaminate or degrade the BAO significance. Given the hierarchical nature of large-scale structure (LSS) correlation functions, systematics that induce a spurious sharp feature at scales in the 4PCF would likely already have a corresponding effect in the 2PCF and 3PCF, and thus be revealed (and removable using) these lower-order statistics. However, the 4PCF likely has a lower signal-to-noise ratio than 2PCF and 3PCF, so could be more adversely affected by issues that would not be noticed in these latter.

We now briefly outline several possible issues to consider when performing the odd-4PCF BAO search.

- (i) Nonlinear evolution and redshift-space distortions (RSD): gravitational evolution introduces nonlinearity in the matter distribution, and the dewiggling method outlined in Sec. IV does not address this. However, it is important to note that such nonlinear effects are important primarily on scales well below the BAO scale, and in studies of the 2PCF have not been found to greatly influence either the BAO detection significance or its use as a standard ruler (for instance [46], Sec. III). Furthermore, to undo the nonlinear gravitational evolution, density field reconstruction could be added as a preprocessing step [77–82]. Doing so would offer a more linear field before dewiggling, sharpen the BAO feature, and remove large-scale RSD.
- (ii) Baryon-dark matter relative velocity bias: One of the few known physical effects that can modulate galaxy clustering on BAO scales is the relative velocity between baryons and dark matter, due to the fact that they have different behaviors prior to decoupling [83]. Though decaying after $z \sim 1020$, this relative velocity could impact the formation of the

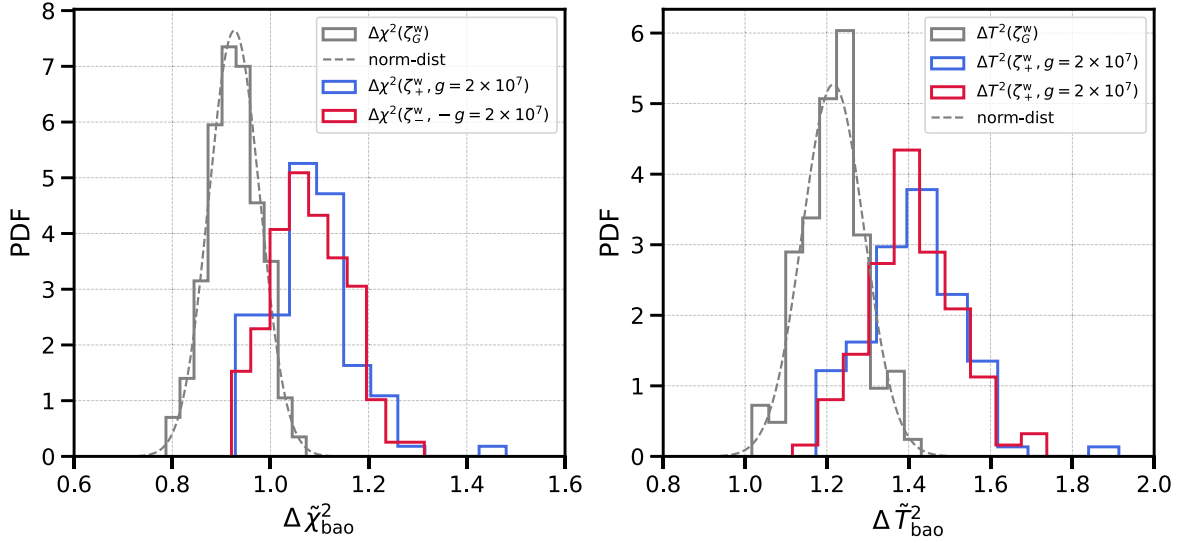


FIG. 6. Distribution of the $\Delta\tilde{\chi}_{\text{BAO}}^2$ Eq. (13); left-hand panel is using the analytic covariance matrix; right-hand is the mock-based one. Each panel shows both positive and negative coupling constant g . In the absence of a true parity-odd 4PCF, the central value of the null distribution is still nonzero (gray curve). However, for both covariance choices, we do obtain a similar excess in $\Delta\tilde{\chi}_{\text{BAO}}^2$ or $\Delta\tilde{T}_{\text{BAO}}^2$ relative to the null distribution.

high-redshift early galaxies that are the ancestors of the galaxies used for late-time spectroscopic samples [84]; this effect is captured by adding a term to the galaxy bias expansion [85,86].

Thus, the BAO feature can be shifted in position and have its shape slightly altered [84,86]. Extensive work has been carried out using the BOSS 3PCF [86,87] and power spectrum [88]. These works did not find a relative velocity bias significantly away from zero in BOSS. They were able to constrain it sufficiently that any effect on the BAO scale measured from BOSS would be far less than the statistical error bars.

- (iii) Neutrinos: Cosmic neutrinos have two impacts on LSS. First, they cause a suppression of the overall clustering amplitude, which is a smooth, broadband effect and does not lead to any sharp features at the BAO scale. Second, prior to the formation of the CMB, the neutrinos in the early Universe are relativistic; they thus alter the phase of the BAO oscillations [89,90]. In position space, this slightly changes the shape of the BAO feature (shown in [91] Fig. 6, lower panel); this is a very small effect, and could certainly be folded into future odd-4PCF BAO analyses if desired.
- (iv) Primordial features: Various primordial features induced by beyond-single-field slow-roll inflationary models could in principle impact the BAO. In particular, isocurvature perturbations, which are fluctuations in the relative number densities of different species (but holding the overall density unchanged), have been considered in this context [92].

While large isocurvature perturbations have been ruled out by the CMB [93,94], relatively significant baryon fluctuations could still exist as long as they are compensated by the dark matter fluctuations. This latter scenario is called compensated isocurvature perturbations (CIP), and could modulate the BAO scale from region to region due to local changes in the sound speed (which depends on the local baryon density) [95].

However, CMB measurements are insensitive to CIP, and as yet no detection of CIP has been reported from galaxy surveys [96]. Certainly this effect could be constrained from standard even-sector BAO searches with 2- and 3-point statistics if it were to become a concern for the odd-4PCF BAO search.

- (v) A late-time parity-violating mechanism: While all primordial parity-violating mechanisms would ensure BAO imprint on the odd sector, this is not necessarily true for late-time mechanisms. For instance, one could imagine a late-time parity-violating mechanism that produces density perturbations that are independent of the normal, adiabatic perturbations. These new perturbations would then not contain BAO, but via gravity, they could still impact the galaxy distribution and create an parity-odd 4PCF. Ultimately this possibility is as yet speculative and requires further theoretical development.

C. BAO from the parity-odd sector as a standard ruler

The BAO have been used as a standard ruler to infer cosmological distances [46,49,97–101]. The precision of

these distance measurements is typically quantified by the error bar on a dilation parameter, α , which relates the BAO scale observed from a galaxy survey to that predicted in a fiducial cosmological model. If the observed BAO scale matches that predicted, $\alpha = 1$. Using this idea, if BAO are detected in the parity-odd sector, they can be used as a standard ruler.

We now examine the expected error bar $\sigma(\alpha)$ on the dilation parameter. We begin with the Fisher matrix, with elements

$$F_{\mu\nu} = \sum_{i,j=1}^N \frac{\partial T_i}{\partial \theta_\mu} \mathbb{C}_{ij}^{-1} \frac{\partial T_j}{\partial \theta_\nu}, \quad (31)$$

where i and j denote trispectrum configurations (i.e. $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$), N is the number of degrees of freedom, and θ_μ and θ_ν are elements of a vector θ made up of the parameters under consideration.

Here we consider only α so the parameter vector has length 1 in Eq. (31), and the matrix $F_{\mu\nu}$ has just one element, which we denote F . The error on α is then

$$\sigma(\alpha) = 1/\sqrt{F}. \quad (32)$$

To obtain F we require $\partial T_i / \partial \alpha$. One can easily show that, for the i th trispectrum configuration,

$$\left[\frac{\partial \ln T}{\partial \alpha} \right]_{\alpha=1} \Big|_i = \sum_{p=1}^4 \left[\frac{\partial \ln \mathcal{T}_\delta(k_p(\alpha))}{\partial \alpha} \right]_{\alpha=1} \Big|_i \quad (33)$$

where $k_p \equiv |\mathbf{k}_p|$ and the notation $[\dots]_i$ means that these four wave-number magnitudes correspond to the i th configuration. The transfer function is a function of the “apparent” wave number, assuming a fiducial cosmology. The apparent wave number, in turn, depends on the dilation parameter α , with $k(\alpha) = \alpha k'$ denoting the scaling of the wave number when applying the dilation parameter to the fiducial grid. Multiplying both sides of the above relation by T_i (evaluated at $\alpha = 1$) then gives the desired $\partial T_i / \partial \alpha$.

We then have

$$F = \sum_{i,j=1}^N T_i \mathbb{C}_{ij}^{-1} T_j \left\{ \sum_{p,q=1}^4 \left[\frac{\partial \ln \mathcal{T}_\delta(k_p(\alpha))}{\partial \alpha} \right]_{\alpha=1} \Big|_i \times \left[\frac{\partial \ln \mathcal{T}_\delta(k_q(\alpha))}{\partial \alpha} \right]_{\alpha=1} \Big|_j \right\}. \quad (34)$$

We notice that the first factor in the outer sum above, $T_i \mathbb{C}_{ij}^{-1} T_j$, in the limit that the quantity in curly brackets went to a constant, would simply lead to F proportional to the *overall* odd-trispectrum detection significance.

In detail, in this limit F would just become χ^2 for the trispectrum [cf. Eq. (12)] with the model set to zero, as

corresponds to our null case of no parity violation. The detection significance is $S_{\text{odd}} \approx \chi^2 / \sqrt{2N}$, since the χ^2 distribution with N degrees of freedom has width $\sqrt{2N}$ [22]. Hence in this limit $F \propto S_{\text{odd}}$. We will return to this point.

We also note that

$$\left. \frac{\partial \ln \mathcal{T}_\delta(k_p(\alpha))}{\partial \alpha} \right|_{\alpha=1} = \left. \frac{\partial \ln \mathcal{T}_\delta(k)}{\partial k} \right|_{k=k_p} k_p. \quad (35)$$

The transfer function may be written [75] (their Eq. 16) as

$$\mathcal{T}_\delta(k) = (1 - f_b) \mathcal{T}_c(k) + f_b \mathcal{T}_b(k), \quad (36)$$

where $f_b \approx 0.2$ is the baryon fraction (with respect to the total matter), $\mathcal{T}_c$ is the cold dark matter transfer function, and $\mathcal{T}_b$ is the baryon transfer function. A very rough form for $\mathcal{T}_c$, motivated by [75], is

$$\mathcal{T}_c(k) \simeq \frac{1}{1 + (k/k_{\text{eq}})^2}, \quad (37)$$

where $k_{\text{eq}} \approx 0.01 \text{ Mpc}^{-1} h$ is the scale of the horizon at matter-radiation equality. Meanwhile, it is also the wave number at which the BAO wiggles in the transfer function begin, and then continue towards larger k .

We then have, dropping the baryon term in Eq. (36) entirely (since $f_b = 0.2$), that

$$\frac{\partial \ln \mathcal{T}_\delta(k)}{\partial k} \approx \frac{\partial \ln \mathcal{T}_c(k)}{\partial k} \simeq -\frac{2k/k_{\text{eq}}^2}{1 + (k/k_{\text{eq}})^2} \rightarrow -\frac{2}{k}, \quad (38)$$

where the final, limiting case is for $k \gg k_{\text{eq}}$.

Inserting this limiting form in Eq. (35) and evaluating it at k_p , we see that the k_p dependence cancels out and we find

$$\begin{aligned} \left. \frac{\partial \ln \mathcal{T}_\delta(k_p(\alpha))}{\partial \alpha} \right|_{\alpha=1} &= \left. \frac{\partial \ln \mathcal{T}_\delta(k)}{\partial k} \right|_{k=k_p} k_p \\ &\approx -\frac{2}{k} \Big|_{k=k_p} k_p = -2. \end{aligned} \quad (39)$$

Interestingly, since the BAO wiggles in the transfer function begin at $k \approx k_{\text{eq}}$ and continue to higher k , and this is also where the transfer function begins to have a nontrivial shape (for $k \ll k_{\text{eq}}$ it is rather flat), this approximation may not be unreasonable to very roughly estimate the precision on α .

Inserting Eq. (39) into Eq. (34), the whole term in curly brackets there will now be a constant and independent of wave number. Given our discussion below Eq. (34), we find that

$$F \propto \sum_{i,j=1}^N T_i \mathbb{C}_{ij}^{-1} T_j \propto S_{\text{odd}}, \quad (40)$$

which implies that F is proportional to our *overall* parity-odd detection significance. Hence, to leading order, the precision on α inferred from Eq. (32) is expected to be

$$\sigma(\alpha) = 1/\sqrt{F} \propto S_{\text{odd}}^{-1/2}; \quad (41)$$

i.e. that the error bar on α will decrease as the square root of our overall detection significance grows.

This formula should be taken simply as a very rough guide; we now offer two caveats. First, we neglected entirely the baryonic component of the transfer function by assuming $f_b \ll 1$; the oscillatory behavior, however, encodes significant additional BAO information, so the estimate above is likely conservative and one might reasonably expect improved precision on $\sigma(\alpha)$ when those features are included. Second, we did not consider any other parameters, such as galaxy bias or H_0 , that could be degenerate with α if broadband shape information is used. Further exploration of these issues is left for future work.

Finally, we highlight that our work here holds for both the parity-even and parity-odd sectors. Further, in the standard cosmological paradigm, there should be no cross covariance between the parity-even and odd sectors. The parity-odd 4PCF can thus add independent information to parameter constraints that harness BAO as a standard ruler.

VI. SUMMARY

In this work, we have demonstrated that BAO can imprint on the parity-odd 4PCF. We then quantified the significance of both the overall parity-odd 4PCF and the BAO in a parity-violating toy model. When considering a toy example with no gravitationally induced non-Gaussianity or observational effects, we also demonstrated that the choice of method used to obtain the covariance matrix has a negligible impact.

We next discuss procedures for BAO dewiggling, which can be implemented either at the field level or at the level of the 4PCF summary statistic. When applying the dewiggling to the 4PCF, we note that a damping function is required to regulate the oscillatory behavior, and can be more straightforwardly applied in the FFT-based algorithms. At present, the FFT-based algorithm does not include survey window function, and future work requires extending the estimator to account for the window function e.g. [73,74]. We also discussed subtleties regarding application of this method in a universe without genuine parity violation, and outlined potential systematics that ought to be considered when performing the odd-parity BAO search on real data.

We then pointed out that BAO in the odd sector can be used as a standard ruler to infer cosmological distances,

finding a simple rough formula relating the precision on the BAO scale to the overall odd-parity detection significance.

We now conclude with a few final important points.

- (1) While this paper focuses on an inflationary origin for the possible parity violation, the approach to significance quantification we present would also apply to postinflationary parity-violating mechanisms, provided these mechanisms involve some transformations of the linear matter density field. Since this field inherently carries the BAO signal, any odd 4PCF generated from such transformations will also retain BAO features.
- (2) As we have seen, the BAO significance depends on the overall detection significance of the odd signal. Therefore, detecting both a parity-odd 4PCF and BAO within it does not automatically imply that systematics or the covariance matrix have been perfectly controlled. An underestimated covariance matrix, for example, could artificially boost the overall detection and also create a spurious BAO feature.
- (3) Here, we have explored the possible significance of BAO in the odd 4PCF using a suite of toy model parity-violating simulations. In the real Universe, however, the significance will depend on the detailed behavior of the underlying model with changes in physical scale, as well as its overall amplitude (this latter effect is present in our work here, through the coupling constant g). Our analysis also assumes linear gravitational evolution and linear galaxy biasing, while the true galaxy bias scheme is likely more complicated (e.g. [102]). Further, we work in real space, not redshift space, so we do not account for the issues peculiar velocities can create. Finally, we do not account for the discrete nature of galaxies, which leads to sampling noise on the observed 4PCF, reflected as an additional set of contributions to the covariance matrix.

In addition to the points mentioned above, future work will address the remaining challenges required to bring our method into practice. The detailed implementation of dewiggling procedure with realistic survey geometry is not addressed in the current work, which uses periodic box simulations, and remains a topic for future investigation. Once these challenges are addressed, the method of this work will be well suited for application to data from SDSS BOSS and extended BOSS (eBOSS), DESI [103], Euclid [104], and future missions such as Spherex [105] and Roman [106].

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DATA AVAILABILITY

The data that support the findings of this article are openly available [107].

APPENDIX A: STANDARD STATISTIC FOR BAO SIGNIFICANCE $\Delta\chi^2_{\text{BAO}}$

In this section, we summarize the statistical properties of the standard $\Delta\chi^2_{\text{BAO}}$ used to quantify the BAO significance, focusing on both its expectation value and variance.

The usual approach to computing the BAO detection significance is to take the χ^2 difference between that for a model without BAO wiggles (in Fourier space), denoted “nw,” and one with, denoted “w” (e.g. [47–50]). We have

$$\Delta\chi^2_{\text{BAO}} = \chi^2_{\text{nw}}(\zeta_m^{\text{nw}}) - \chi^2_{\text{w}}(\zeta_m^{\text{w}}), \quad (\text{A1})$$

where χ^2_{nw} and χ^2_{w} depend, respectively, on the 4PCF model with wiggles, ζ_m^{w} , and the 4PCF model without wiggles,

ζ_m^{nw} . This approach has been used in 2-point and 3-point statistics BAO searches (e.g. [47–50]). It is justified by assuming that the data vector, here, the observed, binned correlation function, follows a multivariate Gaussian. This latter assumption is justified because as long as the bins are not very narrow, the central limit theorem may be invoked on each bin. For the 4-point statistics, we retain this justification, so the binned 4PCF coefficients should follow a multivariate Gaussian and hence a χ^2 test remains appropriate.

1. Expectation values

We first compute $\langle\Delta\chi^2_{\text{BAO}}\rangle$ for the standard BAO search approach. We rewrite the observed 4PCF of the data, ζ_d , as the sum of the with-wiggle model ζ_m^{w} (since in the standard approach, we assume that the BAO are real, so the data vector is a noisy realization of a with-wiggle model) and the statistical noise ϵ^{w} :

$$\zeta_d = \zeta_m^{\text{w}} + \epsilon^{\text{w}}. \quad (\text{A2})$$

Hereafter, we drop the superscript “w” for the noise term. We then find the expectation value for the standard $\Delta\chi^2_{\text{BAO}}$ to be

$$\begin{aligned} \langle\Delta\chi^2_{\text{BAO}}\rangle &= \langle(\zeta_d - \zeta_m^{\text{nw}})\mathbb{C}^{-1}(\zeta_d - \zeta_m^{\text{nw}})^{\text{T}} - (\zeta_d - \zeta_m^{\text{w}})\mathbb{C}^{-1}(\zeta_d - \zeta_m^{\text{w}})^{\text{T}}\rangle \\ &\equiv \langle(\Delta_{\zeta} + \epsilon)\mathbb{C}^{-1}(\Delta_{\zeta} + \epsilon)^{\text{T}} - \epsilon\mathbb{C}^{-1}\epsilon^{\text{T}}\rangle \\ &= \Delta_{\zeta}\mathbb{C}^{-1}\Delta_{\zeta}^{\text{T}} \equiv \Sigma_{\text{BAO}}^2, \end{aligned} \quad (\text{A3})$$

where the BAO detection significance Σ_{BAO} is defined in Eq. (30). In the last line above we defined

$$\Delta_{\zeta} \equiv \zeta_m^{\text{w}} - \zeta_m^{\text{nw}}. \quad (\text{A4})$$

We also dropped the expectation value brackets because the wiggle and no-wiggle models are deterministic.

We now make a remark on the reason that, in the standard approach, the detection significance is quantified as $\Sigma_{\text{BAO}} \equiv \sqrt{\Delta\chi^2_{\text{BAO}}}$. If the “signal” is defined as the difference between the wiggle and no-wiggle models, i.e. Δ_{ζ} , then Eq. (A3) is simply the square of the associated signal-to-noise ratio. The signal-to-noise ratio is then simply $\sqrt{\Delta\chi^2_{\text{BAO}}}$, and indicates the number of standard deviations σ at which the signal is detected.

In contrast to the expectation value of the modified statistic $\Delta\tilde{\chi}^2_{\text{BAO}}$ in Eq. (29), the standard definition does not receive additional corrections from statistical fluctuations, as it does not alter the statistical noise via the application of the dewiggling function.

2. Variances

In addition to the mean values of the $\Delta\chi^2$, we also consider their variance to gain further insight into the distribution of the χ^2 difference computed from the 4PCF with and without BAO wiggles. The variance is defined as

$$\text{Var}(\Delta\chi^2) \equiv \langle(\Delta\chi^2)^2\rangle - \langle\Delta\chi^2\rangle^2. \quad (\text{A5})$$

To compute the variance, we start by calculating the mean of the square of $\Delta\chi^2_{\text{BAO}}$:

$$\begin{aligned} \langle(\Delta\chi^2_{\text{BAO}})^2\rangle &= \langle((\Delta_{\zeta} + \epsilon)\mathbb{C}^{-1}(\Delta_{\zeta} + \epsilon)^{\text{T}} - \epsilon\mathbb{C}^{-1}\epsilon^{\text{T}})^2\rangle \\ &= \langle(\Sigma_{\text{BAO}}^2 + 2\Delta_{\zeta,i}(\mathbb{C}^{-1})_{ij}\epsilon_j)^2\rangle, \end{aligned} \quad (\text{A6})$$

since the covariance matrix is symmetric $\mathbb{C}_{ij} = \mathbb{C}_{ji}$, we have $\Delta_{\zeta,i}\mathbb{C}^{-1}_{ij}\epsilon_j = \epsilon_i(\mathbb{C}^{-1})_{ij}\Delta_{\zeta,j}$. Next, we apply Isserlis' theorem to evaluate the expectation value and drop the odd moments of the noise-dependent terms. Further, terms involving even moments of noise terms can be evaluated as

$$\langle \Delta_{\zeta,i}(\mathbb{C}^{-1})_{ij}\epsilon_j\epsilon_k(\mathbb{C}^{-1})_{kl}\Delta_{\zeta,l} \rangle = \langle \Delta_{\zeta}\mathbb{C}^{-1}\mathbb{C}\mathbb{C}^{-1}\Delta_{\zeta}^T \rangle = \Sigma_{\text{BAO}}^2. \quad (\text{A7})$$

Since $\mathbb{C}^{-1}\mathbb{C} = \sum_j(\mathbb{C}^{-1})_{ij}\mathbb{C}_{jk} = \delta_{ik}^K$, the number of degrees of freedom does not appear in this cross term. Equation (A6) then becomes

$$\langle (\Delta\chi_{\text{BAO}}^2)^2 \rangle = (\Sigma_{\text{BAO}}^2)^2 + 4\Sigma_{\text{BAO}}^2. \quad (\text{A8})$$

Combining Eq. (A3), the variance of $\Delta\chi_{\text{BAO}}^2$ reads

$$\text{Var}(\Delta\chi_{\text{BAO}}^2) = 4\Sigma_{\text{BAO}}^2. \quad (\text{A9})$$

For the modified statistic, the square of $\Delta\tilde{\chi}_{\text{BAO}}^2$ is

$$\begin{aligned} \langle (\Delta\tilde{\chi}_{\text{BAO}}^2)^2 \rangle &= \langle ((\Delta_{\zeta} + \Delta_{\epsilon})\mathbb{C}^{-1}(\Delta_{\zeta} + \Delta_{\epsilon})^T)^2 \rangle \\ &= \langle (\Sigma_{\text{BAO}}^2 + 2\Delta_{\zeta}\mathbb{C}^{-1}\Delta_{\epsilon}^T + \Delta_{\epsilon}\mathbb{C}^{-1}\Delta_{\epsilon}^T)^2 \rangle \\ &= (\Sigma_{\text{BAO}}^2)^2 + 4\Delta_{\zeta,i}(\mathbb{C}^{-1})_{ij}\langle \Delta_{\epsilon,j}\Delta_{\epsilon,k} \rangle(\mathbb{C}^{-1})_{kl}\Delta_{\zeta,l} \\ &\quad + 2\Sigma_{\text{BAO}}^2(\mathbb{C}^{-1})_{ij}\langle \Delta_{\epsilon,i}\Delta_{\epsilon,j} \rangle \\ &\quad + (\mathbb{C}^{-1})_{ij}(\mathbb{C}^{-1})_{kl}\langle \Delta_{\epsilon,i}\Delta_{\epsilon,j}\Delta_{\epsilon,k}\Delta_{\epsilon,l} \rangle. \end{aligned} \quad (\text{A10})$$

Here we again use the property that $\Delta_{\zeta}\mathbb{C}^{-1}\Delta_{\epsilon}^T = \Delta_{\epsilon}\mathbb{C}^{-1}\Delta_{\zeta}^T$. As previously, we expand the expectation value, use Isserlis' theorem and keep only the even moment of the noise-related terms.

As the dewiggling function acts on the data vector of the 4PCF, it also alters the statistical fluctuation such that $\Delta\epsilon \equiv \epsilon^w - \epsilon^{\text{nw}} \neq 0$. This dewiggling operation corresponds to a multiplication in Fourier space, but translates into a convolution in position space. We denote the difference between the noise terms with and without the BAO wiggle as

$$\Delta\epsilon_i = \epsilon_{ij}\epsilon_j^w, \quad (\text{A11})$$

where ϵ_{ij} arises from the convolutional structure of the dewiggling function in position space and encodes the small modulation due to the BAO in matrix form. As a result,

$$\langle \Delta_{\epsilon,j}\Delta_{\epsilon,k} \rangle = \epsilon_{jm}\epsilon_{kv}\mathbb{C}_{\mu\nu} \equiv \tilde{\mathbb{C}}_{jk}. \quad (\text{A12})$$

Therefore, we can rewrite the second term of Eq. (A10) as

$$\begin{aligned} \Delta_{\zeta,i}(\mathbb{C}^{-1})_{ij}\langle \Delta_{\epsilon,j}\Delta_{\epsilon,k} \rangle(\mathbb{C}^{-1})_{kl}\Delta_{\zeta,l} \\ = \Delta_{\zeta,i}(\mathbb{C}^{-1})_{ij}\tilde{\mathbb{C}}_{jk}(\mathbb{C}^{-1})_{kl}\Delta_{\zeta,l} \\ = \Delta_{\zeta}\mathbb{C}^{-1}\tilde{\mathbb{C}}\mathbb{C}^{-1}\Delta_{\zeta}^T. \end{aligned} \quad (\text{A13})$$

Similarly, the third term becomes

$$\Sigma_{\text{BAO}}^2\langle \Delta_{\epsilon}\mathbb{C}^{-1}\Delta_{\epsilon}^T \rangle = \Sigma_{\text{BAO}}^2\text{Tr}(\mathbb{C}^{-1}\tilde{\mathbb{C}}). \quad (\text{A14})$$

Next, we assume there is no connected term for the statistical fluctuations, and find

$$\langle \Delta_{\epsilon,i}\Delta_{\epsilon,j}\Delta_{\epsilon,k}\Delta_{\epsilon,l} \rangle = 3\epsilon_{i\mu}\epsilon_{j\nu}\epsilon_{k\rho}\epsilon_{l\gamma}\mathbb{C}_{\mu\nu}\mathbb{C}_{\sigma\gamma} = 3\tilde{\mathbb{C}}_{ij}\tilde{\mathbb{C}}_{kl}. \quad (\text{A15})$$

Therefore, the fourth term reads

$$\begin{aligned} (\mathbb{C}^{-1})_{ij}(\mathbb{C}^{-1})_{kl}\langle \Delta_{\epsilon,i}\Delta_{\epsilon,j}\Delta_{\epsilon,k}\Delta_{\epsilon,l} \rangle &= 3(\mathbb{C}^{-1})_{ij}(\mathbb{C}^{-1})_{kl}\tilde{\mathbb{C}}_{ij}\tilde{\mathbb{C}}_{kl} \\ &= [\text{Tr}(\mathbb{C}^{-1}\tilde{\mathbb{C}})]^2. \end{aligned} \quad (\text{A16})$$

As a result, Eq. (A10) becomes

$$\begin{aligned} \langle (\Delta\tilde{\chi}_{\text{BAO}}^2)^2 \rangle &= (\Sigma_{\text{BAO}}^2)^2 + 4\Delta_{\zeta}\mathbb{C}^{-1}\tilde{\mathbb{C}}\mathbb{C}^{-1}\Delta_{\zeta}^T \\ &\quad + 2\Sigma_{\text{BAO}}^2\text{Tr}(\mathbb{C}^{-1}\tilde{\mathbb{C}}) + [\text{Tr}(\mathbb{C}^{-1}\tilde{\mathbb{C}})]^2. \end{aligned} \quad (\text{A17})$$

Inserting Eq. (29), the variance of $\Delta\tilde{\chi}_{\text{BAO}}^2$ is then

$$\text{Var}(\Delta\tilde{\chi}_{\text{BAO}}^2) = 4\Delta_{\zeta}\mathbb{C}^{-1}\tilde{\mathbb{C}}\mathbb{C}^{-1}\Delta_{\zeta}^T. \quad (\text{A18})$$

Assuming that ϵ_{ij} is approximately diagonal with constant diagonal elements of order $\mathcal{O}(1\%)$, we have $\tilde{\mathbb{C}} \approx (4\epsilon)^2\mathbb{C}$ and the variance of the modified statistic becomes $64\epsilon^2\Sigma_{\text{BAO}}^2$.

APPENDIX B: IMPACT OF COVARIANCE MATRIX

Here we show the impact of the choice of covariance matrix on the detection significance. We consider two covariance matrices. The first is from the analytic template based on assuming the density field is a GRF [30]. The second is empirically computed from 5000 Gaussian simulations.

In this work, we do not account for late-time effects such as nonlinear gravitational evolution, nonlinear galaxy biasing, and redshift-space distortions. Therefore, both the analytic template and the Gaussian simulation-estimated covariances should be suitable.

We note that our coupling constant is set to be $g = \pm 2 \times 10^7$ so that it is small enough not to render the likelihood non-Gaussian. We also avoid using overly fine radial bins, setting $\Delta r > 5h^{-1}$ Mpc, so that they are not so narrow as to render the central limit theorem inapplicable when we use it to argue that the coefficient on each bin is Gaussian distributed.

Figure 6 shows the distribution of the $\Delta\tilde{\chi}_{\text{BAO}}^2$, with results from using the analytic covariance matrix in the left-hand panel and from using the mock covariance in the right-hand panel. In both, we display results for both the positive and the negative coupling constant g . In the absence of a parity-odd 4PCF, the null distribution is still not centered at zero (gray curve). Due to the noise in the sample covariance matrix, we

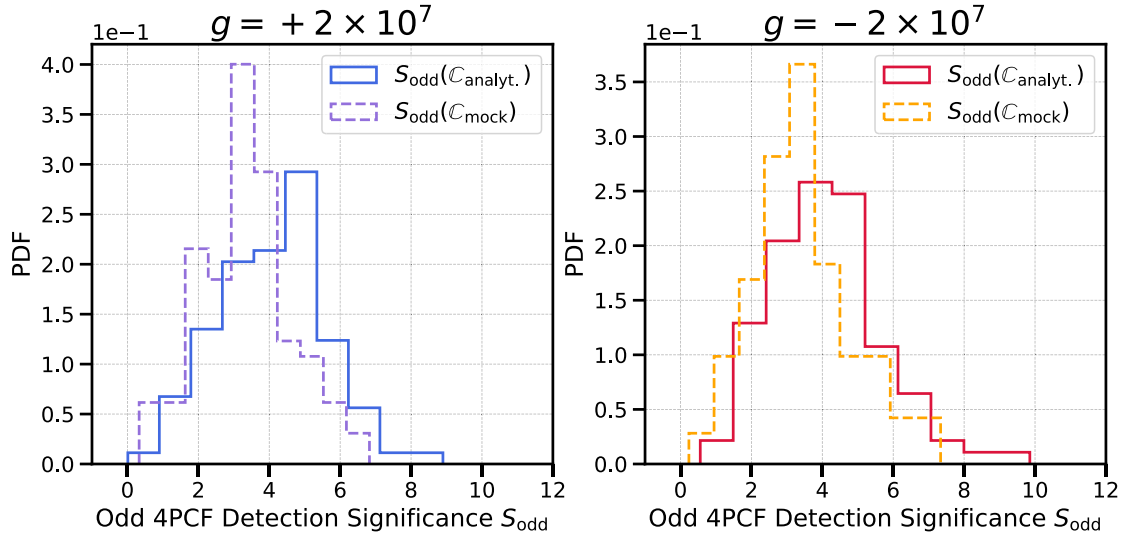


FIG. 7. These panels show how the covariance matrix choice impacts the detection significance for the overall parity-odd 4PCF, for respectively plus (left-hand) and minus (right-hand) coupling constants g . The solid line in each panel is the result using the analytic covariance; the dashed line that using the mock-based. We see that the choice of covariance matrix has only a marginal impact on the recovered significance. Furthermore, the overall parity-odd significance is consistent between the positive and negative coupling constants.

expect that the test statistic will now actually follow a T^2 distribution [60], as discussed in Sec. III B 2; this is confirmed empirically. Using either the analytic covariance matrix or the mock covariance, we obtain a similar excess in $\Delta\tilde{\chi}^2_{\text{BAO}}$ or $\Delta\tilde{T}^2_{\text{BAO}}$ relative to the null distribution.

Figures 7 and 8 show the impact of covariance on the detection significance for the overall parity-odd significance and the imprints of the BAO on the odd 4PCFs. The choice of covariance matrix does not impact the detection significance for either case. We also find the significance to be consistent for both positive and negative coupling constants.

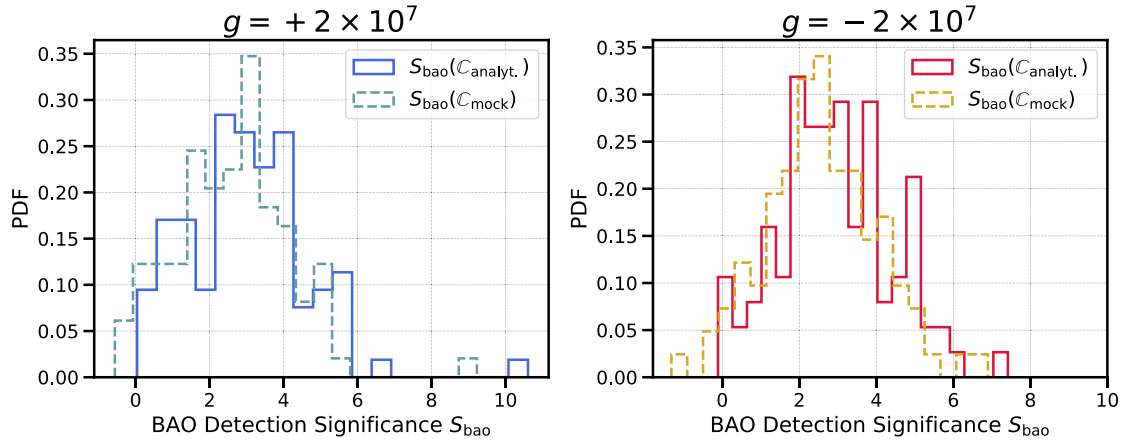


FIG. 8. Similar to Fig. 7, this plot shows the impact of the covariance matrix on the detection significance for the BAO signal on the parity-odd 4PCF. The coupling constants $g = \pm 2 \times 10^7$ are shown in the left and right panels, respectively. We find that the results are independent of the choice of the covariance matrix and consistent for both positive and negative coupling constants.

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