

How Einstein's equations emerge from CFT₂

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The *finiteness* of the entanglement entropies between disjoint subsystems enables us to show that the dynamical equation of the entanglement entropy in CFT₂ is precisely the three-dimensional Einstein equation. We establish a profound relation between the cosmological constant and CFT₂ entanglement entropy. Thus, entanglement entropies induce internal gravitational geometries in CFT₂. Extracting the dual metric from an entanglement entropy becomes a straightforward procedure. Remarkably, we discover that the renormalization group equation is a geometric identity.

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Introduction. The AdS/CFT correspondence [1] has dominated high-energy theory over the past three decades. As a testable realization for the holographic principle [2], it conjectures that a weakly coupled gravitational theory in the bulk of AdS_{d+1} is equivalent to a strongly coupled CFT_d on the conformally flat boundary. Among numerous developments, a central subject is to reconstruct gravity from the dual quantum field theory (QFT), or “derive” Einstein’s equation in QFT.

Since Ryu and Takayanagi (RT) identified the entanglement entropy between adjacent regions of CFT_d with the area of bulk minimal surfaces anchored on the conformal boundary of AdS_{d+1} [3,4], among many approaches, using entanglement entropy to generate the dual geometry has become a very promising direction. For $d = 2$, considered in this paper, the minimal surfaces are geodesics.

In an entangled pure CFT state $\psi_{A\bar{A}}$, entanglement entropy classifies quantum entanglement between complementary parts A and $\bar{A}$. It is defined by the von Neumann entropy: $S_{\text{vN}}(A) = -\text{Tr} \rho_A \log \rho_A$. Here, ρ_A is a reduced density matrix $\rho_A = \text{Tr}_{\bar{A}} |\psi_{A\bar{A}}\rangle \langle \psi_{A\bar{A}}|$. However, as the entangled complementary subsystems are continuously connected, the entanglement entropy diverges due to the very intense entanglement between neighboring fields. This asymptotic behavior makes it very difficult, if not completely impossible, to extract certain exact relations.

It is crucial to note that the entanglement entropy defined above fails to characterize the entanglement in a mixed state, ρ_{AB} . The traditional resolution is to purify the mixed state by adding auxiliary systems, $\bar{A}\bar{B}$. Then, the entanglement of purification (EoP or E_P) [5] is defined by minimizing $S_{\text{vN}}(A\bar{A}:B\bar{B})$ over all possible purifications and quantifies the entanglement between A and B : $S_{\text{vN}}(A:B) := E_P = \min_{\bar{A}\bar{B}} S_{\text{vN}}(A\bar{A}:B\bar{B})$ [6]. However, optimization over purifications is, in practice, intractable in CFTs.

Recently, the SUBTRACTION approach has been proposed to calculate the entanglement entropy $S_{\text{vN}}(A:B)$ between disjoint subsystems A and B in CFT₂ [7–9]. This quantity can be equivalently interpreted as the entanglement entropy in the mixed state ρ_{AB} , as explained in the Appendix. The disconnectedness between subsystems A and B makes $S_{\text{vN}}(A:B)$ finite. This enables us to explicitly show how three-dimensional Einstein gravity emerges from CFT₂ in this paper. Then, we will demonstrate that the renormalization group (RG) equation is surprisingly a geometric identity.

The entanglement entropy for disjoint subsystems in CFT₂. To start with, consider the annular CFT₂. We want to calculate the entanglement entropy between distinct intervals. The pure-state density matrix is shown in Fig. 1.

The entanglement entropy $S_{\text{vN}}(A)$ between A and B is given by

$$S_{\text{vN}}(A) = \lim_{n \rightarrow 1} \frac{1}{1-n} \log \text{Tr}_A \rho_A^n, \quad (1)$$

where $\rho_A = \text{Tr}_B \rho$ and $\text{Tr}_A \rho_A^n = \frac{Z_n}{Z_1^n}$. Here, Z_1 is the partition function on the annulus. Z_n is the partition function on the n -sheeted cover $\mathcal{M}_n$ obtained by cyclically gluing n copies of annuli along A . $S_{\text{vN}}(A)$ for a static time slice is given in Ref. [7]. For generic time-dependent asymmetric cases,

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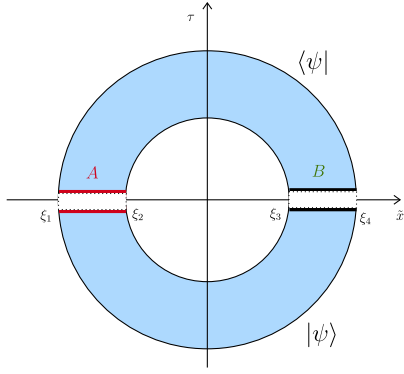


FIG. 1. The Euclidean pure-state density matrix $\rho = |\psi\rangle\langle\psi|$ for the annular CFT₂. $\xi_i = \tilde{x}_i + i\tilde{t}_i$ with $\tilde{t}_i = -i\tau_i$.

$\xi_i = \tilde{x}_i + i\tilde{t}_i$ and $\tilde{t}_i = -i\tau_i$ for Euclidean time, $S_{\text{vN}}(A)$ has been calculated in [8]

$$S_{\text{vN}}(A) = \frac{c}{12} \log \left(\frac{1 + \sqrt{\eta}}{1 - \sqrt{\eta}} \right) + \frac{c}{12} \log \left(\frac{1 + \sqrt{\bar{\eta}}}{1 - \sqrt{\bar{\eta}}} \right), \quad (2)$$

with the cross ratio

$$\eta = \frac{\xi_{21}\xi_{43}}{\xi_{31}\xi_{42}}, \quad \xi_{ij} \equiv \xi_i - \xi_j. \quad (3)$$

As a consequence of the RG equation $\ell \frac{d}{d\ell} S = \frac{c}{6}$ and conformal symmetries, the entanglement entropy [Eq. (2)] turns out to be the universal expression for CFT₂'s. Different CFTs, such as a finite temperature system or a finite size system, are characterized by different cross ratios [8].

Einstein's equation from entanglement entropies in CFT₂. The end points uniquely fix A and B, which amounts to eight free real parameters. However, two of these are gauge freedoms, as shown in Ref. [8]. So, it suffices to consider A and B being collinear, in which six real numbers fix A and B. The disjoint entanglement entropy [Eq. (2)] can be expressed in terms of any equivalent set of linear independent parameters. The physics behind it is the same. For convenience, we regroup the parameters as

$$\begin{aligned} x + it &= \xi_1 + \frac{|\xi_{13}||\xi_{14}|}{|\xi_{12}||\xi_{13}| + |\xi_{34}||\xi_{24}|} \xi_{21}, \\ x' + it' &= \xi_2 + \frac{|\xi_{23}||\xi_{24}|}{|\xi_{13}||\xi_{34}| + |\xi_{12}||\xi_{24}|} \xi_{21}, \\ z &= \frac{|\xi_{14}|\sqrt{|\xi_{12}||\xi_{13}||\xi_{34}||\xi_{24}|}}{|\xi_{12}||\xi_{13}| + |\xi_{34}||\xi_{24}|}, \\ z' &= \frac{|\xi_{23}|\sqrt{|\xi_{12}||\xi_{13}||\xi_{34}||\xi_{24}|}}{|\xi_{13}||\xi_{34}| + |\xi_{12}||\xi_{24}|}. \end{aligned} \quad (4)$$

The cross ratio becomes a real number,

$$\eta = \frac{|\xi_{21}\xi_{43}|}{|\xi_{31}\xi_{42}|} = \frac{\Delta z^2 + \Delta x^2 - \Delta t^2}{(z + z')^2 + \Delta x^2 - \Delta t^2}. \quad (5)$$

In terms of these new coordinates (t, x, z) and (t', x', z') , the entanglement entropy $S_{\text{vN}}(A)$ can be written as

$$S_{\text{vN}}(A) = \frac{c}{6} \cosh^{-1} \left(1 + \frac{\Delta z^2 + \Delta x^2 - \Delta t^2}{2zz'} \right). \quad (6)$$

We define a metric

$$\chi(x, x') := \frac{1}{2} S_{\text{vN}}^2(A), \quad g_{\mu\nu} = -[\chi_{\mu\nu}] := -\lim_{x' \rightarrow x} \partial_{\nu'} \partial_{\mu} \chi, \quad (7)$$

where the bracket $[\dots]$ denotes the coincidence limit $x \rightarrow x'$. Indices in brackets are derivatives with respect to x^μ or x'^μ hereinafter. We introduce

$$R_{\mu\nu} := g^{\rho\sigma} [\nabla_\nu \nabla_\sigma \nabla_\mu \partial_\rho \chi], \quad R := g^{\mu\nu} R_{\mu\nu}, \quad (8)$$

where the covariant derivatives $\nabla_{(\cdot)}$ are defined by $g_{\mu\nu}$. Then, plugging in everything, it is straightforward to verify that

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_c g_{\mu\nu} = 0. \quad (9)$$

This is precisely the three-dimensional Einstein equation with a cosmological constant of

$$\Lambda_c = \frac{R}{6} = \frac{1}{6} [\chi^{\mu\nu'} \chi^{\rho\sigma'} \chi_{\rho\mu\sigma\nu}]. \quad (10)$$

So, the cosmological constant is not really a free parameter, but determined by the CFT entanglement entropy. From Eq. (7), extracting the metric from the entanglement entropy is straightforward, whereas this had been a highly nontrivial task when using the divergent entanglement entropies of connected regions [10–12].

As we explained at the end of the last section, CFT₂'s on other topologies possess the same entanglement entropy expression [Eq. (2)] with different cross ratios which are related by conformal transformations. Since Eqs. (7)–(10) are all tensorial, we immediately conclude that the parameter regrouping in Eq. (4) is *once-for-all* applicable to all CFT₂'s, and the derivation applies to all CFT₂'s. In a companion work [13], we use the thermal CFT as an explicit example to realize ER = EPR.

We have therefore derived Einstein's equations from the quantum entanglement of CFT₂.

Intuitive understanding of the derivation. Although our derivation of Einstein's equations is entirely within the framework of CFT, it is instructive to provide an intuitive picture. To this end, let us turn to geometry. Consider a geodesic $x(\tau)$, $\tau \in [0, t]$, connecting two points $x(\tau = 0)$

and $x'(\tau = t)$. $L(x, x')$ is the geodesic length. Sygne's world function is a biscalar of x and x' , defined as

$$\sigma(x, x') := \frac{1}{2}L^2(x, x') = \frac{1}{2}\Delta\tau \int_0^t d\tau g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}. \quad (11)$$

The metric turns out to be [14]

$$g_{\mu\nu} = -[\sigma_{\mu\nu}] := -\lim_{x' \rightarrow x} \partial_{\nu'} \partial_{\mu} \sigma(x, x'). \quad (12)$$

By default, the indices of σ are ordinary or covariant derivatives. With respect to metric, we have Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \quad (13)$$

We only address vacuum with $\Lambda \neq 0$. Since

$$[\sigma_{\alpha\mu\beta\nu}] := \lim_{x' \rightarrow x} \nabla_\nu \nabla_\beta \nabla_\mu \partial_\alpha \sigma = \frac{R_{\alpha\nu\beta\mu} + R_{\alpha\beta\nu\mu}}{3}, \quad (14)$$

one finds

$$\Sigma_{\mu\nu} := [\sigma^{\alpha\beta} \sigma_{\alpha\mu\beta\nu}] = g^{\alpha\beta} [\sigma_{\alpha\mu\beta\nu}] = \frac{1}{3}R_{\mu\nu} \quad (15)$$

and

$$\Sigma := [\sigma^{\mu\nu} \sigma^{\alpha\beta} \sigma_{\alpha\mu\beta\nu}] = g^{\mu\nu} \Sigma_{\mu\nu} = \frac{1}{3}R. \quad (16)$$

Note that raising or lowering indices and taking the coincidence limit both commute. So, the geodesic version of Einstein's equation is

$$\Sigma_{\mu\nu} - \frac{1}{2}[\sigma_{\mu\nu}]\Sigma + \frac{\Lambda}{3}[\sigma_{\mu\nu}] = 0. \quad (17)$$

Now, considering AdS_3 geometry, on the conformal boundary, two segments A and B are disjoint. Referring to Fig. 2, γ_1 and γ_2 are both geodesics. From the ultraparallel

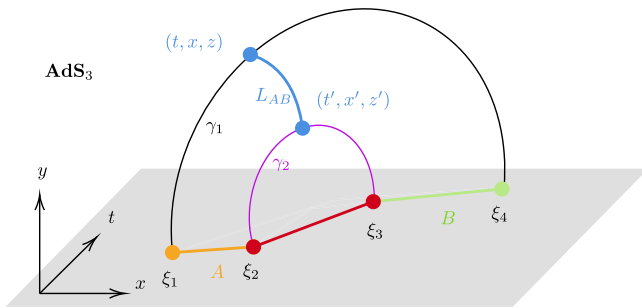


FIG. 2. In AdS_3 , geodesic γ_1 is completely fixed by ξ_1 and ξ_4 , and geodesic γ_2 is completely fixed by ξ_2 and ξ_3 . L_{AB} is a geodesic and unique.

theorem [15] (Theorem 1.1.6) in hyperbolic geometry, the shortest curve connecting γ_1 and γ_2 is a unique geodesic L_{AB} . So, (t, x, z) and (t', x', z') are completely fixed by the four ξ_i 's. The picture also demonstrates explicitly how an extra dimension arises from the boundary theory: four points on the boundary lead to two points in the bulk. Note that the four points on the boundary have eight parameters. However, two of these turn out to be gauge parameters [8]. The remaining six independent parameters exactly match two points in the three-dimensional bulk.

Since L_{AB} has finite length, it must satisfy Einstein's equation (17). On the other hand, one can easily calculate that the length of L_{AB} is precisely equal to $S_{\text{vN}}(A)$ in Eq. (6) with the Brown-Henneaux formula [16] $c = 3R_{\text{AdS}}/2G^{(3)}$ assumed. This is simply the renowned RT formula [3,4]. Therefore, we have reached a holographic description of the CFT Einstein equation. We further identified $\Lambda_c = \Lambda/3$. Note that this holographic description is only for illustrative purposes. It is not a necessary ingredient in our derivation.

A key point to emphasize is that this work is entirely guided by the RT formula. From our derivation, only a finite entanglement entropy is required. There exist several finite measures of entanglement, including the odd entanglement entropy [17], reflected entropy [18], balanced partial entanglement [19], and distillable entanglement [20]. Theoretically, any of these could be used to derive the Einstein equation. However, taking reflected entropy [18] as an example, its holographic dual is a closed curve formed by gluing two identical geodesics. This structure makes it impossible to perform end point derivatives and take the coincidence limit. So, unless obtaining the exact CFT₂ dual—i.e., $S_{\text{vN}}(A)$ in Eq. (1) of the geodesic L_{AB} in Fig. 2—we could not realize how to derive the Einstein equation from the entanglement entropy.

RG equation and Klein-Gordon equation. On the QFT side, we have an entanglement-generated gravity in addition to the CFT itself, whereas on the gravity side, we have spacetime geometry and QFT living on it. The spacetime geometry and entanglement-generated gravity have identical structures.

Basically, there are three fundamental equations: Einstein's equation, the covariant Klein-Gordon equation, and the RG equation. We already know that in bulk gravity, Einstein's equation and the covariant Klein-Gordon equation exist, whereas in the CFT on the boundary, we have a quantum-entanglement-generated Einstein's equation and the RG equation. Therefore, we want to find the covariant Klein-Gordon equation in CFT and the RG equation in gravity.

RG equation as a geometric identity: For a classically scale-invariant theory living on a $d + 1$ -dimensional manifold $\mathcal{M}$ with the metric $ds^2 = \gamma_{ab}dx^a dx^b$, the Callan-Symanzik RG equation is [21]

$$\left[\ell \frac{\partial}{\partial \ell} - 2 \int_{\mathcal{M}} \gamma^{ab} \frac{\delta}{\delta \gamma^{ab}} \right] \log Z_{\text{CFT}} = 0, \quad (18)$$

where ℓ is the length scale. Using the replica trick [22] and Rényi entropy

$$S_{\text{vN}}^{(n)} = \frac{1}{1-n} \log \left[\frac{Z_{\mathcal{M}_n}}{(Z_{\mathcal{M}})^n} \right], \quad (19)$$

where $\mathcal{M}_n$ is the replicated manifold, one can show that the RG equation for the entanglement entropy is [23]

$$\ell \frac{\partial}{\partial \ell} S_{\text{vN}} = \frac{c}{6}. \quad (20)$$

The constant $\frac{c}{6}$ on the right hand side is for $d = 2$.

On the other hand, in *any* geometry, not limited to gravity, there is an identity for the world function (11),

$$g^{\mu\nu} \sigma_{\mu} \sigma_{\nu} = 2\sigma, \quad \sigma(x, x') = \frac{1}{2} L^2(x, x'). \quad (21)$$

On a time slice of AdS_3 , the isometry is $\text{SL}(2, R)$. Those isometries preserve the distance between points. Except for the vertical ones, any other geodesic is a segment of a semicircle,

$$(x - x_0)^2 + z^2 = r^2. \quad (22)$$

We can map a semicircle to a vertical line by the following isometries:

$$(\tilde{x}, \tilde{z}) := \left(\frac{x - x_0 + r}{(x - x_0 + r)^2 + z^2} - \frac{1}{2r}, \frac{z}{(x - x_0 + r)^2 + z^2} \right).$$

In terms of $(\tilde{x}, \tilde{z})$, the semicircle is a vertical line, $\sigma = \sigma(0, \tilde{z}_1; 0, \tilde{z}_2)$. Now, we apply the geometric identity (21):

$$z^2 [(\partial_x \sigma)^2 + (\partial_z \sigma)^2] = \tilde{z}^2 (\partial_{\tilde{z}} \sigma)^2 = 2\sigma. \quad (23)$$

Obviously, $\tilde{z}$ is the renormalization scale ℓ . Including the coupling $c = 3R_{\text{AdS}}/2G^{(3)}$, in either the quantum-entanglement-generated gravity in CFT_2 or the bulk spacetime gravity, $\sigma = \frac{1}{2} L^2 = \frac{1}{2} S_{\text{vN}}^2$, we thus have in both boundary CFT_2 and bulk geometry

$$\ell \frac{\partial}{\partial \ell} S_{\text{vN}} = \frac{c}{6}, \quad (24)$$

which is precisely the RG equation (20).

Covariant Klein-Gordon equation in CFT: Usually, the entanglement entropy is computed by using the replica method. The Rényi entropy is determined by the partition function of the replicated manifold. The replicated partition function equals the correlation function of twisted operators,

$$G(u(\xi_i), v(\xi_i)) = \text{Tr}_A \rho_A^n. \quad (25)$$

In Ref. [7], we have obtained $\text{Tr}_A \rho_A^n = e^{\frac{1}{2}(\frac{1}{n}-n)S_{\text{vN}}(A:B)}$. So,

$$G(u(\xi_i), v(\xi_i)) = e^{\frac{1}{2}(\frac{1}{n}-n)S_{\text{vN}}(A:B)}. \quad (26)$$

Note that $u(\xi_i)$ and $v(\xi_i)$ are basically (t, x, z) and (t', x', z') , respectively, in Fig. 2, but located on the boundary of the cut geometry. Thus, $G(u(\xi_i), v(\xi_i))$ is not the usual CFT propagator but is effectively a four-point function.

To obtain the EOM of this propagator, we refer to the AdS_3 bulk propagator of a scalar with mass $m^2 = \Delta(\Delta - 2)$ and $\Delta = \frac{c}{12}(n - 1/n)$,

$$G_{\text{Bulk}}(u, v) = \frac{e^{-\Delta L(u, v)}}{1 - e^{-2L(u, v)}}, \quad (27)$$

which satisfies the bulk covariant Klein-Gordon equation. Under the semiclassical limit, $\Delta \rightarrow \infty$, keeping Δ/c fixed, this propagator reduces to the CFT one [Eq. (26)]. Therefore, we have

$$(\nabla^2 - m^2)G(u(\xi_i), v(\xi_i)) = \delta_{\mathcal{M}}^3(\xi_i - \xi_j) \quad (28)$$

at the large- c limit.

Summary and discussion. We demonstrated that the dynamics of entanglement entropy in CFT_2 are precisely the three-dimensional Einstein equations. A profound relation between the cosmological constant and CFT entropy is also given. We thus uncovered a hidden gravitational structure within CFT_2 . We confirmed that in addition to the Einstein equation and the covariant Klein-Gordon equation, in the bulk, a geometrical identity is the RG equation. On the CFT side, in addition to the quantum-entanglement-generated Einstein equation, we gave the covariant Klein-Gordon equation and showed that the RG equation is a geometric identity of the emerged gravity.

Since the ordinary QFT and the induced gravity are linked through entanglement entropy, it is now possible to unify the black hole thermodynamics and Einstein's equations.

In this paper, we have focused on AdS . Nevertheless, the derivation is applicable to dS in the same fashion.

We want to emphasize some important features indicated by the formula $g_{\mu\nu} = -[\frac{1}{2}(S_{\text{vN}}^2)_{\mu\nu}]$:

- (1) We showed that the metric is a derived but not a fundamental quantity. Perhaps this is why gravity is nonrenormalizable and we should not try to quantize the metric itself. Furthermore, S_{vN} is already a quantum quantity, so why bother to perform further quantization?
- (2) Note the explicit local/global duality as shown by the formula.
- (3) Obviously, spacetime singularities are phase transitions from this formula.

(4) The lhs is geometric, whereas the rhs is algebraic. We have addressed the vacuum Einstein equation in CFT₂. Including matter sources is absolutely a critical question. Perhaps the most important future direction is to study CFT_d for $d > 2$.

In physics, energy dominates and links everything. But now, we tend to believe that the fundamental object in physics is entropy.

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Data availability. No data were created or analyzed in this study.

Appendix. In this appendix, we will show that the $S_{\text{vN}}(A:B)$ in Eq. (1) calculated by the subtraction approach in CFT₂ coincides with the entanglement of purification (EoP) $E_p(A:B)$.

First, note that our purpose is to compute the entanglement between disjoint segments A and B . In the procedure of the EoP, one first obtains a mixed state ρ_{AB} after tracing out C and D . Then, the auxiliary systems $\bar{A}$ and $\bar{B}$ are introduced to purify ρ_{AB} . This is totally equivalent to replacing C and D with auxiliary systems from the very outset, in which no mixed state appears as an intermediate step and we always work on pure states.

In order to compute the entanglement between disjoint segments A and B , it is necessary to eliminate the overcounted degrees of freedom (denoted as D_{oc}), which originate from the entanglement between the combined subsystem AB and its complement CD . For instance, the EoP replaces CD with auxiliary systems $\bar{A}\bar{B}$; its built-in minimization procedure guarantees the cancellation of D_{oc} . In any case, the purpose is to remove the effects caused by CD in a controlled manner. To this end, why not seek a covariant approach to remove CD within the path integral, thereby directly eliminating D_{oc} ? This is precisely what the SUBTRACTION procedure does. As illustrated in Fig. 3, after removing segments C and D with two discs in the infinite system, we get the annular CFT in Fig. 1.

Notably, a striking advantage of taking the annular CFT₂ as the starting point is that we can extend the method to higher dimensions straightforwardly. Specifically, in a recent paper [24], by replacing A and B with $(D-1)$ -dimensional disks, we consider the solid torus $\mathbb{B}^{D-1} \times S^1$. We are then able to provide a very simple field-theoretic method to calculate the entanglement entropies of CFT in all dimensions.

We now provide specific arguments about the equivalence between $S_{\text{vN}}(A:B)$ and the EoP.

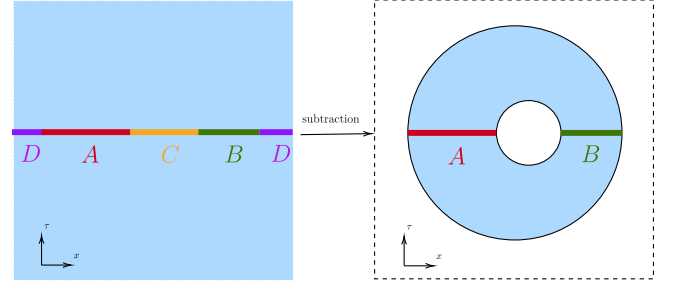


FIG. 3. After subtracting segments C and D with two discs in the infinite system, we obtain an asymmetric annular region in which A and B are in a pure entangled state ψ_{AB} , which is identical to the annular CFT₂.

The entanglement of purification $E_p(A:B)$ is defined as

$$E_p(A:B) = \min_{\bar{A}, \bar{B}} S_{\text{vN}}(A\bar{A}:B\bar{B}), \quad (\text{A1})$$

where $\bar{A}, \bar{B}$ are auxiliary systems, and the minimization is over all possible purifications $|\Psi_{A\bar{A}B\bar{B}}\rangle$ of ρ_{AB} .

Consider a static CFT₂ at $t = 0$ with subsystems $A = (0, l_A)$ and $B = (l_A + d, l_A + l_B + d)$. First, we add an auxiliary system $\bar{A} = (l_A, l_A + l_{\bar{A}})$ (left panel of Fig. 4). From the results in Ref. [7], it is straightforward to get

$$\frac{\partial S_{\text{vN}}(A\bar{A}:B)}{\partial l_{\bar{A}}} = \frac{c}{6(d - l_{\bar{A}})} \sqrt{\frac{l_B(d + l_A)}{(l_{\bar{A}} + l_A)(d - l_{\bar{A}} + l_B)}} \geq 0, \quad (\text{A2})$$

with $0 < l_{\bar{A}} < d$. One can also choose another auxiliary system $\bar{A} = (-l_{\bar{A}}, 0)$, as shown in the right panel of Fig. 4, where we still have

$$\begin{aligned} \frac{\partial S_{\text{vN}}(A\bar{A}:B)}{\partial l_{\bar{A}}} &= \frac{c}{6(d + l_{\bar{A}} + l_A + l_B)} \\ &\times \sqrt{\frac{l_B(d + l_B)}{(l_{\bar{A}} + l_A)(d + l_{\bar{A}} + l_A)}} \geq 0, \end{aligned}$$

with $l_{\bar{A}} > 0$.

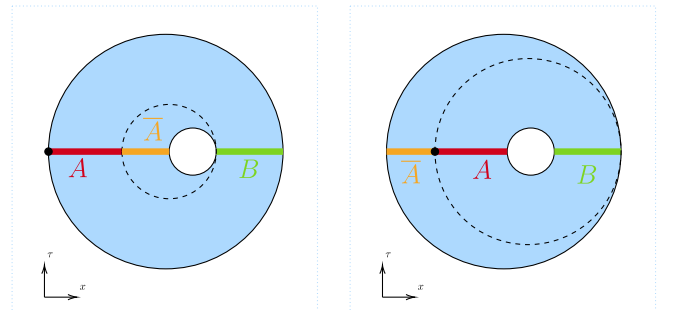


FIG. 4. Two configurations for adding the auxiliary system $\bar{A}$. Each black point denotes the origin.

Treating $A\bar{A}$ as a composite subsystem and adding $\bar{B}$ adjacent to B , with the same pattern, it follows that

$$\frac{\partial S_{\text{vN}}(A\bar{A}:B\bar{B})}{\partial l_{\bar{B}}} \geq 0 \quad \text{and} \quad \frac{\partial S_{\text{vN}}(A\bar{A}:B\bar{B})}{\partial l_{\bar{A}}} \geq 0. \quad (\text{A3})$$

These inequalities imply that $S_{\text{vN}}(A\bar{A}:B\bar{B})$ is minimized when $l_{\bar{A}} \rightarrow 0$ and $l_{\bar{B}} \rightarrow 0$. Consequently, the entanglement of purification reduces to

$$E_p(A:B) = S_{\text{vN}}(A:B).$$

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