

Singularity-free Feynman integral bases

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Standard integration-by-parts (IBP) reduction methods typically yield Feynman integral bases where the reduction of some integrals gives rise to coefficients singular as the dimensional regulator $\epsilon \rightarrow 0$. These singular coefficients can also appear in scattering amplitudes, obscuring their structure, and rendering their evaluation more complicated. We investigate the use of bases in which the reduction of any integral is free of singular coefficients. We present two general algorithms for constructing such bases. The first is based on sequential $D = 4$ IBP reduction. It constructs a basis iteratively by projecting onto the finite part of the set of IBP relations. The second algorithm performs Gaussian elimination within a local ring forbidding division by ϵ while permitting division by polynomials in ϵ finite at $\epsilon = 0$. We study the application of both algorithms to a pair of two-loop examples, the planar and nonplanar double-box families of integrals. We also explore the incorporation of finite Feynman integrals into these bases. In one example, the resulting basis provides a simpler and more compact representation of a scattering amplitude.

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I. INTRODUCTION

Scattering amplitudes beyond tree level can be expressed as sums over Feynman integrals with coefficients rational in kinematic quantities. The complete set of Feynman integrals relevant to any given scattering amplitude is subject to a large set of linear identities. We can obtain these identities systematically using integration-by-parts (IBP) techniques [1,2]. Eliminating the identities reduces the set of all Feynman integrals to a basis, but a given basis

is of course not unique. Some bases are better than others. Indeed, different bases may be adapted to different purposes: generating and solving a set of differential equations to obtain explicit expressions for the integrals; finding a compact representation for the amplitude (and thereby minimizing computational effort in evaluating it); or manifesting certain properties in special limits.

In practical applications, the systems of equations that emerge from IBP are large, and require specialized codes to solve them. Solution methods for these systems of equations have been a subject of ongoing research since the pioneering algorithm of Laporta [3]. Researchers have developed a variety of general-purpose software packages implementing the algorithm and related ideas over the years, including AIR [4], FIRE [5–8], LiteRed [9,10], Reduze [11,12], Kira [13–17], Blade [18–20], and NeatIBP [21,22]. In related developments, new finite-field techniques for functional reconstruction and solving large systems of linear equations based on computation in finite fields have helped increase the power of IBP solvers [23]. These techniques have been implemented in specialized tools such as FiniteFlow [24], FireFly [25,26], and

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RaTracer [27]. Other recent work has examined choosing master integrals to avoid mixed poles involving both the dimensional regulator and the Mandelstam invariants [28,29], and ways of simplifying IBP coefficients [30,31]. Researchers have also developed algorithms to find uniform-transcendentality bases [32–36], useful in setting up systems of differential equations for integrals. An approach quite distinct from IBP systems—intersection theory—has also provided new insights into reduction of Feynman integrals [37–39].

With a basis in hand for a specific process at a desired loop order, any Feynman integral that may arise in computing the amplitude can be expressed in terms of it. Canonical bases [40,41], for example, are well-suited to computing the integrals via the method of differential equations [42–45]. Such a basis, however, may not necessarily be best-suited to obtaining a compact representation for scattering amplitudes. Unpleasant features in standard bases may include the presence of integral coefficients singular in the dimensional regulator ϵ , or cancellations of divergences between different integrals. These may arise in the reduction of some integrals, and may also be present in the expression for scattering amplitudes.

In this article, we study the utility of *singularity-free* bases. These are bases where no explicit singularities in ϵ appear in the coefficients of basis integrals in the reduction of any integral, and no explicit singularities in ϵ are present in the definition of any basis integral. All singularities in ϵ then arise solely from the loop integrations. Such a basis is expected to improve numerical evaluation and stability [46]. We also conjecture that it can simplify expressions for physical quantities.

We present two algorithms for obtaining such bases. We also examine the possibility of combining the criterion of singularity freedom with the use of locally finite Feynman integrals. Such integrals were considered in Refs. [47–49], and most recently by Gambuti, Novichkov, Tancredi, and one of the authors (GKNT) [50]. These authors presented a comprehensive approach to finding and organizing the set of locally finite Feynman integrals. The GKNT approach relies on classifying singular surfaces in loop-momentum space using the Landau equations. De La Cruz, Novichkov, and one of the authors [51] presented an equivalent approach relying on Newton polytopes. The resulting sets of finite integrals are not independent under IBP, but a subset could be included as basis integrals. Including finite integrals in the basis may simplify expressions for scattering amplitudes and reduce the extent of cancellations between different divergent terms.

In the next section, we review the appearance of singular coefficients in integral reduction. In Sec. III, we give two algorithms for obtaining a singularity-free basis starting with a set of integrals and IBP relations between them. In Sec. IV, we explore two examples, that of the planar and nonplanar double box integrals. Our conclusions are in Sec. V. Two Appendixes give technical details on one of the

algorithms, and the coefficients of the subleading-color terms in the all-plus two-loop amplitude in a modified basis.

II. SINGULAR COEFFICIENTS IN INTEGRAL REDUCTION

In general, we will want to consider Feynman integrals with nontrivial numerators,

$$I[\mathcal{N}(\ell_i)] = \int \prod_{i=1}^L d^D \ell_i \frac{\mathcal{N}(\ell_i)}{\mathcal{D}_1 \cdots \mathcal{D}_E}. \quad (2.1)$$

The simplest nontrivial examples of singular coefficients arise at two loops, and we will focus on two-loop integrals for the explicit examples in this article. Let us look at the planar double box as a first example, with

$$\begin{aligned} \mathcal{D}_1 &= \ell_1^2, & \mathcal{D}_2 &= (\ell_1 - k_1)^2, \\ \mathcal{D}_3 &= (\ell_1 - K_{12})^2, & \mathcal{D}_4 &= (\ell_1 - \ell_2)^2, \\ \mathcal{D}_5 &= (\ell_2 - K_{12})^2, & \mathcal{D}_6 &= \ell_2^2, \\ \mathcal{D}_7 &= (\ell_2 - K_{123})^2, \end{aligned} \quad (2.2)$$

where $K_{i\dots j} = k_i + \cdots + k_j$. We denote integrals with these propagators by the subscript “DB.” We define the Gram determinant,

$$G \begin{pmatrix} q_1, \dots, q_m \\ p_1, \dots, p_m \end{pmatrix} = \det(2q_i \cdot p_j), \quad (2.3)$$

where we list only one series of arguments if the two sets are identical. We represent momenta in the arguments by their indices alone, so that, for example,

$$G \begin{pmatrix} \ell_1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \equiv G \begin{pmatrix} \ell_1 & k_2 & k_3 \\ k_1 & k_2 & k_3 \end{pmatrix}. \quad (2.4)$$

Consider two different but standard choices of basis, the Laporta basis [3] and a uniform-transcendentality one (useful for differential equations [40] after adjusting overall normalizations to obtain a canonical basis). We also have two irreducible scalar products (ISPs),

$$\text{ISP}_1 = (\ell_1 + k_4)^2, \quad \text{ISP}_2 = (\ell_2 + k_1)^2. \quad (2.5)$$

The Laporta basis for s -channel integrals has the following elements:

$$\begin{aligned} &\{B_{0,0,1,0,0,1,1,0,0}, B_{0,1,0,0,1,0,1,0,0}, \\ &B_{0,1,0,1,0,1,1,0,0}, B_{0,1,0,1,1,1,1,0,0}, \\ &B_{0,1,1,0,1,1,1,0,0}, B_{1,0,1,1,0,1,0,0,0}, \\ &B_{1,1,1,1,1,1,1,-1,0}, B_{1,1,1,1,1,1,1,0,0}\}, \end{aligned} \quad (2.6)$$

where the indices correspond as usual to the exponents of the denominators in Eq. (2.2), followed by the negative of the exponents of the two ISPs in the numerator (2.5). A possible uniform-transcendentality basis for s -channel integrals has the following elements [40]:

$$\begin{aligned} &\{B_{0,0,2,0,0,1,2,0,0}, B_{0,1,0,1,0,1,2,0,0}, \\ &B_{0,1,1,0,1,1,1,0,0}, B_{0,2,0,0,1,0,2,0,0}, \\ &B_{2,0,1,1,0,2,0,0,0}, B_{1,1,1,0,1,0,2,0,0}, \\ &B_{1,1,1,1,1,1,1,0,0}, B_{1,1,1,1,1,1,0,-1}\}. \end{aligned} \quad (2.7)$$

In the Laporta basis (2.6), we have the following representations for two example integrals:

$$\begin{aligned} I_{\text{DB}}[\mathcal{D}_6^2] &= \frac{3(1-3\epsilon)(2-3\epsilon)}{2st\epsilon^3} B_{0,0,1,0,0,1,1,0,0} + \frac{(3s+t\epsilon)}{t\epsilon} B_{0,1,0,1,1,1,1,0,0} \\ &+ \frac{(3(s+t)-2t\epsilon)}{t(1-2\epsilon)} B_{0,1,1,0,1,1,1,0,0} - \frac{3(1-3\epsilon)}{2t\epsilon^2} B_{0,1,0,1,0,1,1,0,0} \\ &+ \frac{(1-3\epsilon)(2-3\epsilon)(3-2\epsilon)}{2t^2\epsilon^3} B_{0,1,0,0,1,0,1,0,0}, \\ I_{\text{DB}}\left[G\left(\begin{array}{ccc} \ell_1 & 1 & 2 \\ \ell_2 & 3 & 4 \end{array}\right)\right] &= \frac{s^2}{8} B_{1,1,1,1,1,1,1,-1,0} - \frac{(1-\epsilon)(1-3\epsilon)(2-3\epsilon)}{2s\epsilon^3} B_{0,0,1,0,0,1,1,0,0} \\ &+ \frac{s+t}{2} B_{0,1,1,0,1,1,1,0,0} + \frac{(1-2\epsilon)(1-3\epsilon)(2-3\epsilon)}{4t\epsilon^3} B_{0,1,0,0,1,0,1,0,0} \\ &- \frac{5(1-2\epsilon)(1-3\epsilon)}{16\epsilon^2} B_{0,1,0,1,0,1,1,0,0} - \frac{(1-\epsilon)(1-2\epsilon)}{4\epsilon^2} B_{1,0,1,1,0,1,0,0,0}. \end{aligned} \quad (2.8)$$

The first integral diverges as $1/\epsilon^4$ as $\epsilon \rightarrow 0$, but its reduction has singular coefficients. The second integral is in fact finite, but this emerges only through cancellations and in the presence of divergent coefficients.

The expressions in the uniform-transcendentality basis (2.7) are simpler,

$$\begin{aligned} I_{\text{DB}}[\mathcal{D}_6^2] &= -\frac{3s}{4t\epsilon^2(1-2\epsilon)} B_{0,0,2,0,0,1,2,0,0} + \frac{(3(s+t)-2t\epsilon)}{t(1-2\epsilon)} B_{0,1,1,0,1,1,1,0,0} \\ &- \frac{(3s+t\epsilon)}{3\epsilon(1-2\epsilon)} B_{1,1,1,0,1,0,2,0,0} - \frac{(3-2\epsilon)}{4\epsilon^2(1-2\epsilon)} B_{0,2,0,0,1,0,2,0,0} - \frac{B_{0,1,0,1,0,1,2,0,0}}{1-2\epsilon}, \\ I_{\text{DB}}\left[G\left(\begin{array}{ccc} \ell_1 & 1 & 2 \\ \ell_2 & 3 & 4 \end{array}\right)\right] &= -\frac{s^2(1-\epsilon)}{4\epsilon^2(1-2\epsilon)} B_{2,0,1,1,0,2,0,0,0} + \frac{s^2}{8} B_{1,1,1,1,1,1,1,0,-1} \\ &+ \frac{s+t}{2} B_{0,1,1,0,1,1,1,0,0} + \frac{s(1-\epsilon)}{4\epsilon^2(1-2\epsilon)} B_{0,0,2,0,0,1,2,0,0} + \frac{5s}{8\epsilon} B_{0,1,0,1,0,1,2,0,0} - \frac{t}{8\epsilon^2} B_{0,2,0,0,1,0,2,0,0}, \end{aligned} \quad (2.9)$$

but the unwanted singular coefficients are still present.

We would like to find a basis where reductions of integrals have no singular coefficients. We turn to that question in the next section.

III. OBTAINING A SINGULARITY-FREE INTEGRAL BASIS

In this section, we present two algorithms for constructing a singularity-free integral basis. The first is based on the

idea of sequential $D = 4$ IBP reduction; the second on Gaussian elimination in a local ring in ϵ , that is avoiding division by ϵ . Given IBP relations for all Feynman integrals appearing in a given scattering process, our algorithms provide a singularity-free basis for the process.

In the standard Laporta algorithm [3], given M IBP relations in N Feynman integrals, one fixes the ordering of integrals, namely $I_1, \dots, I_N$, based on the integral sector, denominator degree, and numerator degree. Complicated

integrals in the sense of Laporta will have smaller-valued subscripts. This will order them at the beginning of the list, and thereby target them for removal by row reduction. We can organize the coefficients of the integrals in the M IBP relations as an $M \times N$ matrix C , whose j th column corresponds to the integral I_j .

The IBP system could in principle be generated through a Laporta-type algorithm [3], using a template of symbolic IBPs [10], or by other means. In this article, we use IBP relations obtained using a generating-vector approach as implemented in the NeatIBP [21,22] package. The generating-vector approach, as discussed earlier, eliminates integrals with doubled propagators at the very start. It also provides a minimal set of relations, for which $N - M = B$, where B is the number of basis integrals. The approach in NeatIBP is a natural one when a basis free of doubled propagators is desirable (which is almost always the case). These features are not essential to the working of the algorithm, but do make using it more efficient. The number of master integrals is of course independent of the algorithm used to generate the IBP equations.

Should a different algorithm be used, the generated set of IBP relations may be redundant; we assume that redundant relations have been removed before using the algorithms described below. In addition, the initial IBP equations may not suffice to reduce “boundary” integrals (typically at or above the maximal degree in loop momenta in the list of target integrals), but will only suffice to eliminate them from the system. We also assume that this elimination has been done, so that we are left only with target integrals, those whose reductions we need. (We include all possible basis integrals in this set.) The initial IBP relations are necessarily linear in ϵ , but the removal of “boundary” integrals may raise the degree in ϵ . The algorithms do not assume linearity in ϵ , though some steps will be simpler if the IBP system remains linear.

Implementations of the algorithms described in this article are available at [52].

A. Reduction by projection

Our starting point is a system of IBP equations for Feynman integrals,

$$\sum_{j=1}^{N^{[0]}} C_{ij}^{[0]} I_j^{[0]} = 0, \quad i = 1, \dots, M^{[0]}, \quad (3.1)$$

where $N^{[0]}$ is the initial number of integrals appearing in the IBP relations. We have added a superscript as the values in this algorithm will change at every iteration.

The list of integrals $I_j^{[0]}$ is initially sorted according to Laporta’s integral ordering. During the algorithm, we will re-sort the integrals. It is possible to keep track of the reduction coefficients to the basis we are constructing, either

analytically or using finite-field reconstruction [26,27,53] with rational values for the Mandelstam invariants. However, it is more efficient to make use of a prior reduction to a Laporta basis, and find the change of basis to the new singularity-free basis afterward. Accordingly, we focus on finding the integrals in the new basis. We therefore choose rational values for all Mandelstam invariants.¹

As our goal is to find a basis where no coefficient has inverse powers of ϵ , we will use Gaussian elimination avoiding all divisions by ϵ . There is one exception: if a row is proportional to ϵ , we may still divide it out. Our algorithm is iterative; the superscripts in $C^{[w]}$, $I^{[w]}$, and $N^{[w]}$ record the iteration number. The algorithm proceeds as follows:

- (1) Initialize $w = 0$.
- (2) Define $\tilde{C}^{[w]}$ via

$$\tilde{C}_{ij}^{[w]} = C_{ij}^{[w]}|_{\epsilon=0}. \quad (3.2)$$

Both $\tilde{C}^{[w]}$ and $C^{[w]}$ are $M^{[w]} \times N^{[w]}$ matrices. The rank of $\tilde{C}^{[w]}$ may be smaller than that of $C^{[w]}$, because in setting $\epsilon = 0$, we are discarding combinations of linear relations which have overall factors of ϵ . The relations expressed by $\tilde{C}^{[w]}$ are not exact, but hold up to corrections of $\mathcal{O}(\epsilon)$.

- (3) Perform Gaussian elimination on $\tilde{C}^{[w]}$. Denote the independent integrals by $I_i^{[w+1]}$; they are candidate basis integrals. This procedure yields a possibly *overcomplete* set of master integrals, with $i = 1, \dots, N^{[w+1]} \leq N^{[w]}$. It may be overcomplete because we have discarded combinations of relations proportional to ϵ in step 2. We will denote by $\tilde{I}_j^{[w]}$ the integrals which are reducible using the result of this Gaussian elimination. The expressions for these reducible integrals obtained here will of course only be valid up to corrections² of $\mathcal{O}(\epsilon)$.
- (3') (optional, preparatory to step 6') If using the alternative step 6' and if $C^{[w]}$ is linear in ϵ , extract the terms linear in ϵ ,

$$d_{ij}^{[w]} = \frac{1}{\epsilon} (C_{ij}^{[w]} - \tilde{C}_{ij}^{[w]}). \quad (3.3)$$

¹We must take care that these rational values be generic and in particular not yield a *special kinematic* configuration, that is a kinematic configuration for which any of the *letters* [54–56] of the Feynman integrals vanish. For six-point or higher integrals, we must also ensure that the chosen values satisfy the Gram determinant identities enforcing the four-dimensionality of external momenta.

²This can include reductions setting integrals to zero, where the true expression for the integral may simply be $\epsilon \times I_r^{[w+1]}$, for example.

If $C^{[w]}$ is not linear in ϵ , construct and retain the row-reduction matrix $R^{[w]}$, which carries out the Gaussian elimination on $\tilde{C}^{[w]}$ by left multiplication. For example, this can be achieved using Gauss-Jordan elimination.

- (4) Check if $N^{[w+1]} = B$. If so, we have obtained a new basis, and the algorithm terminates. If $N^{[w+1]} > B$, continue with the next step.
- (5) We reorder the integrals so that the independent integrals obtained in step 2 are ordered last. We do this by defining a permutation matrix $S^{[w]}$ such that

$$(S^{[w]})_{ij} I_j^{[w]} = (\bar{I}^{[w]}, I^{[w+1]})_i. \quad (3.4)$$

The orderings within the $\bar{I}^{[w]}$ and $I^{[w+1]}$ subsets are left unchanged; this defines $S^{[w]}$ uniquely. The IBP relations must be permuted by the inverse

permutation: $C^{[w]}(S^{[w]})^{-1}$, so we are left with the following set of IBP relations to consider:

$$\sum_{j,k=1}^{N^{[w]}} C_{ik}^{[w]} ((S^{[w]})^{-1})_{kj} (\bar{I}^{[w]}, I^{[w+1]})_j = 0, \quad i = 1, \dots, M^{[w]}. \quad (3.5)$$

This set is equivalent to the original set of IBP relations (3.1), once the previously removed relations $U^{[0]}, \dots, U^{[w-1]}$ (defined below) are added back.

- (6) We now perform Gaussian elimination on $C^{[w]}(S^{[w]})^{-1}$. The resulting matrix still has $M^{[w]}$ rows. (Were we to have started with IBP relations without removing redundant equations, we would have zero rows here which we would then have to discard.) We can identify four submatrices in it,

$$\begin{array}{c} N^{[w]} - N^{[w+1]} \\ N^{[w+1]} - B \end{array} \left\{ \begin{array}{c} \left[\begin{array}{cccc} 1 & 0 & \dots & 0 \\ & 1 & \dots & 0 \\ & & \ddots & 0 \\ & & & 1 \end{array} \right] & \begin{array}{c} N^{[w+1]} \\ * \\ r^{[w+1]} \end{array} \end{array} \right\} \left[\begin{array}{c} \bar{I}^{[w]} \\ I^{[w+1]} \end{array} \right] = 0. \quad (3.6)$$

The upper-left submatrix is the identity; the upper-left and upper-right parts of the matrix together reduce the integrals in $\bar{I}^{[w]}$ to linear combinations of $I^{[w+1]}$ s. These parts together form the matrix $U^{[w]}$. By construction, this matrix is free of singularities in ϵ . The lower-left part of the matrix is identically zero. So long as the set of $I^{[w+1]}$ s is overcomplete, the lower-right part of the matrix will be non-trivial and will provide further identities between them. In general, its elements may be divergent as $\epsilon \rightarrow 0$. We denote the most negative power of ϵ in a given row $N^{[w]} - N^{[w+1]} < j \leq N^{[w]} - M$ by $-h_j$. We define the $(N^{[w+1]} - B) \times N^{[w+1]}$ coefficient matrix $C^{[w+1]}$ used in the next iteration in terms of the lower-right hand matrix, removing the singularities in ϵ . We can formally define it as follows:

$$C_{ij}^{[w+1]} = \epsilon^{h_i} r_{i+\delta, j+\delta}^{[w+1]}, \quad 1 \leq i \leq N^{[w+1]} - B, 1 \leq j \leq N^{[w+1]}, \quad (3.7)$$

where $\delta = N^{[w]} - N^{[w+1]}$. This matrix is now nonsingular as $\epsilon \rightarrow 0$. If the set of $I^{[w+1]}$ is not overcomplete, $N^{[w+1]}$ will be equal to B , and the algorithm will terminate on the next

iteration. If the set of $I^{[w+1]}$ is overcomplete, at least one row will contain one or more elements nonvanishing as $\epsilon \rightarrow 0$, which will provide a new reduction, ensuring that $N^{[w+2]}$ will be strictly smaller than $N^{[w+1]}$. (Recall that we assumed removal of redundant IBP identities prior to starting the algorithm.) This ensures the algorithm will terminate with no redundant integrals.

We have now obtained a new, smaller, IBP system for $I^{[w+1]}$,

$$\sum_{j=1}^{N^{[w+1]}} C_{ij}^{[w+1]} I_j^{[w+1]} = 0, \quad i = 1, \dots, M^{[w+1]}. \quad (3.8)$$

This description assumes a symbolic treatment of ϵ . As we have chosen rational values for the Mandelstam invariants, it is the only symbol left. We could speed up this step by choosing finite-field values for the Mandelstam invariants, using a set of different values for ϵ , and using finite-field reconstruction techniques to find the (rational) functional dependence on ϵ .

- (6') (alternative to step 6) Gaussian elimination on $C^{[w]}(S^{[w]})^{-1}$, with generic ϵ -dependent entries, can be computationally expensive. We can instead perform this alternative step. If $C^{[w]}$ is linear in ϵ , we can form a new $M \times 2N^{[w]}$ matrix by placing $d^{[w]}$ to the right of $\tilde{C}^{[w]}$ after right-transforming both by $(S^{[w]})^{-1}$,

$$(\tilde{C}^{[w]}(S^{[w]})^{-1} | d^{[w]}(S^{[w]})^{-1}), \quad (3.9)$$

and performing Gaussian elimination on it. We then rebuild $C^{[w+1]}$ by examining each row, with v_1 given by the first $N^{[w]}$ elements, and v_2 by the remaining $N^{[w]}$ elements. The output row is

$$\begin{aligned} v_1 + \epsilon v_2, & \quad v_1 \neq 0, \\ v_2, & \quad v_1 = 0. \end{aligned} \quad (3.10)$$

If $C^{[w]}$ is not linear in ϵ , we instead construct the matrix

$$R^{[w]} C^{[w]} (S^{[w]})^{-1}. \quad (3.11)$$

The last $N^{[w+1]} - B$ rows of this matrix will all be proportional to positive powers of ϵ by construction. Indeed, if we had used $\tilde{C}^{[w]}$ in Eq. (3.11), rather than $C^{[w]}$, such rows would be identically zero. We can now build the diagonal matrix $P^{[w]}$, whose entries are

$$P_{ii}^{[w]} = \frac{1}{\epsilon^{l_i}}, \quad \text{and} \quad P_{ij}^{[w]} = 0 \quad \text{for } i \neq j, \quad (3.12)$$

where l_i is the *lowest overall power* of ϵ in the i th row. Finally, we define

$$C^{[w+1]} = P^{[w]} R^{[w]} C^{[w]} (S^{[w]})^{-1}. \quad (3.13)$$

- (7) We increment the iteration number, and continue with step 2 for the new coefficient matrix and set of integrals.

The algorithm terminates at step 4, when the new basis is no longer overcomplete. At that iteration, the set of master integrals is the desired singularity-free basis.

B. Gaussian elimination in a local ring

Our second algorithm performs Gaussian elimination in a *local ring*. Formally, a local ring $R \equiv \mathbb{F}[x]_{\mathbf{m}}$ is a

polynomial ring $\mathbb{F}[x]$ *localized* on a maximal ideal $\mathbf{m}$,

$$R = \left\{ \frac{f(x)}{g(x)} \middle| f(x), g(x) \in \mathbb{F}[x], g(x) \notin \mathbf{m} \right\}. \quad (3.14)$$

In other words, the local ring R consists of rational functions whose denominators are *not* in $\mathbf{m}$. We refer to the standard textbook [57] for the details about the concept and related computations in local rings.

In our application, we use the local ring $\mathbb{Q}[\epsilon]_{\langle \epsilon \rangle}$. The maximal ideal $\langle \epsilon \rangle$ is the set of all polynomials in ϵ vanishing at $\epsilon = 0$. The ring $\mathbb{Q}[\epsilon]_{\langle \epsilon \rangle}$ consists of all rational functions in ϵ which are finite at $\epsilon = 0$. We consider the Gaussian elimination of IBP identities, and demand that all the coefficients in the intermediate steps and the final results are in the local ring $\mathbb{Q}[\epsilon]_{\langle \epsilon \rangle}$. This forbids division by ϵ , while division by polynomials that are finite at $\epsilon = 0$ is allowed.

This method has the advantage that this form of Gaussian elimination is very similar to the standard one. Available Gaussian elimination packages can easily be adapted for our purpose with minor adjustments.

Using the standard Laporta ordering, the Gaussian elimination of $C^{[0]}$ (without any column swaps) would implement IBP reduction and give us the set of master integrals. However, in general, this process may yield poles in ϵ in the reduction coefficients. This happens because pivots proportional to ϵ will result in division by ϵ and thence poles in ϵ further along in the pivot row. A pivot proportional to a power of ϵ will avoid generating poles only if all other row elements are also proportional to the same power of ϵ .

In order to avoid generating poles in ϵ , we must avoid pivots proportional to ϵ (after removing common factors of ϵ). Could we simply forbid terms proportional to ϵ in selecting a pivot? In general, when we disallow column swaps, all pivot candidates are in the leading column past already-reduced rows. If all nonzero entries in this column are proportional to ϵ , then forbidding such pivots will cause the Gaussian elimination procedure to terminate too early. The IBP reduction will then be incomplete.

Instead, we modify the standard procedure as follows: we allow column swaps and forbid pivots proportional to ϵ . At each step of the Gaussian elimination, we divide out any overall power of ϵ in every remaining row. We then swap the columns to select a pivot which is not proportional to ϵ . We keep track of these swaps, as they will lead to a new choice of master integrals. In general, it will be different from that obtained by the usual Laporta procedure.

As in the sequential algorithm described in the previous subsection, we start with a set of integrals ordered using the Laporta ordering, with “complicated” integrals ordered earlier than “simple” ones.

The steps of the algorithm are as follows:

- (1) Define an index p and initialize it, $p = 1$; define a list $l^{[1]}$ containing the relevant sorted integrals, $l^{[1]} = [I_1, I_2, \dots, I_N]$. As above, the matrix $C^{[1]}$ contains the coefficients of the integrals in the N input IBP relations. The index p is also the iteration counter.
- (2) Divide each row i ($p \leq i \leq M$) of $C^{[p]}$ by the greatest common divisor of the row's elements.
- (3) Find a candidate pivot: scan all elements of $C^{[p]}$, columns v from the p th rightward, rows u from the p th to bottom within each column.
Select (u, v) such that (a) $C_{uv}^{[p]}$ is nonzero and *not* proportional to ϵ , (b) $l_v^{[p]}$ is the integral with the *smallest subscript* (that is, the earliest position in the original list $l^{[1]}$), and (c) the u th row is the sparsest row (or the first of a set of equally sparsest rows). This corresponds to a so-called “minimal swap” strategy. Other strategies are possible, and will be discussed below.
- (4) Pivot: swap the p th and u th rows and the p th and v th columns of $C^{[p]}$. Swap the p th and v th element of $l^{[p]}$ to get a new list $l^{[p+1]}$.
- (5) Normalize: divide the p th row of $C^{[p]}$ by $C_{pp}^{[p]}$.
- (6) Reduce: subtract appropriate multiples of the p th row of $C^{[p]}$ to eliminate all leading entries of rows $p+1, \dots, M$. Denote the resulting new matrix by $C^{[p+1]}$.
- (7) Increment p by one. If $p \leq M$, continue at step 2.

The resulting choice of master integrals is given by the last $N - M$ elements in the list l^{M+1} . By construction, the reductions of all other integrals have nonsingular coefficients. We demonstrate the correctness of this algorithm in Appendix A.

At the search step (step 3), there are other possible criteria for choosing a pivot. Indeed, if we first assemble a list of all nonzero elements not proportional to ϵ in the submatrix at or below the p th row and at or to the right of the p th column, the list will generally contain more than one element. The “minimal-swap” strategy described above attempts to find the indices of the most “complicated” integral in the sense of the original ordering, so long as the candidate pivot is not proportional to ϵ . That is, the goal is to obtain a singularity-free basis which is as close as possible to the Laporta basis.

In addition to this, we have also examined what we term a “Markowitz” strategy [58]. This strategy attempts to preserve the sparsity of the matrix during Gaussian elimination. We define it as follows: denote by r_i the number of nonzero entries in row i , and by f_j the number of nonzero entries in column j . Define the Markowitz numbers $m_{ij} = (r_i - 1)(f_j - 1)$. For all nonzero elements $C_{ij}^{[p]}$ ($i \geq p$ and $j \geq p$) which are not proportional to ϵ ,

choose the one with minimal Markowitz number to be the pivot.

In practice, we believe the “minimal-swap” strategy is preferable, because it preserves the original ordering of integrals as much as possible. In examples we have studied, it provides simpler IBP reduction coefficients. In contrast, the “Markowitz” strategy mixes integrals from different sectors with different numbers of propagators; and hence leads to more complicated IBP reduction coefficients.

As in the iterative-reduction algorithm discussed in the previous section, when our sole purpose is to obtain the set of basis integrals, it is helpful to set all Mandelstam variables and masses to generic rational number values.

IV. EXAMPLES

In this section, we apply the algorithms presented above to two integral families: the two-loop massless planar and nonplanar double boxes. In both cases, we start with all (“generic”) integrals with up to the fourth power of ISPs alongside the set of finite integrals determined in Ref. [50].³ We start with the IBP system for the generic integrals as found using generating-vector method as implemented in NeatIBP. We also construct a set of relations between the generic and finite integrals obtained by expanding the numerators of the finite integrals. As described in Sec. III, we sort the generic integrals with decreasing Laporta weight. When adding the set of finite integrals, we place them at the very end, effectively assigning them the lowest Laporta weights. This fixes the matrix of integral coefficients $C^{[0]}$.

A. Two-loop planar double box

We begin by running both algorithms solely on the generic integrals. We find the same set of master integrals,

$$\{B_{0,1,1,1,1,0,1,0,0}, B_{1,0,1,1,1,0,1,0,0}, \\ B_{1,1,0,1,1,0,1,0,0}, B_{1,1,1,1,0,1,1,0,0}, \\ B_{1,1,1,1,1,0,1,0,0}, B_{1,1,1,1,1,1,0,0,0}, \\ B_{1,1,1,1,1,1,1,-1,0}, B_{1,1,1,1,1,1,1,0,0}\}. \quad (4.1)$$

Let us look at the decomposition of the two example integrals from Eqs. (2.8) and (2.9) into this new basis,

³We omit the integrals that vanish upon symmetrization.

$$\begin{aligned}
I_{\text{DB}}[\mathcal{D}_6^2] &= \frac{\epsilon(2s+t)}{s(1-2\epsilon)} B_{0,1,1,1,1,0,1,0,0} - \frac{\epsilon(3s+t)}{s(1-2\epsilon)} B_{1,1,0,1,1,0,1,0,0} \\
&+ \frac{(6s+2t\epsilon)}{6(1-2\epsilon)} B_{1,1,1,1,1,0,1,0,0} - \frac{2s\epsilon}{3(1-2\epsilon)} B_{1,1,1,1,0,1,1,0,0} \\
&+ \frac{2s\epsilon}{3(1-2\epsilon)} B_{1,1,1,1,1,0,0,0,0} + \frac{3\epsilon}{1-2\epsilon} B_{1,0,1,1,1,0,1,0,0}, \\
I_{\text{DB}}\left[G\begin{pmatrix} \ell_1 & 1 & 2 \\ \ell_2 & 3 & 4 \end{pmatrix}\right] &= -\frac{s^2(3+2\epsilon)}{24(1-2\epsilon)} B_{1,1,1,1,0,1,1,0,0} - \frac{s^2(3-8\epsilon)}{24(1-2\epsilon)} B_{1,1,1,1,1,1,0,0,0} \\
&+ \frac{s^2}{8} B_{1,1,1,1,1,1,1,-1,0} + \frac{s+t}{2} B_{1,1,0,1,1,0,1,0,0} - \frac{s(1-6\epsilon)}{4(1-2\epsilon)} B_{1,0,1,1,1,0,1,0,0} - \frac{t}{2} B_{0,1,1,1,1,0,1,0,0}. \quad (4.2)
\end{aligned}$$

Unlike the reductions of these two integrals given in Eqs. (2.8) and (2.9), the coefficients of the reductions here are not singular as $\epsilon \rightarrow 0$. (The basis integrals appearing here still are singular.) These reductions are examples of why the basis in Eq. (4.1) is a singularity-free basis.

We can go further. The second integral is one of the finite integrals constructed in Ref. [50]; we can consider including it as a basis integral in place of one of the integrals in Eq. (4.1). To implement this improvement in the basis, add the set of finite integrals at the end of list of generic integrals, and also add the relations expressing them in terms of generic integrals to the list of IBP identities. The placement at the end effectively assigns them lower Laporta weights, favoring their appearance over divergent integrals. We then run the algorithms on this extended set of integrals. Both yield the same new basis in which the two $\mathcal{O}(1/\epsilon^4)$ -divergent double-box integrals [the two last entries in Eq. (4.1)] are replaced by the two finite double boxes with degree-two numerators,

$$\begin{aligned}
F_1 &= I_{\text{DB}}\left[G\begin{pmatrix} \ell_1 & 1 & 2 \\ \ell_2 & 3 & 4 \end{pmatrix}\right], \\
F_2 &= I_{\text{DB}}\left[G\begin{pmatrix} \ell_1 & 1 & 2 & 3 \\ \ell_2 & 1 & 2 & 3 \end{pmatrix}\right]. \quad (4.3)
\end{aligned}$$

For the planar double box, this replacement retains the finiteness of the integral coefficients. The resulting basis is

$$\begin{aligned}
\{F_1, F_2, B_{1,1,1,1,0,1,1,0,0}, B_{1,1,1,1,1,0,1,0,0}, \\
B_{1,1,1,1,1,0,0,0,0}, B_{1,0,1,1,1,0,1,0,0}, \\
B_{1,1,0,1,1,0,1,0,0}, B_{0,1,1,1,1,0,1,0,0}\}. \quad (4.4)
\end{aligned}$$

These integrals turn out to be the same as the s -channel integrals in the basis simplifying the presentation of the two-loop four-gluon all-plus amplitude in Ref. [50]. The system has 1296 integrals with 1288 independent IBPs. The algorithm of Sec. III B takes 1288 iterations (the same as would be required for standard Gaussian elimination).

B. Two-loop nonplanar double box

We can repeat the construction for the nonplanar double box. The expression for this integral is given by Eq. (2.1) with $\mathcal{D}_j$ replaced by $\mathcal{D}_j^{\text{NP}}$, where

$$\begin{aligned}
\mathcal{D}_1^{\text{NP}} &= \ell_1^2, & \mathcal{D}_2^{\text{NP}} &= (\ell_1 - k_1)^2, \\
\mathcal{D}_3^{\text{NP}} &= (\ell_1 - K_{12})^2, & \mathcal{D}_4^{\text{NP}} &= \ell_2^2, \\
\mathcal{D}_5^{\text{NP}} &= (\ell_2 + k_4)^2, \\
\mathcal{D}_6^{\text{NP}} &= (\ell_2 - \ell_1 + K_{124})^2, & \mathcal{D}_7^{\text{NP}} &= (\ell_1 - \ell_2)^2. \quad (4.5)
\end{aligned}$$

The two ISPs here are

$$\text{ISP}_1^{\text{NP}} = (\ell_1 + k_4)^2, \quad \text{ISP}_2^{\text{NP}} = (\ell_2 + k_1)^2. \quad (4.6)$$

The Laporta basis for this integral and its descendants is

$$\begin{aligned}
\{B_{0,0,1,1,0,0,1,0,0}, B_{0,1,0,0,1,0,1,0,0}, \\
B_{0,1,0,1,0,1,0,0,0}, B_{1,0,1,0,1,0,1,0,0}, \\
B_{0,1,0,1,1,1,0,0,0}, B_{0,1,1,1,0,1,1,0,0}, \\
B_{0,1,1,1,1,0,1,0,0}, B_{1,1,1,0,1,0,1,0,0}, \\
B_{1,1,1,1,0,1,0,0,0}, B_{1,0,1,1,1,1,1,0,0}, \\
B_{1,1,1,1,1,1,1,0,0}, B_{1,1,1,1,1,1,1,-1,0}\}. \quad (4.7)
\end{aligned}$$

We run both algorithms on the list of generic integrals supplemented by the finite integrals found in Ref. [50]. Both again yield the same basis,

$$\begin{aligned} &\{B_{0,1,0,1,1,1,1,0,0}, B_{0,1,1,1,1,0,1,0,0}, \\ &B_{1,0,1,1,1,1,0,0,0}, B_{1,1,0,1,1,1,0,0,0}, \\ &B_{0,1,1,1,1,1,0,0,0}, B_{1,0,0,1,1,1,1,0,0}, \\ &B_{1,0,1,1,1,1,1,0,0}, B_{1,1,0,1,1,1,0,1,0,0}, \\ &B_{1,1,1,1,1,1,0,1,0,0}, B_{1,1,1,1,1,1,0,0,0}, \\ &B_{1,1,1,1,1,1,1,0,-1}, B_{1,1,1,1,1,1,1,0,0}\}. \end{aligned} \quad (4.8)$$

In this case, no finite integrals survive in the basis. The system starts with 1431 integrals, and 1419 independent IBP relations. The algorithm of Sec. III B takes 1419 iterations (which again is the same as would be taken by standard Gaussian elimination). The first four integrals in this basis diverge as $1/\epsilon^2$ as $\epsilon \rightarrow 0$,

$$\begin{aligned} B_{0,1,0,1,1,1,1,0,0} &= \frac{[\log(1 + \frac{1}{\chi}) - 2i\pi] \log(\frac{\chi}{1+\chi})}{2s\epsilon^2} + \mathcal{O}(\epsilon^{-1}), \\ B_{0,1,1,1,1,0,1,0,0} &= \frac{\log^2 \chi + \pi^2}{2s(1 + \chi)\epsilon^2} + \mathcal{O}(\epsilon^{-1}), \\ B_{1,0,1,1,1,1,0,0,0} &= \frac{\pi^2}{12s\epsilon^2} + \mathcal{O}(\epsilon^{-1}), \\ B_{1,1,0,1,1,1,0,0,0} &= \frac{[2i\pi - \log(1 + \chi)] \log(1 + \chi)}{2s\chi\epsilon^2} + \mathcal{O}(\epsilon^{-1}), \end{aligned} \quad (4.9)$$

where $\chi = t/s$ and the integrals have been evaluated in the u -channel, i.e., at $s < 0$ and $\chi > 0$. The remaining eight integrals diverge as $1/\epsilon^4$ in the four-dimensional limit.

As an example, consider the reduction of the following integral:

$$\begin{aligned} B_{1,1,1,1,1,1,1,-2,0} &= \frac{(t - 2s + 4t\epsilon - 5s\epsilon)}{(1 + 4\epsilon)(s + t)} B_{0,1,0,1,1,1,1,0,0} + \frac{(s + 2t + 7s\epsilon + 11t\epsilon)}{s(1 + 4\epsilon)} B_{0,1,1,1,1,0,1,0,0} \\ &\quad - B_{1,0,1,1,1,1,0,0,0} + \frac{t(s - 2t + 4s\epsilon - 5t\epsilon)}{s(1 + 4\epsilon)(s + t)} B_{1,1,0,1,1,1,0,0,0} \\ &\quad + \frac{(3s + 2t + 9s\epsilon + 5t\epsilon)}{s(1 + 4\epsilon)} B_{0,1,1,1,1,1,0,0,0} - \frac{3(t + s\epsilon + 4t\epsilon)}{(1 + 4\epsilon)(s + t)} B_{1,0,0,1,1,1,1,0,0} \\ &\quad + \frac{(t - s)}{2} B_{1,0,1,1,1,1,1,0,0} - \frac{t(5s + 2t + 20s\epsilon + 11t\epsilon)}{(1 + 4\epsilon)s(s + t)} B_{1,1,0,1,1,0,1,0,0} \\ &\quad + \frac{(9s - 5t)}{3} B_{1,1,1,1,1,0,1,0,0} + \frac{(4t - 11s)}{6} B_{1,1,1,1,1,1,0,0,0} \\ &\quad + \frac{(2t - 3s)}{2} B_{1,1,1,1,1,1,1,0,-1} + \frac{t(t - s)}{2} B_{1,1,1,1,1,1,1,0,0}. \end{aligned} \quad (4.10)$$

The coefficients are nonsingular as desired. In contrast, in the Laporta basis, some of the reduction coefficients would be singular as $\epsilon \rightarrow 0$.

The simplest application of this integral family is to the subleading-color contributions to the two-loop all-plus four-gluon amplitude in pure Yang–Mills theory. The result for this amplitude was given in the millenium paper [59],

$$\begin{aligned} A_{12;34}^{\text{NP}} &= i \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} s((D_s - 2) I_{\text{NP}} [\lambda_{l_1-l_2}^2 \lambda_{l_2}^2 + \lambda_{l_1-l_2}^2 \lambda_{l_1}^2 + \lambda_{l_1}^2 \lambda_{l_2}^2] \\ &\quad + 16 I_{\text{NP}} [(\lambda_{l_1-l_2} \cdot \lambda_{l_2})^2 - \lambda_{l_1-l_2}^2 \lambda_{l_2}^2]), \end{aligned} \quad (4.11)$$

where I_{NP} is the nonplanar double box, and λ_i^μ is the projection of the vector ℓ^μ on the subspace orthogonal to the external $D = 4$ space spanned by the vectors $k_1^\mu, k_2^\mu, k_3^\mu$ and $v^\mu = \epsilon^{\mu\nu\rho\sigma} k_{1\nu} k_{2\rho} k_{3\sigma} / \sqrt{-G(123)}$. In particular, we have

$$\lambda_p \cdot \lambda_q = \eta_{\perp}^{\mu\nu} p_\mu q_\nu, \quad (4.12)$$

where

$$\eta_{\perp}^{\mu\nu} = \eta^{\mu\nu} - \sum_{i=1}^3 k_i^\mu \hat{k}_i^\nu + v^\mu v^\nu,$$

$$\text{with } \hat{k}_i \cdot k_j = \delta_{ij}. \quad (4.13)$$

We can systematically rewrite the all-plus amplitude as a sum of basis integrals, each multiplied by a coefficient rational in the invariants and ϵ . If we choose the singularity-free basis (4.8), we can pull out an overall factor of ϵ in the

expression for the amplitude. The amplitude is then manifestly no more divergent than $1/\epsilon^3$,

$$A_{12;34}^{\text{NP}} = i \frac{[1\,2][3\,4]}{\langle 1\,2\rangle\langle 3\,4\rangle} \epsilon \sum_{I \in \text{basis}} c_I I. \quad (4.14)$$

This contrasts with the apparent $1/\epsilon^4$ divergence in the Laporta basis. The would-be $1/\epsilon^3$ divergence of course cancels between integrals, leaving an overall $1/\epsilon^2$ divergence.

The appearance of a power of ϵ does give us extra freedom to modify the basis. We can relax our requirement

that all IBP reduction coefficients be free of poles in ϵ to the requirement that they have no poles worse than $1/\epsilon$, and still end up with amplitude expressions that are free of explicit poles in ϵ . As it turns out, we can use this freedom to make the cancellations of the would-be $1/\epsilon^3$ divergences manifest. The best candidates for new elements of the basis are the finite nonplanar double box integrals enumerated in Ref. [50]. We can trade five integrals with $1/\epsilon^4$ divergences for finite integrals, four with degree-three numerators and one with a degree-four numerator,

$$\begin{aligned} & F_1^{\text{NP}} = I_{\text{NP}} \left[G \begin{pmatrix} \ell_1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} G \begin{pmatrix} \ell_2 & 4 \\ 1 & 2 \end{pmatrix} G \begin{pmatrix} \ell_1 - \ell_2 & 3 \\ 1 & 2 \end{pmatrix} \right], \\ & B_{1,1,1,1,1,1,1,-1,0} \quad F_2^{\text{NP}} = I_{\text{NP}} \left[G \begin{pmatrix} \ell_2 & 4 \\ \ell_1 - \ell_2 & 3 \end{pmatrix} G \begin{pmatrix} \ell_1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \right], \\ & B_{1,1,1,1,1,1,1,0,0} \quad \rightarrow \quad F_3^{\text{NP}} = I_{\text{NP}} \left[G \begin{pmatrix} \ell_1 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} G \begin{pmatrix} \ell_2 & 4 \\ 2 & 3 \end{pmatrix} G \begin{pmatrix} \ell_1 - \ell_2 & 3 \\ 2 & 3 \end{pmatrix} \right], \\ & B_{0,1,1,1,1,1,0,0,0} \quad F_4^{\text{NP}} = I_{\text{NP}} \left[G \begin{pmatrix} \ell_2 & 4 \\ 1 & 2 \end{pmatrix} G \begin{pmatrix} \ell_1 & 1 & 2 \\ \ell_1 - \ell_2 & 3 & 1 \end{pmatrix} \right], \\ & B_{1,1,0,1,1,0,1,0,0} \quad F_5^{\text{NP}} = I_{\text{NP}} \left[G(\ell_1 \, 1 \, 2) G \begin{pmatrix} \ell_2 & 4 \\ \ell_1 - \ell_2 & 3 \end{pmatrix} \right]. \end{aligned} \quad (4.15)$$

The choice of integrals on the left-hand side is based on an examination of the pattern of cancellations of the leading singularity in the expression for the subleading-color terms in the amplitude. One can check that the replacement of integrals on the left-hand side by those on the right results in integral-reduction coefficients whose worst divergence is as $1/\epsilon$. The resulting form of the subleading-color contributions is

$$\begin{aligned} A_{12;34}^{\text{NP}} = i \frac{[1\,2][3\,4]}{\langle 1\,2\rangle\langle 3\,4\rangle} \{ & \epsilon c_1 B_{0,1,0,1,1,1,1,0,0} \\ & + \epsilon^2 c_2 B_{0,1,1,1,1,0,1,0,0} + \epsilon^2 c_3 B_{1,0,1,1,1,1,0,0,0} \\ & + \epsilon^2 c_4 B_{1,0,1,1,1,1,1,0,0} + \epsilon^2 c_5 B_{1,1,0,1,1,0,1,0,0} \\ & + \epsilon^2 c_6 B_{1,1,0,1,1,1,0,0,0} + \epsilon^2 c_7 B_{1,1,1,1,1,1,0,0,0} \\ & + c_8 F_1^{\text{NP}} + c_9 F_2^{\text{NP}} + c_{10} F_3^{\text{NP}} \\ & + c_{11} F_4^{\text{NP}} + c_{12} F_5^{\text{NP}} \}. \end{aligned} \quad (4.16)$$

We give explicit expressions for the coefficients c_i in Appendix B.

V. CONCLUSIONS

In this article, we have studied a new type of basis for Feynman integrals, in which singularities in ϵ are absent in the coefficients that emerge from integral reductions. All

divergences then arise solely from the integrals themselves. In Sec. III, we presented two independent algorithms for constructing such bases. We gave examples of their use in the systems of planar and nonplanar double boxes and daughter integrals in Sec. IV. In the case of the planar double box, a singularity-free basis is compatible with choosing some basis integrals to be finite integrals, as given, for example, in Ref. [50]. The basis we find is similar to the modified basis already found to simplify the expression of the leading-color terms in the two-loop all-plus Yang–Mills amplitude [50]. In the case of the nonplanar double box, we cannot choose any basis integral to be a finite integral while maintaining a fully singularity-free basis. Nonetheless, the use of a singularity-free basis does simplify the expression for the subleading-color contributions to the two-loop all-plus amplitude. Trading some of the basis integrals for finite integrals simplifies the expression for the amplitude further. It will be interesting to extend these investigations to the other helicity amplitudes. The results described here suggest that singularity-free bases can play a role in simplifying expressions for scattering amplitudes, and that it would be desirable to find a general criterion to balance judiciously between the singularity-free criterion and the inclusion of finite Feynman integrals in a basis of Feynman integrals.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [52].

APPENDIX A: CORRECTNESS OF LOCAL-RING ALGORITHM

Claim. The Gaussian elimination algorithm given in Sec. III B, for reduction over a local ring generates the maximal number of pivots, the same as the rank of the original matrix of IBP identities. The output matrix C is in row echelon form free of poles in ϵ .

Proof. In each iteration of the algorithm, before step 2, for $p \leq i \leq M$, the i th row must be nonzero, because the matrix has maximal row rank (recall that we require the IBP relations corresponding to the original matrix to be linearly independent); and a row elementary transformation does not change the rank of the matrix. Then after step 2, the i th row $p \leq i \leq M$ must contain at least one nonzero element which is not proportional to ϵ . Therefore step 3 is guaranteed to find a pivot not proportional to ϵ . Accordingly the algorithm finds a pivot in each iteration. The total number of pivots would be M , which is also the rank of the matrix. Therefore the algorithm generates the maximal number of pivots.

In each iteration, after pivoting, the algorithm normalizes the diagonal element to one. All the lower triangular terms are eliminated. Because the algorithm produces the maximal number of pivots, the resulting matrix is in row echelon form.

Because the pivot is not proportional to a power of ϵ , the normalization in step 5 does not create a pole in ϵ . The row elementary transformations do not generate new poles in ϵ . Because the input matrix has no pole in ϵ , the output matrix is then free of ϵ poles. ■

APPENDIX B: INTEGRAL COEFFICIENTS FOR THE SUBLEADING-COLOR CONTRIBUTIONS TO THE TWO-LOOP ALL-PLUS FOUR-POINT AMPLITUDE

Here we list the expansions of the rational coefficients in Eq. (4.16) up to the required order in the ϵ expansion,

$$\begin{aligned}
 c_1 &= (D_s - 2) \left[\frac{2t(s+t)}{3s} - \frac{2(3s^4 - 11s^3t - 17s^2t^2 - 12st^3 - 6t^4)}{9s^3} \epsilon \right] + \mathcal{O}(\epsilon^2), \\
 c_2 &= (D_s - 2) \left[-\frac{4s^5 + 16s^4t + 27s^3t^2 + 18s^2t^3 + 7st^4 + 2t^5}{3s^2t(s+t)} \right] + \mathcal{O}(\epsilon), \\
 c_3 &= (D_s - 2) \left[\frac{2(11s^2 - 2t^2)}{3s} \right] + \mathcal{O}(\epsilon), \\
 c_4 &= (D_s - 2) \left[-\frac{1}{3}s^2 + \frac{(6s^4 - 10s^3t - s^2t^2 + 18st^3 + 9t^4)}{18t(s+t)} \epsilon \right. \\
 &\quad \left. + \frac{255s^4 + 203s^3t + 680s^2t^2 + 954st^3 + 477t^4}{108t(s+t)} \epsilon^2 \right] - 12s^2\epsilon^2 + \mathcal{O}(\epsilon^3), \\
 c_5 &= (D_s - 2) \left[\frac{4t^2}{3s} + \frac{2t(6s^4 + 22s^3t + 31s^2t^2 + 18st^3 + 9t^4)}{9s^3(s+t)} \epsilon \right. \\
 &\quad \left. + \frac{t(255s^4 + 619s^3t + 1096s^2t^2 + 954st^3 + 477t^4)}{27s^3(s+t)} \epsilon^2 \right] - \frac{48t^2}{s}\epsilon^2 + \mathcal{O}(\epsilon^3),
 \end{aligned}$$

$$\begin{aligned}
c_6 &= (D_s - 2) \frac{6s^5 + 22s^4t + 29s^3t^2 + 10s^2t^3 + 3st^4 + 2t^5}{3s^2t(s+t)} + \mathcal{O}(\epsilon), \\
c_7 &= (D_s - 2) \left[\frac{4}{9}(s+t)^2 + \frac{2(s+t)(6s^4 + 22s^3t + 31s^2t^2 + 18st^3 + 9t^4)}{27s^2t} \epsilon \right. \\
&\quad \left. + \frac{(s+t)(255s^4 + 619s^3t + 1096s^2t^2 + 954st^3 + 477t^4)}{81s^2t} \epsilon^2 \right] \\
&\quad - 16(s+t)^2 \epsilon^2 + \mathcal{O}(\epsilon^3), \\
c_8 &= -(D_s - 2) \frac{128(2s^4 - 3s^2t^2 + 2st^3 + t^4)}{3s^4t^2(s+t)^2} + \mathcal{O}(\epsilon), \\
c_9 &= (D_s - 2) \frac{512(s^2 + 3st + 3t^2)}{3st^3(s+t)^2} + \mathcal{O}(\epsilon), \\
c_{10} &= (D_s - 2) \frac{16(2s-t)(3s+t)(s+2t)}{3s^3t(s+t)} + \mathcal{O}(\epsilon), \\
c_{11} &= -(D_s - 2) \frac{64(4s^2 - st - t^2)}{3s^3(s+t)} + \mathcal{O}(\epsilon), \\
c_{12} &= (D_s - 2) \frac{32t(s+t)}{3s^4} + \mathcal{O}(\epsilon). \tag{B1}
\end{aligned}$$

The overall factor $(D_s - 2)$ echos the origin of each term in Eq. (4.11). The coefficients $c_{4,5,7}$ have $\mathcal{O}(\epsilon^4)$ contributions not proportional to $(D_s - 2)$, which cancel in the amplitude through $\mathcal{O}(\epsilon^0)$.

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