

# Functional renormalization group meets computational fluid dynamics: RG flows in a multidimensional field space

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Within the functional renormalization group framework, we present a fluid-dynamical approach to solving flow equations for models living in a multidimensional field space. To this end, the underlying exact flow equation of the effective potential is reformulated as a set of nonlinear advection-diffusion-type equations which can be solved using the Kurganov-Tadmor central scheme, a modern finite-volume discretization from computational fluid dynamics. We demonstrate the effectiveness of our approach by performing explicit benchmark tests using  $(0+0)$ -dimensional models with two discretized field-space directions or two symmetry invariants. Our techniques can be directly applied to flow equations of effective potentials of general (fermion-)boson systems with multiple invariants or condensates, as we also demonstrate for two concrete examples in three spacetime dimensions.

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## I. INTRODUCTION

Within the last decades, lattice Monte Carlo simulations turned out to be one of the most successful methods to study nonperturbative problems in quantum field theory (QFT), such as quantum chromodynamics (QCD). However, Monte Carlo simulations also come with limitations. Especially in the context of critical phenomena, the study of phase transitions and calculations at moderate and high densities, other nonperturbative approaches are required to complement lattice Monte Carlo simulations. One of these methods is the functional renormalization group (FRG) approach which is based on a flow equation for the quantum effective action, the Wetterich equation [1]. The formal derivation of this equation immediately triggered further field-theoretic developments [2–4]. By now, this approach has been successfully applied to a wide range of problems in high-energy, condensed matter, statistical, and gravitational physics [5–11]. One of the main advantages of the FRG is that it provides direct access to the effective action, which is

directly linked to the vertex functions and thermodynamic observables of a system.

A particularly important quantity is the effective potential, which represents the lowest-order contribution of an expansion of the effective action in derivatives of the fields and comprises all orders of local (pointlike) interactions of the (effective) degrees of freedom of the system. Usually, within FRG calculations, the effective potential is calculated by taking advantage of the symmetries of the system in field space. Hence, it oftentimes suffices to solve the renormalization group (RG) flow equations for the effective potential as a function of a single field variable or field invariant. However, in many cases the effective potential exhibits a lower degree of symmetry and the RG flow equations have to be solved in a multidimensional field space. Typical situations arise in systems with multiple order parameters, condensates, and/or field invariants. Selected relevant examples from high-energy physics that have already been addressed with the FRG are mesonic models with chiral and diquark condensates [12–15], models that comprise strange quarks and their condensation in addition to light quarks [16–19], models that allow for pion and  $\sigma$  condensation [20], and many more. Also in the context of condensed-matter physics, the effective potentials of many systems cannot be reduced to a single field-space direction. Some examples from FRG studies are systems that exhibit inverted phase diagrams and exhibit the Pomeranchuk effect [21] or (frustrated) magnetic

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systems [22–26,26–30]. Other examples are models for  $(2+1)$ -dimensional Dirac materials such as graphene [31,32]. All of them can be approximately described via  $O(N) \times O(M)$ -symmetric models or, in general, models with two invariants. However, it should also be mentioned that probably the first FRG calculations with several invariants originated from the field of statistical physics; in particular, studies of matrix models should be mentioned here [33], see also Refs. [34–36] for a more recent analysis.

### A. Motivation and contextualization

Given these many examples, the reader might object that the problem of solving FRG flow equations in multidimensional field space is already solved and it seems unnecessary to address this problem in a separate work again. Indeed, several methods have been developed to study the RG flow equations in multidimensional field space: The most common approach is to use a multidimensional Taylor expansion of the effective potential around a fixed or moving point in field space, which is usually chosen to be the global infrared (IR) or flowing minimum of the effective potential, respectively. Here, one effectively reduces the problem to a set of coupled ordinary differential equations (ODEs) for the Taylor coefficients. Another ansatz is to use a grid-based method where the field space is discretized and the RG flow equations are solved on this grid. Often, “naive” multidimensional finite-difference schemes have been used to solve the partial differential equation (PDE) for the effective potential in the past. However, it should also be mentioned that other approaches have been developed, such as a discretization of field space via spectral methods in terms of Chebyshev polynomials or by approximating the effective potential in terms of splines or other basis functions or a matching of Taylor expansions; see the references above for concrete examples.

Mostly, these approaches are suited to address the respective problem at hand. Especially in the context of fixed-point searches and analyses or studies of second-order phase transitions, some of these techniques are very powerful. Though, in the context of first-order phase transitions and multiple competing minima in the effective potential, most of the methods are no longer unconditionally stable. For example, Taylor expansions certainly fail, if the correct expansion point jumps from one minimum to another during the flow or if the radius of convergence is limited by nonanalyticities in the effective potential. The latter can easily be caused by chemical potentials or other external fields. Also, grid-based methods can fail in such situations, if the discretization scheme is, for example, fundamentally based on the assumption of analyticity and smoothness of the effective potential. However, since all of these challenging situations are of high physical relevance, we decided to address the construction of a powerful and

stable multipurpose discretization scheme for FRG flow equations in multidimensional field space again.

In this spirit, this paper addresses researchers who are interested in the challenges and the numerical techniques that are required to solve FRG flow equations of bosonic potentials with more than one background field. Furthermore, the manuscript addresses readers who are interested in the general connection between FRG flow equations and fluid dynamics. For a more detailed introduction to FRG via  $(0+0)$ -dimensional QFT and recent developments on the connections between FRG and numerical fluid dynamics we refer the interested reader to Refs. [37–39] and some directly related works [40–51].

### B. Research objective

Within this work, we propose to use modern numerical fluid-dynamic techniques to solve FRG flow equations in multidimensional field space. The idea of looking at flow equations from a fluid-dynamic point was proposed in Ref. [40] (based on earlier observations by Refs. [52–55]) and further worked out in Refs. [37–51]. However, so far, the literature mostly focuses on solving FRG problems that can be reduced to a single field-space direction. Here, we extend this approach to the case of multidimensional field space. To this end, the present work might be considered an extension of the ideas and methods presented in the aforementioned references. In particular, we shall demonstrate how to apply a modern finite-volume (FV) discretization scheme—the Kurganov-Tadmor (KT) (central) scheme—to FRG flow equations of effective potentials living in a multidimensional field space. We shall show that this method is stable and accurate by performing multiple benchmark tests. As a testing ground, we shall use  $(0+0)$ -dimensional models with two field-space directions, with and without  $O(2)$  symmetry, as well as  $O(N) \times O(M)$ -symmetric models. Within this and preceding works, we use the term “zero-dimensional” when referring to  $0+0$  spacetime dimensions—no spatial and no temporal dimensions. Note that studies in zero spacetime dimensions come with the advantage that RG time stepping toward the IR is not a relevant problem, see also Ref. [47], such that we can focus on the spatial discretization. Additionally, we can directly compare our results to exact reference values which can be computed exactly from the corresponding partition function. Apart from these formal tests, we shall also discuss general aspects of RG flow equations, such as possible restrictions on initial conditions or the general structure of flow equations in multidimensional field space and their reformulation in terms of a fluid-dynamic diffusion problem.

Ultimately, we shall also demonstrate that our method is indeed applicable to the study of realistic models in nonzero spacetime dimensions. Because the main use case of our method is to consider the actual RG flow and time

stepping from the ultraviolet (UV) to the IR, we consider the following test cases: As a first test model, we use an  $O(2)$ -symmetric model in  $2 + 1$  Euclidean dimensions in the local potential approximation (LPA) in the symmetry-broken phase where the initial condition, i.e., the UV potential, is chosen such that the system comprises competing minima and nonanalyticities. As a second test model, we consider an  $O(N) \times O(M)$ -symmetric model in  $2 + 1$  Euclidean dimensions in LPA. In this case, the UV initial conditions are also chosen such that the system describes a phase with a broken  $O(N) \times O(M)$  symmetry in the ground state in the IR limit.

To summarize, with our present work, we aim to provide a powerful numerical toolbox to solve FRG flow equations for effective potentials that are multidimensional in field space. To be more concrete, our goal is to provide a “black-box solver” for the RG time evolution that is based on robust and stable numerical techniques from computational fluid dynamics (CFD) and tested against exact reference values and can therefore be included in the toolbox of FRG practitioners. Furthermore, we aim to convince the reader through example calculations that our method is directly applicable to realistic models. Last but not least, the here discussed zero-dimensional models may also be of interest to researchers who are developing new methods to solve FRG flow equations in multidimensional field space and want to benchmark their schemes.

Our present work is organized as follows: In Sec. II, we introduce the zero-dimensional QFT of scalar fields, which serves as a testing ground for our numerical method. We provide a brief overview of the key aspects of QFT in zero spacetime dimensions and introduce the effective action and vertex functions. In Sec. III, we present the zero-dimensional version of the Wetterich equation for scalar fields and demonstrate how this equation can be turned into a fluid-dynamic problem in a two-dimensional field space (or the space spanned by suitably chosen field invariants). We then show that our result can be consistently reduced to a flow equation with a single field-space direction, if  $O(2)$  symmetry is assumed. After that, we shall also discuss some additional restrictions on initial conditions for RG flows in two-dimensional field space. Finally, we present the generalization to the case of  $O(N) \times O(M)$ -symmetric models, which can be reduced to a flow equation on a two-dimensional domain spanned by the corresponding field invariants. In Sec. IV, we introduce the KT scheme, a modern FV discretization scheme from CFD that we use to solve the FRG flow equations of the various models considered in our present work. We provide a brief overview of the KT scheme and discuss its adaption to FRG problems.

These more theoretical sections are followed by considerations of appropriate test setups, see Sec. V. After that, we discuss various toy models and the challenges they pose to our numerical approach and numerical methods in general.

To be specific, in Sec. VI, we introduce our  $O(2)$ -symmetric test models, whereas Sec. VII is dedicated to nonsymmetric models. A model with a  $O(N) \times O(M)$  symmetry is then considered in Sec. VIII. Actual numerical results from our test models are presented in Sec. IX, where we also discuss the performance of the KT scheme in the context of FRG flow equations. Before we finally conclude in Sec. XI, we shall discuss selected models in three spacetime dimensions in Sec. X, namely, an  $O(2)$  and an  $O(N) \times O(M)$ -symmetric model in their symmetry-broken phases.

## II. A ZERO-DIMENSIONAL QFT OF SCALAR FIELDS

Since QFT in zero spacetime dimensions provides the testing ground for our numerical method, we give a brief overview of its key aspects in this section. To be more specific, we primarily introduce the fundamental definitions of a zero-dimensional model of  $N$  real scalar (interacting) fields. Additionally, we provide some formulas for correlation and vertex functions that can be utilized to compute high-precision reference values through straightforward numerical integration. For a more comprehensive discussion of QFT in zero spacetime dimensions, we refer the reader to Refs. [37,56–59] and references therein.

### A. Correlation functions

A zero-dimensional QFT model is defined by an action  $\mathcal{S}(\vec{\phi}) = \mathcal{S}(\phi_1, \dots, \phi_N)$  of the fields  $\vec{\phi} = (\phi_1, \dots, \phi_N)^T$ . The fields are plain real numbers without spacetime dependence of any kind as there is no spacetime. The entire field theory can be considered in terms of interacting quantum fields in a single point. Hence, there is also no notion of energies, momenta, etc., and the generating functional of correlation functions reduces to an ordinary function that is defined in terms of an  $N$ -dimensional ordinary integral,

$$\mathcal{Z}(\vec{J}) = \mathcal{N} \int_{-\infty}^{\infty} d^N \phi e^{-\mathcal{S}(\vec{\phi}) + \vec{J}^T \cdot \vec{\phi}}, \quad (1)$$

where  $\vec{J}$  is the vector of source fields and  $\mathcal{N}$  is the normalization of the probability distribution.

In complete analogy to higher-dimensional QFTs and statistical physics, correlation functions can be derived from Eq. (1) by taking derivatives with respect to the components of  $\vec{J}$ . Also, these correlation functions reduce to ordinary integrals, i.e.,

$$\begin{aligned} \langle \phi_{i_1} \cdots \phi_{i_n} \rangle &= \frac{1}{\mathcal{Z}(\vec{J})} \frac{\partial^n \mathcal{Z}(\vec{J})}{\partial J_{i_1} \cdots \partial J_{i_n}} \Big|_{\vec{J}=0} \\ &= \frac{\int_{-\infty}^{\infty} d^N \phi \phi_{i_1} \cdots \phi_{i_n} e^{-\mathcal{S}(\vec{\phi})}}{\int_{-\infty}^{\infty} d^N \phi e^{-\mathcal{S}(\vec{\phi})}}. \end{aligned} \quad (2)$$

It only remains to choose some  $\mathcal{S}(\vec{\phi})$  and we have a full-fledged theory. Here, we are basically free to choose any function of the fields that is

- (1) bounded from below and grows asymptotically at least quadratically in every field-space direction, and
- (2) continuous.

The first restriction ensures convergence of the integrals in Eqs. (1) and (2). The role of the second requirement is solely to maintain contact with higher-dimensional models where potentials are continuous functions of the fields. In zero spacetime dimensions, the classical UV action is identical to a UV potential. Apart from this, there is no need to restrict  $\mathcal{S}$  to smooth or even analytic functions.

### B. The effective action and vertex functions

Of course, we are not interested in calculating the  $N$ -dimensional integrals (2) numerically, but in discussing and testing novel fluid-dynamic methods for FRG flow equations.

The FRG approach is formulated on the level of the effective action [1–7,60]. Therefore, we introduce this quantity in terms of its definition as the Legendre transform of the logarithm of Eq. (1),

$$\Gamma(\vec{\varphi}) \equiv \sup_{\vec{J}} \{ \vec{J}^T \cdot \vec{\varphi} - \ln \mathcal{Z}(\vec{J}) \}, \quad (3)$$

where  $\vec{\varphi}$  are the mean fields. The corresponding observables are the vertex functions, which can be directly calculated from the effective action by taking derivatives with respect to the components of  $\vec{\varphi}$  at the minimum of  $\Gamma(\vec{\varphi})$ ,

$$\Gamma_{\varphi_{i_1} \dots \varphi_{i_n}}^{(n)} = \left. \frac{\partial^n \Gamma(\vec{\varphi})}{\partial \varphi_{i_1} \dots \partial \varphi_{i_n}} \right|_{\vec{\varphi} = \vec{\varphi}_{\min}}, \quad (4)$$

once  $\Gamma(\vec{\varphi})$  is known. Here,  $\vec{\varphi}_{\min}$  denotes its minimum.

In zero spacetime dimensions it is of course possible to calculate  $\mathcal{Z}(\vec{J})$  for a given  $\mathcal{S}(\vec{\phi})$  and perform the Legendre transformation (numerically) to extract the vertex functions. Alternatively, it is straightforward to derive direct relations between (specific) correlation functions (2) and vertex functions (4). For example, for the two-point vertex function, which is the only relevant quantity in this work, one finds [37,57,59]

$$(\Gamma_{\vec{\varphi}\vec{\varphi}}^{(2)})_{ji}^{-1} = \langle \phi_j \phi_i \rangle - \langle \phi_j \rangle \langle \phi_i \rangle. \quad (5)$$

This can be evaluated (numerically) using Eq. (2).

In higher-dimensional QFTs, this is no longer trivial and  $\Gamma(\vec{\varphi})$  and ultimately the vertex functions are often calculated with some other method, e.g., with the FRG approach. In this work, we are doing exactly this: We use the FRG to obtain  $\Gamma(\vec{\varphi})$  including the vertex functions, especially Eq. (5). This is still a challenging task in zero spacetime

dimensions because we have to solve a highly nonlinear PDE; though, in zero spacetime dimensions, we have easily accessible reference values from exact (numeric) computations of the correlation functions (2). Therefore, we can benchmark the (numeric) solution strategy for the PDE in the FRG formalism via Eq. (5). This is rarely possible in higher-dimensional QFTs.<sup>1</sup>

## III. THE ZERO-DIMENSIONAL FRG AND FLUID DYNAMICS

In this section, we briefly present the zero-dimensional version of the FRG and the corresponding zero-dimensional formulation of the Wetterich equation for our QFT of  $N$  real scalar fields. Afterward, we demonstrate how this equation is turned into a fluid-dynamic problem on a two-dimensional spatial domain. We then show that the resulting equation can be consistently reduced to a flow equation in a single dimension, if  $O(2)$  symmetry is assumed. Afterward, we discuss some additional restrictions on initial conditions for RG flows in two-dimensional field space. Before we close the section, we also present the generalization of our considerations to the case of  $O(\bar{N}) \times O(\bar{M})$ -symmetric models (with  $\bar{N} + \bar{M} = N$ ), which can be reduced to a flow equation on a two-dimensional spatial domain spanned by the field invariants.

For comprehensive general introductions to the FRG approach, we refer to Refs. [5–11,65,66]. Detailed introductions to and applications of zero-dimensional systems within the FRG framework can be found in Refs. [37–39,43,44,57,59,67–70].

### A. The zero-dimensional Wetterich equation for scalar fields

The zero-dimensional version of the Wetterich equation [1] for  $N$  real scalar fields simply reads

$$\partial_t \bar{\Gamma}(t, \vec{\varphi}) = \text{tr} \left[ \left( \frac{1}{2} \partial_t R(t) \right) (\bar{\Gamma}^{(2)}(t, \vec{\varphi}) + R(t))^{-1} \right]. \quad (6)$$

It is an exact evolution equation for the effective average action  $\bar{\Gamma}(t, \vec{\varphi})$  in field space, which is spanned by  $\vec{\varphi} \in \mathbb{R}^N$ , and the RG time  $t \in [0, \infty)$ . We define the RG time  $t$  to be manifestly positive. It runs from  $t = 0$ , which we refer to as the UV, to  $t \rightarrow \infty$ , which is the zero-dimensional analog of the IR limit. In the IR limit, the effective average action  $\bar{\Gamma}(t, \vec{\varphi})$  approaches the quantum effective action  $\Gamma(\vec{\varphi})$  that we are interested in, whereas the UV initial condition is given by the classical action  $\mathcal{S}(\vec{\varphi})$  evaluated on the mean fields  $\vec{\varphi}$ , i.e.,

$$\bar{\Gamma}(t = 0, \vec{\varphi}) = \mathcal{S}(\vec{\varphi}), \quad \lim_{t \rightarrow \infty} \bar{\Gamma}(t, \vec{\varphi}) = \Gamma(\vec{\varphi}). \quad (7)$$

<sup>1</sup>For example, exceptions are QFTs in certain limits, such as the infinite- $N$  limit, where theories sometimes become integrable, cf. Refs. [40,61–64].



The matrix valued function  $R(t)$  is the regulator, which we choose<sup>2</sup> to be diagonal in field space with an exponentially monotonically decreasing regulator-shape function  $r(t)$ ,

$$R(t) = \mathbb{1}_{N \times N} r(t), \quad r(t) = \Lambda e^{-t}. \quad (8)$$

Here,  $\Lambda$  is the zero-dimensional version of the UV cutoff, which needs to be much greater than the typical “scales” in the classical action  $\mathcal{S}(\vec{\varphi})$ .<sup>3</sup>

Typically, in higher-dimensional FRG applications it is not possible to solve Eq. (6) exactly and approximations are necessary. In zero dimensions, there is no need for truncations and one finds that the only possible “ansatz” for the effective average action is an RG-time-dependent potential,<sup>4</sup>

$$\bar{\Gamma}(t, \vec{\varphi}) = U(t, \vec{\varphi}). \quad (9)$$

Still, Eq. (6) constitutes a highly nonlinear PDE for  $U(t, \vec{\varphi})$ , whose spatial domain is the  $N$ -dimensional field space and whose temporal domain is the RG time. It is therefore not feasible to solve this equation for some large  $N$  without imposing additional constraints.

Luckily, we can often employ symmetries of the classical action  $\mathcal{S}(\vec{\varphi})$  in field space, which transfer to  $U(t, \vec{\varphi})$  by construction, to reduce the computational domain to a subspace of  $\mathbb{R}^N$ . For example, for an  $O(N)$ -symmetric model, the minimally required subspace is  $\mathbb{R}_{(\geq 0)}$  and one dimensional. This is achieved by choosing the  $O(N)$ -invariant  $\varrho = \frac{1}{2}\vec{\varphi}^2$  or functions thereof as the spatial domain. Similarly, for  $O(\bar{N}) \times O(\bar{M})$ -symmetric models, with  $\bar{N} + \bar{M} = N$ , the  $N$ -dimensional computational domain  $\mathbb{R}^N$  can be reduced to  $\mathbb{R}_{(\geq 0)} \times \mathbb{R}_{(\geq 0)}$ , that is spanned by the two invariants  $\varrho_1 = \frac{1}{2}\vec{\varphi}_1^2$  and  $\varrho_2 = \frac{1}{2}\vec{\varphi}_2^2$ , where  $\vec{\varphi}_1 = (\varphi_1, \dots, \varphi_{\bar{N}})^T$  and  $\vec{\varphi}_2 = (\varphi_{\bar{N}+1}, \dots, \varphi_N)^T$ .

Oftentimes, however, for more complicated problems or symmetry groups, one remains with a multidimensional domain, cf. Sec. I, such that a PDE for  $U(t, \vec{\varphi})$  is still a flow equation in more than one/two field-space dimensions. Thus, it is also useful to artificially simulate this situation by simply considering a model of two interacting real scalar fields with—*a priori*—no additional symmetry assumptions. The above mentioned cases are certainly the minimal setup to test FRG flow equations in more than one field-space dimension. This is the main topic of this work.

<sup>2</sup>If there are no symmetries in the system, we could, in principle, choose a regulator that is nondiagonal.

<sup>3</sup>For a detailed discussion of suitable choices of the UV cutoff in the context of zero-dimensional QFT, we refer to Refs. [37,71] and references therein for higher-dimensional systems.

<sup>4</sup>Of course, it is still possible to artificially impose truncations in zero dimensions, cf. Refs. [57,59,67,72]. For example, in Ref. [37], the convergence of the FRG vertex/Taylor expansion was tested in zero spacetime dimensions.

## B. Flow equation of a zero-dimensional model with two fields

We now derive the flow equation for the effective potential of a zero-dimensional interacting QFT of two ( $N = 2$ ) real scalar fields from the Wetterich equation (6). Recall that the effective potential is identical to the effective action in zero spacetime dimensions. In a subsequent step, we then show how this flow equation can be recast as a fluid-dynamic problem in terms of two nonlinear diffusion-type equations.

### 1. The flow equation of the effective potential

First, we set  $N = 2$  and therefore consider a two-dimensional Euclidean field space with coordinates  $\varphi_1, \varphi_2 \in \mathbb{R}$  and  $\vec{\varphi} = (\varphi_1, \varphi_2)^T$ . For the sake of readability, we use the following abbreviations:

$$U = U(t, \vec{\varphi}), \quad r = r(t). \quad (10)$$

In general, we only have to insert our ansatz (9) in the Wetterich equation (6) and evaluate the trace on the rhs. For the sake of clarity, we proceed step by step and start with the calculation of the full two-point function,

$$\bar{\Gamma}^{(2)}(t, \vec{\varphi}) + R(t) = \begin{pmatrix} r + \partial_{\varphi_1}^2 U & \partial_{\varphi_1} \partial_{\varphi_2} U \\ \partial_{\varphi_2} \partial_{\varphi_1} U & r + \partial_{\varphi_2}^2 U \end{pmatrix}, \quad (11)$$

which is simply a two-dimensional matrix in field space. This matrix needs to be inverted to obtain the full propagator. The inverse is

$$(\bar{\Gamma}^{(2)}(t, \vec{\varphi}) + R(t))^{-1} = \frac{\text{adj}(\bar{\Gamma}^{(2)}(t, \vec{\varphi}) + R(t))}{\det(\bar{\Gamma}^{(2)}(t, \vec{\varphi}) + R(t))}, \quad (12)$$

where  $\text{adj}$  denotes the adjugate matrix,

$$\text{adj}(\bar{\Gamma}^{(2)}(t, \vec{\varphi}) + R(t)) = \begin{pmatrix} r + \partial_{\varphi_2}^2 U & -\partial_{\varphi_1} \partial_{\varphi_2} U \\ -\partial_{\varphi_2} \partial_{\varphi_1} U & r + \partial_{\varphi_1}^2 U \end{pmatrix} \quad (13)$$

and the determinant is

$$\det(\bar{\Gamma}^{(2)}(t, \vec{\varphi}) + R(t)) = (r + \partial_{\varphi_1}^2 U)(r + \partial_{\varphi_2}^2 U) - (\partial_{\varphi_1} \partial_{\varphi_2} U)(\partial_{\varphi_2} \partial_{\varphi_1} U). \quad (14)$$

It should be noted that the inversion is only possible and the Wetterich equation is only well defined if this determinant is nonzero for all RG times  $t$  and all points in field space. We return to this delicate issue below in Sec. III E. In addition, we emphasize that we do not necessarily have  $\partial_{\varphi_i} \partial_{\varphi_j} U = \partial_{\varphi_j} \partial_{\varphi_i} U$  since  $U$  does not need to be analytic.

For the moment, we combine our intermediate results and arrive at the flow equation for the effective potential,

$$\partial_t U = \frac{(\frac{1}{2}\partial_t r)(2r + \partial_{\varphi_1}^2 U + \partial_{\varphi_2}^2 U)}{(r + \partial_{\varphi_1}^2 U)(r + \partial_{\varphi_2}^2 U) - (\partial_{\varphi_1}\partial_{\varphi_2} U)(\partial_{\varphi_2}\partial_{\varphi_1} U)}. \quad (15)$$

This is a nonlinear PDE, up to second order in spatial (field) derivatives and first order in (RG) time, on the two-dimensional noncompact domain  $\mathbb{R}^2$ . Structurally, flow equations for effective bosonic potentials in higher space-time dimensions for fermion-boson systems with, for example, two possible condensate directions are usually rather similar; see our discussion in Sec. I.

Note that only second derivatives of  $U$  with respect to the fields appear on the rhs of Eq. (15). In addition, the actual value of a potential has no relevance and physical quantities correspond to relative differences or derivatives of potentials. These observations already suggest to study the field-space derivatives of  $U$  instead of  $U$  itself.

## 2. A fluid-dynamic reformulation

It was shown in Refs. [40,54,55] that taking a field-space derivative of the flow equation of an effective potential recasts the corresponding PDE into its conservative formulation. This reformulation and its consequences were further worked out in Refs. [37–39,41–44,47]. Here, we shall only demonstrate how this reformulation is done for the RG flow equation at hand.

Taking spatial derivatives with respect to each spatial coordinate of Eq. (15), without executing these derivatives on the rhs, we obtain,

$$\begin{aligned} \partial_t(\partial_{\varphi_i} U) \\ = \frac{d}{d\varphi_i} \left[ \frac{(\frac{1}{2}\partial_t r)(2r + \partial_{\varphi_1}^2 U + \partial_{\varphi_2}^2 U)}{(r + \partial_{\varphi_1}^2 U)(r + \partial_{\varphi_2}^2 U) - (\partial_{\varphi_1}\partial_{\varphi_2} U)(\partial_{\varphi_2}\partial_{\varphi_1} U)} \right]. \end{aligned} \quad (16)$$

Next, we rename the two orthogonal field-space derivatives of  $U$  as follows:

$$u = \partial_{\varphi_1} U, \quad v = \partial_{\varphi_2} U, \quad (17)$$

where we again use shorthand notations  $u = u(t, \vec{\varphi})$  and  $v = v(t, \vec{\varphi})$ . Expressing Eq. (16) in terms of these new variables, we find a set of two coupled PDEs, i.e.,

$$\partial_t u = \frac{d}{d\varphi_1} \left[ \frac{(\frac{1}{2}\partial_t r)(2r + \partial_{\varphi_1} u + \partial_{\varphi_2} v)}{(r + \partial_{\varphi_1} u)(r + \partial_{\varphi_2} v) - (\partial_{\varphi_1} v)(\partial_{\varphi_2} u)} \right], \quad (18)$$

$$\partial_t v = \frac{d}{d\varphi_2} \left[ \frac{(\frac{1}{2}\partial_t r)(2r + \partial_{\varphi_1} u + \partial_{\varphi_2} v)}{(r + \partial_{\varphi_1} u)(r + \partial_{\varphi_2} v) - (\partial_{\varphi_1} v)(\partial_{\varphi_2} u)} \right]. \quad (19)$$

This system can be rearranged in vector notation and we arrive at a “conservation law”

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \partial_{\varphi_1} \begin{pmatrix} Q \\ 0 \end{pmatrix} + \partial_{\varphi_2} \begin{pmatrix} 0 \\ Q \end{pmatrix}. \quad (20)$$

In compact notation, this equation reads

$$\partial_t \vec{u}^T = \vec{\nabla}^T \cdot \mathbf{Q}, \quad \mathbf{Q} \equiv \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}, \quad (21)$$

where  $\vec{u} \equiv (u, v)^T$ ,  $\vec{\nabla} \equiv (\partial_{\varphi_1}, \partial_{\varphi_2})^T$ , and the nonlinear diffusion flux is defined as

$$Q = \frac{(\frac{1}{2}\partial_t r)(2r + \partial_{\varphi_1} u + \partial_{\varphi_2} v)}{(r + \partial_{\varphi_1} u)(r + \partial_{\varphi_2} v) - (\partial_{\varphi_1} v)(\partial_{\varphi_2} u)}. \quad (22)$$

Some comments are in order:

- (1) In this conservative form, we can identify  $u$  and  $v$  with two fluid fields, which are evolving in the two-dimensional field space with RG time.
- (2) The denomination of the equations as nonlinear diffusion-type equations with a diffusion flux becomes clear when the derivatives on the rhs of Eqs. (18) and (19). One then observes terms that are proportional to second derivatives of  $u$  and  $v$ . Interpreting their coefficients as highly nonlinear diffusion coefficients, we are confronted with a quasi parabolic problem—a nonlinear diffusion equation.
- (3) Structurally, Eq. (21) is a conservation law. Therefore, methods from the field of CFD are the appropriate choice to approach it numerically [73–76]. Because of the highly nonlinear character of the equation, nonanalytic behavior may emerge during the RG flow, such that expansion schemes or numerical methods that cannot cope with nonanalyticities are inappropriate. In Sec. IV, we briefly sketch a possible choice for a numeric FV scheme that can be used to solve equations of the type of Eq. (20).

## C. $O(2)$ symmetry in the Wetterich equation: The one-dimensional reduction as a consistency check

As already stated above, a multidimensional formulation of the field dependence is not required for models that exhibit an additional symmetry in field space. For example, if the action in the path integral of our field theory with  $N = 2$  is symmetric under  $O(2)$  transformations, the effective average action inherits this symmetry by construction. This implies that the RG-time-dependent potential is only a function of the  $O(2)$ -invariant  $\varrho = \frac{1}{2}\vec{\varphi}^2$  constructed from the background fields rather than a function of two independent fields  $\varphi_1$  and  $\varphi_2$ , i.e.,

$$U(t, \vec{\varphi}) = \tilde{U}(t, \varrho). \quad (23)$$

It follows that the field-space derivatives on the rhs of Eq. (15) should be rewritten in terms of  $\varrho$ . We simply use the chain rule and find

$$\partial_{\varphi_i} U(t, \vec{\varphi}) = \varphi_i \partial_{\varphi} \tilde{U}(t, \varrho), \quad (24)$$

and

$$\partial_{\varphi_i} \partial_{\varphi_j} U(t, \vec{\varphi}) = \delta_{ij} \partial_{\varphi} \tilde{U}(t, \varrho) + \varphi_i \varphi_j \partial_{\varphi}^2 \tilde{U}(t, \varrho), \quad (25)$$

where  $i, j \in \{1, 2\}$ . Inserting this explicitly in Eq. (15) and using Eq. (23) on the lhs, we obtain the RG flow equation of the potential expressed in terms of  $\varrho$ ,

$$\partial_t \tilde{U} = \frac{\frac{1}{2} \partial_t r}{r + \partial_{\varrho} \tilde{U}} + \frac{\frac{1}{2} \partial_t r}{r + \partial_{\varrho} \tilde{U} + 2\varrho \partial_{\varrho}^2 \tilde{U}}. \quad (26)$$

Readers who are familiar with common FRG literature on  $O(N)$  models in higher dimensions will immediately recognize the generic structure of this flow equation as a LPA, which is, however, not an approximation in zero dimensions but exact.

Thus, we have successfully demonstrated that the version of the flow equation in two field-space dimensions, where we do not use the symmetry to restrict the spatial domain of the PDE, correctly reduces to the one-dimensional PDE, where the symmetry is encoded in field space. Here, the spatial domain is parametrized by the coordinate  $\varrho \in \mathbb{R}_{\geq 0}$ .

In Ref. [37], it is discussed in great detail why another formulation of this PDE may be better suited for practical numerical implementations. We do not repeat this discussion here but only show how we can arrive at this formulation. To this end, we introduce the background field  $\sigma$ , which can be viewed without loss of generality as a field configuration  $\vec{\varphi} = (\sigma, 0)$ , such that

$$\varrho = \frac{1}{2} \sigma^2 \quad \Leftrightarrow \quad \sigma = \pm \sqrt{2\varrho}. \quad (27)$$

This coordinate transformation results in the following flow equation:

$$\partial_t U = \frac{\frac{1}{2} \partial_t r}{r + \frac{1}{\sigma} \partial_{\sigma} U} + \frac{\frac{1}{2} \partial_t r}{r + \partial_{\sigma}^2 U}, \quad (28)$$

which has a spatial domain spanned by  $\sigma \in \mathbb{R}$  and where  $U(t, \sigma) = U(t, -\sigma)$ .

### D. A one-dimensional advection-diffusion equation

It was also shown in Ref. [37] how Eq. (28) can be recast into conservative form and solved numerically. Again, defining  $u = \partial_{\sigma} U$  with  $u = u(t, \sigma) = -u(t, -\sigma)$ , we obtain a conservation law,

$$\partial_t u = \frac{d}{d\sigma} \left( \frac{\frac{1}{2} \partial_t r}{r + \frac{1}{\sigma} u} + \frac{\frac{1}{2} \partial_t r}{r + \partial_{\sigma} u} \right), \quad (29)$$

which presents as a highly nonlinear advection-diffusion equation. Again, some comments are in order:

- (1) It was explicitly demonstrated in Refs. [37,44] that this equation can be solved with modern schemes from CFD. To be specific, Ref. [44] uses discontinuous Galerkin methods, whereas Ref. [37] presents benchmark tests for a FV method—the KT central scheme—as also presented below.
- (2) It is remarkable that the formal description of a system of two highly nonlinear diffusion-type equations (18) and (19) in a two-dimensional spatial domain with initial conditions that include the  $O(2)$  symmetry in some way is equivalent to a highly nonlinear advection-diffusion equation (29), where the  $O(2)$  symmetry is imprinted in the spatial coordinate. In any case, since the mathematical concepts are clear, it only remains to compare performance of the two approaches on the numerical level, see below.

### E. Initial conditions and well posedness

Before we present the numeric implementation and study explicit examples, let us revisit the derivation of the flow equation in Sec. III B 1. Specifically, we analyze the invertibility of the full two-point function (11). If this matrix is not invertible for all RG times  $t \in [0, \infty)$  and positions in field space  $\vec{\varphi} \in \mathbb{R}^2$ , the Wetterich equation becomes ill defined. In particular, it has to be invertible in the UV for the initial potential.<sup>5</sup> Invertibility of this matrix is guaranteed when its determinant (14). Moreover, both eigenvalues and the determinant of Eq. (11) should always be positive. This is because the scale-dependent version of the Legendre transformation (3) is only well defined for a convex scale-dependent effective action

$$\Gamma(t, \vec{\varphi}) = \bar{\Gamma}(t, \vec{\varphi}) + \frac{1}{2} \vec{\varphi}^T R(t) \vec{\varphi}. \quad (30)$$

Positivity of the determinant also ensures a well-behaved diffusion flux (22). Loosely speaking, in QFT language, we should never “overshoot” the pole of the propagator in Eq. (15) at any RG time and any position in field space.

An obvious question is whether these constraints are always fulfilled for any initial potentials that are continuous, bounded from below, and come with at least a quadratic asymptotic behavior in both field directions. The answer to this question is not straightforward. In any case, this class of potentials leads to well-defined converging integrals and field expectation values when we directly operate on the level of the partition function.

<sup>5</sup>In general, this should be sufficient to also guarantee invertibility and well posedness for the entire RG flow, at least for untruncated flows.

Before we present an explicit counterexample, which violates the above constraints, let us also provide the eigenvalues of the full two-point function (11), assuming  $\partial_{\varphi_1} \partial_{\varphi_2} U = \partial_{\varphi_2} \partial_{\varphi_1} U$  for the moment. These are

$$\lambda_{1/2} = \frac{1}{2}(2r + \partial_{\varphi_1}^2 U + \partial_{\varphi_2}^2 U \pm \sqrt{4(\partial_{\varphi_1} \partial_{\varphi_2} U)^2 + (\partial_{\varphi_1}^2 U - \partial_{\varphi_2}^2 U)^2}). \quad (31)$$

Considering now the initial potential

$$U(\vec{\varphi}) = \frac{1}{2} \varphi_1 \varphi_2^2 + \frac{1}{4!} (\varphi_1^4 + \varphi_2^4), \quad (32)$$

it can be checked by numeric integration that this potential leads to well-defined correlation functions (2) and vertex functions, e.g., via Eq. (5). However, the determinant of the full UV two-point function is

$$\det(\bar{\Gamma}^{(2)}(0, \vec{\varphi}) + r(0)) = \left( \Lambda + \frac{1}{2} \varphi_1^2 \right) \left( \Lambda + \varphi_1 + \frac{1}{2} \varphi_2^2 \right) - \varphi_2^2, \quad (33)$$

and the corresponding UV eigenvalues of the two-point function are

$$\lambda_{1/2} = \frac{1}{2} \left( 2\Lambda + \varphi_1 + \frac{1}{2} (\varphi_1^2 + \varphi_2^2) \pm \sqrt{4\varphi_2^2 + \frac{1}{4} (\varphi_2^2 - \varphi_1(\varphi_1 - 2))^2} \right). \quad (34)$$

From this, we deduce that, for  $\varphi_2 = 0$ , the determinant and the eigenvalue  $\lambda_2$  become negative at  $\varphi_1 = -\Lambda$ . This implies that this eigenvalue as well as the determinant cannot be regularized by increasing  $\Lambda$ . The problematic region in field space is simply moved to smaller  $\varphi_1$ . Additionally, employing the one-loop effective action

$$\Gamma_\Lambda(\vec{\varphi}) = \mathcal{S}(\vec{\varphi}) + \frac{1}{2} \ln \det \mathcal{S}^{(2)}(\vec{\varphi})|_{\vec{\varphi}=\vec{\varphi}} \quad (35)$$

as the initial condition at the UV scale  $\Lambda$ , as is oftentimes done while working with a finite UV cutoff, will not solve the issue since the logarithm of the determinant suffers from these problems. The same applies to choosing another regulator function that is still quadratic in the fields, but, e.g., involves off-diagonal terms in Eq. (8). Field-dependent regulators, cf. Refs. [77–79], might solve the problem of the unregulated eigenvalue but generically lead to modifications of the Wetterich equation.

Hence, we come to the conclusion that there are UV actions/initial potentials which, in general, lead to an ill-posed initial value problem for the PDE (6) with standard mass-type regulators, while being well defined on the level

of the path integral (also in the presence of the regulator).<sup>6</sup> The same problem is also present in (truncated) FRG flow equations in higher dimensions.

We refrain from discussing this issue any further here and defer it to future work. In our present work, we focus on well-posed problems and only remark that one should always explicitly monitor the determinant/eigenvalues of the full two-point function at the beginning and during RG flows over the entire field space. In addition, we note that potentials with an  $O(2)$ - or at least an  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric asymptotic behavior do not feature the above problems and it is always possible to start with a sufficiently large  $\Lambda$  that ensures positivity of both eigenvalues for all RG times.<sup>7</sup> As can be seen from Eqs. (14) and (31), explicit symmetry-breaking terms linear in  $\varphi_1$  or  $\varphi_2$  do not cause any problems. We conjecture that the problematic initial conditions are directly linked to higher-order interaction terms that break the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry and are finite at scales  $k \neq \Lambda$ . We add that this problem might therefore be solved by suitable modifications of the initial conditions of the Wetterich equation in cases where the RG flows are initialized at finite UV scales.

## F. Generalization to $O(\bar{N}) \times O(\bar{M})$ -symmetric RG flow equations

Let us now also consider situations where the  $N$ -dimensional field space can be divided into two subspaces of dimension  $\bar{N}$  and  $\bar{M}$ , such that  $\bar{N} + \bar{M} = N$  and where the model is separately invariant under  $O(\bar{N})$  and  $O(\bar{M})$  transformations of the fields in the respective subspaces. Hence, we study a zero-dimensional model whose classical action, (scale-dependent) effective (average) action, and the (scale-dependent) potential are functions of the two corresponding invariants,

$$\rho_1 = \frac{1}{2} \vec{\phi}_1^2, \quad \rho_2 = \frac{1}{2} \vec{\phi}_2^2, \quad (36)$$

where the entire field-space vector is split into two field-space vectors  $\vec{\phi} = (\phi_1, \dots, \phi_N)^T = (\vec{\phi}_1, \vec{\phi}_2)^T$ ,

$$\vec{\phi}_1 = (\phi_1, \dots, \phi_{\bar{N}})^T, \quad \vec{\phi}_2 = \underbrace{(\phi_{\bar{N}+1}, \dots, \phi_N)^T}_{\bar{M}}. \quad (37)$$

From the perspective of the generating functional, the calculation of (connected) correlation functions and also vertex functions is still possible via simple numerical integration, see Eq. (2) and Sec. II. This allows us to

<sup>6</sup>It is straightforward to construct additional counterexamples.

<sup>7</sup>It is even possible to construct potentials that are defined piecewise with an outer symmetric region and an inner region without symmetry that do not violate the above conditions, see below.



compute exact reference values for our flow equation studies.

### 1. Flow equation of the effective potential

From the perspective of our RG approach, the most general ansatz for the Wetterich equation (6) is the scale-dependent effective potential

$$\tilde{\Gamma}(t, q_1, q_2) = \tilde{U}(t, q_1, q_2). \quad (38)$$

The initial condition is given by  $\tilde{\Gamma}(t=0, q_1, q_2) = \tilde{S}(t, q_1, q_2)$ . Theoretically, we have the option to choose two different regulator-shape functions for the two subspaces, such that the regulator insertion is also only  $O(\bar{N}) \times O(\bar{M})$  symmetric. Instead, for the sake of simplicity, we work with an  $O(N)$ -symmetric regulator insertion that provides the same regulator-shape function for all fields, see Eq. (8). Basically, these are all ingredients that are required to derive the RG flow equation for the potential  $\tilde{U}(t, q_1, q_2)$  and we can follow the same steps as in Sec. III B 1.

It is convenient to derive all quantities in terms of the field-space invariants  $q_1$  and  $q_2$  instead of the individual fields  $\vec{\varphi}_1$  and  $\vec{\varphi}_2$ . Furthermore, without loss of generality, we can choose a particular background-field configuration that simplifies the calculation of the flow equation. For example, we can choose  $\vec{\varphi}_1 = (0, \dots, \sigma_1)$  and  $\vec{\varphi}_2 = (\sigma_2, \dots, 0)$ , such that the invariants are  $q_1 = \frac{1}{2}\sigma_1^2$  and  $q_2 = \frac{1}{2}\sigma_2^2$ . It follows that the field-dependent two-point function (evaluated on this

background-field configuration) reads

$$\tilde{\Gamma}^{(2)}(t) + R(t) = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & A_2 \end{pmatrix}, \quad (39)$$

with

$$A_{1/2} = \text{diag}(\underbrace{r + \partial_{q_{1/2}} \tilde{U}, \dots, r + \partial_{q_{1/2}} \tilde{U}}_{\bar{N}/\bar{M}}), \quad (40)$$

and

$$B = \begin{pmatrix} r + \partial_{q_1} \tilde{U} + 2q_1 \partial_{q_1}^2 \tilde{U} & \sigma_1 \sigma_2 \partial_{q_1} \partial_{q_2} \tilde{U} \\ \sigma_2 \sigma_1 \partial_{q_2} \partial_{q_1} \tilde{U} & r + \partial_{q_2} \tilde{U} + 2q_2 \partial_{q_2}^2 \tilde{U} \end{pmatrix}. \quad (41)$$

Because of the block-diagonal structure, an inversion of the regularized and field-dependent two-point function is possible and the Wetterich equation is well defined, provided that the determinant of the block matrices are separately nonzero for all  $t$  and  $q_{1/2}$ . In the following, we shall assume that this is the case for the considered initial conditions. Inverting this two-point function and inserting the result together with the regulator (8) in the Wetterich equation (6), we obtain the flow equation for the effective potential,

$$\begin{aligned} \partial_t \tilde{U} = & \frac{(\bar{N}-1)(\frac{1}{2}\partial_t r)}{r + \partial_{q_1} \tilde{U}} + \frac{(\bar{M}-1)(\frac{1}{2}\partial_t r)}{r + \partial_{q_2} \tilde{U}} \\ & + \frac{(\frac{1}{2}\partial_t r)(2r + \partial_{q_1} \tilde{U} + 2q_1 \partial_{q_1}^2 \tilde{U} + \partial_{q_2} \tilde{U} + 2q_2 \partial_{q_2}^2 \tilde{U})}{(r + \partial_{q_1} \tilde{U} + 2q_1 \partial_{q_1}^2 \tilde{U})(r + \partial_{q_2} \tilde{U} + 2q_2 \partial_{q_2}^2 \tilde{U}) - 4q_1 q_2 (\partial_{q_1} \partial_{q_2} \tilde{U})(\partial_{q_2} \partial_{q_1} \tilde{U})}. \end{aligned} \quad (42)$$

Reformulated in terms of the background fields  $\sigma_1$  and  $\sigma_2$ , which are associated with the directions of possibly existing condensates in higher-dimensional problems, we have

$$\begin{aligned} \partial_t U = & \frac{(\bar{N}-1)(\frac{1}{2}\partial_t r)}{r + \frac{1}{\sigma_1} \partial_{\sigma_1} U} + \frac{(\bar{M}-1)(\frac{1}{2}\partial_t r)}{r + \frac{1}{\sigma_2} \partial_{\sigma_2} U} \\ & + \frac{(\frac{1}{2}\partial_t r)(2r + \partial_{\sigma_1}^2 U + \partial_{\sigma_2}^2 U)}{(r + \partial_{\sigma_1}^2 U)(r + \partial_{\sigma_2}^2 U) - (\partial_{\sigma_1} \partial_{\sigma_2} U)(\partial_{\sigma_2} \partial_{\sigma_1} U)}. \end{aligned} \quad (43)$$

The reason for choosing the background fields  $\sigma_1$  and  $\sigma_2$  rather than the invariants  $q_1$  and  $q_2$  to span our spatial computational domain is to ensure a more straightforward handling of the boundary conditions in field space [37]; see

also our discussion above. Note that, for  $\bar{N} = \bar{M} = 1$ , we recover the flow equation (15) for the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric case with two fields.

In general, the situation is now the same as in Sec. III B: We find a PDE for  $U(t, \sigma_1, \sigma_2)$  whose spatial domain is  $\sigma_{1/2} \in [0, \infty)$ ,  $\mathbb{R}_{(\geq 0)} \times \mathbb{R}_{(\geq 0)}$  with a temporal evolution from  $t = 0$  to  $t \rightarrow \infty$ . In total, this constitutes a  $(2+1)$ -dimensional PDE problem. Again, we shall not solve this problem for given UV initial conditions for  $U$  but consider the derivatives of  $U$  with respect to the two field directions as evolving fluid fields.

### 2. A fluid-dynamic reformulation

As above, we start our fluid-dynamic reformulation by defining

$$u = \partial_{\sigma_1} U, \quad v = \partial_{\sigma_2} U, \quad (44)$$

and taking derivatives of Eq. (43) with respect to  $\sigma_1$  and  $\sigma_2$  to obtain two conservation laws for  $u$  and  $v$ ,

$$\begin{aligned} \partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \partial_{\sigma_1} \begin{pmatrix} f^x \\ 0 \end{pmatrix} + \partial_{\sigma_2} \begin{pmatrix} 0 \\ f^y \end{pmatrix} \\ = \partial_{\sigma_1} \begin{pmatrix} Q^x \\ 0 \end{pmatrix} + \partial_{\sigma_2} \begin{pmatrix} 0 \\ Q^y \end{pmatrix}. \end{aligned} \quad (45)$$

Here,

$$Q^{x/y} = \frac{(\frac{1}{2}\partial_t r)(2r + \partial_{\sigma_1} u + \partial_{\sigma_2} v)}{(r + \partial_{\sigma_1} u)(r + \partial_{\sigma_2} v) - (\partial_{\sigma_1} v)(\partial_{\sigma_2} u)} \quad (46)$$

is again identified as a diffusion flux, see above, while

$$f^{x/y} = -\frac{(\bar{N} - 1)(\frac{1}{2}\partial_t r)}{r + \frac{1}{\sigma_1} u} - \frac{(\bar{M} - 1)(\frac{1}{2}\partial_t r)}{r + \frac{1}{\sigma_2} v} \quad (47)$$

is identified as an advection flux. The explicit field dependence, which is an explicit position dependence on the level of the PDEs, causes the advection flux to contain some contributions from the source. Both are seen best by executing the  $\sigma_{1/2}$  derivatives in Eq. (45) and comparing the result to standard advection-diffusion equations with sources/sinks. For a detailed discussion, we refer to Ref. [37].

#### IV. NUMERIC APPROACH

Once the PDE problem is formulated in a conservative form, see Eqs. (21) and (45), we are free to choose the numerical method as long as we stick to the highly developed toolbox of numerical fluid dynamics and ensure the applicability of the respective scheme.

For example, modern discontinuous Galerkin methods (and first-order upwind schemes for particular applications) have been successfully applied to flow equations formulated in a conservative form [40,41,43,44,47]. Loosely speaking, these methods are a variant of so-called finite-element methods. Here, we opt for the also well-established and related finite-volume methods in terms of the KT central scheme [80], which was already used and tested in detail in our previous works [37–39,42]. In particular, we apply the two-dimensional version of this scheme to Eqs. (20) and (45).

Readers who are not interested in the explicit implementation of this scheme may skip the remainder of this section.

##### A. The Kurganov-Tadmor central scheme

Here, we briefly recapitulate the two-dimensional KT central scheme as well as some minor but relevant modifications of the scheme. For details, we refer to the

original work by Kurganov and Tadmor [80] and to Ref. [37] for its one-dimensional version in the context of the FRG approach. All details of its explicit two-dimensional implementation, which is used in this work, are presented in Appendix B, while all necessary underlying formulas are provided in the following, amended by short explanations but without derivation.

In general, the two-dimensional version of the KT central scheme is tailor-made for the solution of fluid-dynamic PDEs of the advection-diffusion type,

$$\begin{aligned} \partial_t \vec{u} + \frac{d}{dx} \vec{f}^x[\vec{u}] + \frac{d}{dy} \vec{f}^y[\vec{u}] \\ = \frac{d}{dx} \vec{Q}^x[\vec{u}, \partial_x \vec{u}, \partial_y \vec{u}] + \frac{d}{dy} \vec{Q}^y[\vec{u}, \partial_x \vec{u}, \partial_y \vec{u}], \end{aligned} \quad (48)$$

where  $\vec{u}$  is the vector of fluid fields,  $t$  is the temporal coordinate, and  $x$  and  $y$  denote the Cartesian spatial coordinates.<sup>8</sup> Furthermore,  $\vec{f}^x$  and  $\vec{f}^y$  are the advection fluxes and  $\vec{Q}^x$  and  $\vec{Q}^y$  are diffusion-type fluxes. All fluxes can be (highly) nonlinear functions of their arguments. Equations of this type therefore tend to form nonanalytic structures in the (weak) solution for  $\vec{u}$  [73–76,81], such as shock waves. These have to be handled by a numerical scheme that can, for example, be done via modern FV methods.

FV methods are based on a partitioning of the spatial computational domain of the PDE into small cells of finite volumes. In our calculations, we shall use a rectangular regular mesh of equally sized volume cells of size  $\Delta x \cdot \Delta y$ . The cell centers are located at positions  $(x_{j_x}, y_{j_y})$  with cell boundaries at

$$x_{j_x \pm \frac{1}{2}} = x_{j_x} \pm \frac{\Delta x}{2} \quad \text{and} \quad y_{j_y \pm \frac{1}{2}} = y_{j_y} \pm \frac{\Delta y}{2}, \quad (49)$$

see Fig. 1. This partitioning of the computational domain can be used to turn the PDE (48) into its weak integral form for each volume cell and time step. To this end, one first computes the spatial volume integral over both sides of Eq. (48) and defines the cell average of the fluid fields  $\vec{u}$  in a cell with cell center  $(x_{j_x}, y_{j_y})$  as follows:

$$\bar{\vec{u}}_{j_x, j_y}(t) = \frac{1}{\Delta x \Delta y} \int_{x_{j_x - \frac{1}{2}}}^{x_{j_x + \frac{1}{2}}} dx \int_{y_{j_y - \frac{1}{2}}}^{y_{j_y + \frac{1}{2}}} dy \vec{u}(t, x, y). \quad (50)$$

This already turns the PDE in Eq. (48) into a coupled set of ODEs for the cell averages. However, in order to perform a single time step of the ODE system from  $t^i$  to some  $t^{i+1} = t^i + \Delta t$  with some small  $\Delta t$  via some ODE integrator, one has to evaluate the spatial integrals and the temporal

<sup>8</sup>In principle, this can be straightforwardly generalized to higher-dimensional systems.

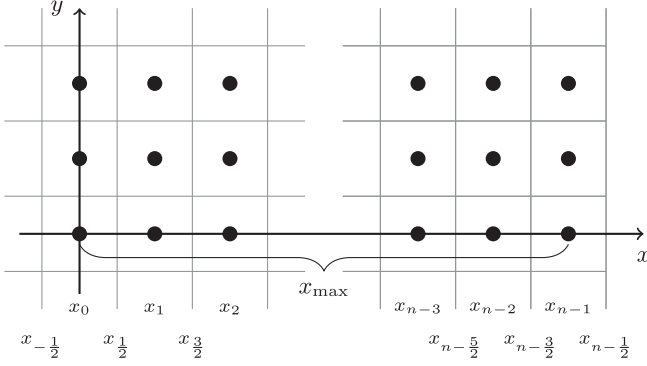


FIG. 1. Sketch of the two-dimensional FV discretization of the computational domain. Cell centers are marked with bullets.

integration from  $t^i$  to  $t^{i+1}$  over the advection and diffusion fluxes. This evaluation of the flux integrals or their approximate evaluation is at the heart of any FV scheme and constitutes a so-called Riemann problem at each cell interface for each time step that has to be solved [73–76,81].

There are FV schemes that are based on the exact or approximate solution of these Riemann problems. Interestingly, the KT scheme does not rely on such a Riemann solver or the characteristic decomposition of the (advection) fluxes. The only information that is required is the spectral radius of the Jacobian of the advection fluxes to estimate the characteristic velocities at each cell interface.

Luckily, we do not have to construct the KT scheme here or recapitulate its entire construction. We can simply use the results from Ref. [80]. First of all, the result is a fully discrete scheme for the computation of time steps of size  $\Delta t$ , which is second-order accurate in  $\Delta x$  and  $\Delta y$ . The particular appealing aspect of the KT scheme is that there is a semidiscrete reduction of the scheme (a controlled limit  $\Delta t \rightarrow 0$ )—being continuous in time, though discrete and still second-order accurate in space. This makes the scheme a ready-made black-box solver that can be combined with basically any ODE time steppers.<sup>9</sup> Furthermore, there exist several improved or adapted versions of this semidiscrete scheme that yield higher-order accuracy under certain circumstances [82–86]. However, these are not discussed in this work.

For this work, we simply need the following formulas, which are all continuous in time and discrete in space. The corresponding semidiscrete version of the PDE (48), i.e., the ODEs for the cell averages read

$$\partial_t \bar{u}_{j_x, j_y} = - \frac{\bar{H}_{j_x+\frac{1}{2}, j_y}^x - \bar{H}_{j_x-\frac{1}{2}, j_y}^x}{\Delta x} - \frac{\bar{H}_{j_x, j_y+\frac{1}{2}}^y - \bar{H}_{j_x, j_y-\frac{1}{2}}^y}{\Delta y} + \frac{\bar{P}_{j_x+\frac{1}{2}, j_y}^x - \bar{P}_{j_x-\frac{1}{2}, j_y}^x}{\Delta x} - \frac{\bar{P}_{j_x, j_y+\frac{1}{2}}^y - \bar{P}_{j_x, j_y-\frac{1}{2}}^y}{\Delta y}. \quad (51)$$

<sup>9</sup>For recent advances on the problem of time stepping of RG flow equations at late RG times (the deep IR), we again refer to Ref. [47].

The corresponding numerical advection fluxes to the adjacent cells in  $x$  and  $y$  direction are

$$\bar{H}_{j_x+\frac{1}{2}, j_y}^x = \frac{1}{2} \left( \bar{f}^x[\bar{u}_{j_x+\frac{1}{2}, j_y}^+] + \bar{f}^x[\bar{u}_{j_x+\frac{1}{2}, j_y}^-] \right) - \frac{1}{2} a_{j_x+\frac{1}{2}, j_y}^x \cdot \left( \bar{u}_{j_x+\frac{1}{2}, j_y}^+ - \bar{u}_{j_x+\frac{1}{2}, j_y}^- \right), \quad (52)$$

$$\bar{H}_{j_x, j_y+\frac{1}{2}}^y = \frac{1}{2} \left( \bar{f}^y[\bar{u}_{j_x, j_y+\frac{1}{2}}^+] + \bar{f}^y[\bar{u}_{j_x, j_y+\frac{1}{2}}^-] \right) - \frac{1}{2} a_{j_x, j_y+\frac{1}{2}}^y \cdot \left( \bar{u}_{j_x, j_y+\frac{1}{2}}^+ - \bar{u}_{j_x, j_y+\frac{1}{2}}^- \right), \quad (53)$$

whereas the  $x$ - and  $y$ -numerical diffusion fluxes are given by

$$\bar{P}_{j_x+\frac{1}{2}, j_y}^x = \frac{1}{2} \left( \bar{Q}^x \left[ \bar{u}_{j_x, j_y}, \frac{\bar{u}_{j_x+1, j_y} - \bar{u}_{j_x, j_y}}{\Delta x}, (\partial_y \bar{u})_{j_x, j_y} \right] + \bar{Q}^x \left[ \bar{u}_{j_x+1, j_y}, \frac{\bar{u}_{j_x+1, j_y} - \bar{u}_{j_x, j_y}}{\Delta x}, (\partial_y \bar{u})_{j_x+1, j_y} \right] \right), \quad (54)$$

$$\bar{P}_{j_x, j_y+\frac{1}{2}}^y = \frac{1}{2} \left( \bar{Q}^y \left[ \bar{u}_{j_x, j_y}, (\partial_x \bar{u})_{j_x, j_y}, \frac{\bar{u}_{j_x, j_y+1} - \bar{u}_{j_x, j_y}}{\Delta y} \right] + \bar{Q}^y \left[ \bar{u}_{j_x+1, j_y}, (\partial_x \bar{u})_{j_x+1, j_y}, \frac{\bar{u}_{j_x+1, j_y+1} - \bar{u}_{j_x+1, j_y}}{\Delta y} \right] \right). \quad (55)$$

The calculation of these numerical fluxes first requires the value of the fluid vector  $\bar{u}$  on the cell interfaces. This is reconstructed from the cell averages via a piecewise linear reconstruction from both sides of the corresponding cell interface,

$$\bar{u}_{j_x+\frac{1}{2}, j_y}^\pm = \bar{u}_{j_x+1, j_y} \mp \frac{\Delta x}{2} (\partial_x \bar{u})_{j_x+\frac{1}{2}, j_y}, \quad (56)$$

$$\bar{u}_{j_x, j_y+\frac{1}{2}}^\pm = \bar{u}_{j_x, j_y+1} \mp \frac{\Delta y}{2} (\partial_y \bar{u})_{j_x, j_y+\frac{1}{2}}. \quad (57)$$

Here and for the calculation of the diffusion fluxes, one needs to estimate the gradient of each fluid field in each cell,  $(\partial_{x/y} \bar{u})_{j_x, j_y}$ . This estimate is also based on the cell averages. Componentwise, we find

$$(\partial_x u^\alpha)_{j_x, j_y} = f_{\text{limiter}} \left( \frac{\bar{u}_{j_x+1, j_y}^\alpha - \bar{u}_{j_x, j_y}^\alpha}{\Delta x}, \frac{\bar{u}_{j_x, j_y}^\alpha - \bar{u}_{j_x-1, j_y}^\alpha}{\Delta x} \right), \quad (58)$$

$$(\partial_y u^\alpha)_{j_x, j_y} = f_{\text{limiter}} \left( \frac{\bar{u}_{j_x, j_y+1}^\alpha - \bar{u}_{j_x, j_y}^\alpha}{\Delta y}, \frac{\bar{u}_{j_x, j_y}^\alpha - \bar{u}_{j_x, j_y-1}^\alpha}{\Delta y} \right). \quad (59)$$

Here,  $f_{\text{limiter}}$  is a flux limiter function that avoids an over- or underestimate of the slopes, which would lead to spurious oscillations in the solution.<sup>10</sup> The explicit choice of the limiter is up to the user and valid limiters are presented in Refs. [74,75,87]. In this work, we simply use the MinMod limiter,

$$f_{\text{MinMod}}(a, b) = \begin{cases} \min(|a|, |b|), & \text{if } a \cdot b > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (60)$$

see Ref. [80].

At this point, we emphasize that we have to slightly adapt the original KT scheme for our purposes. During the benchmark tests of this work, but also in computations in a related work [88], we experienced that using the limited derivatives in Eqs. (58) and (59) in the contributions to the diffusion fluxes in Eqs. (54) and (55) leads to incorrect results in some cases. Spurious oscillations can form in the solution, which may originate from an underestimate of the gradients orthogonal to the direction of the diffusion flux. This leads to an artificial asymmetry of the fluxes and too little diffusion. We solve this by replacing Eqs. (58) and (59) with central difference stencils,

$$(\partial_x u^\alpha)_{j_x, j_y} = \frac{\bar{u}_{j_x+1, j_y}^\alpha - \bar{u}_{j_x-1, j_y}^\alpha}{2\Delta x}, \quad (61)$$

$$(\partial_y u^\alpha)_{j_x, j_y} = \frac{\bar{u}_{j_x, j_y+1}^\alpha - \bar{u}_{j_x, j_y-1}^\alpha}{2\Delta y}, \quad (62)$$

in the diffusion terms in Eqs. (54) and (55) only. However, for the reconstruction of the fluid fields on the cell interfaces in Eqs. (56) and (57), we still use the limited derivatives in Eqs. (58) and (59) which ultimately enter the contributions to the advection fluxes Eqs. (52) and (53).

The final component of our scheme is the advection-velocity estimates of the cell boundaries,

$$a_{j_x+\frac{1}{2}, j_y}^x = \max \left[ \hat{\rho} \left( \frac{\partial \vec{f}}{\partial \vec{u}} [\vec{u}_{j_x+\frac{1}{2}, j_y}^+] \right), \hat{\rho} \left( \frac{\partial \vec{f}}{\partial \vec{u}} [\vec{u}_{j_x+\frac{1}{2}, j_y}^-] \right) \right], \quad (63)$$

$$a_{j_x, j_y+\frac{1}{2}}^y = \max \left[ \hat{\rho} \left( \frac{\partial \vec{g}}{\partial \vec{u}} [\vec{u}_{j_x, j_y+\frac{1}{2}}^+] \right), \hat{\rho} \left( \frac{\partial \vec{g}}{\partial \vec{u}} [\vec{u}_{j_x, j_y+\frac{1}{2}}^-] \right) \right], \quad (64)$$

where  $\hat{\rho}(A) = \max\{|\lambda_1|, \dots, |\lambda_\omega|\}$  is the spectral radius of the matrix  $A$ , with  $\lambda_k$  being the eigenvalues.

Apart from these formulas, no additional information is required to set up the core of our numerical scheme. However, for the numerical implementation on a compact domain, it is more suitable to present this scheme in a matrix-type formulation, as done in Appendix B, instead of the local formulation presented here.

<sup>10</sup>Sometimes one also uses limiters that are additionally functions of the central difference.

## B. Adaptations of the KT scheme to our FRG problem(s)

Next, we comment on some adaptations of the above presented scheme to the FRG flow equations of the effective potential. Therefore, we first identify the field-space variables with the spatial variables, thus  $x = \varphi_1$  and  $y = \varphi_2$ , and the RG time  $t$  with the temporal parameter  $t$ .

Anyhow, the following is clear:

- (1) A PDE problem is exclusively well defined by specifying its boundary conditions and their numerical implementation, which we have not done yet.
- (2) Typical fluxes within the FRG approach, such as Eq. (22), are usually RG-scale/time-dependent [in addition to the dependencies in Eq. (48)] and can comprise explicit dependences on the (field-space) position. Furthermore, in some truncations, one is confronted with additional coupled ODEs.
- (3) Our conservation law (20) matches the form of Eq. (48) for vanishing advection fluxes.

Let us comment on these issues:

- (1) Actually, the PDE in Eq. (20) forms an initial value problem on the noncompact domain  $\mathbb{R}^2$  and boundary conditions are not required to have a well-posed problem. Since we cannot compute numerically on noncompact domains, we restrict ourselves to the compact domain  $[-\varphi_{1,\max}, \varphi_{1,\max}] \times [-\varphi_{2,\max}, \varphi_{2,\max}]$  and impose artificial boundary conditions at the domain boundaries. This is done in the KT scheme by introducing two additional ghost cells at each domain boundary. In Ref. [37] we discussed in great detail that a linear extrapolation of the fluid field  $\vec{u} = (u, v)^T$  currently appears to be a decent choice, provided that the values of  $\varphi_{i,\max}$  are large enough, such that fluxes are either suppressed or the in- and out-flux averages to net zero. In our present work, we use this linear extrapolation from the last two physical cells to the ghost cells and choose  $\varphi_{i,\max}$  according to the empirical knowledge that has been gained in Ref. [37]. For further details on the discussion of boundary conditions, we refer to this reference.

For the PDE in Eq. (45), however, we have additional boundary conditions at  $\varphi_{1/2} = 0 = \sigma_{1/2}$ , whereas the boundary conditions at  $\sigma_{1/2} = \pm\infty$  are the same as for the previous case and also handled in the same way. Using the formulation in  $\sigma_{1/2}$ , the new boundary conditions are easily derived, see again Ref. [37]. The functions  $u$  and  $v$  have the following symmetry properties with respect to the  $\sigma_{1/2}$  axis,

$$u(\sigma_1, \sigma_2) = -u(-\sigma_1, \sigma_2) = u(\sigma_1, -\sigma_2), \quad (65)$$

$$v(\sigma_1, \sigma_2) = -v(\sigma_1, -\sigma_2) = v(-\sigma_1, \sigma_2). \quad (66)$$



These can be directly derived from the symmetry properties of the effective potential  $U$ . On the level of the cell averages of the KT scheme, this is again implemented via the ghost cells, which are given by (see also Fig. 1)

$$\bar{u}_{-1,j_y} = -\bar{u}_{1,j_y}, \quad \bar{u}_{-2,j_y} = -\bar{u}_{2,j_y}, \quad (67)$$

$$\bar{v}_{-1,j_y} = \bar{v}_{1,j_y}, \quad \bar{v}_{-2,j_y} = \bar{v}_{2,j_y}, \quad (68)$$

$$\bar{u}_{j_x,-1} = \bar{u}_{j_x,1}, \quad \bar{u}_{j_x,-2} = \bar{u}_{j_x,2}, \quad (69)$$

$$\bar{v}_{j_x,-1} = -\bar{v}_{j_x,1}, \quad \bar{v}_{j_x,-2} = -\bar{v}_{j_x,2}. \quad (70)$$

In addition, by symmetry, we know that the following cell averages are also fixed:

$$\bar{u}_{0,j_y} = 0, \quad \bar{v}_{j_x,0} = 0. \quad (71)$$

We emphasize that these boundary conditions pose a challenge: Consider, for example, the numerical advection flux in  $x$  direction ( $\sigma_1$  corresponds to  $x$  direction) in Eq. (52) with Eq. (47). Because of the explicit position dependence, it is impossible to numerically evaluate this flux at the boundary cells with  $j_y = 0$  because we would have to divide the cell averages  $\bar{v}_{j_x,0} = 0$  by  $y_0 = 0$ .<sup>11</sup> The same happens along the  $\sigma_2 = y$  axis for the flux in  $y$  direction and the first term in Eq. (47). We solve this issue by using the estimate

$$\lim_{\sigma_1 \rightarrow 0} \frac{1}{\sigma_1} u \simeq \lim_{\sigma_1 \rightarrow 0} \partial_{\sigma_1} u, \quad (72)$$

motivated by l'Hôpital's rule. However, this converts the respective part of the advection fluxes into a diffusion flux. Hence, for all cells along the domain boundaries at  $\sigma_{1/2} = 0$ , we modified the diffusion and advection fluxes (46) and (47) as follows:

$$f^x|_{\sigma_2=0} = -\frac{(\bar{N}-1)(\frac{1}{2}\partial_t r)}{r + \frac{1}{\sigma_1}u}, \quad (73)$$

$$Q^x|_{\sigma_2=0} = \frac{(\frac{1}{2}\partial_t r)(2r + \partial_{\sigma_1}u + \partial_{\sigma_2}v)}{(r + \partial_{\sigma_1}u)(r + \partial_{\sigma_2}v) - (\partial_{\sigma_1}v)(\partial_{\sigma_2}u)} + \frac{(\bar{M}-1)(\frac{1}{2}\partial_t r)}{r + \partial_{\sigma_2}v}, \quad (74)$$

and similarly for the fluxes in  $y$  direction. Even though this seems cumbersome, we did not find another

solution, e.g., by “moving” the grid. Note that the sign in front of the last term in the diffusion flux stems from moving the term to the other side of the equation, see Eq. (45).

In general, the above formulation of the KT scheme does not explicitly include the boundary conditions and ghost cells. Therefore, it is better for the numerical implementation to discuss the KT scheme in a matrix formulation in terms of pseudocode, where the ghost cells are explicitly included, see Appendix B.

- (2) Indeed, the original KT scheme was presented for fluxes that exclusively depend on the fluid fields and their derivatives; although, there is no reason why the scheme could be spoiled, if the fluxes are  $t$  dependent or additional ODEs are coupled to the PDE. However, if the (advection) fluxes gain explicit position dependences, the situation is different because fundamental properties of the KT scheme, such as being total variation diminishing/total variation nonincreasing get formally lost. In our previous works [37–39], however, we experienced by explicit benchmark tests that even position-dependent advection fluxes do not seem to invalidate the applicability of the scheme. Nevertheless, great caution and detailed testing is in order when numerical methods for nonlinear PDEs are run at the edge of their applicability.
- (3) Of course, the conservation law (20) is represented as the viscous limit of Eq. (48) without any advection fluxes. However, the scheme is still applicable, as is demonstrated by the benchmark tests in Ref. [37] and, e.g., tests for the heat equation.

Finally we comment on the initial condition for the PDEs. From Eq. (7) we deduce that the initial condition of the coupled PDEs (18) and (19) at  $t = 0$  is given by the field-space derivatives of the classical action (the UV potential). On the level of the cell averages in Eq. (50), this implies that the discretized initial condition of the FV scheme is calculated as follows [e.g., for  $v(t = 0, \vec{\varphi})$ ]:

$$\begin{aligned} \bar{v}_{j_x,j_y}(0) &= \frac{1}{\Delta x \Delta y} \int_{x_{j_x-\frac{1}{2}}}^{x_{j_x+\frac{1}{2}}} dx \int_{y_{j_y-\frac{1}{2}}}^{y_{j_y+\frac{1}{2}}} dy v(0, x, y) \\ &= \frac{1}{\Delta x \Delta y} \int_{x_{j_x-\frac{1}{2}}}^{x_{j_x+\frac{1}{2}}} dx \int_{y_{j_y-\frac{1}{2}}}^{y_{j_y+\frac{1}{2}}} dy \partial_y U(0, x, y) \\ &= \frac{1}{\Delta x \Delta y} \int_{x_{j_x-\frac{1}{2}}}^{x_{j_x+\frac{1}{2}}} dx \left[ U(0, x, y_{j_y+\frac{1}{2}}) - U(0, x, y_{j_y-\frac{1}{2}}) \right], \end{aligned} \quad (75)$$

and analogously for  $u(t = 0, \vec{\varphi})$ . Hence, we can and should make use of the fact that we have direct access to  $U(t = 0, \vec{\varphi})$ ; see also Ref. [37]. Usually, the remaining integral in Eq. (75) has to be evaluated with high precision, which is, however, not a substantial problem.

<sup>11</sup>At  $j_x = 0$ , there is no problem for the flux in  $x$  direction because the fluxes are only evaluated at the cell boundaries that are not located at  $x_0 = 0$ , see Fig. 1.

## V. GENERAL TEST SETUP

Our present work deals with the development and tests of a fluid-dynamic formulation of FRG problems that are two-dimensional in field space. Explicitly, we want to present a proof of concept that the numeric approach, which was presented in Ref. [37], namely, the application of the 1D KT central scheme to the  $O(N)$ -symmetric flow equation of the effective potential, can be generalized and applied to flow equations of effective potentials that are two-dimensional in field space. In complete analogy to Ref. [37], we therefore need benchmark tests that demonstrate that the spatial discretization scheme is applicable and performs well but also clearly shows its limitations at the same time.

By taking over some findings from our earlier works concerning decent UV and IR cutoffs and appropriate sizes of the artificial computational domain in field space, our present study can mostly focus on the discretization errors stemming from the KT scheme. These errors are directly caused by the finite resolution  $\Delta x$  and  $\Delta y$  in field space as well as a possible artificial breaking of a continuous symmetry in field space by our Cartesian discretization scheme.

Therefore, we basically consider three testing scenarios:

- (1) We study our field theory of two scalar fields under the additional assumption of  $O(2)$  symmetry in field space. Still, we stick to the flow equations (18) and (19), which are two-dimensional in field space and solve the full two-dimensional PDE problem. From the IR solution, we can then extract  $\Gamma^{(2)}$  at the IR minimum and compare this result for different grid spacings  $\Delta x$ ,  $\Delta y$  against the exact result from the path integral as well as against the results from the one-dimensional reduction, see Sec. III C. This is particularly interesting because we know that there is no dynamical symmetry breaking in zero-dimensional QFTs, such that the IR minimum is always trivial, i.e.,  $\vec{\varphi} = 0$ , see Ref. [37]. Hence, the results are not contaminated by errors that emerge from a location of the minimum in field space. Furthermore, it can be shown that the IR potential has to be convex and smooth and should, of course, still be globally  $O(2)$  symmetric. Whether this is still (approximately) the case on a rectangular grid is the subject of our investigations. Note that all these aspects are totally independent of the specific choice of the UV action/potential.

More specifically, if we choose the FV grid such that there is a cell with cell center exactly at  $\varphi_1 = \varphi_2 = 0$ , we can extract  $\Gamma^{(2)}$  from the solution via the finite-difference stencil

$$\Gamma^{(2)} = \begin{cases} \partial_{\varphi_1} u(t, \vec{\varphi})|_{t \rightarrow \infty, \vec{\varphi}_{\min}=0} \simeq \frac{\bar{u}_{1,0} - \bar{u}_{0,0}}{\Delta x} + \mathcal{O}(\Delta x), \\ \partial_{\varphi_2} v(t, \vec{\varphi})|_{t \rightarrow \infty, \vec{\varphi}_{\min}=0} \simeq \frac{\bar{v}_{0,1} - \bar{v}_{0,0}}{\Delta y} + \mathcal{O}(\Delta y). \end{cases} \quad (76)$$

The results can then be compared with the exact results, as well as the results from the one-dimensional formulation of the problem; see Sec. III C below.

The quality of the  $O(2)$  symmetry is tested with the help of the  $L^1$  and  $L^\infty$  norms/errors,

$$\mathcal{O}_{U,L^1} = \frac{1}{\mathbf{n}(i,j)} \sum_{i,j} (|\text{rot}_{90^\circ}(\bar{U})_{i,j} - \bar{U}_{i,j}|), \quad (77)$$

$$\mathcal{O}_{u,L^1} = \frac{1}{\mathbf{n}(i,j)} \sum_{i,j} (|\text{rot}_{90^\circ}(\bar{u}/x)_{i,j} - \bar{u}_{i,j}/x_i|), \quad (78)$$

and

$$\mathcal{O}_{U,L^\infty} = \max_{i,j} (|\text{rot}_{90^\circ}(\bar{U})_{i,j} - \bar{U}_{i,j}|), \quad (79)$$

$$\mathcal{O}_{u,L^\infty} = \max_{i,j} (|\text{rot}_{90^\circ}(\bar{u}/x)_{i,j} - \bar{u}_{i,j}/x_i|), \quad (80)$$

respectively. Here,  $\bar{U}_{i,j}$  is calculated in the IR via a simple Riemann sum<sup>12</sup> from the cell averages  $\bar{u}_{i,j}$  and  $\bar{v}_{i,j}$  and  $\text{rot}_{90^\circ}(\cdot)$  means a  $90^\circ$  rotation around the origin, i.e.,  $i = j = 0$ . Furthermore,  $\mathbf{n}(i,j)$  is the number of cells included in the sum. Since we are using a Cartesian discretization, we are restricted to  $90^\circ$  rotations. The first observable is based on the potential, whereas the second one is based on the cell averages  $\bar{u}_{i,j}$ . Therefore, the latter observable is slightly more interesting as it only comprises errors from the violation of the  $O(2)$  symmetry from the KT method. Additional errors from the numeric integration can be excluded. For details, we refer to the discussion in Sec. IX A 2.

All these tests are based on some conventional and some uncommon choices of UV initial potentials, which were presented in Ref. [37] as hard tests for the spatial discretization scheme. For comparison, we

<sup>12</sup>We use the following algorithm for the Riemann summation for the calculation of  $\bar{U}_{i,j}$ : (i) Set  $\bar{U}_{0,j}^u = 0$  and  $\bar{U}_{i,0}^v = 0$  for all  $i, j$ . (ii) Define

$$\bar{U}_{i,j}^u = \begin{cases} \bar{U}_{i-1,j}^u + \frac{\Delta x}{2}(\bar{u}_{i-1,j} + \bar{u}_{i,j}) & \text{for } i > 0, \\ \bar{U}_{i+1,j}^u - \frac{\Delta x}{2}(\bar{u}_{i+1,j} + \bar{u}_{i,j}) & \text{for } i < 0, \end{cases} \quad (81)$$

and

$$\bar{U}_{i,j}^v = \begin{cases} \bar{U}_{i,j-1}^v + \frac{\Delta y}{2}(\bar{v}_{i,j-1} + \bar{v}_{i,j}) & \text{for } j > 0, \\ \bar{U}_{i,j+1}^v - \frac{\Delta y}{2}(\bar{v}_{i,j+1} + \bar{v}_{i,j}) & \text{for } j < 0. \end{cases} \quad (82)$$

(iii) Combine  $\bar{U}_{i,j} = \bar{U}_{i,j}^u + \bar{U}_{i,j}^v$  and  $\bar{U}_{i,j} = \bar{U}_{i,0}^u + \bar{U}_{i,j}^v$ . Both procedures are equivalent and the results are identical.

also show results from the dimensionally reduced formulation of the two-dimensional problem as presented in Ref. [37].

- (2) We study the field theory with minimal additional symmetry and simply consider UV initial potentials of two interacting scalar fields. The potentials are asymptotically still at least  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetric to ensure well-posedness of the problem (i.e., well-behaved initial conditions), cf. Sec. III E, but contain also nonsymmetric interactions for small  $|\vec{\varphi}|$ .

One of the initial potentials is constructed such that we have a nonvanishing expectation value for  $\langle \vec{\phi} \rangle$ . Here, we test the following aspects of the two-dimensional PDE approach to our FRG framework: (i) The precision of computations of the location of the IR minimum  $\vec{\varphi}_{\min}$  of the potential in field space. To that end, we use  $\vec{\varphi}_{\min} = \langle \vec{\phi} \rangle$  from the path integral as reference. (ii) We again test the precision of our numerical scheme by calculating the two-point functions  $\Gamma_{\varphi_1\varphi_1}^{(2)}, \Gamma_{\varphi_1\varphi_2}^{(2)} = \Gamma_{\varphi_2\varphi_1}^{(2)}$ , and  $\Gamma_{\varphi_2\varphi_2}^{(2)}$  at the minimum of the potential and benchmark these values against the exact result from Eq. (5).

Another example is a UV potential, which has again a trivial IR minimum and a global  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry. However, by misaligning the discretization axes with the symmetry axes, we can test the violation of such symmetries by the KT scheme. Furthermore, we test how the KT scheme performs in the presence of a nonanalytic UV potential.

- (3) Both of the previous test setups are based on the “diffusion-only” setup, i.e., the advection fluxes are zero for the case with two scalar fields. To test the application of the full KT scheme, we therefore also consider a problem with  $O(\tilde{N}) \times O(\tilde{M})$  symmetry, see Sec. III F. Here, we again know that the IR minimum is trivial, but for  $\tilde{N} > 1$  and/or  $\tilde{M} > 1$  we have nontrivial advection fluxes. This benchmark test also includes the calculation of  $\Gamma^{(2)}$  at the trivial IR minimum and the comparison against the exact result from the path integral. Here, we can again use Eq. (76) to extract  $\Gamma^{(2)}$  from the solution.

In summary, these are valuable tests for assessing the quality of numerical results that will be obtained in future applications of our scheme to higher-dimensional models. In particular, our error estimates will help to rank the relevance of systematic errors from truncation schemes, approximations, cutoffs, the time stepping, and the numerical errors that are exclusively linked to the (spatial) discretization. Note that we shall already present a first application of our scheme to a two-dimensional problem in three-dimensional spacetime, which is closer to realistic applications.

Before we present the results from all these tests, let us introduce our explicit test models in the next section.

## VI. ZERO-DIMENSIONAL TEST MODELS WITH $O(2)$ SYMMETRY

In this section, we introduce our first test environment, the  $O(2)$ -symmetric model.

### A. Reference values from the path integral formalism

Assuming that the classical action  $\mathcal{S}(\vec{\phi})$  and the partition function (1) at vanishing source fields  $\vec{J} = 0$  are invariant under  $O(2)$  transformations of the quantum fields,

$$\vec{\phi} \mapsto \vec{\phi}' = O\vec{\phi} \quad (83)$$

with

$$O = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad \alpha \in [0, 2\pi), \quad (84)$$

it is clear that the classical action has to be a function of the  $O(2)$ -invariant  $\rho = \frac{1}{2}\vec{\phi}^2$  rather than the independent fields  $\phi_1$  and  $\phi_2$ ,

$$\mathcal{S}(\vec{\phi}) = \tilde{\mathcal{S}}(\rho). \quad (85)$$

This implies that, using the  $O(2)$  symmetry, all correlation functions (2) can be expressed in terms of one-dimensional integrals,

$$\langle (\vec{\phi}^2)^n \rangle = \frac{2^n \int_0^\infty d\rho \rho^n e^{-\tilde{\mathcal{S}}(\rho)}}{\int_0^\infty d\rho e^{-\tilde{\mathcal{S}}(\rho)}}, \quad (86)$$

see Refs. [37,57,59] for details. In particular, we find that all correlation functions (2) with an odd number of fields vanish. For our present work, the only relevant correlation function is

$$\langle \phi_i \phi_j \rangle = \delta_{ij} \frac{1}{2} \langle \vec{\phi}^2 \rangle. \quad (87)$$

Using Eq. (5), we have

$$\Gamma^{(2)} \equiv \Gamma_{\varphi_i\varphi_i}^{(2)} = \frac{2}{\langle \vec{\phi}^2 \rangle} \quad (88)$$

for the two-point vertex function. We can now simply solve Eq. (21) numerically for  $O(2)$ -symmetric initial conditions, extracting  $\Gamma^{(2)}$ , and comparing this to the reference result from Eq. (88).

However, due to the  $O(2)$  symmetry, we could also reduce the spatial domain of the PDE in Eq. (15) to a one-dimensional domain and solve this one-dimensional fluid-dynamical system numerically as is usually done in FRG literature, cf. Sec. V of Ref. [37]. Results from this



TABLE I. The table lists the (up to numerical-integration errors) exact results for the  $\Gamma^{(2)}$  of the  $O(2)$  model with the initial UV potentials as given in Eqs. (89)–(92). These values have been obtained by a high-precision one-dimensional numerical integration of the expectation values using Eqs. (86) and (88) for  $n=1$  with *Mathematica*'s numerical integration routine *NIntegrate* [89] with a *PrecisionGoal* and *AccuracyGoal* of 10. Here, we present the first ten digits only.

| Test case | $\Gamma^{(2)}$ |
|-----------|----------------|
| I         | 0.295702274(6) |
| II        | 0.316367789(4) |
| III       | 0.178669819(6) |
| IV        | 0.321461336(0) |

formulation provide us with reference values and additional benchmarks.

### B. $O(2)$ -symmetric test cases

In the following, we briefly recapitulate some explicit test cases for our comparisons which were developed in Ref. [37] and comment on the reasoning behind their choice. The corresponding reference values for  $\Gamma^{(2)}$  for each potential are listed in Table I.

#### 1. Test case I: Nonanalytic initial condition

The UV initial potential of test case I, which is plotted in Fig. 2 (upper left panel), reads

$$U(\vec{\phi}) = \begin{cases} -\frac{1}{2}\vec{\phi}^2, & \text{if } |\vec{\phi}| \leq 2, \\ -2, & \text{if } 2 < |\vec{\phi}| \leq 3, \\ \frac{1}{2}(\vec{\phi}^2 - 13), & \text{if } 3 < |\vec{\phi}|, \end{cases} \quad (89)$$

and was chosen for the following reasons:

- (1) Because of the quadratic asymptotic behavior of this potential, the linear extrapolation at large  $|\vec{\phi}|$  for its derivative at the boundary of the computational domain does not generate any errors. Therefore, one can completely focus on the small- $|\vec{\phi}|$  region.
- (2) The nonanalytic points  $|\vec{\phi}| = 2$  and  $|\vec{\phi}| = 3$  give rise to jump discontinuities in the derivatives of the potential, see Fig. 2 (lower left panel), which are the fluid fields in the two PDEs (18) and (19) and therefore represent challenging but manageable tests for modern schemes from CFD. All schemes that cannot cope with discontinuities are bound to fail.

Furthermore, the IR potential of a zero-dimensional QFT has to be smooth, such that discontinuities in all derivatives need to be smeared out during the flow without spurious oscillations.

- (3) The potential comes with an infinite number of nontrivial degenerate minima in the UV, whereas the minimum in the IR has to be unique and the potential needs to become convex [37,57,72]. Thus, symmetry restoration has to be handled numerically in a stable way.

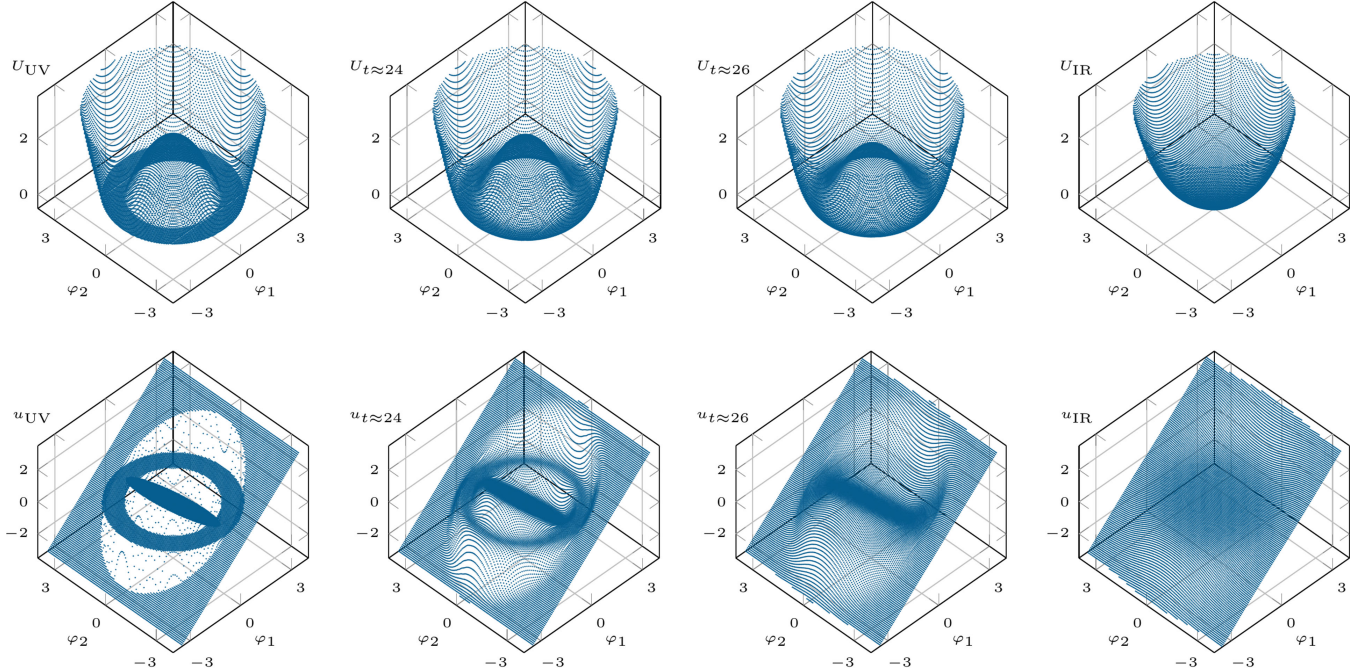


FIG. 2. The RG time evolution of the potential  $U(t, \vec{\phi})$  (upper row) and its  $\phi_1$  derivative  $u(t, \vec{\phi})$  (lower row) from the UV (left column) to the IR (right column) and selected intermediate times for test case I, Eq. (89).



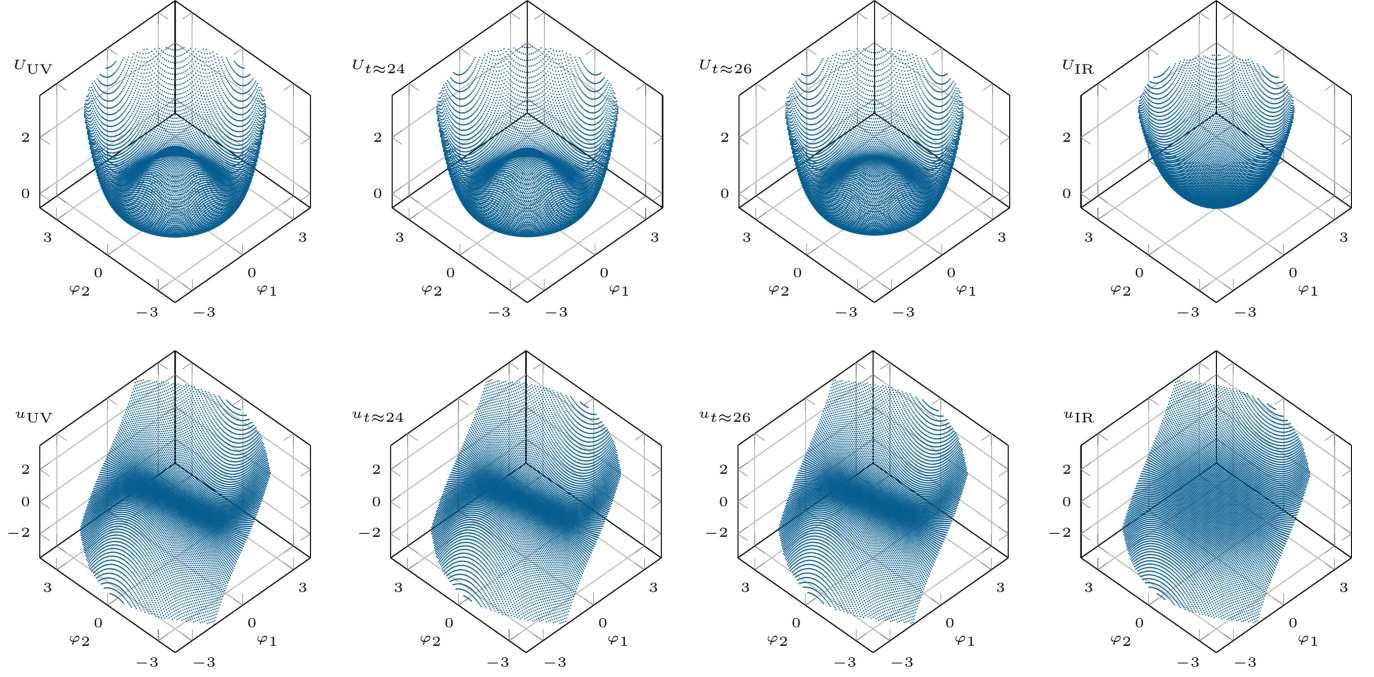


FIG. 3. The RG time evolution of the potential  $U(t, \vec{\varphi})$  (upper row) and its  $\varphi_1$  derivative  $u(t, \vec{\varphi})$  (lower row) from the UV (left column) to the IR (right column) and selected intermediate times for test case II, see Eq. (90).

## 2. Test case II: $\phi^4$ theory

Test case II is the zero-dimensional version of a  $\phi^4$  theory with negative mass term, i.e.,

$$U(\vec{\phi}) = -\frac{1}{2}\vec{\phi}^2 + \frac{1}{4!}(\vec{\phi}^2)^2, \quad (90)$$

see Fig. 3 (upper left panel) for an illustration. From the perspective of the powerful KT scheme, this case is not a challenge, but—being the standard (toy) model of theoretical physics—it has to be included in our analysis. In Ref. [37], however, it was demonstrated that, although the UV potential is smooth, truncations of the potential in terms

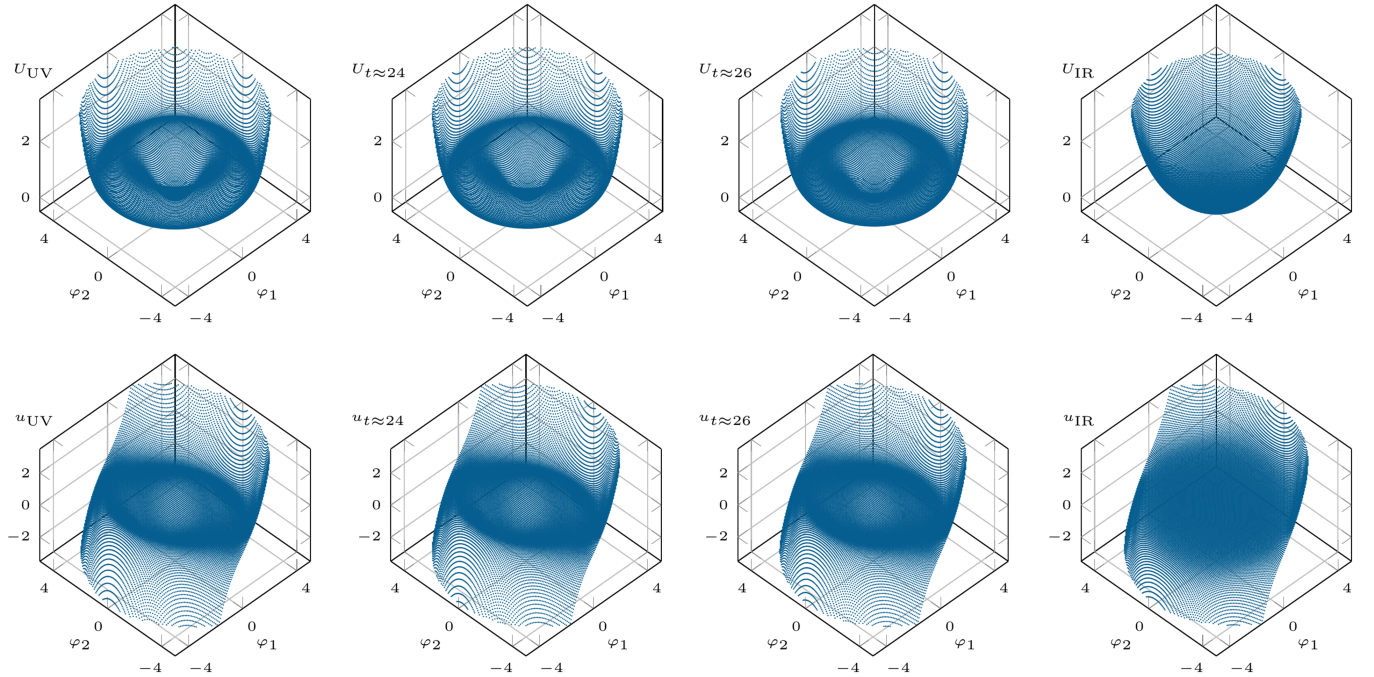


FIG. 4. The RG time evolution of the potential  $U(t, \vec{\varphi})$  (upper row) and its  $\varphi_1$  derivative  $u(t, \vec{\varphi})$  (lower row) from the UV (left column) to the IR (right column) and selected intermediate times for test case III, see Eq. (91).

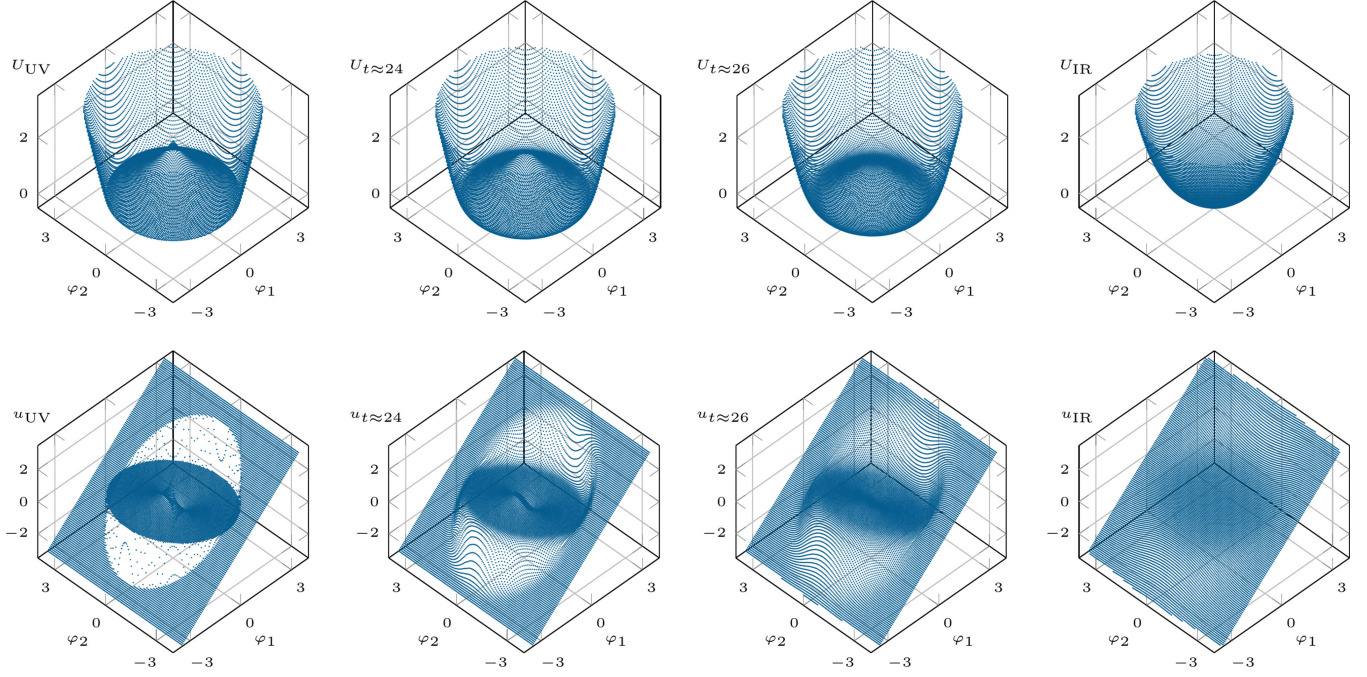


FIG. 5. The RG time evolution of the potential  $U(t, \vec{\phi})$  (upper row) and its  $\phi_1$  derivative  $u(t, \vec{\phi})$  (lower row) from the UV (left column) to the IR (right column) and selected intermediate times for test case IV, see Eq. (92).

of Taylor expansions with fixed expansion point do not converge and a full solution of the PDE is required to obtain small relative errors of vertex functions in the IR limit.

This test case can be used to study symmetry restoration in the RG flow. Furthermore, this potential allows us to test the applicability of the linear extrapolation at the artificial large- $|\vec{\phi}|$  boundary of the computational domain (despite the cubic asymptotic behavior of the derivative of the potential, see lower left panel of Fig. 3) and to find a reasonable size of the computational domain. Here, we do not need to repeat this part of the analysis and simply take suitable values from Ref. [37].

### 3. Test case III: $\phi^6$ theory

The UV potential of test case III reads

$$U(\vec{\phi}) = \frac{1}{2}\vec{\phi}^2 - \frac{1}{20}(\vec{\phi}^2)^2 + \frac{1}{6!}(\vec{\phi}^2)^3 \quad (91)$$

and is illustrated in Fig. 4 (upper left panel). With this potential, the failure of the Taylor expansion of the effective potential can be demonstrated, see also Ref. [37]. A potential of this type, which exhibits several local minima separated by a potential barrier, is expected to describe the dynamics in the vicinity of first-order phase transitions. Therefore, an analysis of its RG flow and the merging of all minima in  $\vec{\phi} = 0$  in the IR is of great relevance also for higher-dimensional applications.

### 4. Test case IV: Pole in the derivative

The fourth test case is given by the UV potential

$$U(\vec{\phi}) = \begin{cases} -(\vec{\phi}^2)^{\frac{1}{3}}, & \text{if } |\vec{\phi}| \leq \sqrt{8}, \\ \frac{1}{2}\vec{\phi}^2 - 6, & \text{if } \sqrt{8} < |\vec{\phi}|. \end{cases} \quad (92)$$

For a visualization of this potential, we refer to Fig. 5 (upper left panel). By inspecting its field-space derivatives, see Fig. 5 (lower left panel), we observe that both  $u = \partial_{\phi_1} U$  and  $v = \partial_{\phi_2} U$  have a pole at  $\phi_{1/2} = 0$ . Poles in the fluid field certainly represent another challenge for our numerical setup in terms of a resolving nonanalytic structures and jump discontinuities.

## VII. ZERO-DIMENSIONAL TEST MODELS WITHOUT $O(2)$ SYMMETRY

We also employ two test models without  $O(2)$  symmetry. Both can simply be understood as field theories of two scalar fields  $\phi_1$  and  $\phi_2$  which are (self-)interacting in a single point via some very complicated mechanism or simply as some statistical models with two random variables. The reason why we define the corresponding initial potentials in terms of piecewise functions, each with quartic asymptotic behavior, is that the artificial boundary conditions of the computational domain at large  $|\vec{\phi}|$  are then essentially irrelevant since linear extrapolation for its field derivatives is justified. This allows us to focus on the small- $|\vec{\phi}|$  region and the handling of the



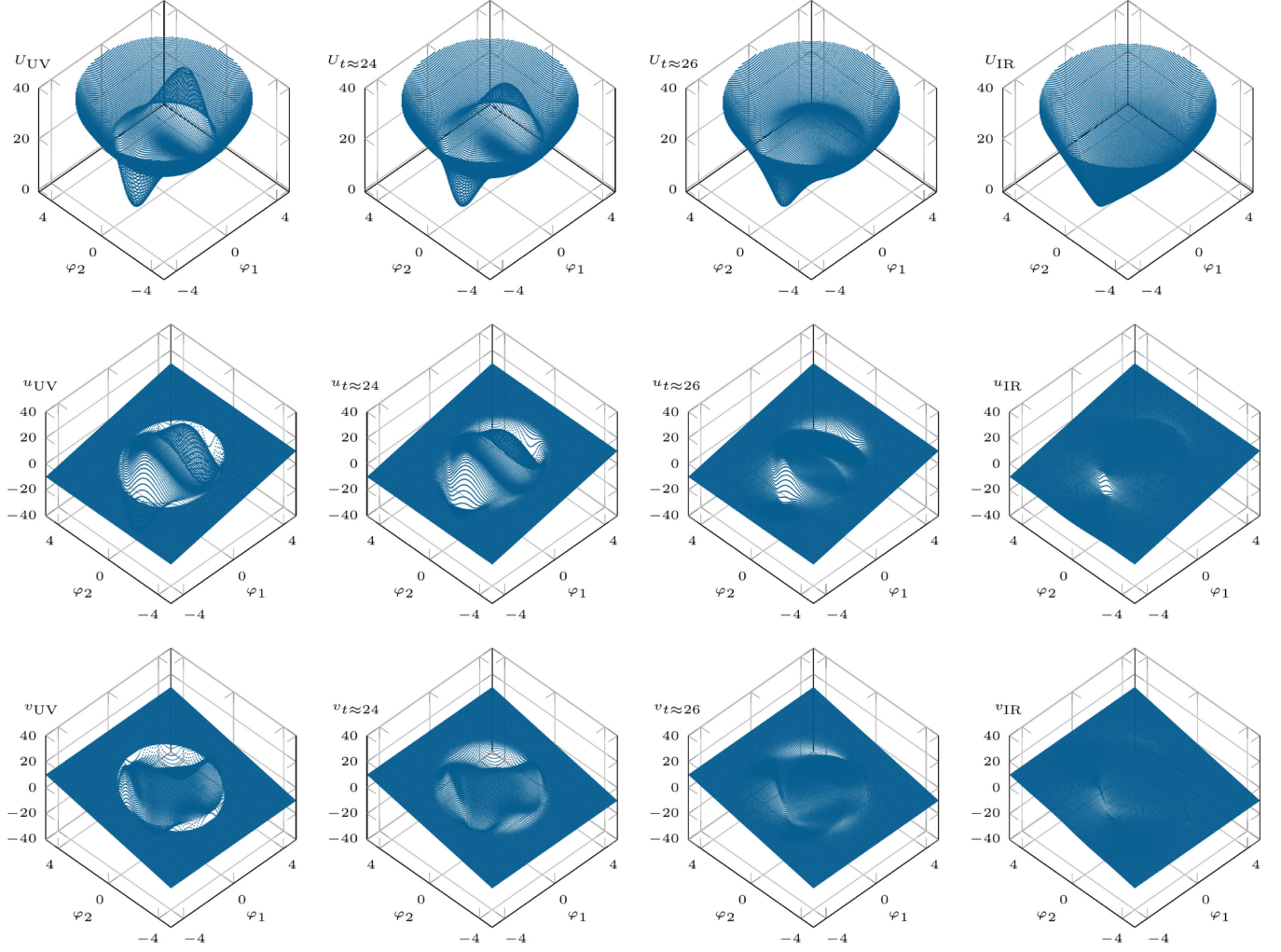


FIG. 6. The RG time evolution of the potential  $U(t, \vec{\phi})$  (upper row), its  $\phi_1$  derivative  $u(t, \vec{\phi})$  (middle row), and  $\phi_2$  derivative  $v(t, \vec{\phi})$  (lower row) from the UV (left column) to the IR (right column) and selected intermediate times for test case V, see Eq. (93).

dynamics of the fluid fields  $u$  and  $v$  during the RG flow via the diffusive part of the KT scheme. Numerical errors should therefore not stem from the boundary conditions but solely from the discretization of the PDE and the time stepping.

#### A. Test case V: Nonvanishing field expectation value

The corresponding UV potential that we created reads

$$U(\vec{\phi}) = \begin{cases} 2(\phi_1^3 + \phi_1\phi_2) \left[ \cos\left(\frac{\pi\phi_2^2}{9}\right) + 1 \right] & \text{if } |\vec{\phi}| \leq 3, \\ \vec{\phi}^2 - 9 & \text{if } |\vec{\phi}| > 3, \end{cases} \quad (93)$$

and is illustrated in Fig. 6 (upper left panel). We constructed this potential for the following reasons: First, its asymptotic behavior ensures that the potential is bounded from below and the probability distribution is well defined. Furthermore, the asymptotic behavior allows for a linear extrapolation of

the field derivatives at the boundary of the computational domain. The regime defined by  $|\vec{\phi}| \leq 3$  represents the actual challenge in calculations. Here, the UV potential has one global minimum at  $\vec{\phi}_{\min} \simeq (-2.07, 0.33)^T$ . Additionally, there is no symmetry in the small- $|\vec{\phi}|$  region, which causes the IR minimum to be nontrivial too. Inspecting the derivatives of the potential, see Fig. 6 (middle left and lower left panel), we also find that these contain cusps as well as rather large gradients, which pose additional challenges in numerical calculations.

Indeed, from the general formula for the correlation functions, Eq. (2), we find by direct numerical integration (using NIntegrate [89] with a PrecisionGoal and AccuracyGoal of 12)

$$\langle \vec{\phi} \rangle = \vec{\phi}_{\min} = \begin{pmatrix} -2.040906130(6) \\ 0.330687529(6) \end{pmatrix} \quad (94)$$

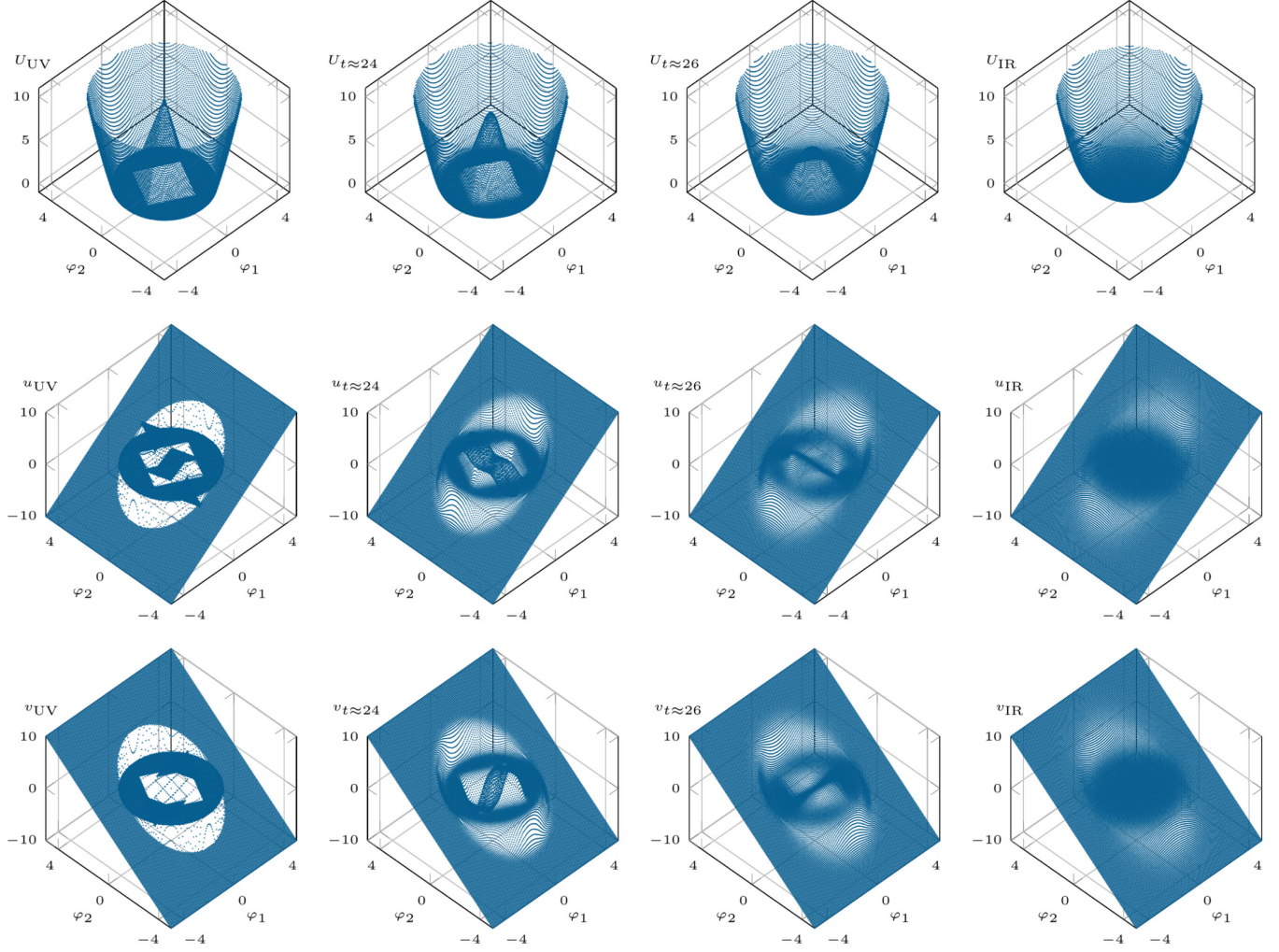


FIG. 7. The RG time evolution of the potential  $U(t, \vec{\varphi})$  (upper row), its  $\varphi_1$  derivative  $u(t, \vec{\varphi})$  (middle row), and  $\varphi_2$  derivative  $v(t, \vec{\varphi})$  (lower row) from the UV (left column) to the IR (right column) and selected intermediate times for test case VI, Eq. (97), for  $\alpha = 0.3$ .

for the expectation values of the fields. The IR minimum of the potential computed with our FRG formalism must be identical to this result. As expected from the UV potential, see Fig. 6 (upper left panel), the field expectation value is at negative  $\varphi_1$  and minimally shifted toward positive  $\varphi_2$  and therefore almost identical to the UV minimum. Hence, the question is whether the field-space resolution of the KT scheme suffices to correctly locate this minimum and therefore to correctly reproduce the result in Eq. (94). From Eq. (2), we can also directly calculate the two-point correlation functions (using the same numerical integration routine as above)

$$\langle \phi_i \phi_j \rangle = \begin{pmatrix} 4.184919188(1) & -0.661996231(7) \\ -0.661996231(7) & 0.188507337(2) \end{pmatrix}_{ij}. \quad (95)$$

Using Eq. (5), this leads us to the following values of the two-point vertex functions at the IR minimum:

$$\Gamma_{ij}^{(2)} = \begin{pmatrix} 57.087319617(0) & -9.308132636(9) \\ -9.308132636(9) & 14.151443165(5) \end{pmatrix}_{ij}. \quad (96)$$

Again, the question is whether it is possible to extract these values with high precision at the minimum of the IR potential that was obtained from the KT scheme in our fluid-dynamical FRG approach. Both the field expectation value and the two-point vertex functions will serve as benchmarks for the KT scheme.

### B. Test case VI: Misalignment of symmetry axes and discretization axes

As another nontrivial test case, we consider a potential that is only asymptotically invariant under  $O(2)$  transformations of the fields, but not in the small- $|\vec{\phi}|$  region, for the same reasons as in the previous test case. However, the potential we consider below comes with a global  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry, which should not be broken by the RG flow. In consequence, we know from Eq. (2) that the IR minimum



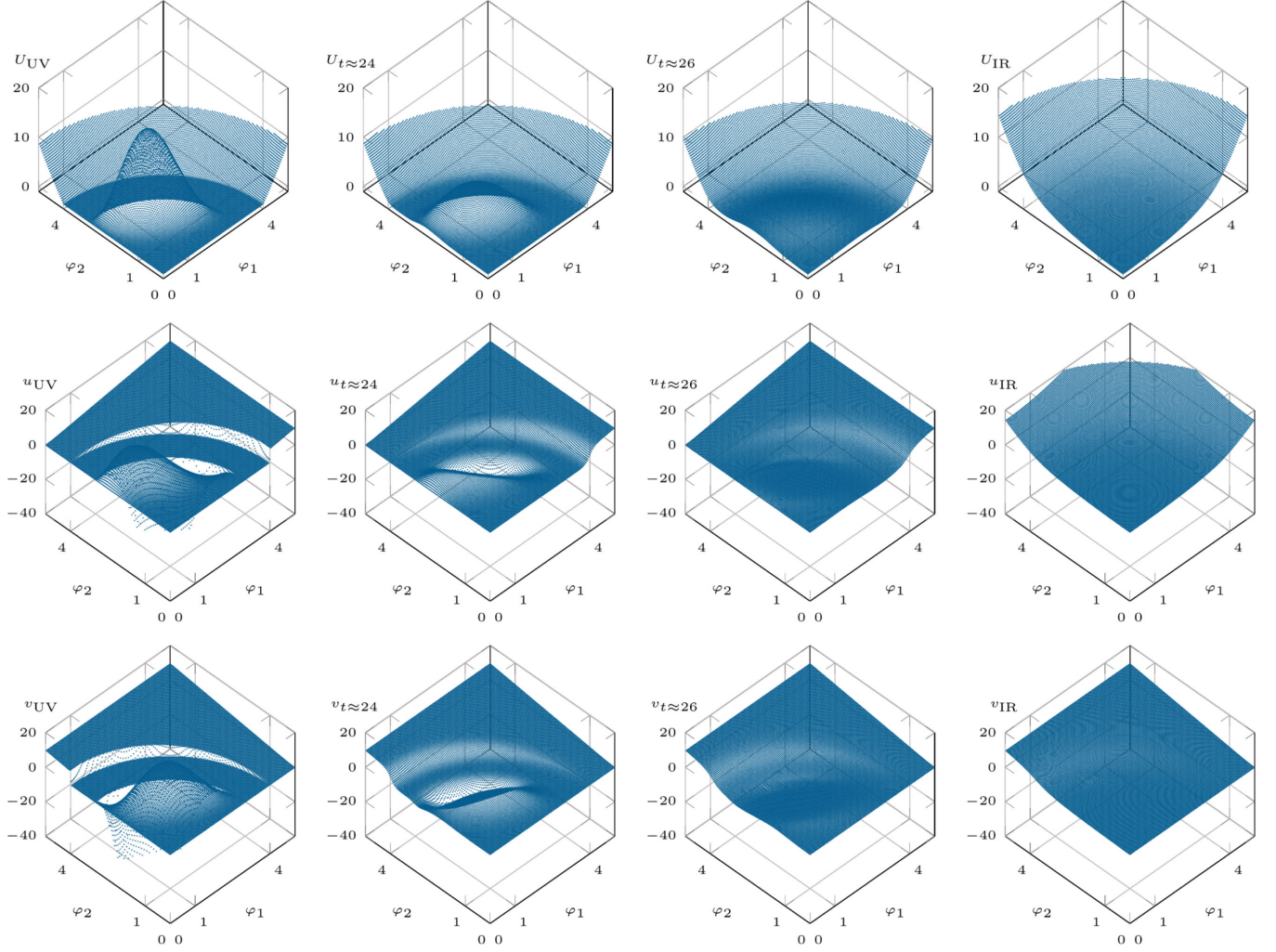


FIG. 8. The RG time evolution of the potential  $U(t, \sigma_1, \sigma_2)$  (upper row), its  $\sigma_1$  derivative  $u(t, \sigma_1, \sigma_2)$  (middle row), and  $\sigma_2$  derivative  $v(t, \sigma_1, \sigma_2)$  (lower row) from the UV (left column) to the IR (right column) and selected intermediate times for test case VII, Eq. (101).

of the potential is at  $\langle \vec{\phi} \rangle = \vec{\phi} = 0$  due to the symmetry. Furthermore, the two-point correlation and vertex functions have to be diagonal.

The way we use this potential to challenge our numerical setup is by misaligning the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry axes of the potential with the discretization axes of the computational domain. On top of that, we introduce jump discontinuities in the derivatives of the potential, which additionally complicates the problem. A particular UV potential that satisfies these specifications is given by

$$U(\vec{\phi}) = 5 \cdot (2 - |\theta_1| - |\theta_2|) \cdot \Theta(2 - |\theta_1| - |\theta_2|) + \Theta(\vec{\phi}^2 - 9) \cdot (\vec{\phi}^2 - 9), \quad (97)$$

where  $\Theta$  is the Heaviside step function and

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \quad (98)$$

For a visualization of the potential, we refer to Fig. 7 (upper left panel). In the middle and lower left panels of the same figure, we present the field derivatives of the potential in the UV, the fluid fields  $u$  and  $v$ , which exhibit several jumps. The angle  $\alpha$  is used to rotate the symmetry axes of the potential relative to the discretization axes of the computational domain. Below, we shall perform tests for different values of  $\alpha \in [0, \frac{\pi}{4}]$  by comparing the two-point vertex function extracted from the KT scheme with the exact values, which are

$$\langle \phi_{1/2}^2 \rangle = 3.041541448(7), \quad (99)$$

$$\Gamma_{11/22}^{(2)} = 0.328780658(4), \quad (100)$$

independent of the choice of  $\alpha$ .

### VIII. ZERO-DIMENSIONAL TEST MODEL WITH $O(\tilde{N}) \times O(\tilde{M})$ SYMMETRY

The last zero-dimensional test model is a potential with  $O(\tilde{N}) \times O(\tilde{M})$  symmetry where  $\tilde{N}$  and  $\tilde{M}$  are arbitrary integers. We consider a model of this type because, on the level of the fluid-dynamical formulation of the FRG, it also introduces advection terms in the PDEs for the fluid fields  $u$  and  $v$  for  $\tilde{N} > 1$  and/or  $\tilde{M} > 1$ , see Eq. (45). To be specific, we consider

$$\tilde{U}(\rho_1, \rho_2) = 4\rho_1\rho_2^2 \sin\left(\frac{2\pi}{9}(\rho_1 + \rho_2)\right) \Theta(4.5 - \rho_1 - \rho_2) + 2(\rho_1 + \rho_2 - 8)\Theta(\rho_1 + \rho_2 - 8), \quad (101)$$

where  $\rho_{1/2}$  are the field invariants. For a visualization of the potential, we refer to Fig. 8 (upper left panel). However, note that the potential is shown as a function of the background field configuration  $\sigma_{1/2}$ , see Sec. III F. The figure also shows the derivatives of the potential with respect to  $\sigma_{1/2}$  in the UV, i.e., the fluid fields  $u$  and  $v$ , see Fig. 8 (middle and lower left panel). Again, we defined the potential piecewise to avoid problems with the boundaries of the computational domain at large  $|\vec{\phi}|$ . On the other hand, for small  $\rho_{1/2}$ , the potential comprises nontrivial  $O(\tilde{N}) \times O(\tilde{M})$ -invariant interactions that are, however, not  $O(N = \tilde{N} + \tilde{M})$  invariant as it is the case for the asymptotic behavior. In contrast to the previous test cases, we require additional boundary conditions at  $\rho_{1/2} = 0 = \sigma_{1/2}$ . Hence, our study of this model implicitly also tests the correct handling of the boundary conditions at  $\rho_{1/2} = 0$ . In order to benchmark our numerical framework, we again employ the two-point vertex functions since the global minimum of the potential is at  $\vec{\phi} = 0$ , as a consequence of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry, see Sec. IV B. Here and in the following, without loss of generality, we consider  $\tilde{N} = 2$  and  $\tilde{M} = 3$ . The exact reference values, which we obtained from a direct numerical integration, read

$$\langle \phi_i^2 \rangle = 2.700144567(9), \quad \Gamma_{ii}^{(2)} = 0.370350540(4), \quad (102)$$

for  $i \in \{1, \dots, \tilde{N}\}$  and

$$\langle \phi_i^2 \rangle = 2.683149014(3), \quad \Gamma_{ii}^{(2)} = 0.372696408(1), \quad (103)$$

for  $i \in \{\tilde{N} + 1, \dots, N = \tilde{N} + \tilde{M}\}$ . Cross-correlations are zero, as expected from the symmetry of the potential.

### IX. FRG RESULTS FOR ZERO-DIMENSIONAL MODELS

In this section, we present our FRG results of the zero-dimensional models introduced above and compare them to the exact results. We start the discussion with the  $O(2)$ -symmetric models from Sec. VI and then proceed with the models without  $O(2)$  symmetry from Sec. VII. Finally, we discuss the  $O(\tilde{N}) \times O(\tilde{M})$ -symmetric model as introduced in Sec. VIII.

#### A. Zero-dimensional test models with $O(2)$ symmetry

Before we dive into the detailed quantitative analysis, we start with a brief qualitative discussion of the RG flows of the  $O(2)$ -symmetric test cases I–IV. Furthermore, we already provide the numerical parameters used for all tests in Table II.

##### 1. Qualitative discussion of RG flows in two field-space dimensions

In Figs. 2–5, we show  $u_{UV} = u_{t=0}$ ,  $u_{IR} = u_{t=60}$ ,  $U_{UV} = U_{t=0}$ , and  $U_{IR} = U_{t=60}$  together with the corresponding figures for two intermediate RG times for the different test cases I–IV with  $n_{\text{cells}} = 175$ . For all these test cases, we find that the IR potential is convex and smooth, and that the global minimum is at the origin, as it should be for every  $O(2)$ -symmetric zero-dimensional model. No spurious oscillations or discontinuities are found in the IR. Furthermore, we see that the KT scheme is indeed able to handle the nonanalytic points  $|\vec{\phi}| = 2$  and  $|\vec{\phi}| = 3$  of test case I as well as the poles of  $u$  and  $v$  at  $\phi_{1/2} = 0$  of test case IV; see Figs. 2 and 5, respectively. In addition, at first glance, it seems that the IR potentials of all  $O(2)$ -symmetric test cases are again  $O(2)$  invariant. Numerically, however, this is not perfectly the case for the initial condition as we shall discuss in detail in the following paragraph.

##### 2. Quantitative benchmark tests

*Error scaling of the two-point vertex function.* In order to estimate discretization errors from the two-dimensional KT scheme, we follow Ref. [37] and study the relative error of

TABLE II. Numerical control parameters used for the various test models in zero spacetime dimensions. For the integration of the flow equations, we have used RK45.

| Test case | $\varphi_{\text{max}}$ | $\Lambda$ | $t_{\text{IR}}$ | $r_{\text{tol}}$ | $a_{\text{tol}}$ |
|-----------|------------------------|-----------|-----------------|------------------|------------------|
| I (89)    | 10                     | $10^{12}$ | 60              | $10^{-10}$       | $10^{-12}$       |
| II (90)   | 10                     | $10^{12}$ | 60              | $10^{-10}$       | $10^{-12}$       |
| III (91)  | 10                     | $10^{12}$ | 60              | $10^{-10}$       | $10^{-12}$       |
| IV (92)   | 10                     | $10^{12}$ | 60              | $10^{-10}$       | $10^{-12}$       |
| V (93)    | 6                      | $10^{12}$ | 60              | $10^{-12}$       | $10^{-12}$       |
| VI (97)   | 10                     | $10^{12}$ | 60              | $10^{-10}$       | $10^{-12}$       |
| VII (101) | 7                      | $10^{12}$ | 60              | $10^{-12}$       | $10^{-12}$       |

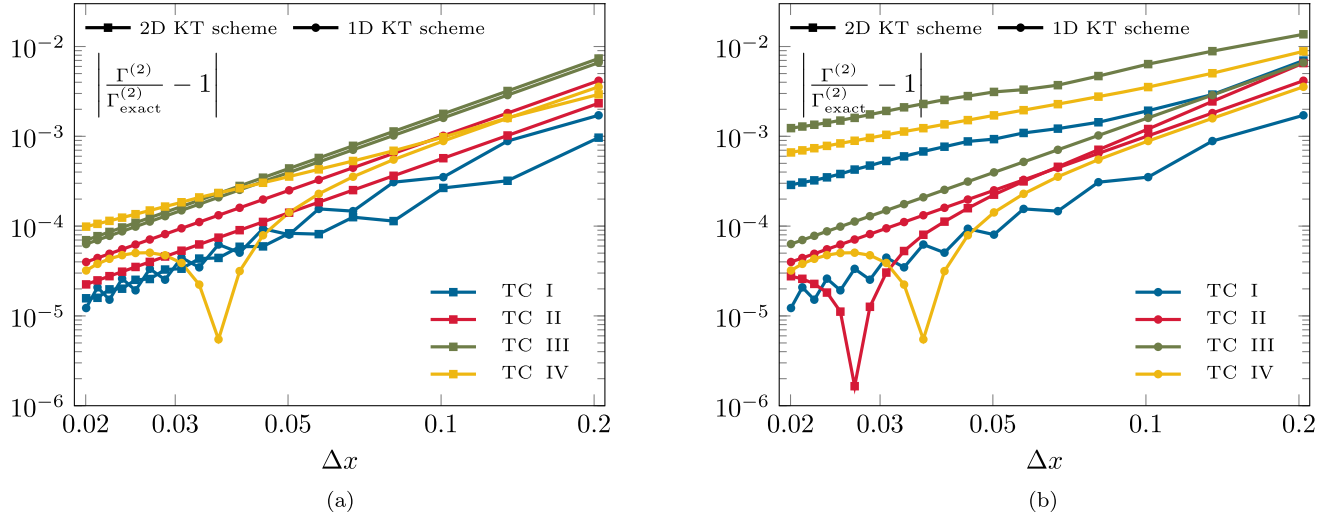


FIG. 9. The relative error (104) of  $\Gamma^{(2)}$  for test cases (TCs) I–IV associated with the UV potentials (89)–(92) as a function of the numerical resolution  $\Delta x$ . The two-point function  $\Gamma^{(2)}$  has been obtained from Eq. (76) with the solution of the one-dimensional PDE in Eq. (29) (1D) and the two-dimensional PDE system in Eqs. (18) and (19) with the 2D KT scheme. The exact results for  $\Gamma^{(2)}$  can be found in Table I. (a) Adapted 2D KT scheme as presented in Sec. IV B for different TCs. (b) Original 2D KT scheme as presented in Ref. [80] for different TCs.

the two-point vertex functions for test cases I–IV, i.e., we study

$$\left| \frac{\Gamma^{(2)}}{\Gamma_{\text{exact}}^{(2)}} - 1 \right| \quad (104)$$

as depicted in Fig. 9. Here,  $\Gamma^{(2)}$  is the two-point vertex function computed from the KT scheme via the finite-difference stencil (76) and  $\Gamma_{\text{exact}}^{(2)}$  is the two-point vertex function that is determined by Eqs. (86) and (88) for  $n = 2$ . The exact values of  $\Gamma^{(2)}$  are listed in Table I. Furthermore, for comparison, we also include the relative errors of the 1D KT scheme for the same test cases in Fig. 9. For both the one- and two-dimensional calculations, we have used the parameters summarized in Table II. Looking at Fig. 9(a), we observe that the relative errors decrease as we increase the number of cells. As one would expect, the relative error of the two-dimensional method slightly deviates from the one-dimensional one for the same field-space resolution  $\Delta x$ , due to a different Taylor coefficient in front of the error scaling but has approximately the same scaling exponent. To be explicit, we find an error scaling for the cases I–IV as

TABLE III. Error-scaling exponent  $n$  extracted from the error scaling  $\Delta x^n$  from Fig. 9. [2D\* denotes the original version of the two-dimensional KT scheme as presented in Ref. [80], see Fig. 9(b)].

|           | TC I      | TC II | TC III | TC IV |
|-----------|-----------|-------|--------|-------|
| $n(1D)$   | 2.00–2.04 | 2.00  | 2.00   |       |
| $n(2D)$   | 1.94–1.95 | 2.00  | 2.00   | 1.47  |
| $n(2D^*)$ | 1.48      |       | 1.07   | 1.05  |

listed in Table III, respectively. These error scalings are in good agreement with the expected error scaling of  $\Delta x^2$ . Note that one usually does not find perfect agreement of the error scaling with the expected one, since the error scaling is only an approximation and the error scaling exponent is not exactly 2. Especially in the presence of, e.g., non-analyticities the error scaling can be of lower order. Interestingly, we observe that the KT scheme in its original form as presented in Ref. [80] systematically leads to an error scaling below the expected one; also for smooth potentials, see Fig. 9(b). We shall even see that the computations with the original KT scheme do not converge at all and spuriously oscillations pop up in the solutions in case of the other test models to be discussed below. Hence, we suggest to use our adapted version of the KT scheme as presented in Sec. IV B for FRG applications (and possibly also for other applications).<sup>13</sup> Note that the number of coupled ODEs for the two-dimensional calculations, in general, increases quadratically as a function of  $n_{\text{cells}}$ , namely, by  $8n_{\text{cells}}^2 \approx 8(\frac{\varphi_{\text{max}}}{\Delta x})^2$  for large  $n_{\text{cells}}$ , as compared with the 1D KT scheme where the number of coupled ODEs increases linearly with  $n_{\text{cells}}$ . For this reason, we have only performed calculations down to the value  $\Delta x = 0.02$ . In fact, already for  $\Delta x = 0.02$  with  $\varphi_{\text{max}} = 10$ , we have a

<sup>13</sup>Note that these adaptations solely affect the diffusion part of the higher-dimensional version of the KT scheme, which was not the main focus of Ref. [80], and that the advection part is not affected. Furthermore, in Ref. [88], where almost linear PDEs have been considered and the code underlying our present studies has been employed, negligible differences between our and the original version of the two-dimensional diffusion has been observed.



system of  $2 \times 10^6$  coupled ODEs and hence the required computing time to solve this set of equations is rather long and will even increase for smaller  $\Delta x$ . For future applications, one might therefore consider parallelization or adaptive mesh refinement with similar discretization schemes. For an estimate of the required run times, see Appendix A.

Overall, the behavior of the relative error, as depicted in Fig. 9(a), indicates that the two-dimensional KT scheme is suitable for studies of  $O(2)$ -symmetric zero-dimensional models, although the 1D KT scheme, as discussed in great detail in Ref. [37], performs much better with respect to run-time and slightly better for test case IV with the pole at  $\vec{\varphi} = 0$ .

*Global error measures.* As already mentioned in the previous sections, there is another issue that we have to address, namely, the artificial breaking of the  $O(2)$  symmetry due to the Cartesian grid. This is the subject of the following paragraphs. Before we return to the observables, we briefly comment on the remnants of the  $O(2)$  symmetry on the level of the cell averages of our Cartesian grid. First, we note that the diffusion fluxes on the rhs of the RG flow equations of  $\partial_t u$  and  $\partial_t v$  are identical, cf. Eqs. (18) and (19). Therefore, for  $O(2)$ -symmetric initial conditions we find a mirror symmetry with respect to the first bisector, namely,

$$\partial_t u(x, y) = \partial_t v(y, x) \quad \Leftrightarrow \quad u(x, y) = v(y, x). \quad (105)$$

This symmetry is still present at the level of the cell averages in the KT scheme and even the Riemann summation or the construction of the initial condition, Eq. (75), do not alter it. Hence, the following equations are trivially fulfilled:

$$\bar{u}_{i,j} = \bar{v}_{j,i}, \quad (106)$$

$$y_j \bar{u}_{j,i} + x_i \bar{v}_{j,i} = x_i \bar{u}_{i,j} + y_j \bar{v}_{i,j}, \quad (107)$$

$$\bar{U}_{i,j}^u = \bar{U}_{j,i}^v, \quad (108)$$

where  $\bar{U}^{u/v}$  denotes the Riemann sum along  $\bar{u}$  and  $\bar{v}$ , respectively.<sup>14</sup> Here, we have chosen the reference point for the integration at some point within the first bisector. Moreover, we find two additional symmetries that are still present in the two-dimensional KT scheme: We have mirror symmetries with respect to the  $x$  and  $y$  direction for  $\bar{v}$  and  $\bar{u}$ , respectively, i.e.,

<sup>14</sup>In the matrix formulation as introduced in Appendix B, we find accordingly

$$\mathbf{u}_i^T = \mathbf{v}_i, (\mathbf{x}_C \cdot \mathbf{u} + \mathbf{y}_C \cdot \mathbf{v})^T = \mathbf{x}_C \cdot \mathbf{u} + \mathbf{y}_C \cdot \mathbf{v},$$

where  $\mathbf{u} = \mathbf{u}[0]$ ,  $\mathbf{v} = \mathbf{u}[1]$ ,  $\mathbf{x}_C = \mathbf{x}_C[0]$ , and  $\mathbf{y}_C = \mathbf{y}_C[0]$ .

$$\bar{u}_{i,j} = \bar{u}_{i,-j} \quad \text{and} \quad \bar{v}_{i,j} = \bar{v}_{-i,j}, \quad (109)$$

and reflection symmetries with respect to the  $x$  and  $y$  direction for  $\bar{u}$  and  $\bar{v}$ , respectively, i.e.,

$$\bar{u}_{i,j} = -\bar{u}_{-i,j} \quad \text{and} \quad \bar{v}_{i,j} = -\bar{v}_{i,-j}. \quad (110)$$

All three symmetries together imply that the whole information of the system is governed by a single quadrant of  $\bar{u}_{i,j}$  or equivalent of  $\bar{v}_{i,j}$ . Furthermore, this analysis reduces the number of proper observables for measuring the deviation of the  $O(2)$  symmetry. For example, the function

$$\mathcal{O} = \max_{i,j} (|\text{rot}_{90^\circ}(\bar{v})_{i,j} - \bar{u}_{i,j}|) \quad (111)$$

is trivially fulfilled, meaning that  $\mathcal{O} \approx 0$ . Another possible observable could be of the form

$$\mathcal{O} = \max_{i,j} (|\text{rot}_{90^\circ}(A)_{i,j} - A_{i,j}|), \quad (112)$$

where  $A_{i,j} = x_i \bar{u}_{i,j} + y_j \bar{v}_{i,j}$ . However, this is not suited for measuring the failure of the  $O(2)$  symmetry since it vanishes also for an elliptic shape of  $A_{i,j}$ . Therefore, the only observables that we are using are Eqs. (79) and (80),

$$\mathcal{O}_{U,L^\infty} = \max_{i,j} (|\text{rot}_{90^\circ}(\bar{U})_{i,j} - \bar{U}_{i,j}|),$$

$$\mathcal{O}_{u,L^\infty} = \max_{i,j} (|\text{rot}_{90^\circ}(\bar{u}/x)_{i,j} - \bar{u}_{i,j}/x_i|),$$

and Eqs. (77) and (78),

$$\mathcal{O}_{U,L^1} = \frac{1}{\mathbf{n}(i,j)} \sum_{i,j} (|\text{rot}_{90^\circ}(\bar{U})_{i,j} - \bar{U}_{i,j}|),$$

$$\mathcal{O}_{u,L^\infty} = \frac{1}{\mathbf{n}(i,j)} \sum_{i,j} (|\text{rot}_{90^\circ}(\bar{u}/x)_{i,j} - \bar{u}_{i,j}/x_i|),$$

respectively. The latter ones are of special interest since they only measure the deviation of the  $O(2)$  symmetry induced by the use of the KT scheme and not from the additional Riemann sum, which is required to reconstruct the potential. It is motivated by Eq. (24): If the underlying potential  $U(t, \vec{\varphi})$  is  $O(2)$  symmetric then  $\frac{1}{\varphi_1} u(t, \vec{\varphi}) = \partial_q \bar{U}(t, q)$  has the same property.

The reason for the implementation of both measures in terms of the  $L^1$  and  $L^\infty$  norms is that the  $L^1$  norm is more sensitive to the average behavior of the solution, whereas the  $L^\infty$  norm is more sensitive to the maximum deviation. If both observables are of the same size, then the deviation of the  $O(2)$  symmetry is uniformly distributed over the whole grid. Note that for the calculation of both observables we have restricted ourselves to the inclusion of cells from the domain  $[-9, 9] \times [-9, 9]$ . Leaving out the outermost 10%

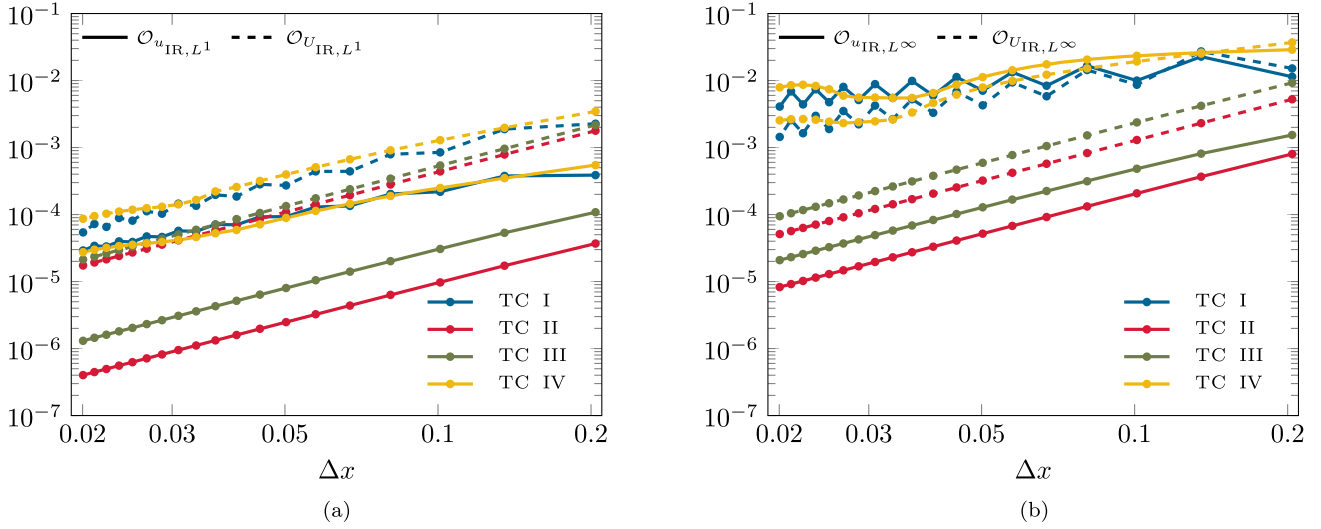


FIG. 10.  $O(2)$ -symmetry observables  $\mathcal{O}_{u_{\text{IR}}}$  (solid lines) and  $\mathcal{O}_{U_{\text{IR}}}$  (dashed lines) as functions of  $\Delta x$  for test cases I–IV, Secs. VIB 1–VIB 4. (a)  $L^1$  norm, Eqs. (77) and (78). (b)  $L^\infty$  norm, Eqs. (79) and (80).

of the cells ensures that boundary effects, which do not propagate into the inner region, are not overemphasized.

In Fig. 10, all four functions, evaluated in the IR, are shown for several values of  $\Delta x$ . Both observables are, in general, nonzero, but decrease for decreasing  $\Delta x$ . This implies that the  $O(2)$  symmetry is, as already anticipated, not perfectly conserved by our numerical scheme but becomes better approximated for smaller  $\Delta x$ . We extracted corresponding approximate error scalings (where appropriate), which we list in Table IV. Again, the ideal expected error scaling is  $\Delta x^2$  and we only find slight deviations from this scaling, especially for the  $L^1$  norms, where we have the averaging over all cells. We conclude that the deviation from perfect  $O(2)$  symmetry is indeed caused by the finite resolution  $\Delta x$  of the Cartesian grid, similar as for spacetime lattices in lattice Monte Carlo simulations.

Still, let us study this effect even further and investigate if symmetry violations actually increase or decrease during the RG flow.

In fact, we find that the  $O(2)$  symmetry is already spoiled in the UV by the initial condition, as can be seen in Fig. 11 for test cases I and IV. The reason for this arises from the fact that we are assuming in all observables that the value of the cell average  $\bar{u}_{i,j}$  is located at the cell center  $(x_i, y_j)$ . However, this is not always fulfilled. In particular,

this is not fulfilled for test cases that come with jumps or discontinuities of  $u$  within a single cell, which is indeed the case, e.g., for test cases I and IV, see Fig. 11. This is also confirmed by Figs. 12 and 13, where we observe that, for increasing number of cells, the observables  $\mathcal{O}_{U_i}$  only decrease during the RG flow. For completeness, we also show these observables for test cases II and III in Figs. 14 and 15. Overall, we find that the deviation from perfect  $O(2)$  symmetry slightly grows during the RG flow, whereas it is, in general, possible to reduce the symmetry violation by increasing the number of cells.<sup>15</sup> Hence, these tests confirm the reliability of the KT scheme for  $O(2)$ -symmetric zero-dimensional models. Moreover, they suggest that our numerical scheme also works reliably for nonsymmetric situations as well as in higher spacetime dimensions because symmetry-breaking artifacts solely introduced by the discretization can be systematically reduced.

## B. Zero-dimensional test models without $O(2)$ symmetry

Now we turn to the zero-dimensional test cases with two field-space directions without  $O(2)$  symmetry in their initial condition, see Sec. VII for their definition.

TABLE IV. Scaling exponent  $n$  extracted from the scaling  $\Delta x^n$  from Fig. 10.

|                                            | TC I      | TC II | TC III | TC IV |
|--------------------------------------------|-----------|-------|--------|-------|
| $n(\mathcal{O}_{u_{\text{IR}}, L^1})$      | 1.28–1.35 | 1.98  | 1.98   |       |
| $n(\mathcal{O}_{U_{\text{IR}}, L^1})$      | 1.78–1.83 | 2.00  | 2.00   |       |
| $n(\mathcal{O}_{u_{\text{IR}}, L^\infty})$ | 0.57–0.66 | 2.00  | 1.98   |       |
| $n(\mathcal{O}_{U_{\text{IR}}, L^\infty})$ | 1.19–1.31 | 2.00  | 2.00   |       |

<sup>15</sup>For RG times  $t \sim 10$ – $30$ , we observe a significant change in the  $O(2)$ -symmetry observables. The precise value of these RG times is eventually determined by our choice for the initial RG scale  $\Lambda$  which sets the scale for the RG time. Early RG times (corresponding to large scales  $k$ ) can be considered as a “warm-up phase” of the flow, where the dynamics is effectively suppressed by the presence of large values of the RG scale  $k$ . At (very) late RG times, the rhs of the flow equation becomes effectively zero since the RG scale then becomes much smaller than the model scales. In between these regimes (corresponding to RG times  $t \sim 10$ – $30$ ), the effective potential changes significantly. This is also reflected in the  $O(2)$ -symmetry observables in, e.g., Fig. 11. Since this dynamical regime is comparatively large, the initial UV scale must be chosen sufficiently large.

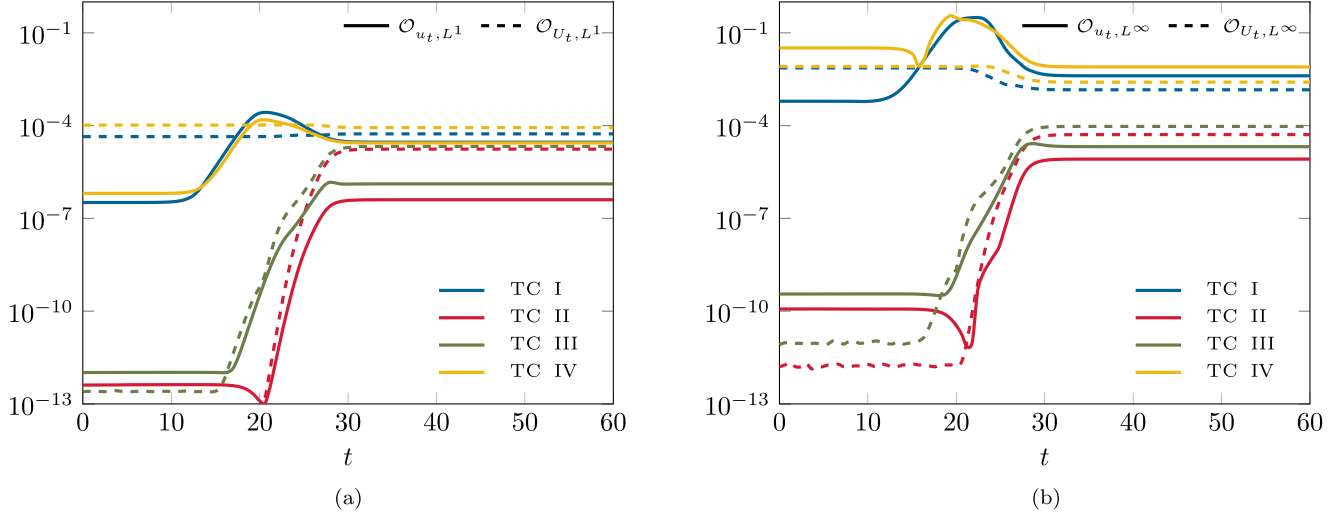


FIG. 11.  $O(2)$ -symmetry observables  $\mathcal{O}_{u_t}$  (solid lines) and  $\mathcal{O}_{U_t}$  (dashed lines) as a function of the RG time for test cases I–IV, Secs. VIB 1–VIB 4, with fixed number of cells  $n_{\text{cells}} = 500$ . (a)  $L^1$  norm, Eqs. (77) and (78). (b)  $L^\infty$  norm, Eqs. (79) and (80).

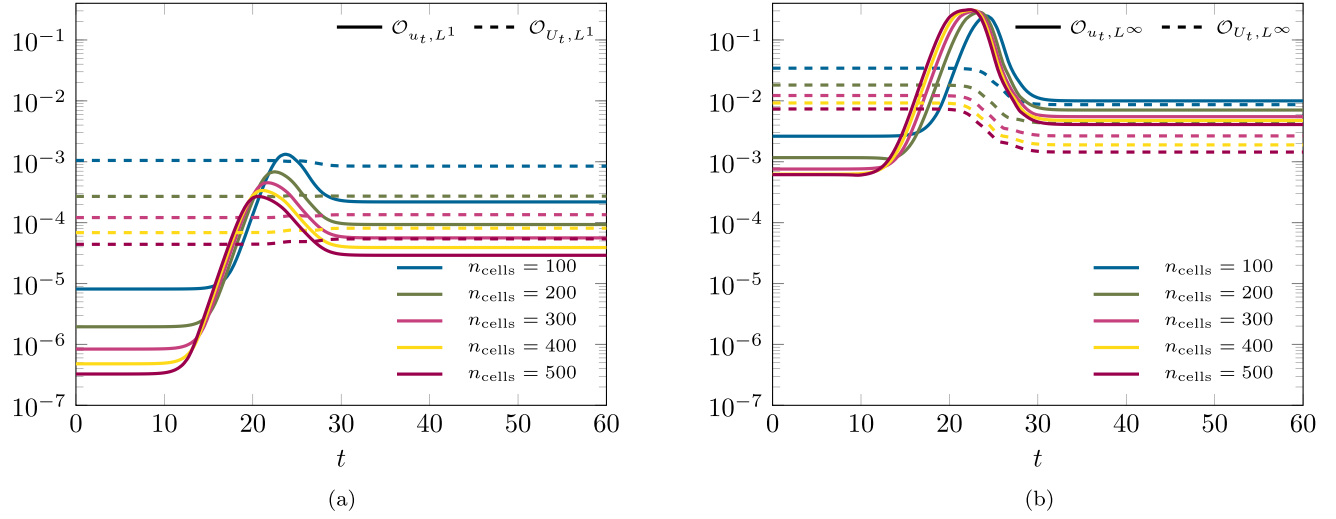


FIG. 12.  $O(2)$ -symmetry observables  $\mathcal{O}_{u_t}$  (solid lines) and  $\mathcal{O}_{U_t}$  (dashed lines) as a function of the RG time for various number of cells for test case I, Sec. VIB 1. (a)  $L^1$  norm, Eqs. (77) and (78). (b)  $L^\infty$  norm, Eqs. (79) and (80).

### 1. Test case V: Nonvanishing field expectation value

We begin with the discussion of the test case V, where the UV potential has no global symmetry anymore. As already discussed in Sec. VII A, this also leads to a nonvanishing expectation value  $\langle \vec{\phi} \rangle$ , i.e., a nontrivial IR minimum. Still, the IR potential has, in general, to be convex and also to be smooth in zero spacetime dimensions. The result is a rather complicated RG flow which is depicted in Fig. 6 for the potential and its derivatives, the “fluid fields.” In general, this setup allows for another test of the KT scheme, namely, the correct location and extraction of the minimum of the IR potential and its comparison with the exact expectation

value of the field  $\vec{\phi}$ . Hence, as a first test, we consider the quantity

$$\left| \frac{\varphi_{\min,i}}{\langle \phi_i \rangle} - 1 \right|, \quad i \in \{1, 2\}, \quad (113)$$

where  $\langle \phi_i \rangle$  is the exact expectation value of the field  $\phi_i$ , Eq. (94), and  $\varphi_{\min,i}$  is the position of the minimum of the IR potential from the KT scheme at resolution  $\Delta x$ . In addition, we analyze the relative error of the two-point vertex function  $\Gamma_{ij}^{(2)}$  extracted from the KT scheme at the IR



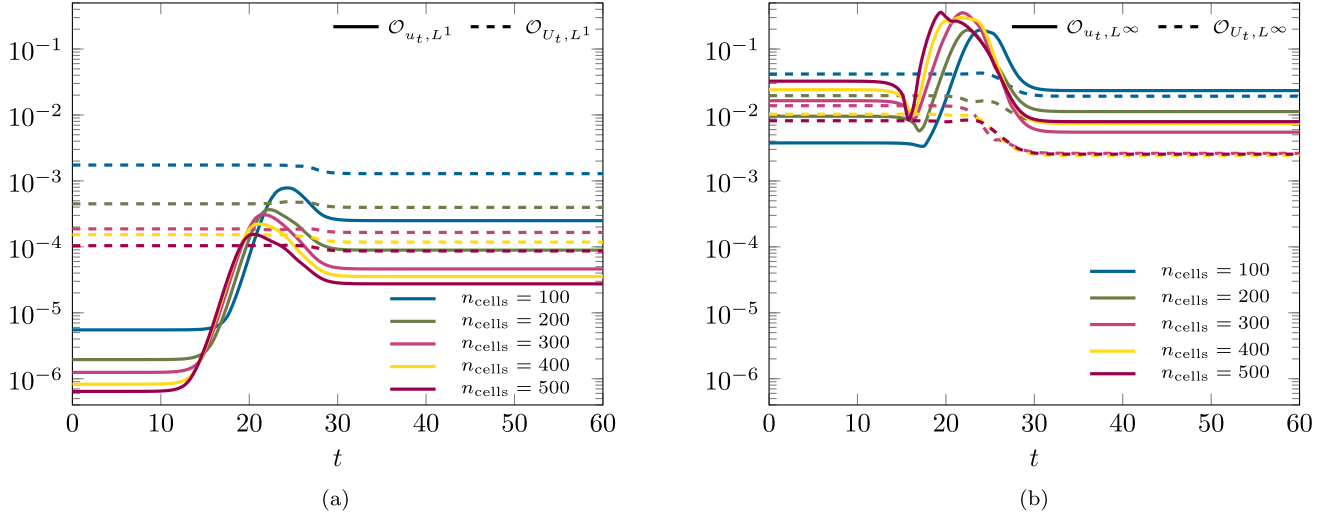


FIG. 13.  $O(2)$ -symmetry observables  $\mathcal{O}_{u_t}$  (solid lines) and  $\mathcal{O}_{U_t}$  (dashed lines) as a function of the RG time for various number of cells for test case IV, Sec. VIB 3. (a)  $L^1$  norm, Eqs. (77) and (78). (b)  $L^\infty$  norm, Eqs. (79) and (80).

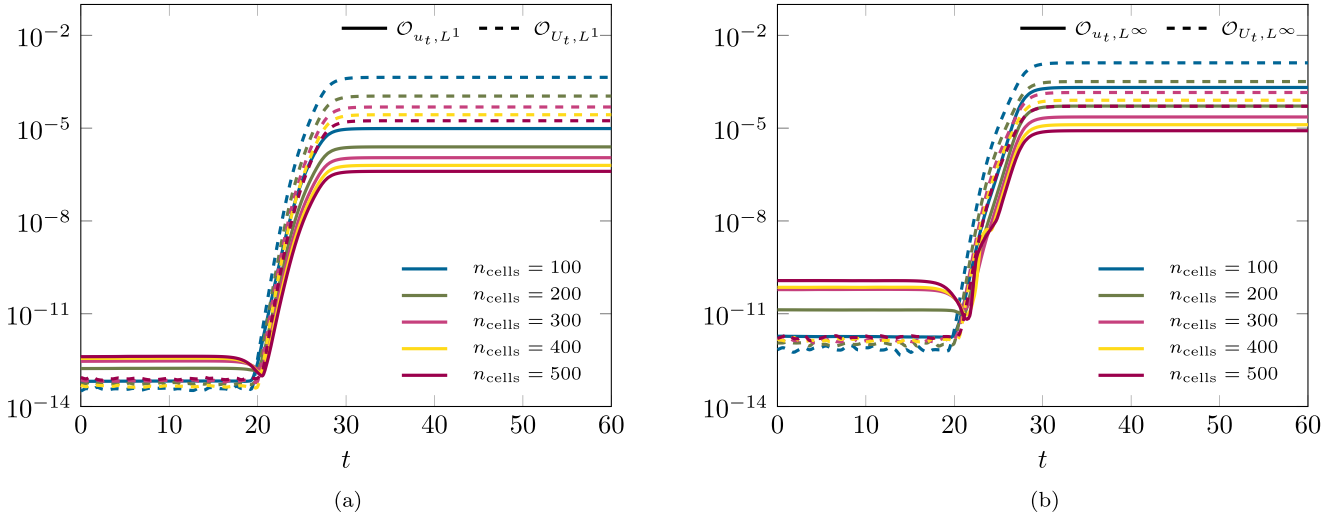


FIG. 14.  $O(2)$ -symmetry observables  $\mathcal{O}_{u_t}$  (solid lines) and  $\mathcal{O}_{U_t}$  (dashed lines) as a function of the RG time for various number of cells for test case II, Sec. VIB 2. (a)  $L^1$  norm, Eqs. (77) and (78). (b)  $L^\infty$  norm, Eqs. (79) and (80).

minimum  $\vec{\phi}_{\min}$  and compare it with the exact  $\Gamma_{ij}^{(2)}$ , see Eq. (96). These tests pose two challenges:

First, the naive extraction of the minimum of the IR potential is only as accurate as the numerical resolution  $\Delta x$ . If  $\langle \phi_i \rangle = \phi_{\min, i}$  is of the same order as  $\Delta x$ , the extraction of the minimum will naturally have a large error. Second, the error from extracting the minimum directly propagates into the error of the two-point vertex function  $\Gamma_{ij}^{(2)}$  because this quantity is evaluated at the minimum of the IR potential.

Often, interpolation of the discrete IR solution of grid-based methods is used to obtain a more accurate estimate of the minimum (and also of the two-point vertex function) or the methods are already based on interpolation right from

the start. However, we remark that interpolation is usually based on splines that require a certain smoothness of the potential about the minimum. In zero spacetime dimensions, this requirement is formally fulfilled but this is not the case for effective potentials in the broken phase in higher-dimensional spacetimes in the IR. In any case, we present both the naive extraction of the minimum and results based on an interpolation with a bivariate spline approximation, `RectBivariateSpline` with spline degree 5 [90].

In Fig. 16, we show the relative error of the minimum of the IR potential and the relative error of the components of the two-point vertex function  $\Gamma_{ij}^{(2)}$  as a function of the numerical resolution  $\Delta x$ . For the only quantity where this is

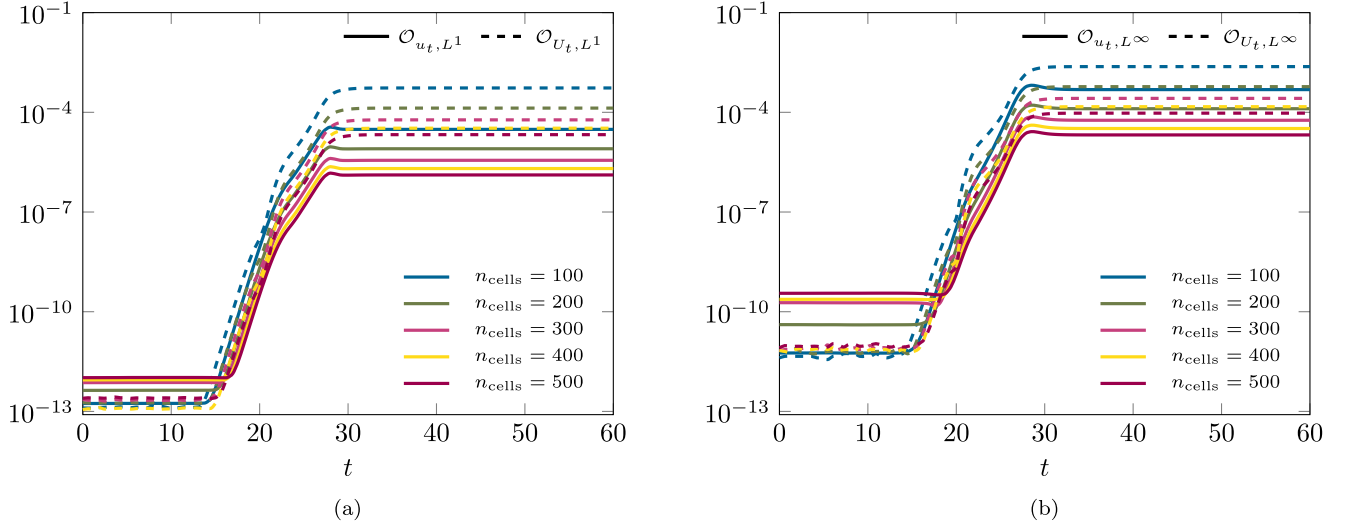


FIG. 15.  $O(2)$ -symmetry observables  $\mathcal{O}_{u_t}$  (solid lines) and  $\mathcal{O}_{U_t}$  (dashed lines) as a function of the RG time for various number of cells for test case III, Sec. VIB 2. (a)  $L^1$  norm, Eqs. (77) and (78). (b)  $L^\infty$  norm, Eqs. (79) and (80).

legitimate, namely,  $\varphi_{\min,1}$ , we can also extract the error scaling exponent. Its value can be found in Table V. Overall, we find that the relative errors of all quantities decrease slowly with increasing numerical resolution  $\Delta x$ . However, as expected for the error of the  $\varphi_1$  component of the minimum, we find a consistent error scaling, which is, however, not as good as the expected  $\Delta x^2$ . For the  $\varphi_2$  component, we indeed observe that the absolute error given by the resolution  $\Delta x$  is of the same order as  $\langle \phi_2 \rangle$  itself, such that the relative error hardly decreases with increasing resolution. As a consequence, the relative errors of the two-point vertex functions  $\Gamma_{ij}^{(2)}$  are rather large. We have improved our results, especially for the relative error of

the position of the minimum, by using an interpolation method, which is also depicted in Fig. 16. However, we would like to emphasize that this interpolation method is only safely applicable in zero spacetime dimensions where the effective potential is smooth about the minimum.

In total, we conclude that the KT scheme is working properly also for zero-dimensional models without  $O(2)$  symmetry. Nevertheless, our tests clearly show that the finite resolution of the grid sets strong limits on the accuracy of observables even if the scheme itself is reliable. For computations in models in higher-dimensional spacetime, this implies that one has to carefully compare the resolution of the grid with the expected scales of the observables.

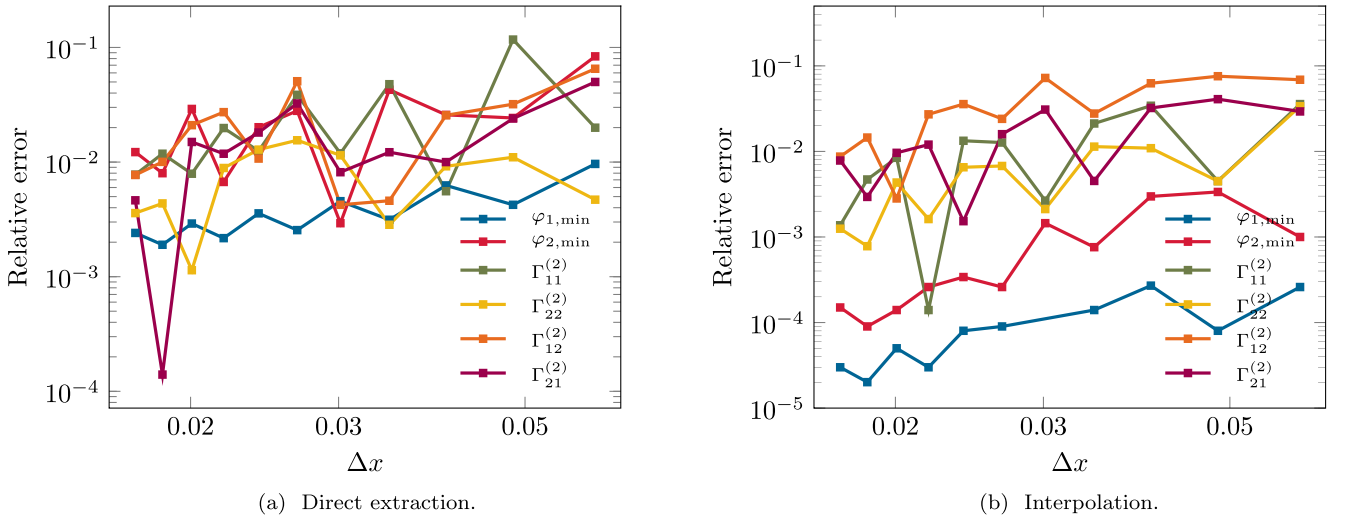


FIG. 16. The relative error for the  $\varphi_1$  and  $\varphi_2$  position of the minimum of the IR potential, Eq. (113) with Eq. (94), for test case V, Sec. VII A, as a function of the numerical resolution  $\Delta x$  as well as the relative error of  $\Gamma_{ij}^{(2)}$ , Eq. (104) with Eq. (96), for the same test case.  $\varphi_{i,\min}$  is extracted from the IR solution of the KT scheme at the cell center with the smallest value of the IR potential (a) or by interpolation (b).

TABLE V. Error scaling exponent  $n$  extracted from the scaling  $\Delta x^n$  corresponding to Fig. 16–18.

| TC                      | $n(\varphi_1)$ | $n(\varphi_2)$ | $n(\Gamma_{11}^{(2)})$ | $n(\Gamma_{22}^{(2)})$ | $n(\Gamma_{12}^{(2)})$ |
|-------------------------|----------------|----------------|------------------------|------------------------|------------------------|
| V                       | 0.8(4)–1.1(0)  |                |                        |                        |                        |
| VI ( $\alpha = 0$ )     |                |                | 1.0(6)                 | 1.0(6)                 | 0                      |
| VI ( $\alpha = 0.3$ )   |                |                | 0.9(6)                 | 0.9(6)                 | 0.4(6)                 |
| VI ( $\alpha = 0.5$ )   |                |                |                        |                        | 0                      |
| VI ( $\alpha = \pi/4$ ) |                |                |                        |                        | 0                      |
| VII                     |                |                | 1.9(0)                 | 1.9(0)                 |                        |

## 2. Test case VI: Misalignment of symmetry axes

The test case VI, see Sec. VII B, might seem extremely artificial for FRG practitioners because piecewise potentials with a pyramid-shaped small- $|\vec{q}|$  region are not expected to appear in any physical situation. However, this test case is ideally suited to investigate how the results from the KT scheme are influenced by a misalignment of the discretization axes associated with the FV discretization and the symmetry axes of the potential. In addition, the chosen potential exhibits edges that cause jump discontinuities in the derivatives. Since these multiple jump discontinuities are also not aligned with the discretization axes, this also poses a test of our construction of the initial condition in terms of cell averages. The misalignment of the symmetry axes is parametrized by the angle  $\alpha$ , see Eq. (98), which we varied in the range  $[0, \pi/4]$ .

As can be seen in Fig. 7, the RG flow associated with this test case is rather involved, while the IR potential is smooth and convex. However, a qualitative analysis of the flow of the potential and its derivatives is not sufficient to judge the quality of the KT scheme. Therefore, we present the relative deviation of the two-point vertex function of the KT scheme from the exact result, see Eq. (100), as a function of the numerical resolution  $\Delta x$ , see Fig. 17. Because of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  symmetry, the IR minimum is trivial and this quantity is extracted at  $\vec{q} = 0$ . Also by symmetry, only the diagonal components of the two-point vertex function are nonvanishing. Nevertheless, we also provide the absolute value of the off-diagonal component in Fig. 17 for the setup, where the symmetry of the pyramid is not aligned with the grid,  $\alpha \neq 0$ . We only show them because the off-diagonal components are numerically not exactly zero, which is a direct consequence of the misalignment and finite numeric resolution, but they have to tend to zero for  $\Delta x \rightarrow 0$ .

In addition to Fig. 17, we provide the error scaling exponents (where appropriate) in Table V. Overall, we find that the relative error of the two-point vertex function decreases with increasing numerical resolution  $\Delta x$ . However, the error scaling is reduced to approximately  $\Delta x^1$ , which is most likely a consequence of the jump discontinuities in the derivatives of the potential. Interestingly, the error scaling exponent is even lower

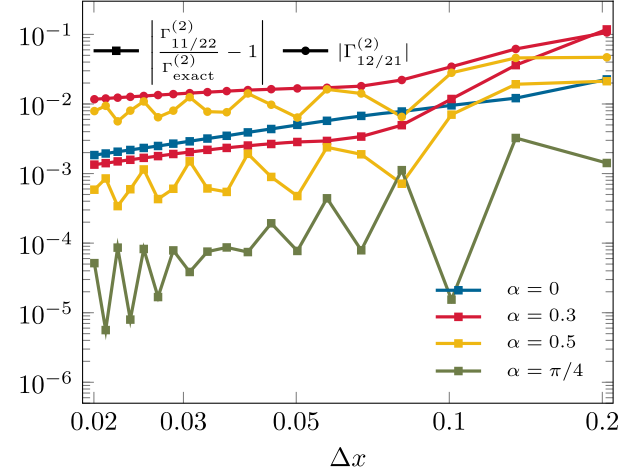


FIG. 17. Relative error (104) of  $\Gamma_{ii}^{(2)}$  for test case VI with the UV potential (97) as a function of the numerical resolution  $\Delta x$  for different misalignments  $\alpha$ , see Eq. (98).  $\Gamma_{ii}^{(2)}$  is obtained via Eq. (76) from the solution of the PDE system with the KT scheme, while the exact  $\Gamma_{ii}^{(2)}$  are taken from Eq. (100). For  $\alpha \neq 0$ , we also provide the absolute value of the off-diagonal component  $\Gamma_{xy}^{(2)}$ .

for the off-diagonal component of the two-point vertex function. In general, this is not a problem but should be kept in mind when applying the KT scheme to models with similar symmetry-breaking features.

In summary, we observe that our numerical framework is still working properly, even under these difficult conditions, but it is very challenging to obtain a very high accuracy for every observable due to very slow convergence with the numerical resolution  $\Delta x$ . However, we would like to emphasize that our results only differ from the exact results on a percentage level, already for the smallest resolution considered here.

## C. Zero-dimensional test models with $O(\bar{N}) \times O(\bar{M})$ symmetry and advection

In our last zero-dimensional test case, which is test case VII, we consider a model with  $O(\bar{N}) \times O(\bar{M})$  symmetry including advection in the PDE, see Sec. VIII. Here, we have two invariants that are the  $O(\bar{N})$  and  $O(\bar{M})$  invariant and two background fields that span the physical and computational domain. For the UV potential, we choose Eq. (101). We consider this test model since so far we have only considered models that are solely driven by the diffusion parts of the 2D KT scheme. The model associated with this test case also has advective contributions from the terms in the RG flow equations that correspond to the Goldstone modes in higher-dimensional systems, see Sec. III F.

As can be seen in Fig. 8, we again find that the RG flow behaves as expected. The  $O(\bar{N}) \times O(\bar{M})$  symmetry is restored in the IR and the potential is smooth and convex.



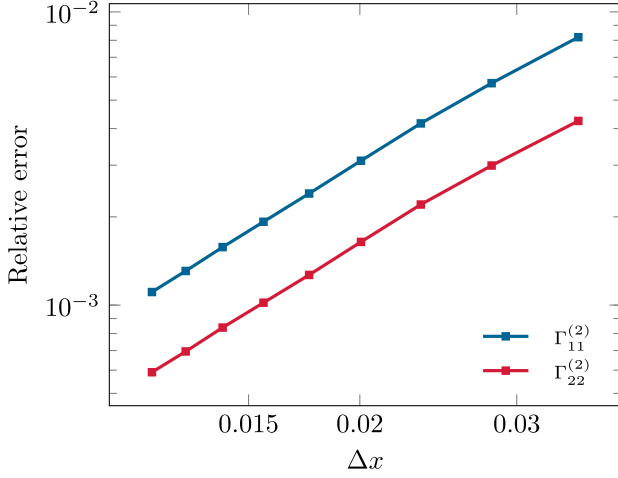


FIG. 18. The relative error (104) of  $\Gamma_{ii}^{(2)}$  for test case VII with UV potential (101) as a function of the numerical resolution  $\Delta x$ .  $\Gamma_{ii}^{(2)}$  is obtained via Eq. (76) from the solution of the PDE system with the KT scheme, while the exact  $\Gamma_{ii}^{(2)}$  are taken from Eqs. (102) and (103).

Also including Goldstone-like contributions (i.e., advection) in the RG flow equation does not pose a general problem to the KT scheme, even in the presence of jump discontinuities in the derivatives or huge gradients. However, without explicit numerical tests, a statement about the accuracy of the scheme is not possible. For this reason, we again consider the errors of the two-point vertex functions of the KT scheme in relation to their exact values, see Eqs. (102) and (103), as a function of the numerical resolution  $\Delta x$  in Fig. 18. Indeed, we find almost perfect agreement of our results with the expected  $\Delta x^2$  scaling of the relative error as listed in the last line of Table V. We find that this also holds for other initial conditions and other values for  $\bar{N}$  and  $\bar{M}$ . Therefore, we conclude that the 2D KT scheme is also perfectly suited to handle two-dimensional FRG problems that involve Goldstone-like modes in terms of advection in the fluid-dynamic reformulation.

## X. SELECTED EXAMPLES IN THREE-DIMENSIONAL EUCLIDEAN SPACETIME

Having extensively tested the 2D KT scheme for zero-dimensional FRG problems, we now turn to two selected sample applications in higher spacetime dimensions.

First, we consider the  $O(2)$  model in three Euclidean dimensions. Here, we compare the 2D KT scheme with the 1D KT scheme at very high resolution which represents another test for the two-dimensional scheme within a truncation. Similar to zero dimensions, this is a diffusion-only problem.

Second, we consider the three-dimensional version of the  $O(\bar{N}) \times O(\bar{M})$  model from the last section. Here, we have advection included in the fluid-dynamical reformulation. However, we cannot benchmark our results against results

from the 1D KT scheme since the problem is inherently two-dimensional because of the two invariants. Nevertheless, we can study convergence of our results by comparing them to results from calculations with different resolutions.

### A. Example I: $O(2)$ model in three dimensions

We start with the well-known  $O(N)$  model with  $N = 2$  in  $d = 3$  Euclidean spacetime dimensions in LPA using the Litim regulator [91,92].

#### 1. Setup

Our ansatz for the effective average action reads

$$\bar{\Gamma}_k[\vec{\varphi}] = \int d^d x \left[ \frac{1}{2} (\partial_\mu \vec{\varphi})^2 + \tilde{U}_k(q) \right] \quad (114)$$

with the  $O(N)$  invariant  $q = \frac{1}{2} \vec{\varphi}^2$ , the RG scale  $k(t) = \Lambda e^{-t}$ , RG time  $t \in [0, \infty)$ , and UV cutoff  $\Lambda$ .

In contrast to zero spacetime dimensions, this is, of course, a truncation of the full effective average action that only contains the scale-dependent effective potential  $\tilde{U}_k(q)$  and second-order derivative terms. Hence, we do not benchmark the results from the solution of the flow equation of the effective potential with the 2D KT scheme against exact results but rather compare them with results from a solution computed with the 1D KT scheme at very high resolution.

To be specific, on the one hand, we use the RG flow equation ( $N = 2$ ,  $d = 3$ )

$$\partial_t U = -A_d k^{d+2} \left( \frac{N-1}{k^2 + \frac{1}{\sigma} \partial_\sigma U} + \frac{1}{k^2 + \partial_\sigma^2 U} \right). \quad (115)$$

To obtain this equation, we have made use of the symmetry in field space and projected onto the field configuration  $\vec{\varphi} = (0, \sigma)^T$ , such that  $U = U(t, \sigma)$ . We add that  $A_d = \frac{\Omega_d}{d(2\pi)^d}$  and  $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ . In conservative form, defining  $u = u(t, \sigma) = \partial_\sigma U$ , this flow equation reads

$$\partial_t u = \frac{d}{d\sigma} \left[ -A_d k^{d+2} \left( \frac{N-1}{k^2 + \frac{1}{\sigma} u} + \frac{1}{k^2 + \partial_\sigma u} \right) \right]. \quad (116)$$

We solve this equation with the 1D KT scheme with  $n_{\text{cells}} = 2001$  on  $\sigma \in [0, \varphi_{\text{max}}]$ , see also Ref. [37]. The solution then serves as the reference solution for our calculations based on the 2D KT scheme.

On the other hand, the flow equation can be kept two dimensional in field space, i.e.,  $U = U(t, \vec{\varphi})$ , similar to the zero-dimensional test cases from Sec. VI. The corresponding flow equation on the computational domain  $[-\varphi_{\text{max}}, \varphi_{\text{max}}] \times [-\varphi_{\text{max}}, \varphi_{\text{max}}]$  is

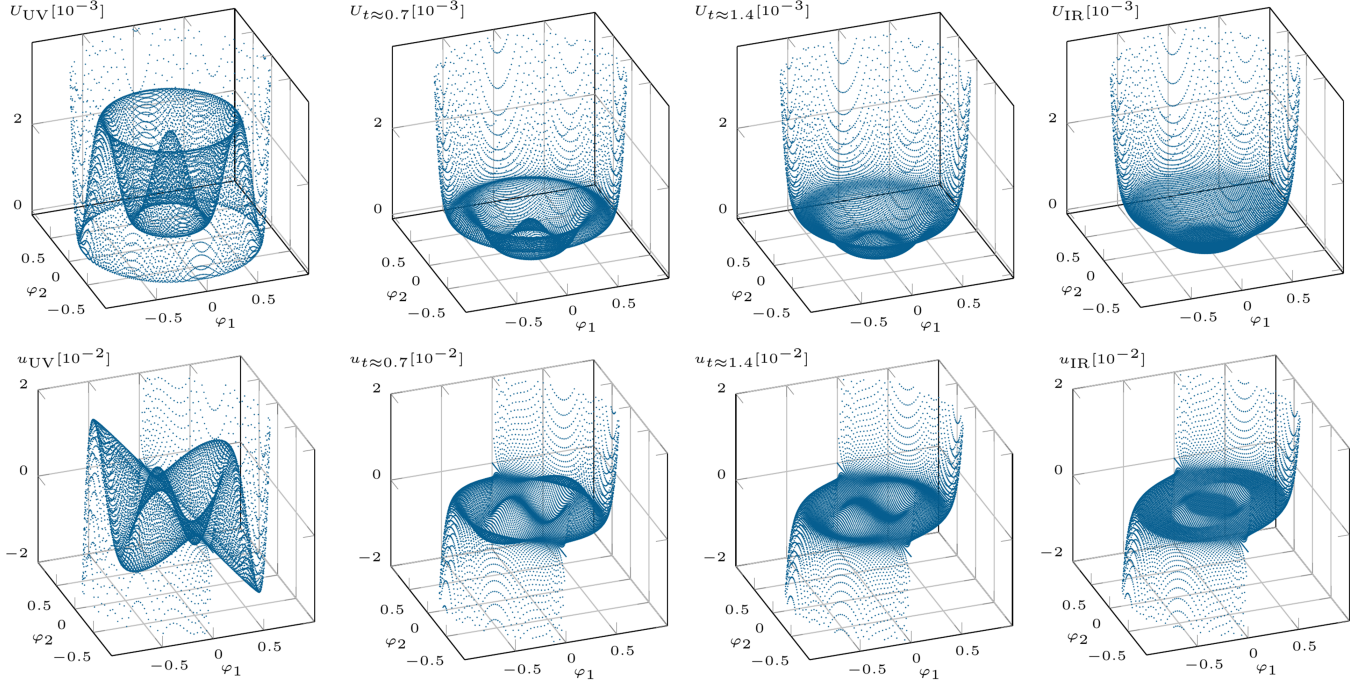


FIG. 19. The RG time evolution of the potential  $U(t, \vec{\varphi})$  (upper row) from the 2D KT scheme and the corresponding  $\varphi_1$  derivative of the potential  $u(t, \vec{\varphi})$  (lower row) from the UV (left column) to the IR (right column) and selected intermediate times for example I, Eq. (120), with  $n_{\text{cells}} = 71$ .

$$\partial_t U = - \frac{A_d k^{d+2} (2k^2 + \partial_{\varphi_1}^2 U + \partial_{\varphi_2}^2 U)}{(k^2 + \partial_{\varphi_1}^2 U)(k^2 + \partial_{\varphi_2}^2 U) - (\partial_{\varphi_1} \partial_{\varphi_2} U)(\partial_{\varphi_2} \partial_{\varphi_1} U)}. \quad (117)$$

Introducing  $u = \partial_{\varphi_1} U$  and  $v = \partial_{\varphi_2} U$ , we find the conservative form of the flow equation of our model,

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \partial_{\varphi_1} \begin{pmatrix} Q \\ 0 \end{pmatrix} + \partial_{\varphi_2} \begin{pmatrix} 0 \\ Q \end{pmatrix}, \quad (118)$$

with the diffusion flux

$$Q = - \frac{A_d k^{d+2} (2k^2 + \partial_{\varphi_1} u + \partial_{\varphi_2} v)}{(k^2 + \partial_{\varphi_1} u)(k^2 + \partial_{\varphi_2} v) - (\partial_{\varphi_1} v)(\partial_{\varphi_2} u)}. \quad (119)$$

This set of equations can be solved with the 2D KT scheme along the lines of the previous sections.

Last, we choose the UV potential to assume the form

$$\tilde{U}(\rho) = 5(\rho - 0.4)(\rho - 0.1)(\rho - 0.3)(\rho - 0.025) \quad (120)$$

at the cutoff scale  $\Lambda = 1$ .<sup>16</sup> Note that we expressed all dimensionful quantities in units of  $\Lambda$ . The UV potential is

<sup>16</sup>Since we are mainly interested in the numerical aspects and comparison of the two KT schemes, we do not discuss any physical implications of the potential and scales. The same applies to the next example.

chosen such that the IR minimum is nonzero and the RG flow therefore ends in the symmetry-broken phase. This implies that we can use the position of the IR minimum as well as the curvature mass at the minimum as observables to compare the 2D with the 1D KT scheme. Furthermore, we introduced a complication by choosing a potential with a nontrivial shape, which is not just the usual mexican hat potential, see Fig. 19 (upper left panel for the potential at the UV scale).

As a consequence of the symmetry breaking, it is no longer possible to flow arbitrarily far into the IR without encountering numerical instabilities due to finite resolution close to the poles of the diffusion flux (119). Because of that, we compare the 2D KT scheme with the 1D KT scheme at the same suitably small chosen IR cutoff scale  $k_{\text{IR}}(t_{\text{IR}})$ .<sup>17</sup> Additionally, the structure of the denominator in the fluxes is simpler, see Eq. (116), such that errors from mixed derivatives  $\partial_{\varphi_i} \partial_{\varphi_j} U$  with  $i \neq j$  are not present. Still, we ensure in our numerical calculations that we are already in the deep IR regime, where the potential is no longer smooth in the symmetry-broken phase but tends to be flat in the small field region and approaches convexity. For a detailed discussion on convexity, we refer to Refs. [93,94] and, for the related issue of time stepping, we refer to

<sup>17</sup>Of course, by using the 1D KT scheme, it is possible to flow to very small RG scales because it is numerically cheaper to work at higher resolution.

TABLE VI. Parameters used for the calculations of Sec. X A. For the integration we used RK45.

| Example | $\sigma_{\max}$ | $\Lambda$ | $t_{\text{IR}}$ | $r_{\text{tol}}$ | $a_{\text{tol}}$ |
|---------|-----------------|-----------|-----------------|------------------|------------------|
| I       | 1               | 1         | 3.5             | $10^{-12}$       | $10^{-12}$       |
| II      | 6               | 40        | 3.69            | $10^{-10}$       | $10^{-12}$       |

Ref. [47]. In any case, the (numerical) parameters of our present study of this model can be found in Table VI.

## 2. Discussion

Let us now discuss the results of our RG flow study of the  $O(2)$  model in three dimensions.

*Qualitative discussion.* At the beginning of the RG flow associated with the UV initial condition (120), the potential well separating the “outer ring” of degenerate minima from the “inner ring” of degenerate minima starts to “melt” see Fig. 19. The same happens for the maximum in the center of the potential. On the level of the derivative of the potential (lower panels), it is clearly visible that the nonlinear diffusion tends to “equilibrate” the inner region(s) of the potential, while the diffusion coefficients/fluxes are already smaller in the outer region. When the flow approaches the IR, the competition between the  $k^2$ -terms and the gradients in the flux (119) sets in. Depending on the sign of the gradient, the diffusion is either enhanced or suppressed. This leads to the formation of the “tilted plateaus” in the derivative of the potential in the IR and the formation of edges. Usually, one would not expect such features in ordinary diffusion-type problems. Here, however, we are confronted with highly nonlinear diffusion with diffusion coefficients that depend on the derivatives of the fluid fields themselves as well as on the time. On the level of the potential itself, this causes the innermost region to be flat (within the numerical resolution) and the potential to approach convexity. A particularly interesting feature is the intermediate region between the flat region and the asymptotic part, which has constant slope in radial direction. Two “rings” (edges) are visible where the regions are connected. As a consequence, in the presence of the linear symmetry-breaking term, one would observe phase transitions at these points. The latter features are also clearly visible in Fig. 20, where we show a cut along the  $\varphi_1$  axis for positive  $\varphi_1$  of the derivative of the potential at selected RG times. Note that this cut essentially corresponds to the computational domain in calculations using the 1D KT scheme. In any case, from a numerical standpoint, it is quite remarkable how the 2D version of the flow equation (117), which solely involves complicated nonlinear diffusion, leads to the same result as the calculations based on the 1D version of the flow equation, see Eq. (116), where the dynamics is driven by an advection-diffusion equation.

*Quantitative discussion.* Returning now to Fig. 20, we observe that the 1D KT scheme and the 2D KT scheme yield the same results for the derivative of the potential at

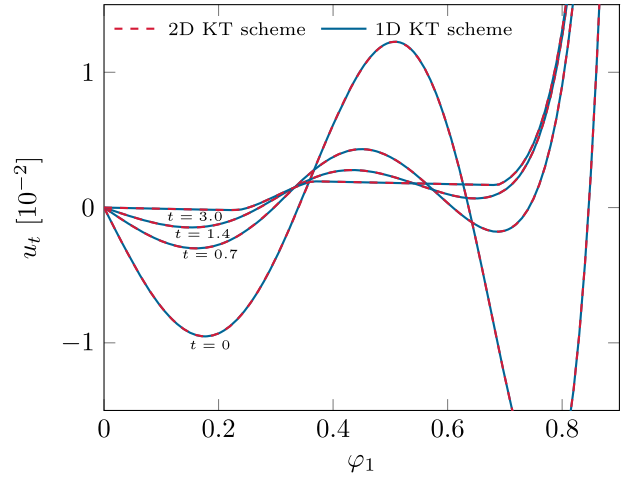


FIG. 20. The RG flow of the derivative of the potential  $u(t, \sigma)$  from the 1D KT scheme (red, dashed) and the  $\varphi_1$  derivative of the potential  $u(t, \vec{\varphi})$  from the 2D KT scheme (blue solid) (evaluated at  $\varphi_2 = 0$ ) at selected RG times. The figure is essentially a section along the  $\varphi_1$  axis for positive  $\varphi_1$  of Fig. 19.

the same IR cutoff scale  $k_{\text{IR}}(t_{\text{IR}})$ . In order to quantify this observation, we also extracted the IR position of the minimum from the 1D and 2D KT scheme as well as the curvature mass  $m^2 = \partial_{\varphi_1}^2 U$  at the minimum. For the sake of simplicity, we employed a sign change in the cell averages  $\bar{u}_i$  to determine the position of the minimum and used the cell center of the positive cell average as the minimum position. The curvature mass is extracted as the right derivative at this cell center. While we extracted the 1D result with a high

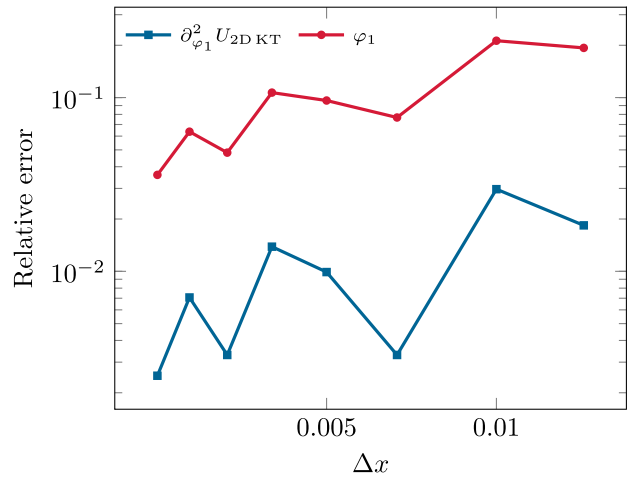


FIG. 21. The relative error between the 2D KT scheme and the 1D KT scheme ( $n_{\text{cells}} = 2001$ ) for the position of the minimum as well as  $\partial_{\varphi_1}^2 U$  for the higher-dimensional example I [see Sec. X A with the UV potential (120)] as a function of the numerical resolution  $\Delta x$ . The curvature mass  $\partial_{\varphi_1}^2 U_{\text{1D KT/2D KT}}$  has been obtained via Eq. (76) from the solution of the PDE system with the 1D/2D KT scheme.

TABLE VII. Error scaling exponent  $n$  extracted from the scaling of observables with  $\Delta x^n$ , corresponding to Figs. 21 and 23.

|          | $n(\varphi_1)$ | $n(\partial_{\varphi_1}^2 U)$ | $n(\sigma_1)$ | $n(\sigma_2)$ | $n(\partial_{\sigma_1}^2 U)$ | $n(\partial_{\sigma_2}^2 U)$ |
|----------|----------------|-------------------------------|---------------|---------------|------------------------------|------------------------------|
| Sec. X A | 0.9(9)         | 0.9(0)                        |               |               |                              |                              |
| Sec. X B |                |                               | 1.7(0)        | 0.8(5)        | 1.6(6)                       | 1.2(4)                       |

spatial resolution in order to use it as our “exact” reference, the 2D results are obtained at different numerical resolutions in order to observe correct error scaling. In Fig. 21, we observe that the numerical error of the 2D KT scheme indeed decreases systematically with increasing numerical resolution, see also Table VII for the corresponding scaling exponents. Hence, we conclude that the 2D KT also performs satisfactorily in higher-dimensional models with  $O(2)$  symmetry and leads to quantitative reliable results which are solely limited by the numerical resolution.

### B. Example II: $O(\bar{N}) \times O(\bar{M})$ model in three dimensions

We now turn to the  $O(\bar{N}) \times O(\bar{M})$  model in three Euclidean dimensions, again considering the LPA and using the Litim regulator. For concreteness, we shall set  $\bar{N} = 2$  and  $\bar{M} = 3$  in all our numerical studies.

#### 1. Setup

Our ansatz for the effective average action is given by

$$\bar{\Gamma}_k[\vec{\varphi}_1, \vec{\varphi}_2] = \int d^3x \left[ \frac{1}{2} (\partial_\mu \vec{\varphi}_1)^2 + \frac{1}{2} (\partial_\mu \vec{\varphi}_2)^2 + \tilde{U}_k(q_1, q_2) \right], \quad (121)$$

where

$$q_1 = \frac{1}{2} \vec{\varphi}_1^2, \quad q_2 = \frac{1}{2} \vec{\varphi}_2^2 \quad (122)$$

are the invariants of the  $O(\bar{N})$  and  $O(\bar{M})$  group, respectively, and

$$\vec{\varphi}_1 = (\varphi_1, \varphi_2, \dots, \varphi_{\bar{N}})^T, \quad (123)$$

$$\vec{\varphi}_2 = (\varphi_{\bar{N}+1}, \varphi_{\bar{N}+2}, \dots, \varphi_{\bar{N}+\bar{M}})^T. \quad (124)$$

Using the same conventions as before, and evaluating the Wetterich equation for the background field configurations

$$\vec{\varphi}_1 = (0, \dots, 0, \sigma_1)^T, \quad \vec{\varphi}_2 = (\sigma_2, 0, \dots, 0)^T, \quad (125)$$

we find the following RG flow equation of the effective potential (see, e.g., Ref. [21]):

$$\partial_t U = -A_d k^{d+2} \left( \frac{(\bar{N}-1)}{k^2 + \frac{1}{\sigma_1} \partial_{\sigma_1} U} + \frac{(\bar{M}-1)}{k^2 + \frac{1}{\sigma_2} \partial_{\sigma_2} U} + \frac{(2k^2 + \partial_{\sigma_1}^2 U + \partial_{\sigma_2}^2 U)}{(k^2 + \partial_{\sigma_1}^2 U)(k^2 + \partial_{\sigma_2}^2 U) - (\partial_{\sigma_1} \partial_{\sigma_2} U)(\partial_{\sigma_2} \partial_{\sigma_1} U)} \right). \quad (126)$$

In complete analogy to Sec. III F, this equation can be brought into the shape of a conservation law by introducing the fields  $u = \partial_{\sigma_1} U$  and  $v = \partial_{\sigma_2} U$ . Here, we refrain from presenting this form of the flow equation as it can be easily derived from a comparison with Eqs. (43) and (45) to (47). For the UV potential in our numerical calculations, we simply choose

$$\tilde{U}(\rho_1, \rho_2) = -10\rho_1^2 - \rho_1\rho_2 - 15\rho_1\rho_2^2 + \frac{1}{4}(\rho_1 + \rho_2)^4. \quad (127)$$

Instead of expressing all quantities in terms of the cutoff scale  $\Lambda$ , we shall use arbitrary units here and set  $\Lambda = 40$ . Note that our choice of the form of the UV potential is not phenomenologically motivated. It is only constructed such that it comprises nontrivial dynamics in the RG flow and ends up in the symmetry-broken regime with a residual  $O(\bar{N}-1) \times O(\bar{M}-1)$  symmetry in the IR, see Fig. 22.

#### 2. Discussion

Let us now discuss the results of the RG flow of the  $O(\bar{N}) \times O(\bar{M})$  model in three dimensions.

*Qualitative discussion.* In the present case, the UV potential has a global  $O(\bar{N}) \times O(\bar{M})$  symmetry. However, there is no  $O(\bar{N} + \bar{M})$  symmetry and also the  $O(\bar{N})$  and the  $O(\bar{M})$  symmetries are separately broken by nontrivial minima. In Fig. 22 (upper left panel), one clearly observes a complicated nonconvex shape of the potential as a function of the two background fields. This can also be seen on the level of the derivatives with respect to the background field configurations, see the middle and lower left panels. Note that the computational domain is greater than the plot region, see Table VI, where we list all numerical control parameters. Of course, despite the large gradients, we also ensured that the initial condition is valid in the sense that it does not overshoot the poles of the propagator already at the UV scale.

In the RG flow from the UV to the IR we find the usual overall behavior of purely bosonic systems. To be more specific, the minima of the potential equalize in their depth and the potential eventually starts to become convex. However, as can be seen from the derivatives of the potential in the lower panels of Fig. 22, the dynamics in the different field/invariant directions sets in at different speeds and RG times. This is due to the different slopes and curvatures of the potential in the different directions. Furthermore, the fact that  $\bar{N} \neq \bar{M}$  also plays a role because



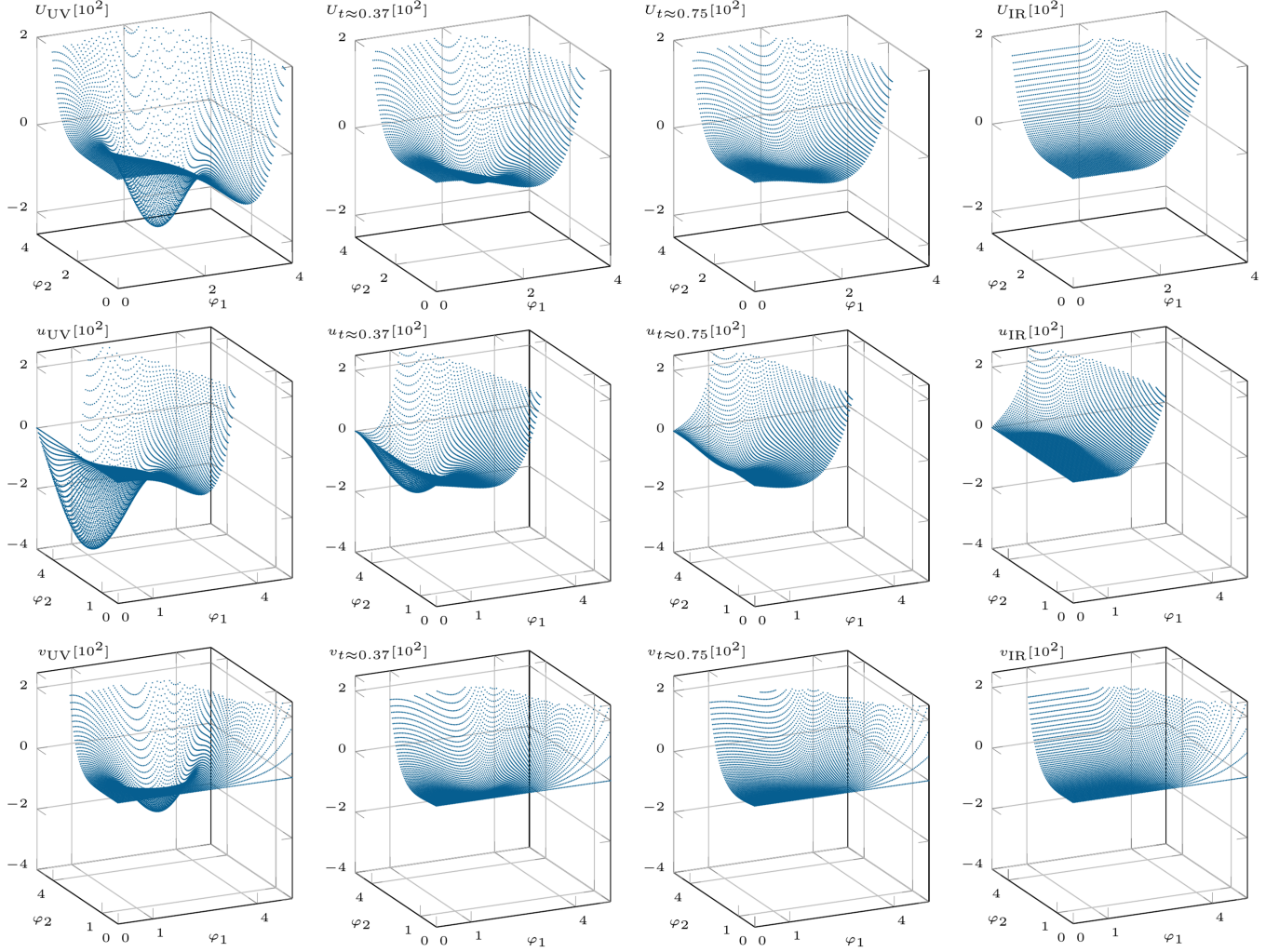


FIG. 22. RG time evolution of the potential  $U(t, \sigma_1, \sigma_2)$  (upper row) from the 2D KT scheme and the corresponding  $\sigma_1$  and  $\sigma_2$  derivative of the potential  $u(t, \sigma_1, \sigma_2)$  and  $v(t, \sigma_1, \sigma_2)$  (middle/lower row) from the UV (left column) to the IR (right column) and selected intermediate times for example II, Eq. (127), with  $n_{\text{cells}} = 151$ .

there is more advection in the direction of the field with larger  $\tilde{N}$  or  $\tilde{M}$ , respectively.

We shall not discuss the details underlying the dynamics of this system further here. The main motivation for considering this particular test case is to demonstrate the capabilities of our 2D KT framework for problems of two invariant directions via discretization in terms of the background fields.

*Quantitative discussion.* As already mentioned above, there are no benchmarks for our results for this model available, neither from the path integral nor from the 1D KT scheme. Thus, the only way to assess the quality of the 2D KT scheme is to study the convergence of the numerical results. To this end, we consider the position of the minimum and the curvature mass of the radial modes. In Fig. 23, we show the relative error for these quantities between the calculation with our highest resolution  $\Delta x = 0.015$  and calculations at lower resolution. Overall, we find

that the relative deviation decreases systematically with increasing resolution. The corresponding scaling exponents can be found in Table VII, which are in good overall agreement with the expected error scaling of the KT scheme. However, we also remark that calculations at even higher resolution are required for an absolutely trustworthy result.

As a second convergence test, we compare the cuts through the potential at the IR cutoff scale  $k_{\text{IR}}(t_{\text{IR}})/\Lambda = 0.025$  for different numerical resolutions. To be specific, we consider  $u(t, \sigma_1, 0)$  and  $v(t, 0, \sigma_2)$ , see Fig. 24. Again, we find that the results converge systematically with increasing resolution, as it should be.

In view of our results for this test case and also the previous one, we conclude that it is promising from a phenomenological and numerical standpoint to generalize our 2D KT framework to more complicated models with more than two invariants.

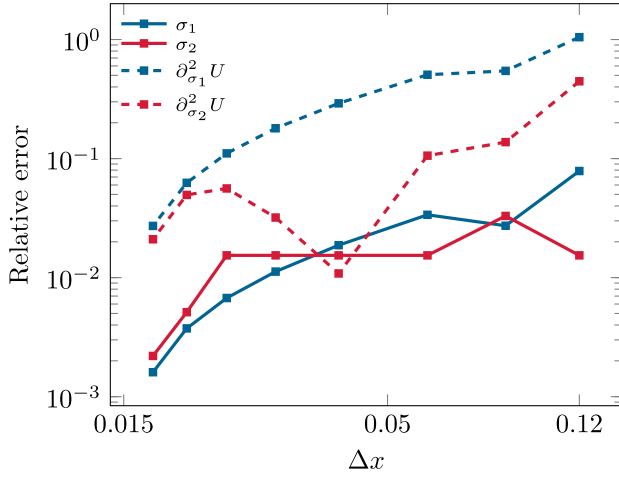


FIG. 23. Relative error of the  $\sigma_{1/2}$  position of the minimum as well as  $\partial^2_{\sigma_{1/2}} U$  computed with  $\Delta x$  and the same quantity at the highest tested resolution ( $\Delta x = 0.015$ ) for Sec. XB with UV potential (127).

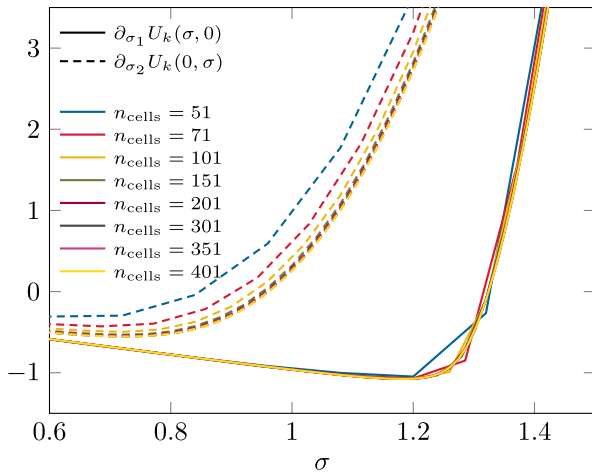


FIG. 24. Comparison of  $u(t, \sigma_1, \sigma_2)$  and  $v(t, \sigma_1, \sigma_2)$  in the IR at  $k_{\text{IR}}(t_{\text{IR}})/\Lambda = 0.025$  for different numerical resolutions (number of cells) for the model presented in Sec. XB with the UV potential (127).

## XI. SUMMARY, CONCLUSIONS, AND OUTLOOK

### A. Summary

In this work, we discussed the numerical treatment of FRG flow equations in situations where the effective potential has to be resolved in more than one field or invariant direction. We have shown that the FRG flow equation of the effective potential of such systems can be reformulated as a fluid-dynamical system of advection-diffusion type for  $O(N)$  and  $O(\bar{N}) \times O(\bar{M})$  models in  $0 + 0$ - and higher-dimensional spacetime. We argued that this also generalizes to other systems which, for example, involve fermions. Furthermore, we have presented a

numerical scheme from CFD, which was developed to solve exactly such fluid-dynamical PDEs, namely, the 2D KT scheme, and we adapted it to the present problem, also noting a possible minor defect in the original implementation by Kurganov and Tadmor. In order to demonstrate the power of our approach, we constructed various models in  $0 + 0$  spacetime dimensions and benchmarked the results from the KT scheme against exact results from the underlying (path) integral. Most importantly, these tests comprised models that involved nonanalyticities in field space, symmetry breaking, multiple minima, and the misalignment of the symmetry axis from the field-space axes. Moreover, we applied our numerical framework to more realistic higher-dimensional models, namely, the  $O(2)$  and the  $O(\bar{N}) \times O(\bar{M})$  model in three spacetime dimensions within the LPA. Also for these cases, we carefully conducted convergence tests and found that the KT scheme is capable of solving these models with satisfying accuracy.

### B. Conclusions

In total, we conclude that the KT scheme can be used to numerically solve FRG flow equations in  $0 + 0$  dimensions and also in higher-dimensional spacetimes. Our numerical framework represents a powerful black-box solver that may also be included in existing FRG codes. All benchmark tests showed error scaling with a numerical resolution that is in agreement with the expected error scaling of the KT scheme. However, we also demonstrated limitations of our approach. Especially in situations with involved symmetry-breaking patterns, rather large numbers of finite-volume cells are required to resolve the location of the minimum and the vertex functions to a satisfactory degree. The same is the case for situations with little symmetry or misalignment of the symmetry axis of the problem with the axes of the discretization. However, this poses a severe though solvable challenge since the number of degrees of freedom that is evolved effectively as ODEs grows quadratically with the number of cells. In the future, this may be tackled by exploiting high-performance computing techniques for an implementation of our present framework.

In addition to the presentation of our test setup and the explicit tests, we also discussed qualitative aspects of flow equations in higher-dimensional field space and details of the numerical implementation. For example, we described restrictions on the UV potentials and hence the microscopic models that can be treated naively with the FRG approach by analyzing the pole structures of the propagators (the fluxes).

### C. Outlook

In our present technical work, we showed that the fluid-dynamical approach to FRG flow equations with a field-space-dependent potential and couplings is a viable and powerful approach that provides much more than just a benchmark for other numerical schemes. In fact, the present

work indeed provides methods that can be used for a wide range of applications relevant for a huge variety of fields, ranging from condensed-matter physics to high-energy physics. One concrete application is the analysis of symmetry-breaking patterns in the QCD phase diagram. Here, the FRG is a very promising tool since it is capable of treating the nonperturbative regime of QCD from first principles and can therefore be used to study the phase diagram in a systematic way; see, e.g., Refs. [95,96]. As the dynamics of QCD is highly involved, the FRG flow equations are rather complicated and involve a large number of degrees of freedom. In particular, at finite density, a reliable numerical framework is therefore required to map out the phase structure, which may indeed be governed by multiple competing condensates (e.g., chiral and diquark condensates) and first-order phase transitions. The framework developed in our present work may represent a valuable tool to provide a deeper qualitative and quantitative understanding of the dynamics of QCD in this regime at low temperatures and high densities.

The authors of this work disclose the use of the GitHub Copilot tool for the preparation of the manuscript. The underlying code is written in PYTHON3 and reference values are calculated with *Mathematica* 14.1.

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### DATA AVAILABILITY

The data that support the findings of this article are openly available [97].

### APPENDIX A: COMPUTATIONAL TIMES

In this appendix, we provide some estimates for the computational wall times required to perform the calculations in two-dimensional field space as presented in the present work. This may serve as reference for future studies

TABLE VIII. Wall times required to run the test cases at a resolution of  $n_{\text{cells}} = 200$  (without ghost cells). For the tests in Secs. VI, VII, and X A, this corresponds to a total number of  $(400 \times 400) - 1$  cells, because  $n_{\text{cells}}$  defines the number of cells in positive  $x$  and  $y$  directions while, for Secs. VIII and X B, this corresponds to  $200 \times 200$  cells in the positive quadrant only.

| Test case  | Wall time (min) |
|------------|-----------------|
| Sec. VIB 1 | <6              |
| Sec. VIB 2 | <4              |
| Sec. VIB 3 | <5              |
| Sec. VIB 4 | <4              |
| Sec. VII A | <197            |
| Sec. VII B | <12             |
| Sec. VIII  | <39             |
| Sec. X A   | <79             |
| Sec. X B   | <135            |

to estimate the computational resources required for similar calculations.

All calculations in the present work have been performed on a single core of an AMD Ryzen Threadripper 3990X 64-core processor CPU with 2.9 GHz and with 128 GB of RAM. In Table VIII, we list the wall times required to run the test cases at a resolution of  $n_{\text{cells}} = 200$  (without ghost cells) with the parameters given in the corresponding sections. As can be seen from the table, wall times can vary significantly between the different test cases and spacetime dimensions, which has to be taken into account when applying the KT scheme to other models.

### APPENDIX B: IMPLEMENTATION OF THE MULTIDIMENSIONAL KT CENTRAL SCHEME

In general, the KT scheme is a FV method tailored for solving fluid-dynamical PDEs of advection-diffusion type like Eq. (48). In Sec. IV, the 2D KT scheme is discussed locally, i.e., the equations are formulated in terms of cells and their adjacent cells. However, for the numerical implementation and, in particular, to get a better understanding of the functioning of the ghost cells, it is instructive to formulate the KT scheme in a matrix formulation. Similar to Sec. IV, we present the two-dimensional semidiscrete version of the scheme, meaning that we treat the temporal direction and the spatial directions as continuous and discrete, respectively. Using the semidiscrete version of the KT scheme with  $N_x$  cells in the  $x$  direction and  $N_y$  cells in the  $y$  direction for the PDE (48), it becomes a set of coupled ordinary differential equations, which can be summarized in the matrix equation

$$\partial_t \hat{u} = \mathcal{F}_{\text{KT}}(t, \hat{u}), \quad (\text{B1})$$

where



$$\hat{\mathbf{u}} = (\hat{\mathbf{u}}_{d\mathbf{y}x})_{d=0,\dots,\text{d.o.f.}-1; y=0,\dots,N_y-1, x=0,\dots,N_x-1}, \quad (\text{B2})$$

i.e.,  $\hat{\mathbf{u}} \in \mathbb{R}^{\text{d.o.f.} \times N_y \times N_x}$ , where “d.o.f.” denotes the number of degrees of freedom—the number of fluid fields. The initial value problem given by Eq. (B1) and some initial condition  $\hat{\mathbf{u}}(t=0) = \hat{\mathbf{u}}_0$  can then be solved numerically.

This appendix is structured as follows: First, we introduce a “slicing operator” for the matrix formulation. Second, we define the two-dimensional spatial grid. Finally, in a third step, we discuss the matrix formulation of the KT scheme, which involves the specification of the rhs of Eq. (B1) denoted as  $\mathcal{F}_{\text{KT}}$ .

### 1. The slicing operator

For the matrix formulation of the 2D KT scheme, it is practical to introduce a so-called slicing operator that is used to systematically cut rows and columns of matrices. This enhances the readability of the formulation, particularly for readers familiar with the PYTHON package NUMPY [98], as this slicing operator functions identically to that of NUMPY. Consequently, all equations in the following sections can be directly implemented in PYTHON.

Let  $\mathbf{A} = (\mathbf{A}_i)_{i=0,1,\dots,N-1} \in \mathbb{R}^N$ , then we define the slicing operator  $[a:b]$  with  $a, b \in \mathbb{Z}$  by

$$\mathbf{A}[a:b] = (\mathbf{A}_i)_{i=\bar{a},\dots,\bar{b}-1}, \quad \text{and} \quad \mathbf{A}[i] = \mathbf{A}_i, \quad (\text{B3})$$

where  $\bar{x} = x$  if  $x \geq 0$ , else  $\bar{x} = N + x$ . For simplicity, if  $a$  or  $b$  is not set, we mean  $a = 0$  or  $b = N$ , respectively. For example, we have  $\mathbf{A}[:] = \mathbf{A}$ . This slicing operator can be simply extended to higher dimensions. For example, let  $\mathbf{A} = (\mathbf{A}_{i_1 \dots i_M})_{i_1=0,\dots,N_1-1; \dots; i_M=0,\dots,N_M-1} \in \mathbb{R}^{N_1 \dots N_M}$ , then we have

$$\begin{aligned} \mathbf{A}[a_1: b_1, \dots, a_M: b_M] \\ = (\mathbf{A}_{i_1 \dots i_M})_{i_1=\bar{a}_1, \dots, \bar{b}_1-1; \dots; i_M=\bar{a}_M, \dots, \bar{b}_M-1} \end{aligned} \quad (\text{B4})$$

and  $\mathbf{A}[i_1, \dots, i_M] = \mathbf{A}_{i_1 \dots i_M}$ .

### 2. The two-dimensional grid

Since the fluid fields are summarized in the multidimensional matrix  $\hat{\mathbf{u}} \in \mathbb{R}^{\text{d.o.f.} \times N_y \times N_x}$  in the following, where  $N_x$  and  $N_y$  are the number of cells in  $x$  and  $y$  direction, respectively, we need grid objects of the same form/dimension. These are generated as follows: A general two-dimensional rectangular grid is completely specified by the locations of (the centers of) the edges parallel to the  $x$  and  $y$  direction. Let  $\mathbf{x}_{\text{edges}}$  be the list of all  $x$  coordinates of edges pointing in  $x$  direction (analogously for  $\mathbf{y}_{\text{edges}}$ ). Then, we can raise those one-dimensional lists to matrices  $\mathbf{x}_{\text{edges}}$  and  $\mathbf{y}_{\text{edges}}$  by

$$\mathbf{x}_{\text{edges}}[i, j, :] = x_{\text{edges}}, \quad (\text{B5a})$$

$$\mathbf{y}_{\text{edges}}[i, :, k] = y_{\text{edges}}, \quad (\text{B5b})$$

for all  $0 \leq i < \text{d.o.f.}$ ,  $0 \leq j < |\mathbf{y}_{\text{edges}}|$ , and  $0 \leq k < |\mathbf{x}_{\text{edges}}|$ . Hence, the  $\mathbf{x}_{\text{edges}}$  object contains  $\text{d.o.f.} \cdot |\mathbf{y}_{\text{edges}}|$  copies of the  $x_{\text{edges}}$  list and the  $\mathbf{y}_{\text{edges}}$  object contains  $\text{d.o.f.} \cdot |\mathbf{x}_{\text{edges}}|$  copies of the  $y_{\text{edges}}$  list. This, however, will turn out to be useful for what follows:

We can now determine the cell centers as well as the cell width in  $x$  and  $y$  directions. For the widths, we find

$$\Delta_x = \mathbf{x}_{\text{edges}}[:, :, -1, :] - \mathbf{x}_{\text{edges}}[:, :, -1, : -1], \quad (\text{B6a})$$

$$\Delta_y = \mathbf{y}_{\text{edges}}[:, 1:, :, -1] - \mathbf{y}_{\text{edges}}[:, :, -1, : -1], \quad (\text{B6b})$$

and for the cell centers we have

$$\mathbf{x}_C = \mathbf{x}_{\text{edges}}[:, :, -1, : -1] + \frac{1}{2} \Delta_x, \quad (\text{B7a})$$

$$\mathbf{y}_C = \mathbf{y}_{\text{edges}}[:, :, -1, : -1] + \frac{1}{2} \Delta_y. \quad (\text{B7b})$$

However, for the KT scheme, we need further locations, e.g., the locations of the cell interfaces of each cell.<sup>18</sup> We define them analogously by

$$\mathbf{x}_W = \mathbf{x}_C - \frac{1}{2} \Delta_x, \quad \mathbf{x}_E = \mathbf{x}_C + \frac{1}{2} \Delta_x, \quad (\text{B8a})$$

$$\mathbf{y}_W = \mathbf{y}_C, \quad \mathbf{y}_E = \mathbf{y}_C, \quad (\text{B8b})$$

$$\mathbf{x}_S = \mathbf{x}_C, \quad \mathbf{x}_N = \mathbf{x}_C, \quad (\text{B8c})$$

$$\mathbf{y}_S = \mathbf{x}_C - \frac{1}{2} \Delta_y, \quad \mathbf{y}_N = \mathbf{y}_C + \frac{1}{2} \Delta_y, \quad (\text{B8d})$$

where we have used the  $N - S - W - E$  convention as it is introduced in Ref. [99–101], see also Fig. 25 for a sketch of the situation of a single cell. However, note that the above formulation allows for a parallel handling of all cells.

### 3. The KT scheme in matrix formulation

With the definition of the two-dimensional grid as it is done in the previous section, we can now discuss the KT scheme implementation in a matrix formulation with an arbitrary number of degrees of freedoms. This means that we have to define/construct the rhs of Eq. (B1),  $\mathcal{F}_{\text{KT}}$ . For that we need at least two boundary layers on each side in the semidiscrete version of the KT scheme, as discussed in detail in Sec. IV B. Here, the number of boundary layers is denoted by  $\mathcal{B} \in \mathbb{N}_{\geq 2}$ . Furthermore, let  $N_x$  ( $\tilde{N}_x$ ) and  $N_y$  ( $\tilde{N}_y$ )

<sup>18</sup>For higher reconstruction orders of the KT scheme, additional positions for the corners of each cell are required, see, for example, Ref. [99].



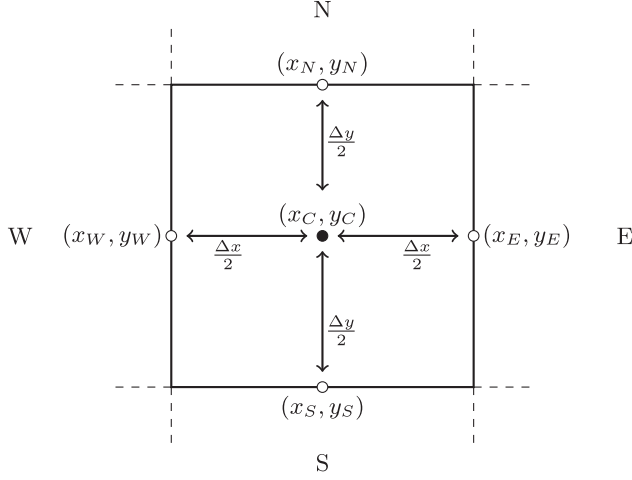


FIG. 25. A sketch of a single fluid cell with the center point  $(x_C, y_C)$  and the interfaces  $(x_N, y_N)$ ,  $(x_S, y_S)$ ,  $(x_W, y_W)$ , and  $(x_E, y_E)$ .

be the number of cells excluding (including) the boundary cells in the  $x$  and  $y$  direction, respectively, meaning that  $\bar{N}_x = |x_{\text{edges}}| - 1$ ,  $\bar{N}_y = |y_{\text{edges}}| - 1$ ,  $N_x = \bar{N}_x - 2\mathcal{B}$  and  $N_y = \bar{N}_y - 2\mathcal{B}$ . Hence, the dimensions of the grid objects are  $\Delta_x, \Delta_y, \mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^{\text{d.o.f.} \times \bar{N}_y \times \bar{N}_x}$  for  $i \in \{N, S, W, E\}$ . In the following, multiplication and division of matrices are understood componentwise.

For a general PDE of the advection-diffusion type, cf. Eq. (48), the rhs of Eq. (B1) has two contributions in the end, one for the advection part and one for the diffusion part, such that the rhs of Eq. (B1) reads

$$\mathcal{F}_{\text{KT}} = -d\mathbf{H} + d\mathbf{Q}, \quad (\text{B9})$$

where  $d\mathbf{H}$  and  $d\mathbf{Q}$  stand for the advection and diffusion part, respectively. Below, the terms  $d\mathbf{H}$  and  $d\mathbf{Q}$  are provided. Note that they must have the same dimension as  $\hat{\mathbf{u}}$ :  $d\mathbf{H}, d\mathbf{Q} \in \mathbb{R}^{\text{d.o.f.} \times N_y \times N_x}$ .

### a. The boundary conditions

First of all, we have to extend the dimension of  $\hat{\mathbf{u}}$  to the dimension of the grid objects. For that, we have to choose some boundary conditions, meaning that we extend  $\hat{\mathbf{u}} \in \mathbb{R}^{\text{d.o.f.} \times N_y \times N_x}$  to  $\mathbf{u} \in \mathbb{R}^{\text{d.o.f.} \times \bar{N}_y \times \bar{N}_x}$  by the boundary layers where the boundary cells are filled according to the boundary conditions.

### b. The flux limiter

As already explained in Sec. IV A, we have to reconstruct the values of the fluid fields at the interfaces of each cell. To this end, one has to estimate a gradient/slope for each fluid field in both directions in every cell. However, since we only have access to cell averages, the estimate can only be

based on those. Using a flux limiter<sup>19</sup>  $f_{\text{limiter}}$ , the slopes are given by<sup>20</sup>

$$f_x = f_{\text{limiter}} \left( \frac{\Delta_{1,0}^x(\mathbf{u})}{\Delta_{1,0}^x(\mathbf{x}_C)}, \frac{\Delta_{0,-1}^x(\mathbf{u})}{\Delta_{0,-1}^x(\mathbf{x}_C)} \right), \quad (\text{B10a})$$

$$f_y = f_{\text{limiter}} \left( \frac{\Delta_{1,0}^y(\mathbf{u})}{\Delta_{1,0}^y(\mathbf{y}_C)}, \frac{\Delta_{0,-1}^y(\mathbf{u})}{\Delta_{0,-1}^y(\mathbf{y}_C)} \right), \quad (\text{B10b})$$

where we have used the difference operators,

$$\begin{aligned} \Delta_{i,j}^x(\mathbf{A}) &= \mathbf{A}[:, \mathcal{B}-1:-\mathcal{B}+1, \mathcal{B}-1+i:\mathcal{B}+1+i+N_x] \\ &\quad - \mathbf{A}[:, \mathcal{B}-1:-\mathcal{B}+1, \mathcal{B}-1+j:\mathcal{B}+1+j+N_x], \end{aligned} \quad (\text{B11a})$$

$$\begin{aligned} \Delta_{i,j}^y(\mathbf{A}) &= \mathbf{A}[:, \mathcal{B}-1+i:\mathcal{B}+1+i+N_y, \mathcal{B}-1:-\mathcal{B}+1] \\ &\quad - \mathbf{A}[:, \mathcal{B}-1+j:\mathcal{B}+1+j+N_y, \mathcal{B}-1:-\mathcal{B}+1], \end{aligned} \quad (\text{B11b})$$

with  $-\mathcal{B} \leq i, j \leq \mathcal{B}$  such that  $\Delta_{i,j}^{x/y}(\mathbf{A}) \in \mathbb{R}^{\text{d.o.f.} \times (N_x+1) \times (N_y+1)}$ . Roughly speaking, the flux limiter compares the left and right gradients at each cell and returns the estimated gradient with which we set the value of the fluid fields at the interfaces. Hence, defining

$$\mathbf{u}_{x/y} = f_{x/y} \cdot \frac{1}{2} \Delta_{x/y}[:, \mathcal{B}-1:-\mathcal{B}+1, \mathcal{B}-1:-\mathcal{B}+1], \quad (\text{B12})$$

we find

$$\mathbf{u}_C = \mathbf{u}[:, \mathcal{B}-1:-\mathcal{B}+1, \mathcal{B}-1:-\mathcal{B}+1], \quad (\text{B13a})$$

$$\mathbf{u}_E = \mathbf{u}_C + \mathbf{u}_x, \quad (\text{B13b})$$

$$\mathbf{u}_W = \mathbf{u}_C - \mathbf{u}_x, \quad (\text{B13c})$$

$$\mathbf{u}_N = \mathbf{u}_C + \mathbf{u}_y, \quad (\text{B13d})$$

$$\mathbf{u}_S = \mathbf{u}_C - \mathbf{u}_y, \quad (\text{B13e})$$

such that  $\mathbf{u}_C, \mathbf{u}_E, \mathbf{u}_W, \mathbf{u}_N, \mathbf{u}_S \in \mathbb{R}^{\text{d.o.f.} \times (N_x+1) \times (N_y+1)}$ .

### c. The advection term

For the advection term, we completely follow Ref. [80] and apply the dimension-by-dimension approach for the

<sup>19</sup>For example, the MinMod limiter is defined by  $f_{\text{MinMod}}(a, b) = \min(|a|, |b|)$  if  $a \cdot b > 0$ , else  $f_{\text{MinMod}}(a, b) = 0$ , see Eq. (60).

<sup>20</sup>The division as well as the evaluation of the function  $f_{\text{limiter}}$  are meant componentwise.

reconstruction of the fluxes, where only information from two cells north/south and two cells east/west of a respective cell is used. For a higher-order reconstruction of fluxes like the second- or third-order genuinely multidimensional central scheme, we refer the reader to Refs. [99,100]. There, also information from the cells that are diagonally adjacent to the cell of interest is used.

For the dimension-by-dimension approach, we need to estimate the advection velocities on the cell interfaces in both directions. They read<sup>21</sup>

$$a_x = \max(\hat{\rho}(t, \mathbf{u}_W[:, :, 1:]), \hat{\rho}(t, \mathbf{u}_E[:, :, : - 1])), \quad (\text{B14a})$$

$$a_y = \max(\hat{\rho}(t, \mathbf{u}_S[:, 1:, :]), \hat{\rho}(t, \mathbf{u}_N[:, :, : - 1])). \quad (\text{B14b})$$

Here, we understand the function  $\hat{\rho}$  componentwise in the  $x$  and  $y$  direction but not in the “d.o.f. direction.” The latter direction is given by  $\hat{\rho}(t, u_0, \dots, u_{\text{d.o.f.}-1}) = (\lambda_{\max}, \dots, \lambda_{\max})$ , where  $\lambda_{\max}$  is the spectral radius of  $\frac{\partial \vec{f}}{\partial \vec{u}}$ . It is determined by  $\lambda_{\max} = \max\{|\lambda_1, \dots, \lambda_\omega|\}$ , where  $\lambda_k$  are the eigenvalues of  $\frac{\partial \vec{f}}{\partial \vec{u}}$  at  $(u_0, \dots, u_{\text{d.o.f.}-1})$ .

With the estimates of the advection velocities on the cell interfaces at hand, we finally find the advection fluxes

$$\mathbf{P}^x = \frac{1}{2}(Q^x(t, \mathbf{u}_C[:, :, : - 1], \mathbf{d}_x \mathbf{u}[:, :, : - 1], \mathbf{f}_y[:, :, : - 1]) + Q^x(t, \mathbf{u}_C[:, :, 1:], \mathbf{d}_x \mathbf{u}[:, :, : - 1], \mathbf{f}_y[:, :, 1:])), \quad (\text{B17a})$$

$$\mathbf{P}^y = \frac{1}{2}(Q^y(t, \mathbf{u}_C[:, : - 1, :], \mathbf{f}_x[:, : - 1, :], \mathbf{d}_y \mathbf{u}[:, : - 1, :]) + Q^y(t, \mathbf{u}_C[:, 1:, :], \mathbf{f}_x[:, 1:, :], \mathbf{d}_y \mathbf{u}[:, : - 1, :])), \quad (\text{B17b})$$

where we used the abbreviations

$$\mathbf{d}_x \mathbf{u} = \frac{\Delta_{1,0}^x(\mathbf{u})}{\Delta_{1,0}^x(\mathbf{x}_C)}, \quad \mathbf{d}_y \mathbf{u} = \frac{\Delta_{1,0}^y(\mathbf{u})}{\Delta_{1,0}^y(\mathbf{y}_C)}. \quad (\text{B18})$$

As we have already discussed in Sec. IV A, this implementation leads to stability issues and a wrong error scaling behavior for certain systems, see Fig. 9. We therefore tested several minor modifications of the original implementation and suggest the following solution: Instead of using the limited slopes  $\mathbf{f}_x$  and  $\mathbf{f}_y$  in Eqs. (B17a) and (B17b), we

$$\begin{aligned} \mathbf{H}^x &= \frac{1}{2}(f^x(t, \mathbf{u}_W[:, :, 1:]) + f^x(t, \mathbf{u}_E[:, :, : - 1])) \\ &\quad + -\frac{1}{2}a_x(\mathbf{u}_W[:, :, 1:] - \mathbf{u}_E[:, :, : - 1]), \end{aligned} \quad (\text{B15a})$$

$$\begin{aligned} \mathbf{H}^y &= \frac{1}{2}(f^y(t, \mathbf{u}_S[:, :, 1:]) + f^y(t, \mathbf{u}_N[:, :, : - 1])) \\ &\quad + -\frac{1}{2}a_y(\mathbf{u}_S[:, :, 1:] - \mathbf{u}_N[:, :, : - 1]). \end{aligned} \quad (\text{B15b})$$

Eventually, we have for the advection contribution of the rhs of Eq. (B1),

$$\begin{aligned} d\mathbf{H} &= \frac{\mathbf{H}^x[:, 1: - 1, 1:] - \mathbf{H}^x[:, 1: - 1, : - 1]}{\Delta_x[:, \mathcal{B} - \mathcal{B}, \mathcal{B} - \mathcal{B}]} \\ &\quad + \frac{\mathbf{H}^y[:, 1:, 1: - 1] - \mathbf{H}^y[:, :, : - 1, 1: - 1]}{\Delta_y[:, \mathcal{B} - \mathcal{B}, \mathcal{B} - \mathcal{B}]}. \end{aligned} \quad (\text{B16})$$

#### d. The diffusion term

In the original paper of the KT scheme [80], the two-dimensional implementation of the diffusion fluxes makes use of the limited slopes/gradients  $\mathbf{f}_x$  and  $\mathbf{f}_y$  from Eqs. (B10a) and (B10b). Hence, the diffusion fluxes are specified as follows:

simply approximate these contributions by central difference stencils

$$\mathbf{d}_x^c \mathbf{u} = \frac{\Delta_{1,-1}^x(\mathbf{u})}{\Delta_{1,-1}^x(\mathbf{x}_C)}, \quad \mathbf{d}_y^c \mathbf{u} = \frac{\Delta_{1,-1}^y(\mathbf{u})}{\Delta_{1,-1}^y(\mathbf{y}_C)}. \quad (\text{B19})$$

Note that these are gradients of the fluid orthogonal to the respective diffusion fluxes. Thus, by inserting Eq. (B19) into Eqs. (B17a) and (B17b), we obtain the following diffusion fluxes:

$$\mathbf{P}^x = \frac{1}{2}(Q^x(t, \mathbf{u}_C[:, :, : - 1], \mathbf{d}_x \mathbf{u}[:, :, : - 1], \mathbf{d}_y^c \mathbf{u}[:, :, : - 1]) + Q^x(t, \mathbf{u}_C[:, :, 1:], \mathbf{d}_x \mathbf{u}[:, :, : - 1], \mathbf{d}_y^c \mathbf{u}[:, :, 1:])), \quad (\text{B20a})$$

<sup>21</sup>Note that by comparing  $\mathbf{u}_W[:, :, 1:]$  and  $\mathbf{u}_E[:, :, : - 1]$ , the locations of the interfaces are identical, i.e.,  $\mathbf{x}_W[:, :, 1:] = \mathbf{x}_E[:, :, : - 1]$  and  $\mathbf{y}_W[:, :, 1:] = \mathbf{y}_E[:, :, : - 1]$ . The same is true for comparing the locations of  $\mathbf{u}_S[:, 1:, :]$  and  $\mathbf{u}_N[:, :, : - 1, :]$ .

$$\mathbf{P}^y = \frac{1}{2}(\mathcal{Q}^y(t, \mathbf{u}_C[:, :, -1, :], \mathbf{d}_x^c \mathbf{u}[:, :, -1, :], \mathbf{d}_y \mathbf{u}[:, :, -1, :]) + \mathcal{Q}^y(t, \mathbf{u}_C[:, 1 :, :], \mathbf{d}_x^c \mathbf{u}[:, 1 :, :], \mathbf{d}_y \mathbf{u}[:, :, -1, :])). \quad (\text{B20b})$$

Eventually, combining Eqs. (B20a) and (B20b), the total diffusion contribution of the rhs in Eq. (B1) reads

$$d\mathcal{Q} = \frac{\mathbf{P}^x[:, 1 : -1, 1 :] - \mathbf{P}^x[:, 1 : -1, : -1]}{\Delta_x[:, \mathcal{B} : -\mathcal{B}, \mathcal{B} : -\mathcal{B}]} + \frac{\mathbf{P}^y[:, 1 :, 1 : -1] - \mathbf{P}^y[:, :, -1, 1 : -1]}{\Delta_y[:, \mathcal{B} : -\mathcal{B}, \mathcal{B} : -\mathcal{B}]}. \quad (\text{B21})$$

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