

Quantum dissipative effects for a real scalar field coupled to a dynamical Neumann surface in $d + 1$ dimensions

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We study dissipative effects for a system consisting of a massless real scalar field satisfying Neumann boundary conditions on a space and time-dependent surface, in $d + 1$ dimensions. We focus on the comparison of the results for this system with the ones corresponding to Dirichlet conditions, and the same surface spacetime geometry. We show that, in $d = 1$, the effects are equal up to second order for rather arbitrary surfaces, and up to fourth order for wavelike surfaces. For $d > 1$, we find general expressions for their difference.

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I. INTRODUCTION

Many interesting macroscopic effects are a result of the interaction of an object that can be accurately described classically, with a quantum field, like the Casimir effect [1,2]. Among those effects, not the least important is the so-called “motion induced radiation,” or dynamical Casimir effect (DCE) [3–8], whereby radiation is emitted when an object imposing boundary conditions on a quantum field is accelerated. The quantum nature of the phenomenon is manifested, for instance, in that the expectation value of the radiation field, linear in the sources, vanishes, while observables (quadratic in the radiation field) do not. Indeed, it is the *correlation* between the fluctuations in the sources, mediated by the quantum field, that makes the effect possible.

In the study of dissipative phenomena, a useful method involves examining the imaginary part of the in-out effective action, which is a functional of the external fields. For instance, in the case of motion-induced radiation, these fields are variables that represent the time-varying geometry of the system in which the fields are present. A path integral approach has been successfully used to study this kind of system [9,10] and a closely related subject, namely, radiation from time-dependent boundary conditions with a fixed geometry [11].

Once obtaining an expression for the effective action, dissipation manifests itself in the existence of nonanalyticities, when the external fields are Fourier transformed to

momentum space. A consequence of this is that one may discard analytic terms when evaluating the dissipative effects, a procedure that greatly simplifies the calculations.

In a previous paper [12], we carried out this study for the dynamical Casimir effect due to a real scalar field in $d + 1$ dimensions, in the presence of a mirror that imposes Dirichlet boundary conditions and undergoes time-dependent motion or deformation. Using a perturbative approach, we expanded in powers of the deviation of the mirror’s surface from a hyperplane, up to fourth order. Here we find expressions for the difference between the Neumann and Dirichlet cases. We explicitly show some interesting phenomena, like the vanishing of the difference between Neumann and Dirichlet for $d = 1$, and the fact that a Neumann condition is, for the same geometry, more effective in producing pairs than a Dirichlet one.

The structure of this paper is as follows: in Sec. II we define the system and introduce the object we plan to evaluate, comparing it to its Dirichlet boundary conditions analog. Then, in Sec. III we evaluate the effective action, focusing on its imaginary part, within the context of a perturbative expansion in powers of the departure of the Neumann surface with respect to a planar one. We do that up to the fourth order in that expansion. Finally, in Sec. IV, we present our conclusions.

II. THE SYSTEM AND ITS EFFECTIVE ACTION

We study a system consisting of a massless real scalar field $\varphi(x)$ in $d + 1$ dimensions, subjected to Neumann boundary conditions on a spacetime surface Σ , but otherwise described by the free action $\mathcal{S}_0(\varphi)$:

$$\mathcal{S}_0(\varphi) = \frac{1}{2} \int d^{d+1}x \partial_\mu \varphi(x) \partial_\mu \varphi(x). \quad (1)$$

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We use letters from the middle of the Greek alphabet ($\mu, \nu, \dots$) to design spacetime indices, running over the values $0, 1, \dots, d$; the 0 index being reserved for the temporal components of an object. Spacetime coordinates are $x^\mu = x_\mu$, x_0 being the imaginary time (we use natural units, such that $\hbar \equiv 1$ and $c \equiv 1$). Since the metric tensor becomes the $(d+1) \times (d+1)$ identity matrix, no meaning is to be ascribed to the position (upper or lower) of a particular index. Finally, the Einstein convention of summation over repeated indices in monomial expressions is also assumed, unless explicitly stated otherwise.

A convenient way to describe the system, while taking into account the boundary conditions, is through the Euclidean vacuum transition amplitude, a functional of Σ :

$$\mathcal{Z}(\Sigma) = \int \mathcal{D}\varphi \delta_\Sigma(\partial_n \varphi) e^{-S_0(\varphi)}, \quad (2)$$

where $\delta_\Sigma(\partial_n \varphi)$ denotes a Dirac δ functional, which imposes Neumann conditions for φ on the surface.

We shall now be more specific about the kind of surface we deal with here, since our treatment relies upon some assumptions regarding that object. Indeed, we assume it to be possible, at least by a proper coordinate system choice, to parametrize Σ by y^μ ($\mu = 0, \dots, d$) as follows:

$$\begin{aligned} \Sigma x_\parallel &\rightarrow y(x_\parallel), & x_\parallel &\equiv (x^\alpha)_{\alpha=0}^{d-1}, & \text{and} \\ y^\mu(x_\parallel) &= \delta_\alpha^\mu x^\alpha + \delta_d^\mu \psi(x_\parallel). \end{aligned} \quad (3)$$

This allows us to represent $\delta_\Sigma(\partial_n \varphi)$ more explicitly, by introducing an auxiliary field $\lambda(x_\parallel)$:

$$\delta_\Sigma(\partial \varphi) = \int \mathcal{D}\lambda e^i \int d^d x_\parallel \sqrt{g(x_\parallel)} \lambda(x_\parallel) n_\mu(x_\parallel) \partial_\mu \varphi(x_\parallel, \psi(x_\parallel)), \quad (4)$$

where $g(x_\parallel) \equiv \det[g_{\alpha\beta}]$, with $g_{\alpha\beta} = \delta_{\alpha\beta} + \partial_\alpha \psi \partial_\beta \psi$, the induced metric tensor on Σ , and $n_\mu(x_\parallel)$ the unit normal vector:

$$n^\mu(x_\parallel) = \frac{N^\mu(x_\parallel)}{||N(x_\parallel)||}, \quad N^\mu(x_\parallel) = \delta_d^\mu - \delta_\alpha^\mu \partial_\alpha \psi(x_\parallel). \quad (5)$$

The factor $\sqrt{g(x_\parallel)}$ in (4), introduced to have a reparametrization invariant expression, is canceled by $||N(x_\parallel)||$, with which it coincides.

The effective action $\Gamma(\Sigma)$, which, given the specific parametrization (3), we also denote by $\Gamma(\psi)$, is obtained by functionally integrating out the quantum field:

$$e^{-\Gamma(\Sigma)} = e^{-\Gamma(\psi)} = \frac{\mathcal{Z}(\Sigma)}{\mathcal{Z}_0}, \quad \mathcal{Z}_0 \equiv \int \mathcal{D}\varphi e^{-S_0(\varphi)}. \quad (6)$$

Integrating out φ , we obtain for Γ an expression¹ as a functional integral over λ :

$$e^{-\Gamma(\psi)} = \int \mathcal{D}\lambda e^{-\frac{1}{2} \int_{x_\parallel, x'_\parallel} \sqrt{g(x_\parallel)} \lambda(x_\parallel) \sqrt{g(x'_\parallel)} \lambda(x'_\parallel) \langle \chi(x) \chi(x') \rangle \Big|_{x_d=\psi(x_\parallel), x'_d=\psi(x'_\parallel)}}, \quad (7)$$

where a new field χ has been defined (for all x) such that $\chi(x) \equiv n_\mu(x_\parallel) \partial_\mu \varphi(x)$ and

$$\langle \dots \rangle \equiv \frac{\int \mathcal{D}\varphi \dots e^{-S_0(\varphi)}}{\mathcal{Z}_0}. \quad (8)$$

From the effective action in its (7) representation, it becomes evident that its similarity to the one for $\Gamma_D(\psi)$ is the effective action for *Dirichlet* boundary conditions:

$$e^{-\Gamma_D(\psi)} = \int \mathcal{D}\lambda e^{-\frac{1}{2} \int_{x_\parallel, x'_\parallel} \sqrt{g(x_\parallel)} \lambda(x_\parallel) \sqrt{g(x'_\parallel)} \lambda(x'_\parallel) \langle \varphi(x) \varphi(x') \rangle \Big|_{x_d=\psi(x_\parallel), x'_d=\psi(x'_\parallel)}}, \quad (9)$$

Indeed, (7) may be interpreted as the effective action, with Dirichlet boundary conditions on Σ , but with the replacement $\varphi(x) \rightarrow \chi(x)$. This property suggests the possibility of comparing the effective actions for Neumann and Dirichlet boundary conditions, to find out the nature and properties of the difference.

To proceed with the calculation of Γ , we now integrate out λ , finding as a result

$$\begin{aligned} \Gamma(\psi) &= \frac{1}{2} \log \det[K(x_\parallel, x'_\parallel)], \\ K(x_\parallel, x'_\parallel) &\equiv N_\mu(x_\parallel x_\parallel) [\Delta_{\mu\nu}(x - x')] \Big|_{x_d=\psi(x_\parallel), x'_d=\psi(x'_\parallel)} N_\nu(x'_\parallel), \end{aligned} \quad (10)$$

with

$$\begin{aligned} \Delta_{\mu\nu}(x - x') &\equiv \partial_\mu \partial'_\nu \Delta(x - x'), \quad \Delta(x - x') \equiv \int_K \frac{e^{ik(x-x')}}{k^2}, \\ [\int_K &\equiv (2\pi)^{-(d+1)} \int_k]. \end{aligned} \quad (11)$$

Thus, what remains is to evaluate the log of the functional determinant of the kernel K or, equivalently,

$$\Gamma(\psi) = \frac{1}{2} \text{Tr} \{ \log[K(x_\parallel, x'_\parallel)] \}. \quad (12)$$

From its very definition, we obtain for the kernel an expression that may be conveniently rendered as follows:

¹We use the shorthand notation $\int d^{d+1}x \dots \equiv \int_{x_\parallel} \dots, \int d^d x_\parallel \dots \equiv \int_{x_\parallel} \dots, \dots$

$$K = K_S + K_T + K_U, \quad (13)$$

with

$$\begin{aligned} K_S(x_{\parallel}, x'_{\parallel}) &= \Delta_{ad}(x_{\parallel} - x'_{\parallel}, \psi(x_{\parallel}) - \psi(x'_{\parallel})), \\ K_T(x_{\parallel}, x'_{\parallel}) &= -\Delta_{da}(x_{\parallel} - x'_{\parallel}, \psi(x_{\parallel}) - \psi(x'_{\parallel})) \partial_{\alpha} \psi(x'_{\parallel}) \\ &\quad - \partial_{\alpha} \psi(x_{\parallel}) \Delta_{ad}(x_{\parallel} - x'_{\parallel}, \psi(x_{\parallel}) - \psi(x'_{\parallel})), \\ K_U(x_{\parallel}, x'_{\parallel}) &= \partial_{\alpha} \psi(x_{\parallel}) \Delta_{\alpha\beta}(x_{\parallel} - x'_{\parallel}, \psi(x_{\parallel}) - \psi(x'_{\parallel})) \partial_{\beta} \psi(x'_{\parallel}). \end{aligned} \quad (14)$$

III. EXPANSION IN POWERS OF ψ

We evaluate Γ by expanding it in powers of ψ . To that end, we first expand $K(x_{\parallel}, x'_{\parallel})$,

$$K(x_{\parallel}, x'_{\parallel}) = K^{(0)}(x_{\parallel}, x'_{\parallel}) + K^{(2)}(x_{\parallel}, x'_{\parallel}) + \dots, \quad (15)$$

where the index denotes order in ψ , and we have incorporated the fact that there are only even powers in the expansion of the kernel.

A. Second order

When expanding in powers of ψ , the departure of Σ with respect to a plane, we shall treat the contributions coming from the three terms in (14) above in turn. To calculate the second order term, we see that

$$\Gamma^{(2)}(\psi) = \frac{1}{2} \text{Tr} \{ [K^{(0)}]^{-1} K^{(2)} \} \quad (16)$$

so that, taking into account (14), there will be three contributions in $\Gamma^{(2)}(\psi)$,

$$\Gamma^{(2)}(\psi) = \Gamma_S^{(2)}(\psi) + \Gamma_T^{(2)}(\psi) + \Gamma_U^{(2)}(\psi), \quad (17)$$

in an obvious notation.

It is convenient in what follows to interpret the kernels above as matrix elements of operators. In a Dirac bra-ket notation, we have

$$K^{(0)}(x_{\parallel}, x'_{\parallel}) = \langle x_{\parallel} | \hat{K}^{(0)} | x'_{\parallel} \rangle = - \int_{\mathcal{P}_{\parallel}} e^{ip_{\parallel}(x_{\parallel} - x'_{\parallel})} \frac{|\hat{p}_{\parallel}|}{2}. \quad (18)$$

Hence,

$$\hat{K}^{(0)} = - \frac{|\hat{p}_{\parallel}|}{2} \quad (19)$$

with $|\hat{p}_{\parallel}| = \sqrt{\hat{p}_{\alpha} \hat{p}_{\alpha}}$, and $\langle x_{\parallel} | \hat{p}_{\alpha} | x'_{\parallel} \rangle = -i \partial_{\alpha} \delta(x_{\parallel} - x'_{\parallel})$. To functions of $x_{\parallel}$ there correspond operators that are diagonal in this representation. For example, to the departure ψ there corresponds an operator $\hat{\psi}$, with

$\langle x_{\parallel} | \hat{\psi} | x'_{\parallel} \rangle = \psi(x_{\parallel}) \delta(x_{\parallel} - x'_{\parallel})$. Besides, we have the fundamental commutation relation $[\hat{x}_{\alpha}, \hat{p}_{\beta}] = i \delta_{\alpha\beta}$.

Now take, for instance, the operator K_S , for which we know $K_S^{(2)}(x_{\parallel}, x'_{\parallel})$ by expanding (10):

$$\begin{aligned} K_S^{(2)}(x_{\parallel}, x'_{\parallel}) &= - \int_{\mathcal{P}_{\parallel}} e^{ip_{\parallel}(x_{\parallel} - x'_{\parallel})} \frac{|p_{\parallel}|^3}{4} (\psi(x_{\parallel})^2 + \psi(x'_{\parallel})^2 \\ &\quad - 2\psi(x_{\parallel})\psi(x'_{\parallel})). \end{aligned} \quad (20)$$

We want to build the operator $\widehat{K}_S^{(2)}$ so that the expression above corresponds to $\langle x_{\parallel} | \widehat{K}_S^{(2)} | x'_{\parallel} \rangle$. To do so, we insert the identity in the $x_{\parallel}$ and $p_{\parallel}$ bases, i.e., $\int_{x_{\parallel}} |x_{\parallel}\rangle \langle x'_{\parallel}|$ and $\int_{p_{\parallel}} |p_{\parallel}\rangle \langle p'_{\parallel}|$, use $\hat{\psi} |x_{\parallel}\rangle = \psi(x_{\parallel}) |x_{\parallel}\rangle$ and $|\hat{p}_{\parallel}| |p_{\parallel}\rangle = |p_{\parallel}| |p_{\parallel}\rangle$, and exploit the relation $\langle x_{\parallel} | p_{\parallel} \rangle = e^{ip_{\parallel} \cdot x_{\parallel}} / (2\pi)^{d/2}$. Then we see that

$$\hat{K}_S^{(2)} = -\frac{1}{4} (\hat{\psi}^2 |\hat{p}_{\parallel}|^3 + |\hat{p}_{\parallel}|^3 \hat{\psi}^2) + \frac{1}{2} \hat{\psi} |\hat{p}_{\parallel}|^3 \hat{\psi}; \quad (21)$$

therefore,

$$\begin{aligned} \Gamma_S^{(2)}(\psi) &= \frac{1}{4} \text{Tr}(|\hat{p}_{\parallel}|^{-1} \hat{\psi}^2 |\hat{p}_{\parallel}|^3) + \frac{1}{4} \text{Tr}(|\hat{p}_{\parallel}|^{-1} |\hat{p}_{\parallel}|^3 \hat{\psi}^2) \\ &\quad - \frac{1}{2} \text{Tr}(|\hat{p}_{\parallel}|^{-1} \hat{\psi} |\hat{p}_{\parallel}|^3 \hat{\psi}). \end{aligned} \quad (22)$$

The first two terms lead to the same result, a divergent contribution, which does not affect the imaginary part of the effective action. Indeed,

$$\begin{aligned} \frac{1}{4} \text{Tr}(|\hat{p}_{\parallel}|^{-1} \hat{\psi}^2 |\hat{p}_{\parallel}|^3) &+ \frac{1}{4} \text{Tr}(|\hat{p}_{\parallel}|^{-1} |\hat{p}_{\parallel}|^3 \hat{\psi}^2) \\ &= \frac{1}{2} \int_{\mathcal{K}_{\parallel}} |k_{\parallel}|^2 \int_{x_{\parallel}} \psi^2(x_{\parallel}), \end{aligned} \quad (23)$$

which corresponds to an infinite renormalization of the mass of the surface. Discarding this term, we are led to

$$\Gamma_S^{(2)}(\psi) = -\frac{1}{2} \text{Tr}(|\hat{p}_{\parallel}|^{-1} \hat{\psi} |\hat{p}_{\parallel}|^3 \hat{\psi}). \quad (24)$$

We now proceed to write the term above in an equivalent way, by noticing that

$$\begin{aligned} \Gamma_S^{(2)}(\psi) &= -\frac{1}{2} \text{Tr}(|\hat{p}_{\parallel}|^{-1} \hat{\psi} \hat{p}_{\parallel}^2 |\hat{p}_{\parallel}| \hat{\psi}) \\ &= -\frac{1}{2} \text{Tr}(|\hat{p}_{\parallel}| \hat{\psi} |\hat{p}_{\parallel}| \hat{\psi}) - \frac{1}{2} \text{Tr}(|\hat{p}_{\parallel}|^{-1} [\hat{\psi}, \hat{p}_{\parallel}^2] |\hat{p}_{\parallel}| \hat{\psi}). \end{aligned} \quad (25)$$

We recall now the Dirichlet case, where one can write the (full) second order contribution to the effective action, $\Gamma_D^{(2)}$,

as follows:

$$\Gamma_D^{(2)}(\psi) = -\frac{1}{2}\text{Tr}(|\hat{p}_\parallel|\hat{\psi}|\hat{p}_\parallel|\hat{\psi}), \quad (26)$$

i.e.,

$$\Gamma_S^{(2)}(\psi) = \Gamma_D^{(2)}(\psi) - \frac{1}{2}\text{Tr}(|\hat{p}_\parallel|^{-1}[\hat{\psi}, \hat{p}_\parallel^2]|\hat{p}_\parallel|\hat{\psi}). \quad (27)$$

After some algebra, we may render the above expression as follows:

$$\begin{aligned} \Gamma_S^{(2)}(\psi) - \Gamma_D^{(2)}(\psi) &= -\text{Tr}(|\hat{p}_\parallel|^{-1}\hat{\partial}_\alpha(\partial_\alpha\hat{\psi})|\hat{p}_\parallel|\hat{\psi}) \\ &\quad - \frac{1}{2}\text{Tr}(|\hat{p}_\parallel|^{-1}(\partial_\alpha\hat{\psi})|\hat{p}_\parallel|(\partial_\alpha\hat{\psi})). \end{aligned} \quad (28)$$

A word about notation: when we put parentheses on a derivative of the operator $\hat{\psi}$, it should be understood as the operator corresponding to the derivative of the function. Otherwise, a derivative means an operator, and it is explicitly denoted as such.

For the other two kernels, we have

$$\begin{aligned} \hat{K}_T^{(2)} &= \frac{i}{2}((\partial_\alpha\hat{\psi})\hat{\psi}|\hat{p}_\parallel|\hat{p}_\alpha - (\partial_\alpha\hat{\psi})|\hat{p}_\parallel|\hat{p}_\alpha\hat{\psi} + \hat{\psi}|\hat{p}_\parallel|\hat{p}_\alpha(\partial_\alpha\hat{\psi}) \\ &\quad - |\hat{p}_\parallel|\hat{p}_\alpha\hat{\psi}(\partial_\alpha\hat{\psi})) \end{aligned} \quad (29)$$

and

$$\hat{K}_U^{(2)} = \frac{1}{2}(\partial_\alpha\hat{\psi})\frac{\hat{p}_\alpha\hat{p}_\beta}{|\hat{p}_\parallel|}(\partial_\beta\hat{\psi}), \quad (30)$$

leading to (discarding terms that do not contribute to the imaginary part)

$$\begin{aligned} \Gamma_T^{(2)}(\psi) &= \text{Tr}[|\hat{p}_\parallel|^{-1}(\partial_\alpha\hat{\psi})|\hat{p}_\parallel|(\partial_\alpha\hat{\psi})] \\ &\quad + \text{Tr}[|\hat{p}_\parallel|^{-1}\hat{\partial}_\alpha(\partial_\alpha\hat{\psi})|\hat{p}_\parallel|\hat{\psi}] \end{aligned} \quad (31)$$

and

$$\Gamma_U^{(2)}(\psi) = -\frac{1}{2}\text{Tr}\left[|\hat{p}_\parallel|^{-1}(\partial_\alpha\hat{\psi})\frac{\hat{p}_\alpha\hat{p}_\beta}{|\hat{p}_\parallel|}(\partial_\beta\hat{\psi})\right]. \quad (32)$$

Summing up the three contributions, we see that the result for $\Gamma^{(2)}$ may be put as follows:

$$\begin{aligned} \delta\Gamma^{(2)}(\psi) &\equiv \Gamma^{(2)}(\psi) - \Gamma_D^{(2)}(\psi) \\ &= \frac{1}{2}\text{Tr}\left[|\hat{p}_\parallel|^{-1}(\partial_\alpha\hat{\psi})|\hat{p}_\parallel|\left(\delta_{\alpha\beta} - \frac{\hat{p}_\alpha\hat{p}_\beta}{|\hat{p}_\parallel|^2}\right)(\partial_\beta\hat{\psi})\right]. \end{aligned} \quad (33)$$

By using the momentum and coordinate representations judiciously, the expression above may be converted into

$$\delta\Gamma^{(2)}(\psi) = \frac{1}{2}\int_{\not p_\parallel} G(k_\parallel)|\tilde{\psi}(k_\parallel)|^2, \quad (34)$$

where

$$G(k_\parallel) = k_\parallel^2 \int_{\not p_\parallel} \frac{|p_\parallel|}{|p_\parallel| + k_\parallel} \left[1 - \left(\frac{p_\alpha k_\alpha}{|p_\parallel||k_\parallel|}\right)^2\right]. \quad (35)$$

When $d = 1$, we note that the integration variable has just one component (a frequency). Therefore, the integrand vanishes, and the difference between Neumann and Dirichlet cases disappears at this order for $d = 1$.

Let us consider now (35), for an arbitrary $d > 1$. We see that

$$G(k_\parallel) = \int_{\not p_\parallel} \frac{p_\parallel^2 k_\parallel^2 - (p_\parallel \cdot k_\parallel)^2}{|p_\parallel||p_\parallel| + k_\parallel}. \quad (36)$$

The integral may be exactly evaluated for any d , leading to

$$G(k_\parallel) = (k_\parallel^2)^{\frac{d+2}{2}}(d-1)\frac{(\Gamma(\frac{d+1}{2}))^2\Gamma(-\frac{d}{2})}{2^{d+1}\pi^{\frac{d+2}{2}}\Gamma(d+1)}, \quad (37)$$

which allows us to compare with the Dirichlet case directly, both for odd and even d , by introducing (37) into (34).

1. Odd d ($d=2q+1$)

Analyzing the case $d = 2q + 1$ for $q > 0$ [since we have already established $\delta\Gamma^{(2)}(\psi) = 0$ for $d = 1$], we notice that the arguments of the involved Gamma functions are all well defined without the need of dimensional regularization, and there is no pole term. The imaginary part of the effective action in Minkowski signature, $\Gamma_M^{(2)}$, comes from the factor $(k_\parallel^2)^{\frac{d+2}{2}}$, which is not analytic in $k_\parallel^2$, so after the Wick rotation involving $k_\parallel^2 \rightarrow -k_\parallel^2$, $i\Gamma_M^{(2)} = -\Gamma^{(2)}$, and $\int_{k_\parallel} \rightarrow -i \int_{k_\parallel}$, this factor develops an imaginary part when $k_\parallel^2 > 0$, i.e., the surface is described by timelike modes.

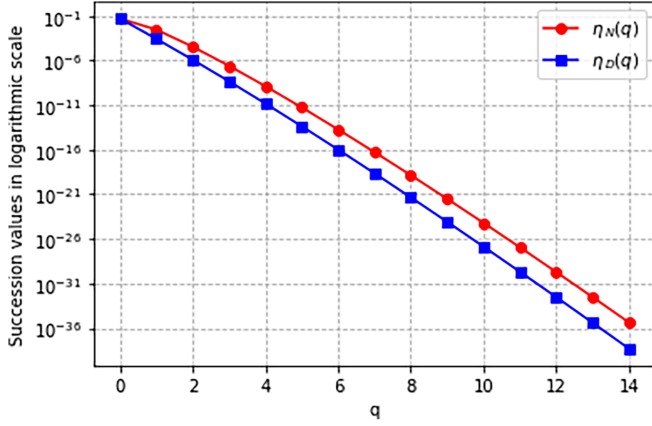
Writing $\text{Im}(\Gamma_M^{(2)})$ for $q = \frac{d-1}{2}$ as

$$\begin{aligned} \text{Im}(\Gamma_M^{(2)}) &= \frac{1}{2}(\eta_D + \eta_\delta) \int_{k_\parallel} \tilde{\psi}(k_\parallel)\tilde{\psi}(-k_\parallel)(k_\parallel^2)^{q+1} \\ &\quad \times \sqrt{(k_0)^2 - |\vec{k}_\parallel|^2} \Theta(|k_0| - |\vec{k}_\parallel|), \end{aligned} \quad (38)$$

where $\eta_D(q)$ corresponds to the contribution in the Dirichlet case,

$$\eta_D(q) = \frac{(q!)^2}{(2\pi)^{q+1}(2q+1)!(2q+3)!!}, \quad (39)$$

and $\eta_\delta(q)$ is the contribution of (37),

FIG. 1. Successions η_D and η_N in logarithmic scale.

$$\eta_\delta(q) = \frac{2q(q+1)(q!)^3}{(\pi)^{q+1}(2q+2)!(2q+1)}, \quad (40)$$

we obtain the succession $\eta_N(q) = \eta_D(q) + \eta_\delta(q)$ for the Neumann case, which is compared to $\eta_D(q)$ in Fig. 1.

2. Even d ($d=2q$)

For $d = 2q$, the factor $\Gamma(-\frac{d}{2})$ yields an infinite counter-term analytic in $k_\parallel^2$ when using dimensional regularization $d = 2q - \epsilon$ and taking the limit $\epsilon \rightarrow 0$. After the Wick rotation, the imaginary part of $\Gamma_M^{(2)}$ comes from the expansion $(k_\parallel^2)^{-\frac{\epsilon}{2}} = 1 - \frac{\epsilon}{2} \log(k_\parallel^2)$, which develops an imaginary part when $k_\parallel$ is timelike. This allows us to write

$$\text{Im}(\Gamma_M^{(2)}) = \frac{1}{2}(\zeta_D + \zeta_\delta) \int_{k_\parallel} \tilde{\psi}(k_\parallel) \tilde{\psi}(-k_\parallel) (k_\parallel^2)^{q+1} \Theta(|k_0| - |\vec{k}_\parallel|), \quad (41)$$

where in a similar notation to the odd d case, $\zeta_D(q)$ is the contribution from the Dirichlet case,

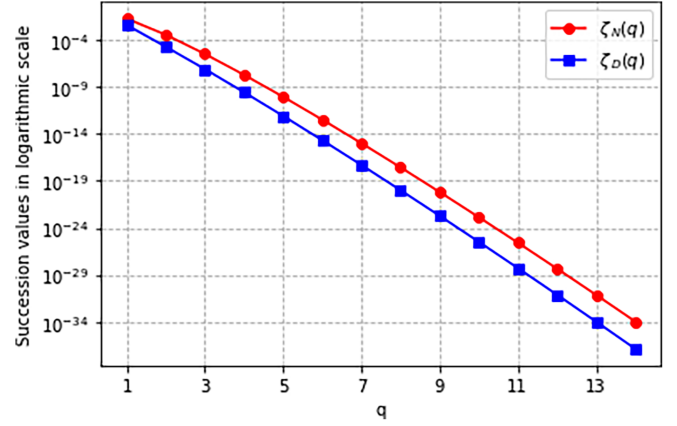
$$\zeta_D(q) = \frac{(2q)!}{\pi^{q-1} 4^{(3q+1)} (q!)^2 (q+1)!}, \quad (42)$$

$\zeta_\delta(q)$ is the contribution from (37),

$$\zeta_\delta(q) = \frac{2(2q-1)(2q)!}{\pi^{(q-1)4^{(3q+1)}} (q!)^3}, \quad (43)$$

and we define $\zeta_N = \zeta_D + \zeta_\delta$ for the Neumann case, which is compared with ζ_D in Fig. 2.

As we see for both odd and even dimensions, the Neumann boundary condition is more effective than the Dirichlet boundary condition when it comes to pair creation, and the probability decreases exponentially with growing dimensions in both cases.

FIG. 2. Successions ζ_D and ζ_N in logarithmic scale.

B. Fourth order

For this order, we have

$$\begin{aligned} \Gamma^{(4)}(\psi) &= \frac{1}{2} \text{Tr}[[K^{(0)}]^{-1} K^{(4)}] - \frac{1}{4} \text{Tr}[[K^{(0)}]^{-1} K^{(2)} [K^{(0)}]^{-1} K^{(2)}] \\ &= \Gamma^{(4,1)}(\psi) + \Gamma^{(4,2)}(\psi), \end{aligned} \quad (44)$$

where we shall treat $\Gamma^{(4,1)}(\psi)$ and $\Gamma^{(4,2)}(\psi)$ separately.

I. $\Gamma^{(4,1)}$

For this contribution, we have

$$\begin{aligned} \widehat{K}_S^{(4)} &= -\frac{1}{48} (\hat{\psi}^4 |\hat{p}_\parallel|^5 + |\hat{p}_\parallel|^5 \hat{\psi}^4) \\ &\quad + \frac{1}{12} (\hat{\psi}^3 |\hat{p}_\parallel|^5 \hat{\psi} + \hat{\psi} |\hat{p}_\parallel|^5 \hat{\psi}^3) - \frac{1}{8} (\hat{\psi}^2 |\hat{p}_\parallel|^5 \hat{\psi}^2), \\ \widehat{K}_T^{(4)} &= \frac{1}{6} (\hat{\psi}^3 (\partial_\alpha \hat{\psi}) \hat{\partial}_\alpha |\hat{p}_\parallel|^3 - (\partial_\alpha \hat{\psi}) \hat{\partial}_\alpha |\hat{p}_\parallel|^3 \hat{\psi}^3) \\ &\quad + \frac{1}{2} (\hat{\psi} (\partial_\alpha \hat{\psi}) \hat{\partial}_\alpha |\hat{p}_\parallel|^3 \hat{\psi}^2 - \hat{\psi}^2 (\partial_\alpha \hat{\psi}) \hat{\partial}_\alpha |\hat{p}_\parallel|^3 \hat{\psi}), \\ \widehat{K}_U^{(4)} &= \frac{1}{4} (\hat{\psi}^2 (\partial_\alpha \hat{\psi}) \hat{p}_\alpha |\hat{p}_\parallel| \hat{p}_\beta (\partial_\beta \hat{\psi})) \\ &\quad + \frac{1}{4} ((\partial_\alpha \hat{\psi}) \hat{p}_\alpha |\hat{p}_\parallel| \hat{p}_\beta (\partial_\beta \hat{\psi}) \hat{\psi}^2) \\ &\quad - \frac{1}{2} (\hat{\psi} (\partial_\alpha \hat{\psi}) \hat{p}_\alpha |\hat{p}_\parallel| \hat{p}_\beta (\partial_\beta \hat{\psi}) \hat{\psi}), \end{aligned} \quad (45)$$

where we have used the symmetry $x_\parallel \rightarrow x'_\parallel$ (when present) beforehand to group operators that give the same trace when multiplied by $[\hat{K}^{(0)}]^{-1}$. This leads to the kernels $\Gamma_S^{(4,1)}$, $\Gamma_T^{(4,1)}$, and $\Gamma_U^{(4,1)}$, respectively.

Following the steps of the second order calculation, we introduce the expression for the Dirichlet equivalent to $\Gamma^{(4,1)}$, namely, $\Gamma_D^{(4,1)}$, as

$$\begin{aligned}\Gamma_D^{(4,1)} = & -\frac{1}{12}\text{Tr}[\hat{p}_\parallel|(\hat{\psi}^3|\hat{p}_\parallel|^3\hat{\psi} + \hat{\psi}|\hat{p}_\parallel|^3\hat{\psi}^3)] \\ & + \frac{1}{8}\text{Tr}[\hat{p}_\parallel|\hat{\psi}^2|\hat{p}_\parallel|^3\hat{\psi}^2],\end{aligned}\quad (46)$$

having discarded terms that do not contribute to the imaginary part. This allows us to write

$$\begin{aligned}\Gamma_S^{(4,1)} - \Gamma_D^{(4,1)} = & -\frac{1}{12}\text{Tr}[\hat{p}_\parallel|^{-1}\hat{\psi}^3|\hat{p}_\parallel|^3[\hat{p}_\parallel^2, \hat{\psi}] \\ & + |\hat{p}_\parallel|^{-1}[\hat{\psi}, \hat{p}_\parallel^2]|\hat{p}_\parallel|^3\hat{\psi}^3] \\ & + \frac{1}{8}\text{Tr}[\hat{p}_\parallel|^{-1}[\hat{\psi}^2, \hat{p}_\parallel^2]|\hat{p}_\parallel|^3\hat{\psi}^2].\end{aligned}\quad (47)$$

Additionally, we have

$$\begin{aligned}\Gamma_T^{(4,1)} = & \frac{1}{6}\text{Tr}[\hat{p}_\parallel|^{-1}(\partial_\alpha\hat{\psi})\hat{\partial}_\alpha|\hat{p}_\parallel|^3\hat{\psi}^3] \\ & - \frac{1}{2}\text{Tr}[\hat{p}_\parallel|^{-1}\hat{\psi}(\partial_\alpha\hat{\psi})\hat{\partial}_\alpha|\hat{p}_\parallel|^3\hat{\psi}^2] \\ & + \frac{1}{2}\text{Tr}[\hat{p}_\parallel|^{-1}\hat{\psi}^2(\partial_\alpha\hat{\psi})\hat{\partial}_\alpha|\hat{p}_\parallel|^3\hat{\psi}]\end{aligned}\quad (48)$$

and

$$\begin{aligned}\Gamma_U^{(4,1)} = & -\frac{1}{2}\text{Tr}[\hat{p}_\parallel|^{-1}\hat{\psi}^2(\partial_\alpha\hat{\psi})\hat{p}_\alpha|\hat{p}_\parallel|\hat{p}_\beta(\partial_\beta\hat{\psi})] \\ & + \frac{1}{2}\text{Tr}[\hat{p}_\parallel|^{-1}\hat{\psi}(\partial_\alpha\hat{\psi})\hat{p}_\alpha|\hat{p}_\parallel|\hat{p}_\beta(\partial_\beta\hat{\psi})\hat{\psi}].\end{aligned}\quad (49)$$

Using $[\hat{\psi}^2, \hat{p}^2] = \{\hat{\psi}, [\hat{\psi}, \hat{p}^2]\}$ and $[\hat{\psi}, \hat{p}^2] = (\partial_\alpha\hat{\psi})\hat{\partial}_\alpha + \hat{\partial}_\alpha(\partial_\alpha\hat{\psi})$, we can write

$$\begin{aligned}\delta\Gamma^{(4,1)} = & \Gamma^{(4,1)}(\psi) - \Gamma_D^{(4,1)}(\psi) \\ = & -\frac{1}{6}\text{Tr}[\hat{p}_\parallel|^{-1}\hat{\partial}_\alpha(\partial_\alpha\hat{\psi})|\hat{p}_\parallel|^3\hat{\psi}^3] + \frac{1}{8}\text{Tr}[\hat{p}_\parallel|^{-1}\{\hat{\psi}, [\hat{\psi}, \hat{p}^2]\}|\hat{p}_\parallel|^3\hat{\psi}^2] \\ & - \frac{1}{2}\text{Tr}[\hat{p}_\parallel|^{-1}\hat{\psi}(\partial_\alpha\hat{\psi})|\hat{p}_\parallel|^3\hat{\partial}_\alpha\hat{\psi}^2] + \frac{1}{2}\text{Tr}[\hat{p}_\parallel|^{-1}\hat{\psi}^2(\partial_\alpha\hat{\psi})|\hat{p}_\parallel|^3\hat{\partial}_\alpha\hat{\psi}] \\ & - \frac{1}{2}\text{Tr}[\hat{p}_\parallel|^{-1}\hat{\psi}^2(\partial_\alpha\hat{\psi})\hat{p}_\alpha|\hat{p}_\parallel|\hat{p}_\beta(\partial_\beta\hat{\psi})] + \frac{1}{2}\text{Tr}[\hat{p}_\parallel|^{-1}\hat{\psi}(\partial_\alpha\hat{\psi})\hat{p}_\alpha|\hat{p}_\parallel|\hat{p}_\beta(\partial_\beta\hat{\psi})\hat{\psi}].\end{aligned}\quad (50)$$

After some algebra, this can be written as

$$\begin{aligned}\delta\Gamma^{(4,1)} = & \text{Tr}\left[|\hat{p}_\parallel|^{-1}\hat{\psi}^2(\partial_\alpha\hat{\psi})|\hat{p}_\parallel|^3\left(\delta_{\alpha\beta} - \frac{1}{2}\frac{\hat{p}_\alpha\hat{p}_\beta}{|\hat{p}_\parallel|^2}\right)(\partial_\beta\hat{\psi})\right] - \text{Tr}\left[|\hat{p}_\parallel|^{-1}\hat{\psi}(\partial_\alpha\hat{\psi})|\hat{p}_\parallel|^3\left(\delta_{\alpha\beta} - \frac{1}{2}\frac{\hat{p}_\alpha\hat{p}_\beta}{|\hat{p}_\parallel|^2}\right)(\partial_\beta\hat{\psi})\hat{\psi}\right] \\ & + \frac{1}{6}\text{Tr}[\hat{p}_\parallel|^{-1}\hat{\psi}^3|\hat{p}_\parallel|^3\hat{\partial}_\alpha(\partial_\alpha\hat{\psi})] - \frac{1}{2}\text{Tr}[\hat{p}_\parallel|^{-1}\hat{\partial}_\alpha\hat{\psi}|\hat{p}_\parallel|^3(\partial_\alpha\hat{\psi})\hat{\psi}^2] \\ & - \frac{1}{4}\text{Tr}[\hat{p}_\parallel|^{-1}(\hat{\partial}_\alpha\hat{\psi}(\partial_\alpha\hat{\psi})|\hat{p}_\parallel|^3\hat{\psi}^2 + \hat{\psi}^2|\hat{p}_\parallel|^3\hat{\partial}_\alpha\hat{\psi}(\partial_\alpha\hat{\psi}))].\end{aligned}\quad (51)$$

For simplicity, as well as to be able to write more explicit expressions, let us analyze the representative case of a wavelike surface:

$$\psi(x_\parallel) = 2A \cos(\omega_0 x_0 - \boldsymbol{\omega}_\parallel \cdot \mathbf{x}_\parallel), \quad (52)$$

which simplifies the outer-momentum structure of the system by imposing

$$\begin{aligned}\tilde{\psi}(k_\parallel) = & A(2\pi)^d[\delta^d(k_\parallel - \omega_\parallel) + \delta^d(k_\parallel + \omega_\parallel)], \\ \widetilde{\psi^2}(k_\parallel) = & A^2(2\pi)^d[\delta^d(k_\parallel - 2\omega_\parallel) + \delta^d(k_\parallel + 2\omega_\parallel) + 2\delta^d(k_\parallel)], \\ \widetilde{\psi^3}(k_\parallel) = & A^3(2\pi)^d[\delta^d(k_\parallel - 3\omega_\parallel) + \delta^d(k_\parallel + 3\omega_\parallel) + 3\delta^d(k_\parallel - \omega_\parallel) + 3\delta^d(k_\parallel + \omega_\parallel)].\end{aligned}\quad (53)$$

Taking the traces in (51), this is reduced to

$$\delta\Gamma^{(4,1)}(\omega_{\parallel}, A) = A^4 (2\pi)^d \delta^d(0) \int_{\mathcal{D}} \omega_{\alpha} \omega_{\beta} (\delta_{\alpha\beta} \ell_{\parallel}^2 - \ell_{\alpha} \ell_{\beta}) \times \left(\frac{|\ell_{\parallel}|}{|\ell_{\parallel} - \omega_{\parallel}|} - \frac{|\ell_{\parallel}|}{|\ell_{\parallel} - 2\omega_{\parallel}|} \right), \quad (54)$$

which vanishes for $d = 1$. Specifically,

$$\frac{\delta\Gamma^{(4,1)}(\omega_{\parallel}, A)}{A^4 (2\pi)^d \delta^d(0)} = (4^{\frac{d+2}{2}} - 1)(d-1) \frac{\Gamma(-\frac{d}{2} - 1) \Gamma(\frac{d+1}{2}) \Gamma(\frac{d+3}{2})}{(4\pi)^{\frac{d+2}{2}} \Gamma(d+2)} \times (\omega_{\parallel}^2)^{\frac{d+4}{2}}. \quad (55)$$

We can perform a Wick rotation $\omega_{\parallel}^2 \rightarrow -\omega^2$ to define $\Gamma_M^{(4,1)}(\omega, A)$ as the Minkowski counterpart of $\Gamma^{(4,1)}(\omega_{\parallel}, A)$, and write

$$\frac{\text{Im}(\Gamma_M^{(4,1)}(\omega, A))}{TV} = \sigma_N A^4 (\omega^2)^{q+\frac{5}{2}} \Theta(|\omega_0| - |\vec{\omega}|) \quad (56)$$

and

$$\frac{\text{Im}(\Gamma_M^{(4,1)}(\omega, A))}{TV} = \kappa_N A^4 (\omega^2)^{q+2} \Theta(|\omega_0| - |\vec{\omega}|), \quad (57)$$

for odd ($d = 2q + 1$) and even ($d = 2q$) dimensions, respectively, where $TV = (2\pi)^d \delta^d(0)$. Similar to the second order, for odd d the imaginary part comes from the nonanalytical momentum expression $(\omega^2)^{1/2}$, while for even d it comes from a logarithmic term using dimensional regularization. The successions σ_N and κ_N are compared to their Dirichlet counterparts, σ_D and κ_D , respectively, in Figs. 3 and 4.

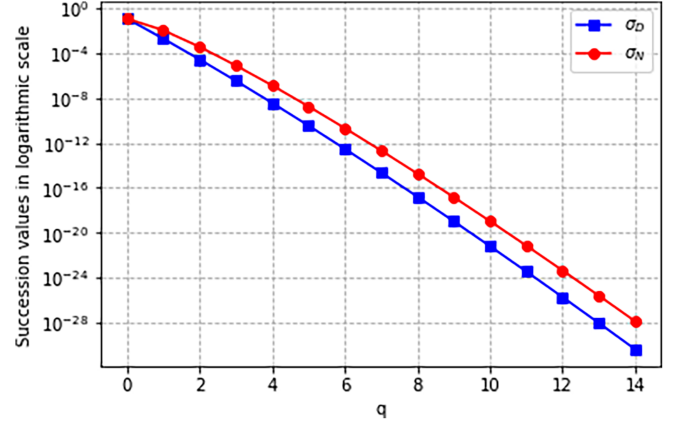


FIG. 3. Successions σ_D and σ_N in logarithmic scale.

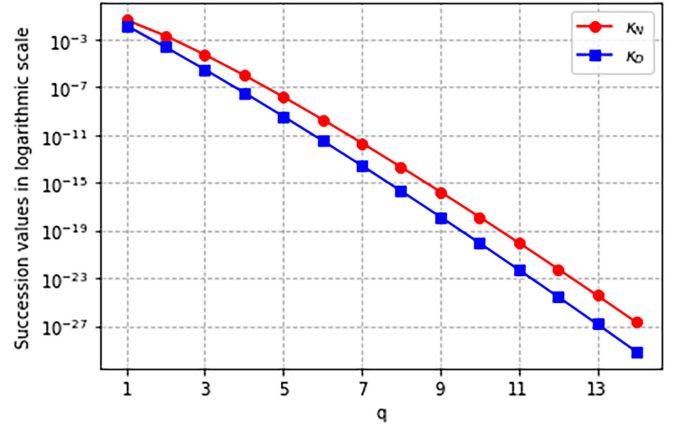


FIG. 4. Successions κ_D and κ_N in logarithmic scale.

2. $\Gamma^{(4,2)}$

For this contribution, we have

$$\begin{aligned} \Gamma^{(4,2)}(\psi) &= -\frac{1}{4} \text{Tr}[[K^{(0)}]^{-1} K^{(2)} [K^{(0)}]^{-1} K^{(2)}] \\ &= -\text{Tr}[\hat{p}_{\parallel}^{-1} \hat{K}_S^{(2)} \hat{p}_{\parallel}^{-1} \hat{K}_S^{(2)}] - \text{Tr}[\hat{p}_{\parallel}^{-1} \hat{K}_T^{(2)} \hat{p}_{\parallel}^{-1} \hat{K}_T^{(2)}] - \text{Tr}[\hat{p}_{\parallel}^{-1} \hat{K}_U^{(2)} \hat{p}_{\parallel}^{-1} \hat{K}_U^{(2)}] \\ &\quad - 2\text{Tr}[\hat{p}_{\parallel}^{-1} \hat{K}_S^{(2)} \hat{p}_{\parallel}^{-1} \hat{K}_T^{(2)}] - 2\text{Tr}[\hat{p}_{\parallel}^{-1} \hat{K}_S^{(2)} \hat{p}_{\parallel}^{-1} \hat{K}_U^{(2)}] - 2\text{Tr}[\hat{p}_{\parallel}^{-1} \hat{K}_T^{(2)} \hat{p}_{\parallel}^{-1} \hat{K}_U^{(2)}], \end{aligned} \quad (58)$$

which we will write as

$$\Gamma^{(4,2)}(\psi) = \Gamma_{SS}^{(4,2)} + \Gamma_{TT}^{(4,2)} + \Gamma_{UU}^{(4,2)} + 2\Gamma_{ST}^{(4,2)} + 2\Gamma_{SU}^{(4,2)} + 2\Gamma_{TU}^{(4,2)} \quad (59)$$

in an obvious notation.

We start with

$$\begin{aligned} \Gamma_{SS}^{(4,2)} &= -\text{Tr}[[K^{(0)}]^{-1} K_S^{(2)} [K^{(0)}]^{-1} K_S^{(2)}] \\ &= -\frac{1}{8} \text{Tr}[\hat{p}_{\parallel}^{-1} \hat{\psi}^2 \hat{p}_{\parallel} | \hat{\psi}^5 \hat{\psi}^2] + \frac{1}{2} \text{Tr}[\hat{p}_{\parallel}^{-1} \hat{\psi} | \hat{p}_{\parallel} | \hat{\psi}^3 \hat{\psi} | \hat{p}_{\parallel} | \hat{\psi}^2 \hat{\psi}^2] - \frac{1}{4} \text{Tr}[\hat{p}_{\parallel}^{-1} \hat{\psi} | \hat{p}_{\parallel} | \hat{\psi}^3 \hat{\psi} | \hat{p}_{\parallel} | \hat{\psi}^{-1} \hat{\psi} | \hat{p}_{\parallel} | \hat{\psi}^3 \hat{\psi}], \end{aligned} \quad (60)$$

and, following the previous procedure, we introduce the Dirichlet counterpart

$$\Gamma_D^{(4,2)} = -\frac{1}{8}\text{Tr}[\hat{p}_\parallel |\hat{\psi}^2| \hat{p}_\parallel^3 \hat{\psi}^2] + \frac{1}{2}\text{Tr}[\hat{p}_\parallel |\hat{\psi}| \hat{p}_\parallel |\hat{\psi}| \hat{p}_\parallel^2 \hat{\psi}^2] - \frac{1}{4}\text{Tr}[\hat{p}_\parallel |\hat{\psi}| \hat{p}_\parallel |\hat{\psi}| \hat{p}_\parallel |\hat{\psi}| \hat{p}_\parallel |\hat{\psi}|]. \quad (61)$$

This allows us to write

$$\begin{aligned} \Gamma_{SS}^{(4,2)} - \Gamma_D^{(4,2)} = & -\frac{1}{8}\text{Tr}[\hat{p}_\parallel^{-1}[\hat{\psi}^2, \hat{p}_\parallel^2]|\hat{p}_\parallel^3 \hat{\psi}^2] + \frac{1}{2}\text{Tr}[\hat{p}_\parallel^{-1}[\hat{\psi}, \hat{p}_\parallel^2]|\hat{p}_\parallel |\hat{\psi}| \hat{p}_\parallel^2 \hat{\psi}^2] \\ & - \frac{1}{2}\text{Tr}[\hat{p}_\parallel^{-1}[\hat{\psi}, \hat{p}_\parallel^2]|\hat{p}_\parallel |\hat{\psi}| \hat{p}_\parallel |\hat{\psi}| \hat{p}_\parallel |\hat{\psi}|] - \frac{1}{4}\text{Tr}[\hat{p}_\parallel^{-1}[\hat{\psi}, \hat{p}_\parallel^2]|\hat{p}_\parallel |\hat{\psi}| \hat{p}_\parallel^{-1}[\hat{\psi}, \hat{p}_\parallel^2]|\hat{p}_\parallel |\hat{\psi}|]. \end{aligned} \quad (62)$$

Furthermore, we have

$$\begin{aligned} \Gamma_{TT}^{(4,2)} = & \text{Tr}[\hat{p}_\parallel^{-1} \hat{\psi} (\partial_\alpha \hat{\psi}) \hat{\partial}_\alpha (\partial_\beta \hat{\psi}) |\hat{p}_\parallel \hat{\partial}_\beta \hat{\psi}] - \text{Tr}[\hat{p}_\parallel^{-1} (\partial_\beta \hat{\psi}) |\hat{p}_\parallel \hat{\partial}_\beta \hat{\psi} \hat{\partial}_\alpha (\partial_\alpha \hat{\psi}) \hat{\psi}] + \frac{1}{2}\text{Tr}[\hat{p}_\parallel^{-1} \hat{\psi} (\partial_\alpha \hat{\psi}) |\hat{p}_\parallel \hat{\partial}_\alpha \hat{\partial}_\beta (\partial_\beta \hat{\psi}) \hat{\psi}] \\ & + \frac{1}{2}\text{Tr}[\hat{p}_\parallel^{-1} \hat{\psi} |\hat{p}_\parallel \hat{\partial}_\beta (\partial_\beta \hat{\psi})| \hat{p}_\parallel^{-1} (\partial_\alpha \hat{\psi}) |\hat{p}_\parallel \hat{\partial}_\alpha \hat{\psi}] - \frac{1}{2}\text{Tr}[\hat{p}_\parallel^{-1} \hat{\psi} |\hat{p}_\parallel \hat{\partial}_\beta (\partial_\beta \hat{\psi})| \hat{p}_\parallel^{-1} \hat{\psi} |\hat{p}_\parallel \hat{\partial}_\alpha (\partial_\alpha \hat{\psi})] \end{aligned} \quad (63)$$

and

$$\Gamma_{UU}^{(4,2)} = -\frac{1}{4}\text{Tr}\left[|\hat{p}_\parallel^{-1} (\partial_\alpha \hat{\psi}) \frac{\hat{p}_\alpha \hat{p}_\beta}{|\hat{p}_\parallel|} (\partial_\beta \hat{\psi}) |\hat{p}_\parallel^{-1} (\partial_\gamma \hat{\psi}) \frac{\hat{p}_\gamma \hat{p}_\sigma}{|\hat{p}_\parallel|} (\partial_\sigma \hat{\psi})\right]. \quad (64)$$

For the nondiagonal terms, we have

$$\begin{aligned} 2\Gamma_{ST}^{(4,2)} = & \frac{1}{2}\text{Tr}[\hat{p}_\parallel^{-1} \hat{\psi} |\hat{p}_\parallel \hat{\partial}_\alpha (\partial_\alpha \hat{\psi})| \hat{p}_\parallel^2 \hat{\psi}^2] + \frac{1}{2}\text{Tr}[\hat{p}_\parallel^{-1} \hat{\psi}^2 |\hat{p}_\parallel^2 \hat{\psi}| \hat{p}_\parallel \hat{\partial}_\alpha (\partial_\alpha \hat{\psi})] \\ & - \frac{1}{2}\text{Tr}[\hat{p}_\parallel^{-1} \hat{\psi}^2 |\hat{p}_\parallel^3 \hat{\partial}_\alpha (\partial_\alpha \hat{\psi}) \hat{\psi}] + \text{Tr}[\hat{p}_\parallel^{-1} \hat{\psi} |\hat{p}_\parallel^3 \hat{\psi} \hat{\partial}_\alpha (\partial_\alpha \hat{\psi}) \hat{\psi}] \\ & - \text{Tr}[\hat{p}_\parallel^{-1} \hat{\psi} |\hat{p}_\parallel^3 \hat{\psi}| \hat{p}_\parallel^{-1} \hat{\psi} |\hat{p}_\parallel \hat{\partial}_\alpha (\partial_\alpha \hat{\psi})], \end{aligned} \quad (65)$$

$$2\Gamma_{SU}^{(4,2)} = \frac{1}{2}\text{Tr}\left[|\hat{p}_\parallel^{-1} \hat{\psi}^2 |\hat{p}_\parallel^2 (\partial_\alpha \hat{\psi}) \frac{\hat{p}_\alpha \hat{p}_\beta}{|\hat{p}_\parallel|} (\partial_\beta \hat{\psi})\right] - \frac{1}{2}\text{Tr}\left[|\hat{p}_\parallel^{-1} \hat{\psi} |\hat{p}_\parallel^3 \hat{\psi}| \hat{p}_\parallel^{-1} (\partial_\alpha \hat{\psi}) \frac{\hat{p}_\alpha \hat{p}_\beta}{|\hat{p}_\parallel|} (\partial_\beta \hat{\psi})\right], \quad (66)$$

and

$$2\Gamma_{TU}^{(4,2)} = -\text{Tr}\left[|\hat{p}_\parallel^{-1} \hat{\psi} (\partial_\alpha \hat{\psi}) \hat{\partial}_\alpha (\partial_\gamma \hat{\psi}) \frac{\hat{p}_\gamma \hat{p}_\sigma}{|\hat{p}_\parallel|} (\partial_\sigma \hat{\psi})\right] + \text{Tr}\left[|\hat{p}_\parallel^{-1} (\partial_\alpha \hat{\psi}) |\hat{p}_\parallel \hat{\partial}_\alpha \hat{\psi}| \hat{p}_\parallel^{-1} (\partial_\gamma \hat{\psi}) \frac{\hat{p}_\gamma \hat{p}_\sigma}{|\hat{p}_\parallel|} (\partial_\sigma \hat{\psi})\right]. \quad (67)$$

Taking the traces for the case where ψ is defined by (53), we identify three different structures that will be presented below. In general, we have two propagator integrals of the forms

$$I_1(\omega_\parallel) = \int_{\mathcal{Q}} \frac{F_1(\ell_\parallel, \omega_\parallel)}{|\ell_\parallel| |\ell_\parallel + \omega_\parallel|} \quad (68)$$

and

$$I_2(\omega_\parallel) = \int_{\mathcal{Q}} \frac{F_2(\ell_\parallel, \omega_\parallel)}{|\ell_\parallel| |\ell_\parallel + 2\omega_\parallel|}, \quad (69)$$

where F_1 and F_2 are functions of $\ell_\parallel^2$, $\omega_\parallel^2$, and $\omega_\parallel \cdot \ell_\parallel$, of degree 6 in momentum powers. The third structure that appears is a three-propagator integral of the form

$$I_3(\omega_{\parallel}) = \int_{\mathcal{Q}} \frac{F_3(\ell_{\parallel}, \omega_{\parallel})}{|\ell_{\parallel}|^2 |\ell_{\parallel} + \omega_{\parallel}| |\ell_{\parallel} - \omega_{\parallel}|}, \quad (70)$$

where F_3 is also a function of $\ell_{\parallel}^2$, $\omega_{\parallel}^2$, and $\omega_{\parallel} \cdot \ell_{\parallel}$, but it has a momentum-power degree of 8. All integrals that appear in this calculation can be put into the form of I_1 , I_2 , or I_3 by multiplying and dividing by $|\ell_{\parallel}|$, $|\ell_{\parallel} \pm \omega_{\parallel}|$, and $|\ell_{\parallel} \pm 2\omega_{\parallel}|$, absorbing all momentum-squared factors into F_1 , F_2 , or F_3 .

Following this procedure for the traces from (62) to (67), we can write

$$\begin{aligned} \delta\Gamma^{(4,2)}(\omega_{\parallel}, A) &= \Gamma^{(4,2)}(\omega_{\parallel}, A) - \Gamma_D^{(4,2)}(\omega_{\parallel}, A) \\ &= A^4 (2\pi)^d \delta^d(0) (I_1(\omega_{\parallel}) + I_2(\omega_{\parallel}) + I_3(\omega_{\parallel})), \end{aligned} \quad (71)$$

with

$$I_1(\omega_{\parallel}) = \int_{\mathcal{Q}} \frac{5\ell_{\parallel}^2(\ell_{\parallel} \cdot \omega_{\parallel})^2 - 5\ell_{\parallel}^4\omega_{\parallel}^2 + \frac{9}{4}(\ell_{\parallel} \cdot \omega_{\parallel})^2\omega_{\parallel}^2 - \frac{9}{4}\ell_{\parallel}^2\omega_{\parallel}^4}{|\ell_{\parallel} + \frac{\omega_{\parallel}}{2}| |\ell_{\parallel} - \frac{\omega_{\parallel}}{2}|}, \quad (72)$$

$$I_2(\omega_{\parallel}) = \int_{\mathcal{Q}} \frac{\ell_{\parallel}^4\omega_{\parallel}^2 - 2\ell_{\parallel}^2(\ell_{\parallel} \cdot \omega_{\parallel})^2 - \frac{1}{2}\ell_{\parallel}^2\omega_{\parallel}^4 + 3(\ell_{\parallel} \cdot \omega_{\parallel})^2\omega_{\parallel}^2 - \omega_{\parallel}^6}{|\ell_{\parallel} + \omega_{\parallel}| |\ell_{\parallel} - \omega_{\parallel}|}, \quad (73)$$

and

$$I_3(\omega_{\parallel}) = \int_{\mathcal{Q}} \frac{-\ell_{\parallel}^4(\ell_{\parallel} \cdot \omega_{\parallel})^2 - (\ell_{\parallel} \cdot \omega_{\parallel})^4 + 2\ell_{\parallel}^6\omega_{\parallel}^2 - 4\ell_{\parallel}^2(\ell_{\parallel} \cdot \omega_{\parallel})^2\omega_{\parallel}^2 + \frac{5}{2}\ell_{\parallel}^4\omega_{\parallel}^4 + \ell_{\parallel}^2\omega_{\parallel}^6}{|\ell_{\parallel}|^2 |\ell_{\parallel} + \omega_{\parallel}| |\ell_{\parallel} - \omega_{\parallel}|}, \quad (74)$$

where the denominators have been put in a symmetric form so that we can discard terms of odd power in $\ell_{\parallel}$.

Notice that all the terms of $I_3(\omega_{\parallel})$, except the one with $(\ell_{\parallel} \cdot \omega_{\parallel})^4$, could be absorbed into $I_2(\omega_{\parallel})$ by canceling the $|\ell_{\parallel}|^2$ propagator in the denominator. However, because of the presence of that term, the quantity $\delta\Gamma^{(4,2)}$ depends on a three-point one-loop function. They have been studied for specific mass and momentum configurations in [13,14]. Naturally, they are considerably more involved than two-propagator integrals, which makes obtaining a numerical result for general d challenging. However, for $d = 1$, we have $(\ell_{\parallel} \cdot \omega_{\parallel})^{2n} = (|\ell_{\parallel}|^2 |\omega_{\parallel}|^2)^n$ for integer n , so (74) can be absorbed completely in the structure of (73). With this condition, we find $I_1(\omega_{\parallel}) = 0$ and $I_2(\omega_{\parallel}) + I_3(\omega_{\parallel}) = 0$, which means $\delta\Gamma^{(4,2)}(\omega_{\parallel}, A) = 0$ for $d = 1$. Since we also had $\delta\Gamma^{(4,1)}(\omega_{\parallel}, A) = 0$ for $d = 1$, this means that $d = 1$ implies $\delta\Gamma^{(4)}(\omega_{\parallel}, A) = 0$.

Thus, we have shown explicitly that, up to the fourth order, the Neumann result agrees for $d = 1$ with the Dirichlet one for wavelike surfaces, and found the explicit form of the difference for other dimensions in terms of general one-loop integrals.

It is worth mentioning that naively performing the loop integrals without first looking for cancellations in $d = 1$ would lead to troublesome expressions involving $\Gamma(d - 1)$ and $\Gamma(\frac{d-1}{2})$, for both the second and the fourth orders. In $d = 1$, those Gamma functions are divergent, and using dimensional regularization $d = 1 - \epsilon$ produces nonanalytic

counterterms. However, we have verified for $\Gamma^{(2)}$ and $\Gamma^{(4,1)}$ that using an auxiliary mass m for the scalar field and taking the limit $m \rightarrow 0$ after performing the integrals yields the same results as in this work, $d = 1$ included.

This apparent divergence can also be seen to emerge if one uses the integration by parts (IBP) relations [15,16] to express loop integrals of the Neumann contributions as functions of known Dirichlet contributions. Specifically, if we define

$$I(\lambda_1, \lambda_2) = \int_{p_{\parallel}} \frac{1}{(p_{\parallel}^2)^{\lambda_1} [(p_{\parallel} + k_{\parallel})^2]^{\lambda_2}}, \quad (75)$$

and consider $I(-\frac{1}{2}, -\frac{1}{2})$ as our master integral, which is the kernel for the Dirichlet case, we can use the IBP relations to solve for one of the kernels of the Neumann case, namely $I(\frac{1}{2}, -\frac{3}{2})$, as $I(\frac{1}{2}, -\frac{3}{2}) = 3 \frac{d+1}{d-1} I(-\frac{1}{2}, -\frac{1}{2})$, which brings problems for $d = 1$ as mentioned. However, $I(\frac{1}{2}, -\frac{3}{2})$ is not the only kernel for the Neumann case, and the divergences cancel when adding all the contributions.

IV. CONCLUSIONS

We have calculated the imaginary part of the effective action, for a Neumann surface in $d + 1$ dimensions that can deform and move, in a time-dependent fashion. The approach relies upon a previous result for a Dirichlet surface, since we have managed to find explicit expressions for the difference between the imaginary parts for both kinds of surfaces.

We have assumed small departures with respect to an average, planar hypersurface, and performed an expansion up to the fourth order in the amplitude of the deformation.

Our results contribute, we believe, to the growing understanding of DCE's and motion-induced radiation phenomena. The explicit dimensional dependence provides benchmarks for future numerical and potentially experimental studies. The exponential suppression of dissipation with increasing dimension, suggests that experimental verification of these effects would be most feasible in low-dimensional systems, consistent with current experimental efforts focusing on quasi-one-dimensional cavity setups and two-dimensional systems.

Several extensions of this work are worth pursuing. First, the calculation could be extended to higher orders to examine whether the systematic difference between Neumann and Dirichlet conditions persists and to determine if there are

more unexpected cancellations. Second, the restriction to wavelike surfaces simplified the momentum structure considerably; a more general analysis of arbitrary surface deformations would provide deeper insight into the dependence on the surface's dynamical properties. Finally, investigating other boundary conditions, such as Robin or mixed conditions, would complete the picture of how quantum fields respond to different types of dynamical boundary conditions.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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