

Determination of the strong coupling constant α_s from inclusive semileptonic B -meson decays

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(Received 6 December 2024; accepted 3 October 2025; published 23 October 2025)

We demonstrate the feasibility of determining the strong coupling constant, α_s , from the inclusive semileptonic decay width of B mesons. We express the semileptonic B decay width as a function of $\alpha_s(5 \text{ GeV})$, the Cabibbo-Kobayashi-Maskawa matrix element $|V_{cb}|$, and b - and c -quark masses in the $\overline{\text{MS}}$ scheme. We fit $\alpha_s(5 \text{ GeV})$ to current world averages of the $B^\pm$ and B^0 semileptonic decay widths. This yields $\alpha_s(5 \text{ GeV}) = 0.245 \pm 0.009$, corresponding to a five-flavor extrapolation of $\alpha_s(m_Z) = 0.1266 \pm 0.0023$. The primary uncertainty contributions arise from the uncertainty on the perturbative expansion and the value of $|V_{cb}|$. Future advancements, including higher-order perturbative calculations and precise measurements of $|V_{cb}|$ and B decay widths from upcoming B and Z factories, could enable this method to determine $\alpha_s(m_Z)$ with a competitive precision of $\Delta\alpha_s(m_Z) \sim 0.0018$. This precision is comparable to the current accuracy of $\alpha_s(m_Z)$ measurements from τ -lepton decays, which is regarded as the most precise experimental approach.

DOI: [10.1103/z6zv-zsyp](https://doi.org/10.1103/z6zv-zsyp)

I. INTRODUCTION

The strong interaction, one of the fundamental interactions in nature, is described by quantum chromodynamics (QCD). The strong coupling constant, $\alpha_s(\mu)$, characterizes the strength of this interaction and exhibits a decreasing trend with increasing energy scale μ . This running behavior is described by the renormalization group equation (RGE) [1], reflecting essential properties of the strong interaction, such as quark confinement at long distances and asymptotic freedom at short distances. Consequently, precise knowledge of $\alpha_s(\mu)$ across the entire range of the energy scale is crucial for a comprehensive understanding and testing of

QCD. α_s at the low-energy scale has been studied through various methodologies, including hadron production in electron-positron annihilation [2], semileptonic charmed meson [3] and τ decays [2,4–9], and inclusive hadronic decay of heavy quarkonia [10,11]. However, there are relatively few measurements of α_s in the energy scale range around 5 GeV.

We consider measuring α_s from the inclusive semileptonic B decay, which corresponds to the energy scale of B -meson masses. Figure 1 shows the Feynman diagram for the inclusive semileptonic B decay ($B \rightarrow X\ell\nu$) at the tree level in the parton model. This process consists of two components: $B \rightarrow X_c\ell\nu$ and $B \rightarrow X_u\ell\nu$, where X_c represents the charmed system and X_u the light hadron system. The ratio of $B \rightarrow X_u\ell\nu$ is approximately 65 times less than the former due to Cabibbo suppression. Using the heavy quark expansion (HQE) method, the branching ratio and the spectral moments of kinematic observables have been parametrized as functions of the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements, $|V_{cb}|$ and $|V_{ub}|$, the

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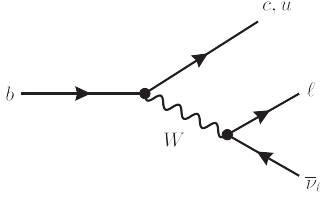


FIG. 1. The parton-level Feynman diagram of semileptonic B decay.

strong coupling constant (α_s), b -quark mass (m_b), c -quark mass (m_c), and nonperturbative HQE parameters [12–21]. On the experimental side, these observables have been measured by *BABAR*, *Belle*, and *Belle II* collaborations over the past two decades [22–25].

The $B \rightarrow X_c \ell \bar{\nu}$ process was used to determine $|V_{cb}|$, m_b , and m_c , with α_s fixed at the value extrapolated from the world average of $\alpha_s(m_Z)$ [19,26–28]. Nowadays, more precise determinations of the $|V_{cb}|$, m_b , and m_c are available, for example, $|V_{cb}|$ from exclusive B decays [29–33] or W decays [34,35], heavy quark masses from lattice QCD [36–40], b - and c -meson masses [41–44], or $e^+e^- \rightarrow \text{hadrons}$ cross section [45–49], etc. Therefore, by fixing the values of $|V_{cb}|$, m_b , and m_c according to those progresses, we could extract α_s using the semileptonic B decay width at the scale around the B -meson masses.

II. THEORETICAL MODEL

Within the HQE framework, the inclusive semileptonic B decay width takes the form shown in (1),

$$\Gamma(B \rightarrow X_c \ell \bar{\nu}_\ell) = \Gamma_0 \left[C_0 - C_{\mu_\pi} \frac{\mu_\pi^2}{2m_b^2} + C_{\mu_G} \frac{\mu_G^2}{2m_b^2} - C_{\rho_D} \frac{\rho_D^3}{2m_b^3} - C_{\rho_{LS}} \frac{\rho_{LS}^3}{2m_b^3} + \dots \right], \quad (1)$$

where $\Gamma_0 \equiv \frac{G_F^2 |V_{cb}|^2 m_b^5 A_{\text{ew}}}{192\pi^3}$, with $G_F = 1.16637886 \times 10^{-5} \text{ GeV}^{-2}$, is the Fermi coupling constant, and $A_{\text{ew}} = 1.014$ is the electroweak correction factor [50]. The coefficients C_i ($i = 0, \mu_\pi, \mu_G$) depend on the ratio of the squared c - and b -quark masses, $\rho = \frac{m_c^2}{m_b^2}$, and have perturbative expansions in α_s . The μ_π^2 , μ_G^2 , ρ_D^3 , and ρ_{LS}^3 are the HQE parameters corresponding to the kinetic, chromomagnetic, Darwin, and spin-orbital terms, respectively. For the purpose of determining α_s , a numerical mapping between $\Gamma(B \rightarrow X_c \ell \bar{\nu}_\ell)$ and α_s has been established, considering the leading-order power correction up to the fourth order, as well as the leading-order contributions from the four terms of high-order power corrections. The precision of the HQE parameters, the next-to-leading-order contributions of the high-order power corrections,

and the contributions of order $\mathcal{O}(1/m_b^{4.5})$ have been included in the error.

The perturbation expansion of C_0 can be expressed as $C_0 = \mathbf{c}_0 + \mathbf{c}_1 \frac{\alpha_s}{\pi} + \mathbf{c}_2 \left(\frac{\alpha_s}{\pi}\right)^2 + \mathbf{c}_3 \left(\frac{\alpha_s}{\pi}\right)^3 + \mathcal{O}(\alpha_s^4)$, where the leading term $\mathbf{c}_0 = 1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2 \ln \rho$ is the tree-level phase space factor [13]. The results for the second-order [15–17] and third-order [20,28] perturbative corrections have been provided in the on-shell scheme. For better perturbative convergence, these perturbative QCD results are reformulated in the $\overline{\text{MS}}$ scheme, at the renormalization scale $\mu = 5 \text{ GeV}$, the mass scale of the decaying B meson. For the purpose of extracting $\alpha_s(5 \text{ GeV})$ in the five-flavor scheme, the perturbative correction C_0 is reformulated consistently in terms of the $\overline{\text{MS}}$ -renormalized quark masses $\bar{m}_b(\mu)$, $\bar{m}_c(\mu)$, and $\alpha_s(\mu)$. The values for the arguments $\bar{m}_b(5 \text{ GeV})$ and $\bar{m}_c(5 \text{ GeV})$ (in the five-flavor scheme) are derived by solving the RGE system with the boundary conditions for $\bar{m}_b(\mu)$, $\bar{m}_c(\mu)$, and $\alpha_s(\mu)$ set as follows: the Particle Data Group (PDG) average values of $\bar{m}_c(\bar{m}_c) = 1.27 \pm 0.02 \text{ GeV}$ and $\bar{m}_b(\bar{m}_b) = 4.18_{-0.02}^{+0.03} \text{ GeV}$ [51], and the sampled values of $\alpha_s(5 \text{ GeV})$ in the fit. In this way, the perturbative correction to $\Gamma(B \rightarrow X_c \ell \bar{\nu}_\ell)$ up to $\mathcal{O}(\alpha_s^3)$ in the leading-order power correction is eventually expressed as a numerical function of $\alpha_s(5 \text{ GeV})$.

The numerical calculations for the coefficients C_{μ_π} and C_{μ_G} were detailed in [52]. For the $1/m_b^3$ power corrections, the coefficients related to the Darwin term and spin-orbital were discussed in [53,54]. The HQE parameters μ_π^2 , μ_G^2 , ρ_D^3 , and ρ_{LS}^3 have been measured through a simultaneous fit using the spectral moments of semileptonic B decays [28]. In addition, the μ_π^2 and μ_G^2 can also be derived from the mass splitting of B mesons [55]. The determination of ρ_D^3 is expected to be achieved with high precision using D decay data from BESIII [56]. Using leading-order approximations of $1/m_b^{2.3}$ power corrections (Eq. (4.1) in [52] and Eq. (23) in [53]) and parameters listed in Table I, we estimate the high-order power corrections to decrease the decay width by $\sim 7\%$, relative to $\Gamma_0 \mathbf{c}_0$.¹

Overall, the numerical mapping from the assumed values of $\alpha_s(5 \text{ GeV})$ to the decay width $\Gamma(B \rightarrow X_c \ell \bar{\nu})$ is shown in Fig. 2. The involved parameters are summarized in Table I. Within the range of $\alpha_s(5 \text{ GeV}) \in [0.16, 0.26]$, the decay width dependence is parametrized by a fifth-order polynomial: $\Gamma(B \rightarrow X_c \ell \bar{\nu}_\ell) = (1.01 \times 10^{-14} \text{ GeV})(-1 + 88.9\alpha_s^{\text{fit}} - 904.8(\alpha_s^{\text{fit}})^2 + 4946.4(\alpha_s^{\text{fit}})^3 - 13467.1(\alpha_s^{\text{fit}})^4 + 15502.7(\alpha_s^{\text{fit}})^5)$, where $\alpha_s^{\text{fit}} \equiv \alpha_s(5 \text{ GeV})$.

III. RESULTS AND DISCUSSION

We fit the value of $\alpha_s(5 \text{ GeV})$ to the inclusive semileptonic decay widths of the $B^\pm$ and B^0 mesons. The experimental decay width can be obtained from measured

¹It is consistent with the numerical values in Eq. (4.1) of [57].

TABLE I. The parameters used during the construction of the theoretical model.

Parameter	Notation	Value and error	Note
Fermi coupling constant	G_F	$1.16637886 \times 10^{-5} \text{ GeV}^{-2}$	[51]
Electroweak correction factor	A_{ew}	1.014	[50]
CKM matrix element	$ V_{cb} $	0.0398 ± 0.0006	[51]
b -quark mass in $\overline{\text{MS}}$	$\bar{m}_b(\bar{m}_b)$	$4.18^{+0.03}_{-0.02} \text{ GeV}$	[51]
c -quark mass in $\overline{\text{MS}}$	$\bar{m}_c(\bar{m}_c)$	$1.27 \pm 0.02 \text{ GeV}$	[51]
HQE parameters	μ_π^2	$0.477 \pm 0.056 \text{ GeV}^2$	[28]
	μ_G^2	$0.306 \pm 0.050 \text{ GeV}^2$	[28]
	ρ_D^3	$0.185 \pm 0.031 \text{ GeV}^2$	[28]
	ρ_{LS}^3	$-0.130 \pm 0.092 \text{ GeV}^2$	[28]
	ρ_{LS}^3	$-0.130 \pm 0.092 \text{ GeV}^2$	[28]
b -quark mass in kinetic scheme	m_b^{kin}	$4.573 \pm 0.012 \text{ GeV}$	[28]

values for the lifetime and semileptonic decay branching ratio,

$$\Gamma_{sl} = \frac{\hbar}{\tau} \mathcal{B}_{sl}. \quad (2)$$

We quote the world averages for the lifetime [51] and the partial branching ratios (with a cut on the lepton energy, $E_l > 0.4 \text{ GeV}$) measured by the Belle experiment [24],

$$\tau_{B^\pm} = 1.638 \pm 0.004 \text{ ps}, \quad \mathcal{B}(B^\pm \rightarrow X_c \ell \nu) = 10.8 \pm 0.4\%,$$

$$\tau_{B^0} = 1.517 \pm 0.004 \text{ ps}, \quad \mathcal{B}(B^0 \rightarrow X_c \ell \nu) = 10.1 \pm 0.4\%.$$

The total branching ratios are obtained by scaling the particle branching ratios by a factor of 1.015, according to [26]. As a result, we obtain $\Gamma(B^\pm) = (4.40 \pm 0.16) \times$

10^{-14} GeV and $\Gamma(B^0) = (4.44 \pm 0.17) \times 10^{-14} \text{ GeV}$. For the related parameters, $|V_{cb}|$ is fixed at the PDG world average value [51], which is extracted from the $\bar{B} \rightarrow D^* \ell \bar{\nu}_\ell$ decays (with $\ell = e, \mu$) along with the lattice QCD calculation of the form factors, independent of the perturbative α_s . The minimum χ^2 fit incorporates the experimental errors in the decay widths, as well as the theoretical uncertainties introduced by parameters and the numerical expression of HQE. These uncertainties are considered to be independent of each other.

The fit yields $\alpha_s(5 \text{ GeV}) = 0.244 \pm 0.013$ from $\Gamma(B^\pm \rightarrow X_c \ell \nu)$, and $\alpha_s(5 \text{ GeV}) = 0.246 \pm 0.013$ from $\Gamma(B^0 \rightarrow X_c \ell \nu)$. Combining the two fits gives

$$\alpha_s(5 \text{ GeV}) = 0.245 \pm 0.009. \quad (3)$$

The combined results are shown and compared with other determinations in Fig. 3.

The uncertainty on α_s arises from three sources: the experimental measurements of the branching ratio and lifetime, the theoretical uncertainty induced by uncertainties on the used parameters, and perturbation expansion for the coefficients of HQE. These terms of the α_s uncertainty are listed in Table II, taking the fit with $\Gamma(B^\pm \rightarrow X_c \ell \nu)$ as an example. The uncertainty in $|V_{cb}|$ propagates to Γ_{sl} via Γ_0 as follows:

$$\left. \frac{\sigma(\Gamma)}{\Gamma} \right|_{|V_{cb}|} = 2 \frac{\sigma(|V_{cb}|)}{|V_{cb}|},$$

contributing a relative uncertainty of 3.0% on Γ_{sl} . The errors induced by the uncertainties on the input values of $\bar{m}_c(\bar{m}_c)$ and $\bar{m}_b(\bar{m}_b)$ are estimated by taking the largest deviations from varying their values within the respective error bounds. The results show that uncertainties in $\bar{m}_c(\bar{m}_c)$ and $\bar{m}_b(\bar{m}_b)$ contribute about 2% and 3% relative uncertainty to the Γ_{sl} prediction, respectively. The uncertainty due to the remnant renormalization scale dependence (R-scale uncertainty) of the leading-order power correction

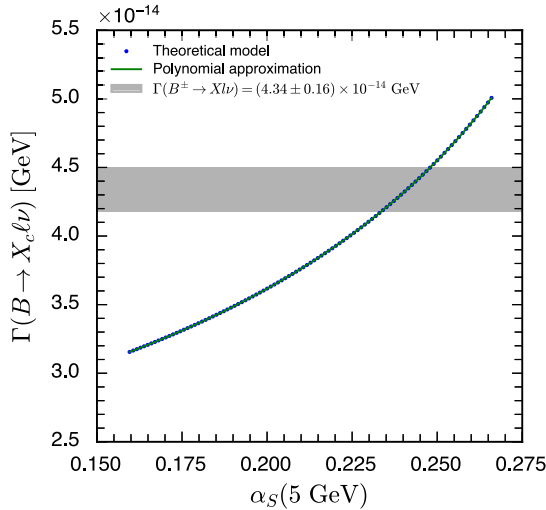


FIG. 2. The numerical function of $\Gamma(B \rightarrow X_c \ell \nu)$ versus $\alpha_s(5 \text{ GeV})$, compared with the $\Gamma(B^\pm \rightarrow X_c \ell \nu) = (4.34 \pm 0.16) \times 10^{-14} \text{ GeV}$ derived from Eq. (2). The numerical function is parametrized by a polynomial function in the $\alpha_s(5 \text{ GeV})$ range from 0.16 to 0.26.

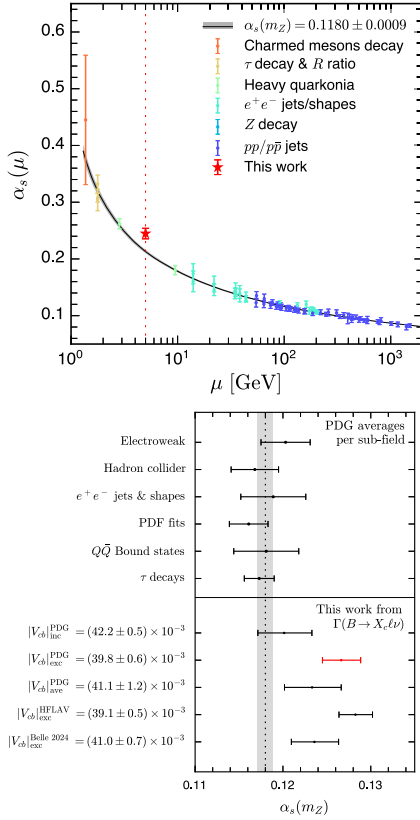


FIG. 3. (Top) The combined $\alpha_s(5 \text{ GeV})$ result [Eq. (3)] compared with the α_s measurements at other energy scales [2,3,8,9,11,58–65]. (Bottom) The comparison of the $\alpha_s(m_Z)$ pre-averages from six experimental subfields in the PDG [51] and the extrapolated values from this work. Additionally, the $\alpha_s(m_Z)$ derived using different values of $|V_{cb}|$ are also compared for reference, including the PDG average [51] for inclusive determination ($|V_{cb}|_{\text{inc}}^{\text{PDG}}$), exclusive determination ($|V_{cb}|_{\text{exc}}^{\text{PDG}}$), their average ($|V_{cb}|_{\text{ave}}^{\text{PDG}}$), exclusive $|V_{cb}|$ from the HFLAV group ($|V_{cb}|_{\text{exc}}^{\text{HFLAV}}$) [66], and recent exclusive determination from Belle ($|V_{cb}|_{\text{exc}}^{\text{Belle 2024}}$) [33]. The extrapolation of α_s along the energy scale is conducted using the RunDec package [67].

is estimated by varying μ from 2.5 GeV to 10 GeV. It leads to about -2% to -4.4% variations in the perturbative corrections relative to the result at $\mu = 5 \text{ GeV}$. The larger variation is taken as the estimation of the uncertainty. The uncertainty of the high-order power corrections is estimated by summing up two contributions quadratically. First, the errors on the μ_π^2 , μ_G^2 , ρ_D^3 , ρ_{LS}^3 , and m_b^{kin} are considered, leading to an uncertainty of about 0.9% on the theoretical prediction for $\Gamma(B \rightarrow X_c \ell \bar{\nu}_\ell)$. The second component is the truncation error. The next-to-leading-order contributions of $C_{\mu_\pi} \frac{\mu_\pi^2}{2m_b^2}$, $C_{\mu_G} \frac{\mu_G^2}{2m_b^2}$, and $C_{\rho_D} \frac{\rho_D^3}{2m_b^3}$ are estimated according to [52,57]. By summing the absolute values of these next-to-leading-order corrections, the truncation errors for $1/m_b^{2,3}$ power corrections are estimated conservatively. Meanwhile, the $O(1/m_b^{4,5})$ order power corrections are

evaluated to cause a reverse influence by a factor of 1.3% in [57], which is also added into the total truncation error. As a result, we assign a truncation error of 2.3% on the high-order power corrections.² The experimental uncertainty of the branching ratio and lifetime also contributes to $\alpha_s(5 \text{ GeV})$ uncertainty.

The result of α_s fit in (3) is extrapolated to the scale of m_Z , $\alpha_s(m_Z) = 0.1266 \pm 0.0023$. As shown in the second plot of Fig. 3, the equivalent $\alpha_s(m_Z)$ from this study exhibits accuracy comparable to the PDG pre-averages from other α_s determination fields. The primary source of the uncertainty arises from the RG-scale uncertainty, which will be refined by future perturbative calculations. Based on the observed reduction in the conventional perturbative QCD-scale uncertainty from $O(\alpha_s^2)$ to $O(\alpha_s^3)$ for $b \rightarrow c \ell \nu$ and $b \rightarrow u \ell \nu$, it is plausible to anticipate that the knowledge of the next order result may halve this perturbative uncertainty. The recent lattice QCD results have determined the quark mass with uncertainties around the 10 MeV level and further improvements are anticipated. The $|V_{cb}|$ measurements from W -boson decays are expected to achieve the accuracy of $\sim 0.7\%$ on the future electron-positron collider [34,35]. The current measurements of the semi-leptonic B decay branching ratios are derived from the 140 fb^{-1} of data collected by Belle [24], with statistical and systematic uncertainties being comparable. Among these, the statistical term will decrease by a factor of approximately 20 when the dataset increases to 50 ab^{-1} on the Belle II data [68]. Assuming the systematic uncertainty remains at the same level, the experimental uncertainty of the B decay width will be $\sim 0.3\%$. All these improvements are scaled to the perspective uncertainties on the Γ_{sl} and $\alpha_s(5 \text{ GeV})$, and marked in parentheses in Table II. Taking into account those advancements, the $\alpha_s(m_Z)$ determination could eventually reach ± 0.0018 , halving the precision conducted by this research. This precision is comparable to the current precision of α_s measurement from τ decays, which is considered one of the most precise approaches.

In addition, this fit uses the exclusive determination of $|V_{cb}|$ as an external input. However, the world average value of the exclusive $|V_{cb}|$ shows 3σ tension with the inclusive one [66]. Consequently, the α_s determination is entangled with this $|V_{cb}|$ puzzle. Figure 3 illustrates the impact of the $|V_{cb}|$ value on the resulting $\alpha_s(5 \text{ GeV})$. More recently, the exclusive $|V_{cb}|$ has been determined to be $|V_{cb}| = (41.0 \pm 0.7) \times 10^{-3}$ using the Belle data [33], which is also compared in Fig. 3.

²During the estimation of each order power correction, results from different schemes are directly used without converting them to a consistent scheme. The effect of this approach is on the next-to-leading order, and included in the uncertainty estimation.

TABLE II. The relative uncertainty contributions to the theoretical prediction of Γ_{sl} and the $\alpha_s(5 \text{ GeV})$ fitting result using $\Gamma(B^\pm \rightarrow X_c \ell \nu)$. Values in the parentheses are the perspective values considering future improvements. The V_{cb} will be measured from W decays [34]. The uncertainty of the branching ratio will be reduced by a factor of about 0.05 considering the 50 ab^{-1} data to be collected by Belle II [68]. The accuracy of b - and c -quark masses has been achieved by lattice QCD calculation [51]. The R-scale μ uncertainty will be further controlled by higher-order perturbative calculations. See the body text for more details.

	Γ_{sl} prediction [%]	$\alpha_s(5 \text{ GeV})$ [%]
$ V_{cb} = 0.0398 \pm 0.0006$	3.0 (1.4)	2.1 (1.0)
$\bar{m}_b(\bar{m}_b) = 4.18^{+0.03}_{-0.02} \text{ GeV}$	3.0 (1.1)	2.1 (0.8)
$\bar{m}_c(\bar{m}_c) = 1.27 \pm 0.02 \text{ GeV}$	2.1 (1.4)	1.4 (1.0)
R-scale $\mu = 5^{+5}_{-2.5} \text{ GeV}$	4.4 (2.2)	3.1 (1.6)
High-order power corrections	2.3 (2.3)	1.6 (1.6)
$\tau_{B^\pm} = 1.638 \pm 0.004 \text{ ps}$	...	0.2
$\mathcal{B}(B^\pm \rightarrow X_c \ell \nu) = 10.8 \pm 0.4\%$	...	3.0 (2.2)
Sum	6.9 (3.2)	5.7 (3.5)

IV. SUMMARY

This manuscript discusses the feasibility to determine $\alpha_s(5 \text{ GeV})$ from the inclusive semileptonic B decay width. The theory model is based on the framework of HQE and includes the $|V_{cb}|$, $\bar{m}_b(\bar{m}_b)$, $\bar{m}_c(\bar{m}_c)$, and four HQE parameters. By constraining the other parameters at external determinations listed in Table I, it is possible to achieve an $\alpha_s(5 \text{ GeV})$ determination of $\alpha_s = 0.245 \pm 0.009$, corresponding to $\alpha_s(m_Z) = 0.1266 \pm 0.0023$.

The uncertainty is estimated to be comparable to the averages of $\alpha_s(m_Z)$ from other experimental methods. The main sources of the uncertainty are estimated in Table II. With further improvements in perturbation calculations and the measurement accuracy of related parameters, the uncertainty of this method could be halved.

It should be noted that as a fundamental parameter, α_s influences QCD predictions through multiple parameters and calculations. A challenge in α_s determination is that theoretical models correlate with prior α_s assumptions. To address this challenge, we use the measurement of $|V_{cb}|$ from exclusive B decays, which is, in principle, independent of the perturbative α_s . We also include the α_s dependency of the scheme transformation and the scale evolution of the quark masses in the α_s fitting. However, the values of $\bar{m}_b(\bar{m}_b)$ and $\bar{m}_c(\bar{m}_c)$ used in our present analysis are taken from the PDG averages, and the HQE parameters are determined from simultaneous fit using spectral moments of inclusive $B \rightarrow X_c \ell \nu$ decays. These parameters depend on the assumptions of the perturbation α_s . To further mitigate this implicit correlation, one potential future avenue is to reformulate the perturbative corrections using the renormalized quark masses defined in the regularization-independent momentum-subtraction schemes [69–73], which can be directly extracted from lattice calculations independently of the perturbative α_s . Alternatively, it is also worth considering a future global fitting of α_s , m_b , and m_c using a broader range of observables, such as spectral moments of semileptonic B and D decays, and the masses of B mesons.

ACKNOWLEDGMENTS

The authors would like to thank Yuming Wang and Lu Cao for the fruitful discussion on the topic of HQE. This study was supported by the National Key R&D Program of China (Grant No. 2022YFE0116900), the National Natural Science Foundation of China (Grants No. 12342502, No. 12325503, No. 12321005, No. 12235008, No. 12205171, and No. 12042507), the Natural Science Foundation of Shandong Province under contracts No. 2024HWYQ-005 and No. tsqn202312052, the United States Department of Energy under contract No. DE-AC02-76SF00515, and the Swiss National Science Foundation (SNF) under contract No. 200020_219367.

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