

Localization in the Dirac spectrum and gauge-field topology

Claudio Bonanno^{1,*} and Matteo Giordano^{2,†}

¹*Instituto de Física Teórica UAM-CSIC, c/ Nicolás Cabrera 13-15,
Universidad Autónoma de Madrid, Cantoblanco, E-28049 Madrid, Spain*

²*Institute of Physics and Astronomy, ELTE Eötvös Loránd University,
Pázmány Péter sétány 1/A, H-1117, Budapest, Hungary*



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We study localization of the low Dirac modes in $3 + 1$ dimensional pure $SU(3)$ gauge theory at zero and nonzero imaginary θ angle, with the aim of better characterizing the relation between low-mode localization and topological features of gauge theories. We show that the mobility edge observed in the deconfined phase at $\theta = 0$ is present also at nonzero θ , appearing exactly at the deconfinement transition. We find that the mobility edge is affected by topology only indirectly, through its effects on the ordering of the Polyakov loop. Moreover, the change in the mobility edge is strongly correlated with the change in the Polyakov-loop expectation value, both as one moves along the critical line, and as one departs from it toward higher temperatures. This further strengthens the connection between low-mode localization and deconfinement, showing in particular the key role played by the ordering of the Polyakov loop.

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I. INTRODUCTION

The discovery of a close relationship between the confining properties and the localization properties of the low Dirac modes in a gauge theory [1–26] (see Ref. [27] for a review) has given a new angle on the finite-temperature transition and on the relation between deconfinement and chiral symmetry restoration. While delocalized in the confined phase, in the deconfined phase of a gauge theory (or more generally in a phase where the Polyakov loop gets ordered) the low Dirac modes are localized, up to a “mobility edge,” λ_c , in the spectrum. This moves toward zero as one approaches the (pseudo)critical temperature, T_c , and vanishes (sometimes abruptly [18,20]) as one crosses over to the confined phase. This happens exactly at the critical temperature when the transition is a genuine thermodynamic transition.¹ This suggests that the localization centers for the low Dirac modes are intimately related to the gauge-field structures responsible for confinement and its loss, as it has been recently demonstrated in the simplest case of $\mathbb{Z}_2$ gauge theory in $2 + 1$ dimensions [26].

The connection between deconfinement and localization could then help in understanding the microscopic mechanism(s) behind confinement in quantum chromodynamics (QCD) and other gauge theories. Given the central role of low Dirac modes in determining the fate of chiral symmetry, it could also help in clarifying the relation between chiral symmetry restoration and deconfinement.

Another aspect of gauge theories that is strongly affected by the finite-temperature transition in QCD and in other gauge theories with gauge group $SU(N_c)$ are the topological features of typical gauge configurations. It is well known that the magnitude and the temperature dependence of the topological susceptibility change dramatically across the transition, both in QCD and in pure $SU(N_c)$ gauge theory, with a strong suppression of nontrivial topology above T_c , getting stronger as the temperature increases [28–42]. Moreover, an ideal instanton-gaslike behavior emerges sufficiently far above T_c [30,31,35,38,39], although with properties different from those of the semiclassical dilute instanton gas [43,44]. This is actually expected in QCD because of the (approximate) restoration of chiral symmetry in the high-temperature phase [45,46].

A third aspect affected by the transition is the behavior of the spectral density of the Dirac operator near zero, that may be related to the change in both the localization properties of Dirac modes and the gauge-field topology. Besides a general depletion of the low-mode region, expected in connection with chiral symmetry restoration, above T_c the near-zero region displays a singular, power-law divergent spectral peak, both in pure gauge [14,20,47–52] and in the presence of dynamical fermions [53–61]. It has been suggested that

*Contact author: claudio.bonanno@csic.es

†Contact author: giordano@bodri.elte.hu

¹An exception is pure $\mathbb{Z}_2$ gauge theory in $2 + 1$ dimensions, where λ_c increases as T_c is approached from above, and disappears abruptly at T_c [26].

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this peak originates from fluctuations of unit charge in the gauge-field topology and the associated (localized) Dirac zero-modes [14,47,52,62] (see Refs. [48–51,59] for an alternative proposal). Moreover, numerical results indicate that the localization properties of the peak modes are nontrivial [50,51,59], with the lowest ones delocalized, or possibly with nontrivial localization properties, and localized higher peak modes. It has been proposed that a mobility edge near zero (but not at zero) separates delocalized peak modes in the immediate vicinity of zero, and localized modes higher up in the spectrum (of course, still below the mobility edge λ_c in the bulk), if $U(1)_A$ remains effectively broken in the chiral limit in the symmetric phase [46].

Deconfinement, chiral symmetry restoration, low-mode localization, and change in topology are all closely connected phenomena, and understanding how they affect each other would help in figuring out what happens at the transition at the microscopic level, and possibly in identifying one of them as being responsible for the others (if there is such a thing). As a first step in this program, in this paper we aim at investigating how the topological features of gauge fields affect the mobility edge λ_c , separating localized low modes from delocalized bulk modes. For simplicity, we do this in the “quenched” limit of pure $SU(3)$ gauge theory, probed with external staggered fermions. The strategy is to artificially increase the instanton content of gauge configurations by switching on an imaginary θ -term, and check how λ_c responds.

Gauge theories and similar systems in the presence of a topological term at imaginary θ -angle, θ_I , have been intensely studied on the lattice [30,40,63–79], mainly in relation with theoretical and phenomenological issues connected to the dependence on a real θ -angle, such as the “ $U(1)_A$ problem” or the “strong- CP problem.” Indeed, this strategy allows one, by means of analytic continuation, to obtain information on the real- θ case, that cannot be directly simulated due to the “complex-action problem,” or “sign problem.”

While in this work we are mainly using the topological term to control the amount of topological excitations in the system, it is worth summarizing the relevant results concerning thermodynamics in its presence. Pure $SU(3)$ gauge theory at nonzero θ_I displays the same kind of first-order transition observed at $\theta_I = 0$ [80], with spontaneous breaking of center symmetry due to ordering of the Polyakov loop. The expectation value of this observable changes discontinuously at the critical temperature, from zero in the low-temperature, confined phase to a nonzero value in the high-temperature, deconfined phase. The critical temperature, $T_c(\theta_I)$, increases with θ_I , as a larger imaginary θ -angle makes it more favorable for the system to remain in the confined phase [72,73]. Accordingly, the magnitude of the Polyakov loop at fixed temperature in the deconfined phase decreases with θ_I [72,73], as the increase

in topological content tends to drive the system back to the confined phase. The phase diagram at physical, real θ -angle can be reconstructed to some extent by means of analytic continuation [72,73,77,79], and more directly by using reweighting methods [73,81] or the subvolume method [82–84]. This interesting but difficult issue is outside of the scope of the present paper. Here we are interested simply in increasing the nontrivial topological content of gauge configurations, which is achieved in a controllable way through an imaginary θ -angle.

In the absence of a topological term, the spectrum of the staggered operator in pure $SU(3)$ gauge theory and the localization properties of its eigenmodes have been studied in Ref. [14]. A similar study for the overlap operator was carried out in Ref. [17]. In both cases a mobility edge was found in the low-lying spectrum, that moves toward the origin as the temperature decreases. Extrapolating a fit to numerical data, λ_c was seen to vanish at a temperature compatible with the critical temperature in both cases. A more detailed study using the overlap operator was carried out exactly at T_c by analyzing separately the configurations belonging to the confined phase and to the real and complex sectors of the deconfined phase [20]. This showed that the mobility edge in the real sector of the deconfined phase is nonzero at T_c (while it is absent in the complex sectors), and disappears abruptly when crossing over to the confined phase.

The observed universal nature of the connection between deconfinement and low-mode localization leads one to expect that a mobility edge is present in the staggered spectrum in the deconfined phase also in the presence of a topological term in the action. The universality of this connection is qualitatively explained by the “sea/islands” picture [22,27,85–88]. According to this picture, low-mode localization in the deconfined phase results from two effects: (1) the opening of a pseudogap (i.e., a region of low spectral density) in the Dirac spectrum as a consequence of the ordering of the Polyakov loop; and (2) the presence of localized “islands” of gauge-field fluctuations in the “sea” of ordered Polyakov loops that are favorable for low modes.

From this point of view, the deconfined phase of $SU(3)$ gauge theory at nonzero imaginary θ is no different than any other deconfined phase, and low-mode localization should take place. We expect then that a mobility edge appears in the real Polyakov-loop sector at temperatures on the critical line, $T_c(\theta_I)$; and that it moves up in the spectrum as the temperature increases, i.e., as the distance from the critical line increases. If this is actually the case, it is an interesting question whether a suitable definition of this distance would fully capture the dependence of the mobility edge on thermal effects. In this context, the most natural choice is the reduced temperature $t(T, \theta_I) \equiv [T - T_c(\theta_I)]/T_c(\theta_I)$.

Determining the θ dependence of λ_c does not present any particular difficulty in itself. On the other hand, actually

understanding how the θ -term affects λ_c seems hopeless, since its introduction affects several features at the same time: (1) the critical temperature of the system, that increases with θ_1 ; (2) the ordering of the Polyakov loop, that plays an essential role in low-mode localization; and (3) the amount of local thermal gauge-field fluctuations as well as of local topological fluctuations, that is likely to affect the amount of favorable localization centers.

Luckily enough, the answer turns out to be quite simple. As we will demonstrate by means of numerical simulations, the dependence of λ_c on θ_1 and on the temperature T decomposes into the sum of two terms, $\lambda_c|_{T,\theta_1} \approx \lambda_c|_{T=T_c(\theta_1),\theta_1} + F(t(T,\theta_1))$. The first term is the position of the mobility edge at the θ_1 -dependent critical temperature, $T_c(\theta_1)$. This is obtained following the approach used in Ref. [20] at $\theta_1 = 0$, by separating confining gauge configurations and nonconfining gauge configurations in the real Polyakov-loop sector. As in that case, a mobility edge is found only in the nonconfining configurations. The second term is a function of T and θ_1 that depends only on the reduced temperature t , and vanishes at $t = 0$. This shows how indeed the reduced temperature is the appropriate variable to describe the influence of thermal fluctuations on the mobility edge. Moreover, we found that the two terms are strongly (directly) correlated with the corresponding change in the Polyakov-loop expectation value, from $T_c(0)$ to $T_c(\theta_1)$ along the critical line for the first term, and from the critical line ($t = 0$) along the temperature axis ($t > 0$) for the second term. This indicates that the increase with θ_1 in local topology at a given temperature, T , affects λ_c only indirectly, through its (dis)ordering effect on the Polyakov loop configuration.

The plan of the paper is as follows. Sections II and III are devoted to introductory material, with a brief description of pure SU(3) gauge theory with an imaginary θ -term in Sec. II, and of localization of staggered eigenmodes in Sec. III. In Sec. IV we present our numerical results, that we discuss in Sec. V. Finally, in Sec. VI we draw our conclusions and discuss future directions for our studies.

II. LATTICE SU(3) GAUGE THEORY WITH A θ -TERM

We discretize SU(3) gauge theory with a topological term on a hypercubic, $N_s^3 \times N_t$ lattice. Sites are denoted as $n = (\vec{x}, t)$, with $0 \leq x_i \leq N_s - 1$ for $i = 1, 2, 3$, and $0 \leq t \leq N_t - 1$. Link variables $U_\mu(n) \in \text{SU}(3)$, $\mu = 1, \dots, 4$, are assigned to the oriented lattice edges connecting sites n and $n + \hat{\mu}$, where $\hat{\mu}$ are the unit vectors parallel to the lattice axes; $U_{-\mu}(n) = U_\mu(n - \hat{\mu})^\dagger$ are assigned to the oppositely oriented lattice edges. Periodic boundary conditions are imposed in all directions. The action is

$$S = S_W - i\tilde{\theta}_L Q_L, \quad (1)$$

where S_W is the Wilson discretization of the Yang–Mills action,

$$S_W = \beta \sum_n \sum_{1 \leq \mu < \nu \leq 4} \left(1 - \frac{1}{3} \text{Re tr } U_{\mu\nu}(n) \right), \quad (2)$$

where β is the lattice coupling, the sums are over all lattice sites and over pairs of (positive) directions, respectively, and $U_{\mu\nu}(n)$ is the usual plaquette variable,

$$U_{\mu\nu}(n) = U_\mu(n) U_\nu(n + \hat{\mu}) U_{-\mu}(n + \hat{\mu} + \hat{\nu}) U_{-\nu}(n + \hat{\nu}); \quad (3)$$

and Q is the topological charge, defined here, as in Refs. [70,72,73], as the clover-discretized version of the field-theoretic continuum expression, i.e., $Q_L = \sum_n q_L(n)$ with

$$q_L(n) = -\frac{1}{2^9 \pi^2} \sum_{\mu,\nu,\rho,\sigma=\pm 1}^{\pm 4} \tilde{\epsilon}_{\mu\nu\rho\sigma} \text{tr}[U_{\mu\nu}(n) U_{\rho\sigma}(n)], \quad (4)$$

where $\tilde{\epsilon}_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}$ is the usual totally antisymmetric Levi-Civita tensor if all indices are positive, and $\tilde{\epsilon}_{(-\mu)\nu\rho\sigma} = -\tilde{\epsilon}_{\mu\nu\rho\sigma}$. Expectation values in the model defined by Eq. (1) are denoted with $\langle \dots \rangle$.

The lattice operator $q_L(n)$ is related to the topological charge density operator $q(x)$ in the continuum as $q_L(n) \sim a^4 Z_Q(\beta) q(an) + O(a^6)$ [89], where a is the lattice spacing and $Z_Q(\beta)$ a suitable renormalization constant that tends to 1 in the continuum limit, $\lim_{\beta \rightarrow \infty} Z_Q(\beta) = 1$. The physical topological charge Q is defined as the integer closest to Q_L measured after cooling [90,91], and the renormalization constant Z_Q is obtained as $Z_Q = \langle Q Q_L \rangle / \langle Q^2 \rangle$ [70], where the expectation values are computed at $\tilde{\theta}_L = 0$. Values of Z_Q for the range of lattice couplings considered in this paper can be found in Ref. [72]. We work here at imaginary $\tilde{\theta}_L = -i\theta_L$. Accounting for the renormalization of the topological charge, the relevant physical imaginary angle at a given lattice spacing is related to θ_L and β as $\theta_1 = Z_Q(\beta)\theta_L$.

The SU(3) gauge theory with an imaginary θ -term displays a first-order transition at a critical lattice coupling β_c , dependent on N_t and θ_1 . At $\beta < \beta_c$ the system is in a confined phase with unbroken center symmetry, i.e., it is invariant under the center transformation

$$U_4(\vec{x}, N_t - 1) \rightarrow z U_4(\vec{x}, N_t - 1), \quad \forall \vec{x}, \quad (5)$$

where $z \in \mathbb{Z}_3$ is an element of the gauge group center. In this phase, the Polyakov loop,

$$P(\vec{x}) \equiv \frac{1}{3} \text{tr} \left(\prod_{t=0}^{N_t-1} U_4(\vec{x}, t) \right), \quad (6)$$

has vanishing expectation value. At $\beta > \beta_c$ center symmetry breaks down spontaneously, due to the Polyakov loops $P(\vec{x})$ ordering along one of the center elements $\{1, e^{\pm i\frac{2\pi}{3}}\}$.² The real sector with $P(\vec{x})$ favoring the value 1 is the sector that would be selected by heavy fermions in the “quenched” limit of infinite mass, and for this reason it is sometimes referred to as the “physical” sector. The critical coupling $\beta_c(N_t, \theta_1)$ has been determined in Refs. [72,73] for several N_t and a range of θ_1 . At β_c the expectation value of the magnitude of the Polyakov loop, $\langle |P| \rangle$, where $P \equiv \frac{1}{N_s} \sum_{\vec{x}} P(\vec{x})$, changes discontinuously from zero to a nonzero, presumably θ_1 -dependent value. In the deconfined phase $\langle |P| \rangle$ decreases quadratically with θ_1 for small θ_1 at fixed T [72,73]. This means that the introduction of an imaginary θ -term makes the Polyakov loop less ordered.

III. LOCALIZATION OF STAGGERED EIGENMODES

The staggered discretization of the Dirac operator is

$$D = \frac{1}{2} \sum_{\mu=1}^4 \eta_{\mu} (U_{\mu} T_{\mu} - T_{\mu}^{\dagger} U_{\mu}^{\dagger}), \quad (7)$$

where T_{μ} is the translation operator in direction μ , with boundary conditions periodic in space and antiperiodic in time understood; and η_{μ} are diagonal matrices, trivial in color space, with entries the usual staggered phases $\eta_{\mu}(n) = (-1)^{\sum_{\alpha < \mu} n_{\alpha}}$. The staggered operator is anti-Hermitian, and obeys $\{D, \varepsilon\} = 0$, where ε is a diagonal, color-trivial matrix with entries $\varepsilon(n) = (-1)^{\sum_{\alpha} n_{\alpha}}$. Its eigenvalues are therefore purely imaginary, and its spectrum is symmetric about zero. In fact, if ψ_l is an eigenvector of D with eigenvalue $ia\lambda_l$, then $\varepsilon\psi_l$ is an eigenvector with eigenvalue $-ia\lambda_l$.

The localization properties of the staggered eigenmodes are determined by the volume scaling of their “size,” suitably averaged over gauge configurations. A practical definition of the mode size is obtained starting from the inverse participation ratio [96–98],

$$\text{IPR}_l \equiv a^{-4} \sum_n \|\psi_l(n)\|^4, \quad (8)$$

where $\|\psi_l(n)\|^2 = \sum_c |(\psi_l)_c(n)|^2$, with the sum running over the color index, $c = 1, 2, 3$, is the gauge-invariant

local magnitude of the eigenmode. The inverse of the IPR is a good definition of the mode size: it is easy to check that if a mode has magnitude uniformly distributed in a region of four-volume V_0 , then its IPR satisfies $(\text{IPR})^{-1} = V_0$. The localization properties of the modes in a given spectral region are determined by the volume scaling of the average IPR computed locally in the spectrum, $\overline{\text{IPR}}(\lambda, L_s)$,³

$$\overline{\text{IPR}}(\lambda, L_s) \equiv \frac{1}{\frac{V}{T} \rho(\lambda)} \left\langle \sum_l \delta(\lambda - \lambda_l) \text{IPR}_l \right\rangle, \quad (9)$$

where $L_s = aN_s$, $V = L_s^3$ is the physical volume, and $\rho(\lambda)$ is the spectral density normalized by the four-volume,

$$\rho(\lambda) \equiv \frac{T}{V} \left\langle \sum_l \delta(\lambda - \lambda_l) \right\rangle. \quad (10)$$

For large L_s one has $\overline{\text{IPR}}(\lambda, L_s) \sim L_s^{-D_2(\lambda)}$, where D_2 is the fractal dimension of the modes. If $D_2(\lambda) = 0$ the mode size is volume independent, and the modes are localized; if $D_2(\lambda) = 3$ the modes spread out over the whole lattice and are delocalized. For intermediate values the modes are known as “critical” in the condensed-matter literature: this is the kind of behavior found at a mobility edge, i.e., a point in the spectrum separating localized and delocalized modes, where the localization length diverges and a phase transition (“Anderson transition”) is encountered along the spectrum [99].

A practically more convenient way to detect localization is through its connection with the (bulk) statistical properties of the spectrum [100]. In fact, localized modes fluctuate independently of each other under local changes in the gauge configuration, and so the corresponding eigenvalues are expected to obey Poisson statistics. On the other hand, fluctuations of delocalized modes under changes in the gauge configuration are strongly correlated, and one expects the corresponding eigenvalues to obey the same statistics as those of the appropriate ensemble of random matrix theory (RMT) [101,102], as determined by the symmetry class of the system. In the case of the staggered operator in the background of SU(3) gauge fields, these are the unitary class, and the Gaussian unitary ensemble [102].

The statistical properties of the spectrum are most easily unveiled by making use of the unfolding procedure, that reveals their universal features. Unfolding is the monotonic mapping $\lambda_n \rightarrow x_n$ of the spectrum defined by

$$x_n = \frac{V}{T} \int_{\lambda_{\min}}^{\lambda_n} d\lambda \rho(\lambda), \quad (11)$$

²For a study of the relation between deconfinement and ordering of the Polyakov loop in higher-order representations in pure gauge theory and at the physical point see Refs. [92–95].

³The dependence of the various expectation values on T and θ_1 is left implicit. As the normalized spectral density is expected to depend mildly on the spatial volume, we leave its dependence on L_s implicit, too.

that makes the density $r_u(x)$ of unfolded eigenvalues identically equal to 1, i.e., $r_u = \frac{V}{T} \rho \frac{d\lambda}{dx} = 1$. In particular, the probability distribution $p(s)$ of the unfolded spacings, i.e., of the spacing between consecutive unfolded eigenvalues, $s_i \equiv x_{i+1} - x_i$, is known for both Poisson and RMT statistics. For Poisson statistics, $p_P(s) = e^{-s}$; for RMT statistics a closed form is not available, but a good approximation is provided by the Wigner surmise $p_{\text{RMT}}(s) = c_1 s^{\beta_D} e^{-c_2 s^2}$, with $\beta_D = 2$ for the unitary class, and c_1 and c_2 determined by the normalization condition $\int_0^\infty ds p(s) = 1$, and by the fact that $\int_0^\infty ds p(s) s = 1/r_u = 1$.

To check the localization properties of the modes one can pick a feature of $p(s)$, e.g., its second moment, and monitor how its average over configurations, computed locally in the spectrum, changes as one moves along it. A numerically convenient choice in this regard [103] is the integrated probability density, I_{s_0} ,

$$I_{s_0}(\lambda, L_s) \equiv \int_0^{s_0} ds p(s; \lambda, L_s),$$

$$p(s; \lambda, L_s) \equiv \frac{1}{\frac{V}{T} \rho(\lambda)} \left\langle \sum_i \delta(\lambda - \lambda_i) \delta(s - s_i) \right\rangle, \quad (12)$$

with $s_0 \simeq 0.508$ chosen to maximize the difference between the expectations for Poisson and (unitary) RMT statistics, i.e., $I_{s_0, P} \simeq 0.398$ and $I_{s_0, \text{RMT}} \simeq 0.117$. As the system size increases, $I_{s_0}(\lambda, L_s)$ tends to $I_{s_0, P}$ or $I_{s_0, \text{RMT}}$, depending on the localization properties of the modes in the spectral region under scrutiny. At a mobility edge, λ_c , $I_{s_0}(\lambda, L_s)$ is volume-independent and takes a universal “critical value,” $I_{s_0, c}$, that depends on the symmetry class of the system; for the unitary class this has been obtained in Ref. [9].

This allows one to determine λ_c quite accurately and very efficiently as the point where $I_{s_0}(\lambda_c, L_s) = I_{s_0, c}$ [14].

IV. NUMERICAL RESULTS

We simulated pure SU(3) gauge theory at finite temperature on $N_s^3 \times N_t$ hypercubic lattices for temporal size $N_t = 4$ and spatial size $N_s = 24$, for various values of the lattice coupling β and of the imaginary lattice θ -parameter, θ_L . The temperature $T = T(\beta, N_t) = 1/[a(\beta)N_t]$ is determined by the inverse temporal size and by the lattice coupling β . For the scale setting we used the subpercent determinations of $a(\beta)/r_0$ of Ref. [104] (or interpolations thereof), where r_0 is the Sommer scale [105]. We chose for the bare angle the values $\theta_L = 0, 15, 20, 25$, and chose values of β so that the dimensionless reduced temperature

$$t(T, \theta_L) = \frac{T - T_c(\theta_L)}{T_c(\theta_L)} \quad (13)$$

where $\theta_L = Z_Q(\beta)\theta_L$, lied on trajectories of constant $t = 0, 0.07, 0.12, 0.18$. For the renormalization constant $Z_Q(\beta)$, for the critical coupling $\beta_c(\theta_L)$, that determines the critical temperature $T_c(\theta_L) = 1/[a(\beta_c(\theta_L))N_t]$, and for the ratio $T/T_c(\theta_L)$ we used the values found in Refs. [72,73] (or interpolations thereof). Our simulation parameters are summarized in Table I.

For each choice of (β, θ_L) we then obtained the low-lying spectrum of the staggered Dirac operator, computing its lowest 200 positive eigenvalues (230 for $\theta_L = 25$) and corresponding eigenvectors on 600 configurations, using the PARPACK routine [106,107]. Configurations were separated by 100 update steps, one step consisting of 1 heat-bath [108,109] and 4 over-relaxation [110] sweeps of

TABLE I. Summary of simulation parameters, and corresponding estimates of the mobility edge (see Sec. IV C).

N_t	N_s	β	θ_L	$\theta_L = Z_Q(\beta)\theta_L$	$t(T, \theta_L) = T/T_c(\theta_L) - 1$	$T_c(\theta_L)/T_c(0)$	$a\lambda_c$
4	24	5.6911	0	0	0.00	1	0.202(16)
		5.7092	15	1.142(19)		1.0395(11)	0.220(13)
		5.7248	20	1.566(23)		1.0746(10)	0.235(14)
		5.7447	25	2.025(30)		1.1209(10)	0.246(16)
4	24	5.7204	0	0	0.07	1	0.2900(21)
		5.7391	15	1.207(20)		1.0395(11)	0.2961(34)
		5.7550	20	1.663(20)		1.0746(10)	0.3038(18)
		5.7762	25	2.177(19)		1.1209(10)	0.3129(20)
4	24	5.7407	0	0	0.12	1	0.3213(20)
		5.7597	15	1.260(14)		1.0395(11)	0.3251(18)
		5.7762	20	1.742(15)		1.0746(10)	0.3314(15)
		5.7970	25	2.282(18)		1.1209(10)	0.3368(18)
4	24	5.7644	0	0	0.18	1	0.3438(21)
		5.7838	15	1.329(18)		1.0395(11)	0.3525(25)
		5.8007	20	1.841(14)		1.0746(10)	0.3567(17)
		5.8220	25	2.417(19)		1.1209(10)	0.3635(19)

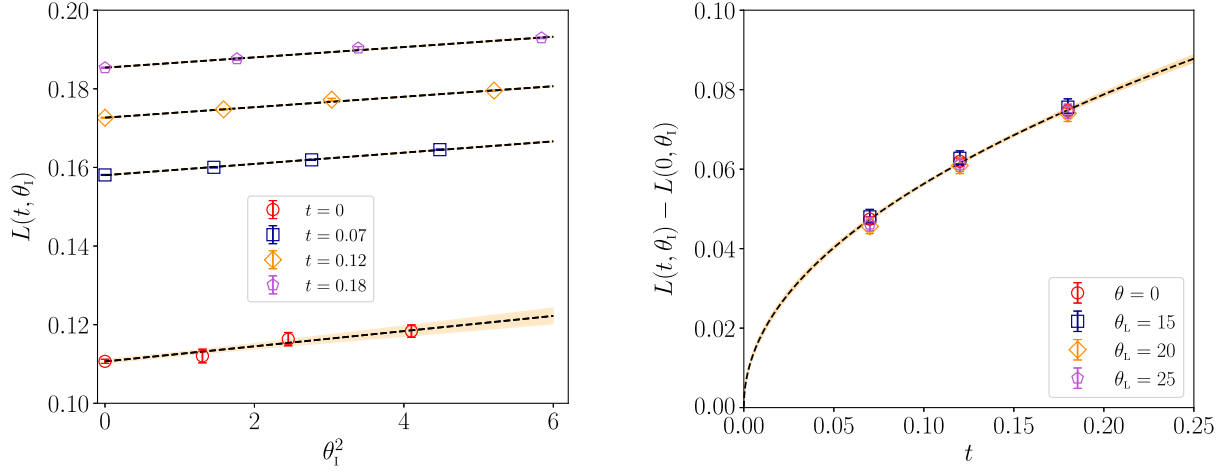


FIG. 1. Left panel: expectation value of the modulus of the spatially averaged Polyakov loop [see Eq. (14)] as a function of θ_1^2 . Lines correspond to separate linear fits of data at constant t . Right panel: expectation value of the modulus of the spatially averaged Polyakov loop for a given θ_1 minus its value at the corresponding critical temperature, $T_c(\theta_1)$, as a function of the reduced temperature, t , for the available values of θ_1 . The dashed line is a common fit with the power-law function at^b to all the data. Our best fit yields $a = 0.172(5)$ and $b = 0.48(2)$, with a reduced chi-squared of $3.0/10.0$.

the lattice. Both local updates have been implemented *à la* Cabibbo—Marinari [111], i.e., updating all the 3 diagonal SU(2) subgroups of SU(3).

To unfold the spectrum we ranked all the eigenvalues of all the configurations in the ensemble and replaced them by their rank divided by the number of configurations [8]. To check the reliability of the method we monitored the average spacing computed locally in the spectrum and verified that it was equal to 1 within errors in the relevant spectral region where the mobility edge is found. This also shows that the effects of taste symmetry, leading to the formation of quartets of staggered eigenvalues, are negligible for what concerns the determination of the mobility edge for our lattice setup (see Ref. [25] for a detailed discussion). These effects show up only at the lowest end of the spectrum, where the average unfolded spacing deviates from 1.

A. Polyakov loop

As a preliminary analysis, in the left panel of Fig. 1 we show the expectation value of the magnitude of the spatially averaged Polyakov loop,

$$L(t, \theta_1) \equiv \langle |P| \rangle|_{T=(1+t)T_c(\theta_1), \theta_1}, \quad (14)$$

as a function of θ_1 for the available values of t . This shows that while this quantity decreases with θ_1 at fixed temperature, T , as shown in Refs. [72,73], it grows with θ_1 at fixed reduced temperature, t , including along the critical line ($t = 0$). The dependence on θ_1 is clearly quadratic in the explored range. In the right panel of Fig. 1 we show L as a function of t after subtracting its value at the critical temperature, $L(t, \theta_1) - L(0, \theta_1)$. Since Z_Q depends on β

we do not have data at the same exact value of $\theta_1 = Z_Q \theta_L$ for different temperatures, and so we actually subtracted the value of L at the critical temperature corresponding to the same bare angle, θ_L . In spite of this, the collapse of data points on a single, power-law curve is quite clear, well within the numerical uncertainties. The reduced temperature gives then a good measure of the ordering of the system above T_c , independently of θ_1 .

B. Spectral density

In Fig. 2 we show the spectral density ρ , Eq. (10), divided by $T_c(0)^3$ to make it dimensionless, against the dimensionless ratio $\lambda/T_c(0)$. Here we use the fixed scale $T_c(0)$ in order not to introduce additional dependences on T and θ_1 .

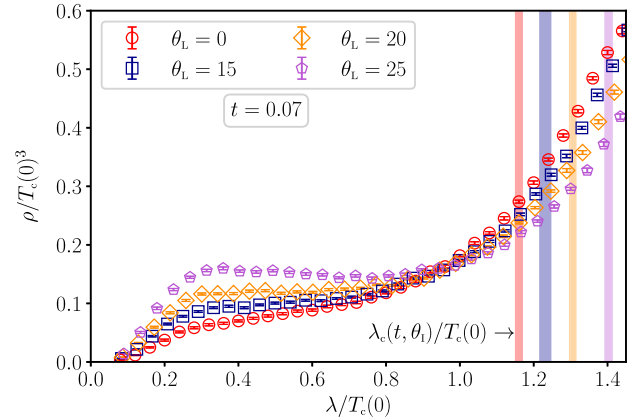


FIG. 2. Spectral density at reduced temperature $t = 0.07$ for all the available θ_L . The corresponding mobility edges and their uncertainties (see Sec. IV C) are shown by vertical bands of the same color as the data points.

The most interesting feature is the emergence of a (rather broad) peak at the low end of the spectrum as one increases θ_L . While nothing more than a barely visible shoulder at $\theta_L = 0$, the structure is already quite clear at $\theta_L = 20$. This structure is the counterpart for staggered fermions on coarse lattices of the sharp near-zero peak observed with overlap valence fermions in the background of pure SU(3) gauge configurations in Refs. [20,47–52], originating from the mixing of the zero modes associated with weakly interacting topological objects [14,47,52,62]. While invisible on $N_t = 4$ lattices at $\theta_L = 0$, it was shown in Ref. [14] that the peak structure is already visible at $N_t = 6$ for $T = 1.05T_c(0)$, becoming sharper as N_t increases and the lattice becomes finer at fixed T , with the staggered operator becoming more accurate at resolving topological effects on the spectrum. Here the emergence of the structure is instead driven by the increase of θ_1 and so of the density of topological objects in the system, further supporting the topological origin of the near-zero peak. Notice that the peak structure is well below the mobility edge (determined in Sec. IV C).

C. Mobility edge

To demonstrate the presence of localized modes at the low end of the spectrum, in Fig. 3 we show the average IPR computed locally in the spectrum, Eq. (9), in units of T^4 , for $t = 0.07$ and all available θ_L . The normalization is chosen in order to compare the size of the modes with the natural scale set by the inverse temperature. While, strictly speaking, the localized nature of modes is revealed by how their size scales with the spatial size of the system, it is clear that near-zero modes have a much larger IPR and so are much smaller than bulk modes. Interestingly, the IPR does not depend monotonically on λ , with a θ_L -dependent maximum in the range $\lambda/T_c(0) \sim 0.5 - 0.8$, that increases in

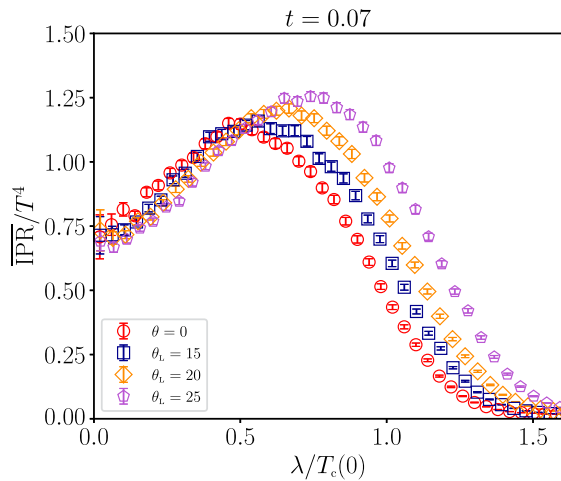


FIG. 3. IPR at reduced temperature $t = 0.07$ for all the available θ_L .

height and moves up in the spectrum as θ_L increases. The smallest modes become then smaller and are found farther away from zero, with a spatial size ℓ of about $\ell T \equiv (T^4/\overline{IPR})^{1/4} \sim 0.93 - 0.96$, i.e., of the order of $1/T$. The lowest modes have a smaller IPR and so a larger size ($\ell T \sim 1.1 - 1.13$), that depends mildly on θ_L . Such a mild θ_L -dependence is generally displayed by modes in the region of the spectral peak seen in Fig. 2.

We then proceeded with a systematic study of localization across the available range of values of θ_1 and t , listed in Table I. As a first step we studied localization at $t = 0$, i.e., at $T = T_c(\theta_1)$. We followed the approach of Ref. [20], separating the nonconfining gauge configurations in the real Polyakov-loop sector from the confining ones (see Fig. 4, left panels), and identifying λ_c as the crossing point of I_{s_0} with its critical value (see Fig. 4, center panels). For the confining configurations $I_{s_0} = I_{s_0,\text{RMT}}$ within errors in the whole explored spectral range, indicating the absence of localized low modes and of a mobility edge. In the nonconfining configurations, instead, for the low modes I_{s_0} rises well above $I_{s_0,\text{RMT}}$ toward $I_{s_0,\text{Poisson}}$, and crosses the critical value, $I_{s_0,c}$, showing the presence of localized low modes and of a mobility edge. The uncertainty on λ_c was then estimated as the half-width of the interval between the crossing points of the upper and lower ends of the error band of I_{s_0} with $I_{s_0,c}$. This whole procedure requires imposing a somewhat arbitrary cutoff on the value of $\text{Re}P$, near the peak in its distribution corresponding to the real sector of the deconfined phase, to safely label a certain configuration as nonconfining, while avoiding configurations intermediate between the confined and the deconfined phase that play no role in the thermodynamic limit. To check for systematic effects we have repeated the analysis changing the value of the cutoff around the center of this peak (see Fig. 4, right panels). The results are within the error band of λ_c obtained at the peak center, that we then took with the corresponding error as our final estimate.

We then proceeded to measure the mobility edge above $T_c(\theta_1)$ on lines of constant reduced temperature t , again using the crossing point of I_{s_0} with its critical value for its determination. We will denote the mobility edge at constant reduced temperature as $\lambda_c(t, \theta_1) \equiv \lambda_c|_{T=(1+t)T_c(\theta_1), \theta_1}$. Our results for the mobility edge in lattice units, $a\lambda_c$, are reported in Table I, including those at $t = 0$ discussed above. In Fig. 5 (left panel) we show the dimensionless ratio $\lambda_c(t, \theta_1)/T_c(0)$ against θ_1^2 , for all the available values of t . The linear trend is clear. A quadratic dependence of $\lambda_c(t, \theta_1)/T_c(0)$ on θ_1 is in line with the usual behavior expected of observables at finite imaginary θ -angle, under the assumption that CP is not spontaneously broken at $\theta = 0$. However, this dependence is not at all obvious, as the mobility edge is not an observable in the conventional sense (i.e., it is not some local, or even nonlocal functional

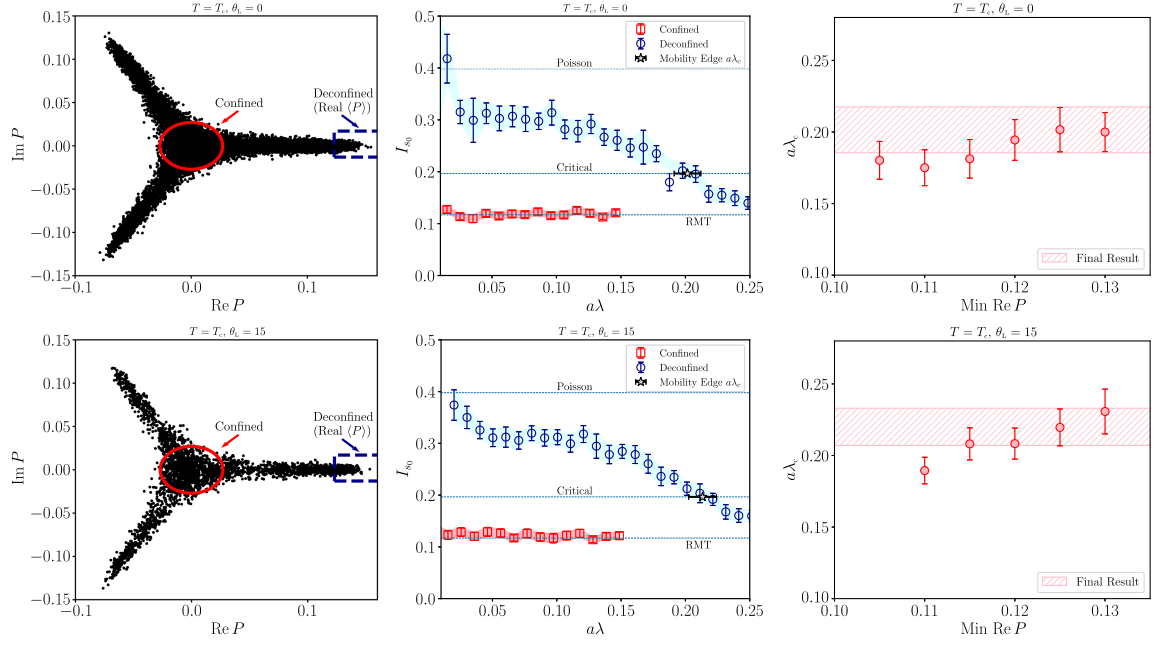


FIG. 4. Appearance of the mobility edge at the critical temperature $T_c(\theta_l)$ for $\theta_L = 0$ (top panels) and $\theta_L = 15$ (bottom panels). The left panels show how we separated configurations belonging to the confined phase from those belonging to the real sector of the deconfined phase. The center panels show the absence of a mobility edge in the confining configurations, and how we determined λ_c from I_{s_0} for the deconfining ones for a specific choice of cutoff (here the center of the peak of the Polyakov-loop distribution at positive $\text{Re } P$). The right panels show how our estimates of the mobility edge depend on the cutoff imposed on the Polyakov loop to assign a configuration to the deconfined phase.

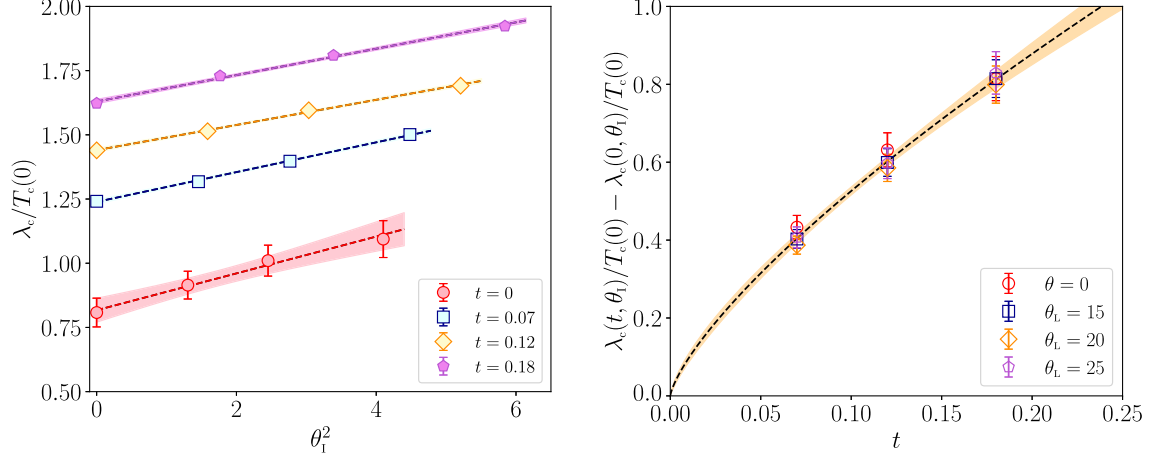


FIG. 5. Left panel: mobility edge in units of $T_c(0)$ against θ_l^2 . Lines correspond to separate linear fits of data at constant reduced temperature t . Right panel: mobility edge in units of $T_c(0)$ minus its value at $T_c(\theta_l)$ against the reduced temperature t . The curve corresponds to a fit of all the data points with the power-law function at^b . Our best fit yields $a = 2.89(30)$ and $b = 0.74(5)$ with a reduced chi-squared of 2.4/10.

of the gauge fields), and even under this assumption it need not be an analytic even function of θ_l .

Coherently with the increase of $T_c(\theta_l)$ and of $L(t, \theta_l)$ as functions of θ_l , also the mobility edge at fixed t increases as a function of θ_l , with approximately the same rate of increase observed for all the explored values of t . For each

θ_l we then subtracted the value of the mobility edge obtained at the corresponding critical temperature, $\lambda_c^c(\theta_l) \equiv \lambda_c|_{T=T_c(\theta_l), \theta_l} = \lambda_c(0, \theta_l)$, to check if the result had a simple dependence on the reduced temperature t . Subtractions were made again between quantities measured at the same θ_L rather than at the same θ_l , which accounts for at least

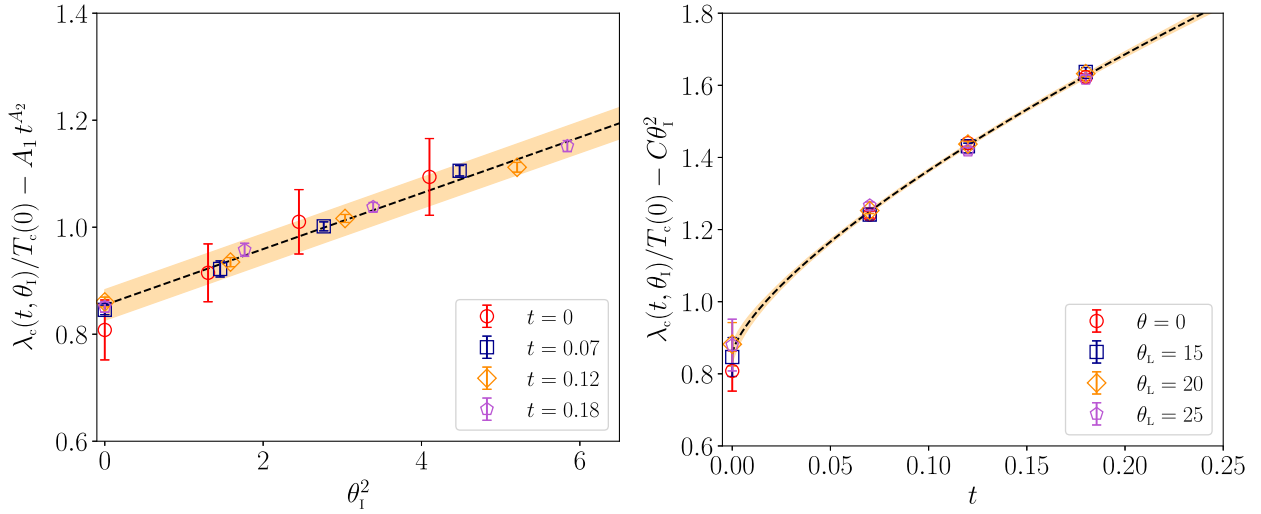


FIG. 6. Global fit of the data for the mobility edge using the function in Eq. (15). After subtracting the temperature-dependent term $A_1 t^{A_2}$, all data appear to collapse on a common curve as a function of θ_1 (left panel). Analogously, after subtracting the correction $C\theta_1^2$ all data collapse on a common curve as a function of t (right panel).

part of the residual scatter of data points.⁴ In spite of this, the collapse of data points on a single, power-law curve as a function of t is quite clear (see Fig. 5, right panel). This shows that the change of the mobility edge from its value at criticality is a good, θ_1 -independent measure of the ordering of the system—basically as good as the reduced temperature.

Motivated by these findings, we performed a global fit of all our numerical results for the mobility edge divided by $T_c(0)$ with an ansatz of the general form $\lambda_c(t, \theta_1)/T_c(0) = \lambda_c^c(\theta_1)/T_c(0) + f(t)$, choosing specifically the functional form

$$\frac{\lambda_c(t, \theta_1)}{T_c(0)} = \frac{\lambda_c^c(0)}{T_c(0)} (1 + C\theta_1^2) + A_1 t^{A_2}. \quad (15)$$

This choice corresponds to assuming that the mobility edge is determined by two contributions. The first contribution comes from the mobility edge at the θ_1 -dependent critical temperature, $\lambda_c^c(\theta_1)$, that is taken to depend quadratically on θ_1 . The other contribution comes from the deviation from the critical line measured by the reduced temperature, t , and is assumed to be a power law in t vanishing at $t = 0$. Our best fit yields the following parameters,

$$\begin{aligned} \frac{\lambda_c^c(0)}{T_c(0)} &= 0.859(29), & C &= 0.0608(28), \\ A_1 &= 2.61(10), & A_2 &= 0.713(40). \end{aligned} \quad (16)$$

⁴Note that the subtraction of the mobility edge at $t = 0$ is mostly meant for illustrative purposes and to motivate the fit with Eq. (15). In this fit the systematic effects due to the mismatch in θ_1 , that are at worst comparable with the statistical errors, are obviously irrelevant. Similar considerations apply for the Polyakov loop, discussed above in Sec. IV A.

The best fit has a reduced chi-squared of $\simeq 0.89$ with 12 degrees of freedom, corresponding to a p -value of $\simeq 56\%$. The fitted value of $\lambda_c^c(0)/T_c(0)$ is in good agreement with its direct determination, $\lambda_c^c(0)/T_c(0) = 0.808(64)$ (see Table I). This fit is pretty robust under the addition of further $\mathcal{O}(\theta_1^2)$ corrections to the A_1 and A_2 parameters, as such terms turn out to be compatible with zero within errors. Moreover, their inclusion does not change the values of $\lambda_c^c(0)/T_c(0)$, C , A_1 , and A_2 within errors. The results of this fit are illustrated in Fig. 6.

It is worth noting that Ref. [20] found $\lambda_c^c(0)/T_c(0) = \lambda_c|_{T=T_c(0), \theta_1=0}/T_c(0) \sim 1$ in pure SU(3) gauge theory, analyzing $N_t = 8$ gauge configurations with valence overlap fermions. Given that these results for the mobility edge are not renormalized,⁵ and given that they are obtained at different lattice spacings and for two different discretizations of the lattice Dirac operator, these two numbers do not need to agree. Nevertheless, they fall in the same ballpark.

D. Correlation of the mobility edge with the Polyakov loop and the topological charge

As the mobility edge is expected to be “dragged” by the pseudogap opened in the spectrum by the ordering of the Polyakov loop, it is interesting to study the correlation between λ_c and the expectation value of the magnitude of the spatially averaged Polyakov loop, L . In Fig. 7 we show (with empty points) λ_c against L , after subtracting from both quantities their value on the critical line, i.e., $\lambda_c(t, \theta_1) - \lambda_c(0, \theta_1)$ is plotted against $L(t, \theta_1) - L(0, \theta_1)$. (Once more, subtractions are made between quantities

⁵The mobility edge renormalizes like the quark mass, see Refs. [8, 112].

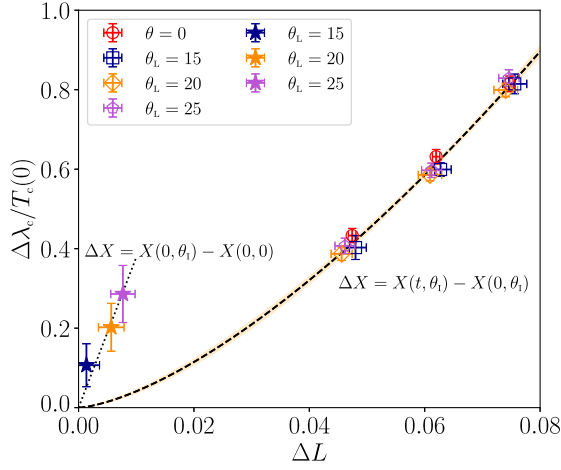


FIG. 7. Correlation between the variations $\Delta\lambda_c/T_c(0)$ of the mobility edge and ΔL of the expectation value of the modulus of the spatially averaged Polyakov loop [see Eq. (14)], relative to the critical line for all $t \neq 0$ (empty points), and relative to $\theta_L = 0$ along the critical line $t = 0$ for all $\theta_L \neq 0$ (filled points). The dashed and dotted lines show the results of a power-law and a linear fit to all the data, respectively [see Eqs. (18) and (19)].

measured at the same θ_L rather than at the same θ_L .) The strong correlation between the two quantities is clear. Notice that the main source of uncertainty comes from the values of both λ_c and L on the critical line (see Figs. 1, left panel, and 5, left panel).

In contrast, no such strong correlation is found between the mobility edge and the average topological charge. This is shown in Fig. 8, where we plot λ_c against

$$\langle Q_L^{(\text{cool})} \rangle(t, \theta_L) \equiv \langle Q_L^{(\text{cool})} \rangle \Big|_{T=(1+t)T_c(\theta_L), \theta_L}, \quad (17)$$

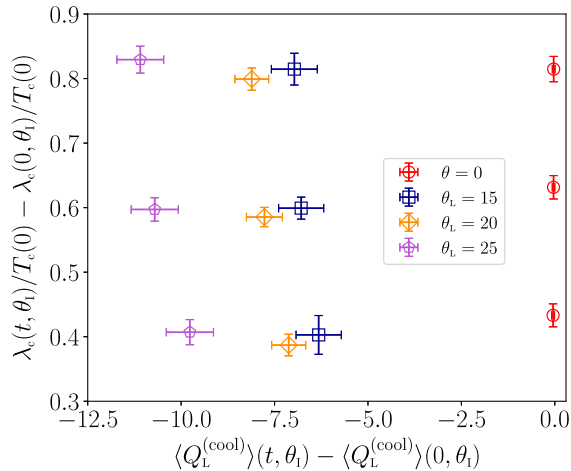


FIG. 8. Relation between the mobility edge and the expectation value of the topological charge minus their respective values on the critical line.

again after subtracting from both quantities their value on the critical line [of course $\langle Q_L^{(\text{cool})} \rangle(0, 0) = 0$]. Here we use $Q_L^{(\text{cool})}$, obtained by measuring Q_L [see Eq. (4)] after 30 cooling steps, without rounding to the nearest integer. On the contrary, the correlation between the two quantities is rather weak at fixed t : indeed, the three sets of data points lying approximately on the same horizontal line correspond to the same t .

Since both λ_c and L , after subtracting their value at criticality, grow as powers of t (see Figs. 1 and 5), the relation between them will also be a power law. A fit of the data in Fig. 7 with a power law gives

$$\frac{\lambda_c(t, \theta_L)}{T_c(0)} - \frac{\lambda_c(\theta_L)}{T_c(0)} = a[L(t, \theta_L) - L(0, \theta_L)]^b, \quad (18)$$

with $a = 37.9(5.7)$ and $b = 1.482(55)$, and a reduced chi-squared of $\simeq 7.3/10$. The exponent is in agreement with those obtained fitting λ_c and L against t separately (see Figs. 1 and 5).

Along the critical line, $T_c(\theta_L)$, the mobility edge depends linearly on the Polyakov loop as a consequence of the fact that both quantities are quadratic in θ_L in the explored range (see Fig. 7, filled points). This is not as trivial as it seems, since the quadratic dependence of $\lambda_c^c(\theta_L)/T_c(0)$ on θ_L is not dictated by symmetry and analyticity arguments, as pointed out above. The relation is in this case

$$\frac{\lambda_c^c(\theta_L)}{T_c(0)} - \frac{\lambda_c^c(0)}{T_c(0)} = s[L(0, \theta_L) - L(0, 0)], \quad (19)$$

with $s = 37.8(6.9)$ obtained by a linear fit (see Fig. 7).

These results are quite remarkable, and show that the mobility edge is essentially dragged by the Polyakov loop getting more ordered, both when moving along the critical line and when departing upward in temperature from it. It also shows, however, that the response of the mobility edge to Polyakov-loop ordering is different in the two cases, with a higher sensitivity along the critical line than when moving away from it (see Fig. 7). This likely reflects the different response of the spectrum to the ordering of the Polyakov loop when this is accompanied by an increase (along the critical line) or a decrease (away from it) in the topological content of gauge configurations—recall that for small θ_L one has

$$\langle Q_L^{(\text{cool})} \rangle(t, \theta_L) = \theta_L \langle Q_L^{(\text{cool})2} \rangle \Big|_{T=(1+t)T_c(0), \theta_L=0} + O(\theta_L^3), \quad (20)$$

with $\langle Q_L^{(\text{cool})2} \rangle|_{T, \theta_L=0}$ decreasing with increasing T . We elaborate further on this point in the next section. Here

we want to stress that in both cases the effects of topology on λ_c are only indirect, and mediated by the ordering of the Polyakov loop. This is the main conclusion of our analysis.

V. DISCUSSION

Figures 5 and 6 show that regarded as a function of θ_1 and the reduced temperature t , rather than θ_1 and the temperature T , the dimensionless mobility edge $\lambda_c(t, \theta_1)/T_c(0)$ is given by two independent pieces, $\lambda_c(t, \theta_1)/T_c(0) = \lambda_c^c(\theta_1)/T_c(0) + f(t)$, each dependent on a single variable. This separation into two parts, one corresponding to the behavior of the system at criticality for a given θ_1 , and one depending on the “effective distance” of the thermal state of the system from its critical state at that θ_1 , as measured by $t(T, \theta_1)$, reflects an analogous separation in the expectation value of the Polyakov loop (see Fig. 1). At fixed θ_1 , the reduced temperature, the Polyakov-loop expectation value, and the position of the mobility edge are then all good measures of the distance from criticality, at least in the explored range of θ_1 and t .

The observed increase with θ_1 of the mobility edge at criticality, $\lambda_c^c(\theta_1)$, follows linearly an analogous increase in the discontinuity of the Polyakov-loop expectation value at the transition (an increase that we demonstrate here quantitatively for the first time, to the best of our knowledge). Moreover, the increase of λ_c with the reduced temperature at fixed θ_1 , encoded in the function f , is proportional to a power of the corresponding increase of the Polyakov-loop expectation value. This shows how close the connection between the mobility edge and the Polyakov loop is: in practice, they provide similar measures of the ordering of the system.

The relation between topology, Polyakov-loop ordering, and behavior of the mobility edge deserves further discussion. First of all, we note that topology does not directly increase the ordering of the Polyakov loop, quite the opposite: increasing topology by increasing θ_1 at fixed T disorders the Polyakov loop. This results in an increase of $T_c(\theta_1)$ with θ_1 , as the transition is delayed in temperature; and in the reduction of the effects of increasing the temperature on the thermal state of the system, by reducing its effective distance (measured by the reduced temperature, t) from the critical state. The increase of the expectation value of the Polyakov loop, both along the critical line and when moving upward in temperature from it, is instead driven by the increase in T . Along the critical line the increase of the Polyakov loop is a side effect of the delayed transition, that takes place at a higher T leading to a higher jump in its expectation value. Moving away from criticality by increasing t , the Polyakov loop obviously increases as a thermal effect, but the increase in topology at nonzero θ_1 slows down its ordering with temperature, reducing the rate of increase of the Polyakov loop with T by reducing the relevant thermal variable, t .

What an increase in θ_1 directly does along the critical line is increase the amount of topological objects present at the transition, an amount that is then reduced by increasing t . This should explain the different sensitivity of the mobility edge to the ordering of the Polyakov loop when moving in these two directions. Proceeding along the critical line, the larger amount of topological objects is expected to increase the density of near-zero (“peak”) modes, rearranging the spectrum by removing modes from the bulk. This rearrangement of the spectrum is clearly seen in the data (the behavior for $t = 0$ is similar to that for $t = 0.07$ shown in Fig. 2). This reduces the bulk’s resistance to the Polyakov loop dragging up the mobility edge, thus making λ_c more sensitive to the Polyakov-loop ordering. Moving away from the critical line, instead, the amount of topological objects decreases and the density of near-zero modes with it, the bulk opposes more resistance to the push of the Polyakov loop toward a larger mobility edge, and the effectiveness of Polyakov-loop ordering in dragging up the mobility edge is reduced.

It is worth noting in this context that both low-mode localization and the appearance of a near-zero spectral peak crucially depend on the opening of the pseudogap, and so on deconfinement. Concerning localization, the pseudogap provides a region of low mode density similar to a spectrum edge, where localized modes start to appear as soon as some disorder is present in the system. The reason is that eigenvalues within the pseudogap correspond to wave functions supported on rare, localized fluctuations, which do not mix easily with modes above the pseudogap for “energetic” reasons, and do not mix easily with other modes of similar eigenvalue due to the typically large spatial separation. For the spectral peak, the opening of the pseudogap prevents mixing of the approximate zero modes associated with localized topological objects with nontopological modes, again due to energetic reasons.

VI. CONCLUSIONS

In this paper we have shown that the connection between deconfinement and localization of the low-lying Dirac modes observed in a large variety of gauge theories [1–27] exists also in the presence of a topological term in the action. This allowed us to study quantitatively the effects of topology on the localization properties of the Dirac modes, by changing the amount of topological excitations in the system.

The main conclusion of this study is that the effects of increasing the topological content of gauge configurations by dialing up an imaginary θ -angle clearly separates into two components. At constant reduced temperature $t(T, \theta_1) = T/T_c(\theta_1) - 1$, the effect of θ_1 is to drag λ_c up in the spectrum, following linearly the Polyakov loop as this gets more ordered. On top of this, the mobility edge further increases with $t(T, \theta_1)$, again due to the ordering of the

Polyakov loop but this time with temperature, and again following the ordering of the Polyakov loop, although with a somewhat reduced sensitivity. In both cases, then, the effect of topology on the mobility edge is only indirect, and mediated by the ordering of the Polyakov loop.

There are several lessons one can draw from this about the relation between localization and topology, both at zero and nonzero θ_1 . The first one is that the appearance of a mobility edge in the bulk of the spectrum at the deconfinement transition is very unlikely to be caused by the corresponding change in the local topology. This is not really surprising, as one already expects from the sea/islands picture of localization that the appearance of this mobility edge is mostly due to the ordering of the Polyakov loop, which depletes the spectrum near the origin (except for the singular peak) and makes it qualitatively similar to a spectrum edge, where eigenmode localization is generally expected. This is also suggested by the appearance of a mobility edge at the deconfinement transition in gauge theories without a nontrivial topological charge, such as $2+1$ dimensional pure SU(3) gauge theory [16], or theories with discrete gauge group [19,22,26].

The other, more interesting lesson drawn from the influence of θ_1 on the mobility edge being only indirect is that fluctuations in the local topology are also unlikely to significantly affect the localization properties of modes below λ_c but well above the near-zero peak. The existence of instantons is then not only not necessary for low-mode localization in gauge theories, as already remarked above, but it has also little to no direct effect on the mobility edge. The relevance of topological fluctuations for the physics of localization near the mobility edge λ_c stems mostly from their general disordering effect on the Polyakov loop, and so local topological fluctuations are not expected to play any distinguished role in supporting localized modes near λ_c . This agrees with the suggestion that these fluctuations affect mostly the near-zero, singular-peak region of the Dirac spectrum [52], and with the modeling of this region in terms of the mixing of the zero modes associated with a dilute gas of topological objects [14,47,52,62]. This is also supported by our findings concerning the enhancement of the near-zero spectral density as the topological content of configurations is increased.

Our results justify *a posteriori* the use of staggered fermions to study the effects of topology on the near-bulk region of the Dirac spectrum: although staggered fermions are known to have “bad” chiral and topological properties, meaning that they are not ideal to detect the fine details of chiral and topological effects unless the lattice spacing is

sufficiently small,⁶ the decoupling of topology from the near-bulk localization physics indicates that this is not a problem for a correct description of the behavior of the mobility edge.

Our study provides also one more piece of evidence for the close connection between localization and deconfinement, with a mobility edge appearing precisely at the deconfinement temperature, $T_c(\theta_1)$, and confirms once more the expectations of the sea/islands picture of localization [22,27,85–88]. In particular, the strong correlation between the position of the mobility edge and the Polyakov-loop expectation value provides clear evidence of the fundamental role played by the Polyakov loop and its ordering in the localization of low Dirac modes: an increase in Polyakov-loop expectation value widens the spectral pseudogap, and drives the mobility edge up in the spectrum.

The natural extension of this study is to look in the place in the spectrum where topological effects are instead expected to play a central role, namely the near-zero region where a singular peak in the spectral density has been observed [14,20,47–61]. This is left for future work.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

⁶This does not mean that staggered fermions are unsuitable to study the topological properties of gauge fields. See, e.g., Ref. [113] for a calculation of the topological susceptibility in zero-temperature pure SU(3) gauge theory from the low-lying staggered spectrum.

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