

Plasmon damping in a charged Bose-Einstein condensate model

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In this work we consider the calculation of the imaginary part of the dispersion relations of the propagating modes in a model of a charged scalar Bose-Einstein (BE) condensate, as well as the contribution to the imaginary part of the longitudinal component of the photon polarization tensor and the dielectric constant. In that model, two modes correspond to the transverse photon polarizations, while the other two modes are combinations of the longitudinal photon and the massive scalar field, which we denote as the $(\pm)$ modes. The dispersion relations of the transverse modes have the usual form for transverse photons in a plasma, and we do not consider them here any further. In a previous work we determined the real part of the dispersion relations of the $(\pm)$ modes, as well as the real part of the longitudinal component of the polarization tensor and dielectric constant, which have some unique features. In the appropriate limit, those results reproduce the results obtained for the dielectric constant and dispersion relation in nonrelativistic models of the BE condensation of charged scalars. Here we determine, in the same model, their imaginary part. The results can be useful in physical contexts involving the electrodynamics of a charged scalar BE condensate, and can serve as benchmark results for further exploration.

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I. INTRODUCTION AND MOTIVATION

The properties of a particle that propagates in a medium, such as its dispersion relation, depend crucially on the composition and conditions of the background. Models that extend the standard electroweak theory usually include additional scalar particles, neutral and charged, that interact with the standard model particles. Such new scalar particles and interactions can have implications in nuclear, astrophysics, and cosmological plasma systems, which may contain a background of the scalar particles [1–4]. Thus, for example, the intrinsic properties of neutrinos and photons propagating in such a background medium, such as the dispersion relations and/or electromagnetic properties, can be modified in special ways that lead to significant and observable effects. Under the appropriate conditions, the

scalar background may form a Bose-Einstein (BE) condensate. This is the case, for example, in the models involved in the hypothesis that the dark matter is a self-interacting BE condensate [5–9].

Since the BE condensate is a very special state of matter, the properties of the particles propagating in such a background, such as the dispersion relations, are different from those in normal matter. The propagation of photons in a medium that contains a BE condensate has been studied using various approaches and in various contexts, for example the electrodynamics of fermions and charged scalar bosons at low temperature [10], heavy-ion collisions [11], exciton-polaritons [12], the BE condensation of photons and the Cerenkov effect [13], and the reduction of the photon velocity [14–16] in atomic gases.

Motivated by these developments, in a previous work [17] we presented a model for the propagation of photons in a thermal background that contains a BE condensate of a charged scalar. The model is in turn based on a model we had considered earlier [18] to study the propagation of fermions in a BE condensate. The field theoretical method we used in Refs. [17,18] to treat the BE condensate, which we use here, has been discussed by various authors [19–22].

In Ref. [17] we determined the dispersion relations of the propagating modes after the symmetry breaking associated

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with the BE condensation in the model. In that model, two modes correspond to the transverse photon polarizations, while the other two modes are combinations of the longitudinal photon and the massive scalar field, which we denote as the $(\pm)$ modes. While the dispersion relations of the transverse modes have the usual form for transverse photons in a plasma, the dispersion relations of the $(\pm)$ modes reveal some unique features. In our previous work cited above, we determined and analyzed the real part of the dispersion relations of the $(\pm)$ modes, as well as the real part of the contribution to the longitudinal component of the polarization tensor and the corresponding contribution to the dielectric constant. In particular, in the appropriate limit, the dispersion relation of the $(-)$ mode reduces to the expression for the dispersion relation that has been obtained in nonrelativistic models of the BE condensation of charged scalars [23,24].

The focus of the present work is the calculation of the imaginary part of the dispersion relations, as well as the corresponding contributions to the imaginary part of the longitudinal component of the polarization tensor and the dielectric constant, in the same model. These parameters are important because they determine the energy loss and absorption effects when the system is subject to external perturbations. The main result is a set of explicit formulas for those quantities, obtained by the one-loop calculations of the self-energy of the propagating $(\pm)$ modes in the context of thermal field theory (TFT). These results can be useful in the context of the various applications that have been considered in the references mentioned above, and in the works related to dark-matter searches using materials with special optical properties [25].

To illustrate some features of the results, we consider in some detail the long wavelength limit, including specifically the high and low frequency limits, and obtained simple explicit expressions for the damping rate as well as the background contribution of the longitudinal polarization tensor and dielectric constant in those cases. These results are obtained in a general way, within the context of the model, but not tied to any specific application, and as such can be useful and pave the way for further development in various contexts and applications. In the following sections we present the details of the calculations, which we summarize as follows.

In Sec. II we summarize for reference the essential features of the scalar BE condensation model we use, followed in Sec. III by a review the basic elements of the model with the coupling to the photon. In Sec. III we focus our attention on the relevant properties regarding the $(\pm)$ modes, in particular the dispersion relations and related quantities. In Sec. IV we obtain the expressions for the thermal propagators of the $(\pm)$ modes, and in Sec. IV E we state precisely the one-particle approximation that we used in the calculations of the damping terms. In Sec. V we discuss the framework we use for the calculation, including

the couplings that are required to perform the one-loop calculation of the damping terms. The details of the calculation itself are given in Sec. VI. There we obtain the expression for the imaginary part of the longitudinal component of the polarization tensor and the dielectric constant, as well as the decay rate of the $(+)$ mode into two $(-)$ modes. Explicit formulas for those quantities are given by considering limiting cases of the general expressions, such as the long-wavelength limit, and in the high or low frequency limits. Further discussion and examples are given in Sec. VII, in particular the correspondence with the results obtained for the real part of the dielectric constant and dispersion relation in nonrelativistic models of the BE condensation of charged scalars [23]. Some details of the calculations are given in the Appendixes. Our closing statements are given in Sec. VIII. In short, we believe that our work and results presented here open the path for further exploration of the model, which can be relevant for applications related to the optical properties of this or similar systems.

II. THE BE CONDENSATE MODEL

For reference we first summarize the essential features of the scalar BE condensation model we use, and then review the basic elements of the model with the coupling to the photon.

The scalar model for the ϕ , discussed in detail in Ref. [18], is based on the Lagrangian

$$L_\phi = (d^\mu \phi)^* (d_\mu \phi) - V_\phi, \quad (2.1)$$

where

$$d_\mu \equiv \partial_\mu - i\mu u_\mu, \quad (2.2)$$

and

$$V_\phi = m^2 \phi^* \phi + \lambda (\phi^* \phi)^2. \quad (2.3)$$

The parameter μ is to be identified with the chemical potential of the ϕ , and u^μ is to be identified with the velocity four-vector of the medium,

$$u^\mu = (1, \vec{0}), \quad (2.4)$$

in the medium's own rest frame.

As discussed in Ref. [18] the motivation for considering this model is based on the following result: the calculation of the effective potential $V_{\text{eff}}(T, \mu)$ for ϕ can be carried out in TFT using $\mu = 0$ in the partition (and/or distribution) function, but using the μ -dependent Lagrangian density L_ϕ defined in Eq. (2.1). Then neglecting the T -dependent terms (that is, at zero temperature), $V_{\text{eff}}(0, \mu)$ is simply the U potential, given below in Eq. (2.8). In other words, calculations with the ϕ propagators, ϕ Lagrangian density, and partition function for ϕ ,

$$Z = e^{-\beta H + \alpha Q}, \quad (2.5)$$

give the same result as calculating them with the Lagrangian density for ϕ' obtained by using $\partial_\mu \rightarrow D_\mu = \partial_\mu - i\mu u_\mu$ and using

$$Z = e^{-\beta H'}. \quad (2.6)$$

A simple proof is given in Ref. [17].

Expanding the d term in Eq. (2.1),

$$L_\phi = (\partial^\mu \phi)^* (\partial_\mu \phi) + i[\phi^* (v \cdot \partial \phi) - (v \cdot \partial \phi^*) \phi] - U(\phi), \quad (2.7)$$

where

$$U \equiv V_\phi - \mu^2 \phi^* \phi = -(\mu^2 - m^2) \phi^* \phi + \lambda (\phi^* \phi)^2, \quad (2.8)$$

with

$$v_\mu = \mu u_\mu. \quad (2.9)$$

The case in which $\mu^2 < m^2$ corresponds to a standard massive complex scalar with mass $m^2 - \mu^2$. On the other hand, if $\mu^2 > m^2$, the minimum of the potential is not at $\phi = 0$. In this situation ϕ develops a nonzero expectation value and the $U(1)$ symmetry is broken, which is identified as corresponding to the BE condensation phase.

We assume the situation is such that

$$\mu^2 > m^2. \quad (2.10)$$

Then putting

$$\phi = \frac{1}{\sqrt{2}} (\phi_0 + \phi_1 + i\phi_2), \quad (2.11)$$

where

$$\langle \phi \rangle \equiv \frac{1}{\sqrt{2}} \phi_0 \quad (2.12)$$

is the minimum of

$$U_0 = -\frac{1}{2}(\mu^2 - m^2) \phi_0^2 + \frac{1}{4} \lambda \phi_0^4, \quad (2.13)$$

gives

$$\phi_0^2 = \frac{\mu^2 - m^2}{\lambda}. \quad (2.14)$$

The Lagrangian density for the scalar excitations $\phi_{1,2}$, obtained by substituting Eqs. (2.11) and (2.14) in Eq. (2.1), becomes

$$L_\phi = \frac{1}{2}[(\partial^\mu \phi_1)^2 + (\partial^\mu \phi_2)^2] + \phi_2 v \cdot \partial \phi_1 - \phi_1 v \cdot \partial \phi_2 - U(\phi), \quad (2.15)$$

where

$$U(\phi) = -\frac{1}{2}(\mu^2 - m^2)[(\phi_0 + \phi_1)^2 + \phi_2^2] + \frac{1}{4} \lambda [(\phi_0 + \phi_1)^2 + \phi_2^2]^2. \quad (2.16)$$

Using Eq. (2.14), it follows that only ϕ_1 appears in the quadratic part,

$$U_2 = \frac{1}{2} m_1^2 \phi_1^2, \quad (2.17)$$

with

$$m_1^2 = 2(\mu^2 - m^2), \quad (2.18)$$

while ϕ_2 does not appear. However, ϕ_1 and ϕ_2 are mixed by the v^μ term.

A convenient way to diagonalize the bilinear part of L_ϕ and determine the modes that have definite dispersion relations, is to use matrix notation and write

$$\hat{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \quad (2.19)$$

In momentum space, the bilinear part of L_ϕ can then be written in the form

$$L_\phi^{(2)}(k) = \frac{1}{2} \hat{\phi}^*(k) \Delta_\phi^{-1}(k) \hat{\phi}(k), \quad (2.20)$$

where

$$\Delta_\phi^{-1}(k) = \begin{pmatrix} k^2 - m_1^2 & 2iv \cdot k \\ -2iv \cdot k & k^2 \end{pmatrix}. \quad (2.21)$$

The classical field equations are

$$\Delta_\phi^{-1}(k) \hat{\phi} = 0, \quad (2.22)$$

and the dispersion relations of the eigenmodes are obtained by solving

$$(\omega^2 - \kappa^2)(\omega^2 - \kappa^2 - m_1^2) - 4\mu^2 \omega^2 = 0. \quad (2.23)$$

In Eq. (2.23) we have used Eq. (2.9) and defined the variables

$$\begin{aligned}\omega &= u \cdot k, \\ \kappa &= \sqrt{\omega^2 - k^2}.\end{aligned}\quad (2.24)$$

In the rest frame of the medium

$$k^\mu = (\omega, \vec{\kappa}), \quad (2.25)$$

with $\kappa = |\vec{\kappa}|$. The dispersion relations can be written in the form

$$\omega_\pm^2(\kappa) = \kappa^2 + \frac{1}{2}m_\rho^2 \pm \sqrt{\frac{1}{4}m_\rho^4 + 4\mu^2\kappa^2}, \quad (2.26)$$

where

$$m_\rho^2 = m_1^2 + 4\mu^2 = 6\mu^2 - 2m^2. \quad (2.27)$$

It follows that

$$\begin{aligned}\omega_+(0) &= m_\rho, \\ \omega_-(0) &= 0.\end{aligned}\quad (2.28)$$

The combinations of $\phi_{1,2}$ that have the definite dispersion relations $\omega_\pm$ can be obtained from Eq. (2.22), but we will not need them here and therefore do not discuss them further. We mention that, while the mode corresponding to ω_- is the realization of the Goldstone mode associated with the breaking of the global $U(1)$ symmetry, its dispersion relation is not the usual $\omega = \kappa$, but the one given by $\omega_-(\kappa)$ in Eq. (2.26).

III. CHARGED BE CONDENSATE MODEL

As discussed in Ref. [17], to study the propagation of photons in the BE condensate background, we extended the model by considering a ϕ field that is electrically charged and interacts with the electromagnetic field in the standard way. The Lagrangian density for the model is then

$$L = -\frac{1}{4}F^2 + L_{\phi A}, \quad (3.1)$$

where $F_{\mu\nu}$ is the electromagnetic field tensor,

$$L_{\phi A} = (\hat{D}^\mu \phi)^* (\hat{D}_\mu \phi) - V_\phi, \quad (3.2)$$

with

$$\hat{D}_\mu = d_\mu + iqA_\mu = \partial_\mu - iv_\mu + iqA_\mu, \quad (3.3)$$

while V_ϕ and v^μ are defined in Eqs. (2.3) and (2.9), respectively. Notice that, while it would seem that the presence of μ in the Lagrangian can be eliminated by letting

$$A_\mu = A'_\mu - \frac{1}{q}v_\mu, \quad (3.4)$$

this would actually restore the presence of μ in the partition function, which defeats the purpose of using the transformed field ϕ' .

For our purposes, it is more convenient to adopt the unitary gauge and parametrize ϕ in the form

$$\phi = \frac{1}{\sqrt{2}}(\phi_0 + \rho)e^{i\theta/\phi_0}. \quad (3.5)$$

The field θ does not appear in V_ϕ , and by a gauge transformation it disappears also from the kinetic term. To indicate clearly that we are using the unitary gauge, we denote by V_μ the transformed vector potential $V_\mu = A_\mu + \frac{1}{q\phi_0}\partial_\mu\theta$. With this notation, $L_{\phi A}$ is written in the form

$$L_{\phi A} = \frac{1}{2}(\partial\rho)^2 + \frac{1}{2}(v - qV)^\mu(v - qV)_\mu(\phi_0 + \rho)^2 - V_\phi. \quad (3.6)$$

Proceeding as in the previous section, in particular assuming again that

$$\mu^2 > m^2, \quad (3.7)$$

it follows that ϕ_0 is given as in Eq. (2.14). Furthermore, the bilinear part of L , including the F^2 term, is given by

$$L^{(2)} = -\frac{1}{4}F^2 + \frac{1}{2}m_V^2V^2 + \frac{1}{2}(\partial\rho)^2 - \frac{1}{2}m_1^2\rho^2 - 2m_V(v \cdot V)\rho, \quad (3.8)$$

where

$$m_V^2 = q^2\phi_0^2 = \frac{q^2}{\lambda}(\mu^2 - m^2), \quad (3.9)$$

and m_1 is defined in Eq. (2.18).

The physical picture is that we end up with the two fields V and ρ . The field θ becomes the longitudinal component of V , which is mixed with the scalar field ρ , by the term $\rho v \cdot V$ in the Lagrangian. As a consequence of this, the propagating modes with definite dispersion relations are superpositions of the longitudinal component of V and the ρ . Finding the dispersion relations, and the corresponding *mixing* combinations of the longitudinal component of V and the ρ was the subject of Ref. [17]. Our goal here is to determine the imaginary part, or damping term, of the dispersion relations.

For this purpose, below we summarize the salient features of the results we obtained for the dispersion relations, and more importantly for the details of the

present work, the corresponding mixing parameters that determine the propagating modes in terms of the fields ρ and the longitudinal component of V .

A. Mixing equations

In momentum space $L^{(2)}$ becomes

$$L^{(2)}(k) = -V^{*\mu}[k^2\tilde{g}_{\mu\nu} - m_V^2 g_{\mu\nu}]V^\nu + \rho^*(k^2 - m_1^2)\rho - 2\mu m_V[(u \cdot V)^*\rho + \text{c.c.}], \quad (3.10)$$

where

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}. \quad (3.11)$$

The first two terms are the usual terms for a vector with squared mass m_V^2 and a scalar with squared mass m_1^2 , while the last term mixes the scalar with the longitudinal components of V^μ . We decompose V^μ in the form

$$V^\mu = V_T^\mu + V_L e_3^\mu + \frac{k^\mu}{\sqrt{k^2}} V_k, \quad V_T^\mu = \sum_{i=1,2} V_i e_i^\mu \quad (3.12)$$

where

$$e_{1,2}^\mu = (0, \vec{e}_{1,2}), \quad (\vec{e}_{1,2} \cdot \vec{k} = 0), \quad e_3^\mu = \frac{\tilde{u}^\mu}{\sqrt{-\tilde{u}^2}}, \quad (3.13)$$

with

$$\tilde{u}_\mu = \tilde{g}_{\mu\nu} u^\nu = u_\mu - \frac{(k \cdot u)k_\mu}{k^2}, \quad \sqrt{-\tilde{u}^2} = \frac{\kappa}{\sqrt{k^2}}. \quad (3.14)$$

Substituting Eq. (3.12) in Eq. (3.10), it follows that the component V_k does not have a kinetic energy term, and therefore it is not a dynamical variable. Eliminating it from the Lagrangian by using its Lagrange equation,

$$V_k = \frac{2\mu(k \cdot u)}{m_V \sqrt{k^2}} \rho, \quad (3.15)$$

and substituting it in Eq. (3.10),

$$L^{(2)}(k) = -(k^2 - m_V^2)V_T^* \cdot V_T + (k^2 - m_V^2)V_L^* V_L + \left(k^2 - m_1^2 - \frac{4\mu^2(k \cdot u)^2}{k^2}\right)\rho^* \rho + 2\mu m_V \sqrt{-\tilde{u}^2}[V_L^* \rho + \text{c.c.}], \quad (3.16)$$

Thus, the transverse modes V_T have a standard degenerate dispersion relation for the two polarizations, and we will not consider them further.

The remaining terms mix the longitudinal mode V_L with the scalar ρ . Introducing the notation

$$\xi = \begin{pmatrix} \rho \\ V_L \end{pmatrix}, \quad (3.17)$$

they can be written in the form

$$L_\xi^{(2)}(k) = \xi^\dagger (\Delta_{F0}^{(\xi)})^{-1} \xi, \quad (3.18)$$

where

$$(\Delta_{F0}^{(\xi)})^{-1} = \begin{pmatrix} k^2 - m_\rho^2 - \frac{4\mu^2 \kappa^2}{k^2} & \frac{2\mu m_V \kappa}{\sqrt{k^2}} \\ \frac{2\mu m_V \kappa}{\sqrt{k^2}} & k^2 - m_V^2 \end{pmatrix}. \quad (3.19)$$

In writing Eq. (3.19) we have used the relation $(k \cdot u)^2 = k^2 + \kappa^2$ as well as Eq. (2.27). The field equations, which determine the dispersion relations and mixing parameters, are

$$(\Delta_{F0}^{(\xi)})^{-1} \xi = 0. \quad (3.20)$$

The notation $(\Delta_{F0}^{(\xi)})^{-1}$ in Eq. (3.19) is introduced in anticipation that $\Delta_{F0}^{(\xi)}$ is identified with the propagator obtained from the bilinear part of the Lagrangian, in analogy with the identification of the tree-level propagator in field theory, which is the subject of Sec. IV.

B. Dispersion relations

From Eq. (3.20), the equation for the dispersion relations is

$$(k^2 - m_V^2)(k^2 - m_\rho^2) - 4\mu^2 \kappa^2 = 0. \quad (3.21)$$

Writing the momentum of each propagating mode ($s = \pm$) as

$$k^\mu = (\omega_r^{(s)}, \vec{k}), \quad (3.22)$$

the dispersion relations are given by

$$\omega_r^{(s)}(\kappa) = \sqrt{\kappa^2 + \sigma_s(\kappa)}, \quad (3.23)$$

where

$$\sigma_s(\kappa) = \frac{1}{2}(m_\rho^2 + m_V^2) + \frac{s}{2}\sigma(\kappa), \quad (3.24)$$

with

$$\sigma(\kappa) = [(m_\rho^2 - m_V^2)^2 + 16\mu^2\kappa^2]^{\frac{1}{2}}. \quad (3.25)$$

As mentioned in Sec. I, throughout this work we are assuming that

$$m_\rho > m_V. \quad (3.26)$$

In this case, at zero momentum,

$$\sigma(0) = m_\rho^2 - m_V^2, \quad (3.27)$$

and therefore

$$\begin{aligned} \omega_r^{(+)}(0) &= m_\rho, \\ \omega_r^{(-)}(0) &= m_V. \end{aligned} \quad (3.28)$$

This is an indication that, at $\kappa = 0$, the (+) mode is a pure ρ while the (−) mode is a pure V_L , as we will confirm below.

C. Mixing: Wave functions

Up to a normalization factor, the corresponding eigenvectors can be written in the form

$$\begin{aligned} u_+ &= \begin{pmatrix} \Lambda \\ -m_V\gamma_+ \end{pmatrix}, \\ u_- &= \begin{pmatrix} m_V\gamma_- \\ \Lambda + \gamma_-^2 \end{pmatrix}. \end{aligned} \quad (3.29)$$

where

$$\begin{aligned} \gamma_\pm &= \frac{2\mu\kappa}{\sqrt{\sigma_\pm}}, \\ \Lambda &= \frac{1}{2}(m_\rho^2 - m_V^2) + \frac{1}{2}\sigma. \end{aligned} \quad (3.30)$$

In the limit $\kappa = 0$, these solutions reduce to

$$u_+ \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (3.31)$$

$$u_- \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.32)$$

In other words, at $\kappa = 0$ the (+) mode is pure ρ while the (−) mode is a pure V_L , in agreement with Eq. (3.28).

D. Polarization tensor and longitudinal dielectric constant

The contribution to the longitudinal dielectric constant can be obtained as follows. The procedure is to start from

Eq. (3.16) and integrate out the variable ρ , which is the tree-level analog of calculating the one-loop self-energy, leaving only the dynamical variables of V , and then identifying the polarization tensor by comparing the result with

$$L_{\text{eff}} = -V^{*\mu}[(k^2 - \pi^{(T)})R_{\mu\nu} + (k^2 - \pi^{(L)})Q_{\mu\nu}]V^\nu, \quad (3.33)$$

where $R_{\mu\nu}$ and $Q_{\mu\nu}$ are defined by

$$\begin{aligned} Q_{\mu\nu} &= \frac{\tilde{u}_\mu \tilde{u}_\nu}{\tilde{u}^2}, \\ R_{\mu\nu} &= \tilde{g}_{\mu\nu} - Q_{\mu\nu}. \end{aligned} \quad (3.34)$$

In terms of the variables $V_{T,L}$ we have defined in Eq. (3.12),

$$L_{\text{eff}} = -(k^2 - \pi^{(T)})V_T^* \cdot V_T + (k^2 - \pi^{(L)})V_L^* V_L. \quad (3.35)$$

In general, when the thermal corrections are taken into account, $\pi^{(T,L)}$ develop an imaginary part, which we indicate by writing

$$\pi^{(T,L)} = \pi_r^{(T,L)} + i\pi_i^{(T,L)}. \quad (3.36)$$

Solving Eq. (3.20) for ρ and substituting the solution back in Eq. (3.16) reproduces Eq. (3.35) with

$$\pi_r^{(L)} \equiv m_V^2 + \frac{4\mu^2 m_V^2 \kappa^2}{k^2(k^2 - m_\rho^2) - 4\mu^2 \kappa^2}. \quad (3.37)$$

The solutions of the equation

$$k^2 - \pi_r^{(L)} = 0 \quad (3.38)$$

reproduce the dispersion relations given in Sec. III B, as it should be.

For our current purposes, one merit of Eq. (3.37) is that it gives the contribution to the longitudinal dielectric function. Using [26],

$$\epsilon_\ell = 1 - \frac{\pi^{(L)}}{k^2}, \quad (3.39)$$

we then obtain

$$\text{Re}\epsilon_\ell = 1 - \frac{m_V^2}{k^2} \left[1 + \frac{4\mu^2 \kappa^2}{k^2(k^2 - m_\rho^2) - 4\mu^2 \kappa^2} \right]. \quad (3.40)$$

As we discuss in some detail in Sec. VII, in the large m_ρ limit, stated precisely there, the dispersion relation of the (−) mode, given by $\omega_r^{(-)}$ in Eq. (3.23), as well as the expression for $\text{Re}\epsilon_\ell$ given above in Eq. (3.40), reproduce the results obtained in nonrelativistic models of BE condensation of charged scalars [23].

The imaginary part of the longitudinal polarization tensor, which we denote by $\pi_i^{(L)}$, arises at one loop, and it is the subject of the present work. The corresponding contribution to the imaginary part of the longitudinal dielectric constant is then given by

$$\text{Im } \epsilon_\ell = -\frac{\pi_i^{(L)}}{k^2}. \quad (3.41)$$

The details of their calculation are presented in the sections that follow.

IV. PROPAGATORS

A. Zero temperature ($\pm$) propagators

The zero temperature propagators of the $s = (\pm)$ modes can be written in the standard form

$$\Delta_{F0}^{(s)}(k) = \frac{1}{k^2 - \sigma_s(\kappa) + i\epsilon}, \quad (4.1)$$

where the $\sigma_s(\kappa)$ is given in Eq. (3.24). For notational purposes in the discussions that follow, it is convenient to generalize the expression above and write the propagator for any momentum vector (e.g., q) in the form

$$\Delta_{F0}^{(s)}(q) = \frac{1}{q^2 - \hat{\sigma}_s(q) + i\epsilon}, \quad (4.2)$$

where

$$\hat{\sigma}_s(q) = \sigma_s(Q) \quad (4.3)$$

and

$$Q = \sqrt{(q \cdot u)^2 - q^2}. \quad (4.4)$$

B. Finite temperature ($\pm$) propagators

The damping of each mode ($s = \pm$) is obtained from the imaginary part of the self-energy, which in turn is determined from the $(\pi_s)_{12}$ component of the thermal self-energy of each mode, calculated, in our case, to one loop. One simplifying feature of the ($\pm$) propagators is the fact that σ_s is a function of Q but not of $q \cdot u$, and therefore the usual derivation of the thermal propagators can be applied here as well. The relevant components of the thermal propagator matrices, which enter in the one-loop calculation of $(\pi_s)_{12}$, are [27]

$$\begin{aligned} \Delta_{21}^{(s)}(q) &= -2\pi i \delta(q^2 - \hat{\sigma}_s(q)) [\eta_B(q) + \theta(q)], \\ \Delta_{12}^{(s)}(q) &= -2\pi i \delta(q^2 - \hat{\sigma}_s(q)) [\eta_B(q) + \theta(-q)], \end{aligned} \quad (4.5)$$

where

$$\eta_B(q) = \theta(k) n_B(x_q) + \theta(-q) n_B(-x_q), \quad (4.6)$$

with

$$x_q = \beta q \cdot u. \quad (4.7)$$

Further,

$$\theta(q) = H(q \cdot u), \quad (4.8)$$

where $H(z)$ is the step function, and n_B is the boson distribution function

$$n_B(x) = \frac{1}{e^x - 1}. \quad (4.9)$$

C. Zero temperature flavor propagators

From Eq. (3.19) it follows that the flavor propagator is given by

$$\Delta_{F0}^{(\xi)}(k) = \frac{C^{(\xi)}}{D^{(\xi)}}, \quad (4.10)$$

where

$$C^{(\xi)} = \begin{pmatrix} k^2 - m_V^2 & -\frac{2\mu m_V \kappa}{\sqrt{k^2}} \\ -\frac{2\mu m_V \kappa}{\sqrt{k^2}} & k^2 - m_\rho^2 - \frac{4\mu^2 \kappa^2}{k^2} \end{pmatrix}, \quad (4.11)$$

and

$$\begin{aligned} D^{(\xi)} &= (k^2 - m_V^2) \left(k^2 - m_\rho^2 - \frac{4\mu^2 \kappa^2}{k^2} \right) - \frac{4\mu^2 m_V^2 \kappa^2}{k^2} \\ &= (k^2 - m_V^2)(k^2 - m_\rho^2) - 4\mu^2 \kappa^2. \end{aligned} \quad (4.12)$$

The equation $D^{(\xi)} = 0$ gives us the dispersion relations that we have already discussed, and in fact we can write

$$D^{(\xi)} = d_+ d_-, \quad (4.13)$$

where

$$d_s = k^2 - \sigma_s, \quad (4.14)$$

and σ_s has been defined in Eq. (3.24).

Using the relations

$$\begin{aligned} k^2 - m_V^2 &= d_- + \sigma_- - m_V^2, \\ k^2 - m_\rho^2 - \frac{4\mu^2 \kappa^2}{k^2} &= d_+ + \sigma_+ - m_\rho^2 - \frac{4\mu^2 \kappa^2}{k^2}, \end{aligned} \quad (4.15)$$

and

$$\frac{1}{d_+ d_-} = \frac{1}{\sigma_+ - \sigma_-} \left(\frac{1}{d_+} - \frac{1}{d_-} \right) = \frac{1}{\sigma} \left(\frac{1}{d_+} - \frac{1}{d_-} \right), \quad (4.16)$$

the propagator in Eq. (4.10) can be written in the form

$$\Delta_{F0}^{(\xi)}(k) = \sum_s \Delta_{F0}^{(s)}(k) C_s, \quad (4.17)$$

where $\Delta_{F0}^{(s)}$ is given in Eq. (4.1), and

$$C_+ = \frac{1}{\sigma} \begin{pmatrix} \sigma + \sigma_- - m_V^2 & -\frac{2\mu m_V \kappa}{\sqrt{k^2}} \\ -\frac{2\mu m_V \kappa}{\sqrt{k^2}} & \sigma_+ - m_\rho^2 - \frac{4\mu^2 \kappa^2}{k^2} \end{pmatrix},$$

$$C_- = \frac{1}{\sigma} \begin{pmatrix} -(\sigma_- - m_V^2) & \frac{2\mu m_V \kappa}{\sqrt{k^2}} \\ \frac{2\mu m_V \kappa}{\sqrt{k^2}} & \sigma - \left(\sigma_+ - m_\rho^2 - \frac{4\mu^2 \kappa^2}{k^2} \right) \end{pmatrix}. \quad (4.18)$$

Although we are not indicating it explicitly here, it should be kept in mind that the matrices $C_\pm$ are functions of ω and κ .

D. Finite temperature flavor propagators

Equation (4.17) is the formula for the full flavor propagators at zero temperature. Their finite temperature counterparts are given by the same expression but replacing the zero temperature $\Delta_{F0}^{(s)}$ by their finite-temperature counterparts, given in Eq. (4.5). Explicitly,

$$\Delta_{ab}^{(\xi)}(k) = \sum_s \Delta_{ab}^{(s)}(k) C_s \quad (4.19)$$

where a and b stand for the thermal indices 1,2, as used in the real-time formulation of thermal field theory that we are adopting, as already mentioned [27].

For sufficiently small κ , the $C_\pm$ matrices become diagonal, $C_+ \rightarrow \text{diag}(1, 0)$, $C_- \rightarrow \text{diag}(0, 1)$, and therefore

$$\Delta_{F0}^{(\xi)}(k) \rightarrow \begin{pmatrix} \Delta_{F0}^{(+)}(k) & 0 \\ 0 & \Delta_{F0}^{(-)}(k) \end{pmatrix}, \quad (4.20)$$

and correspondingly for the thermal propagator matrix,

$$\Delta_{ab}^{(\xi)}(k) \rightarrow \begin{pmatrix} \Delta_{ab}^{(+)}(k) & 0 \\ 0 & \Delta_{ab}^{(-)}(k) \end{pmatrix}. \quad (4.21)$$

That is, the flavor propagator matrix becomes diagonal with ρ identified with the (+) mode and V_L with the (−) mode, as we have already anticipated.

E. One-particle approximation

The one-particle approximation consists in making the replacements

$$C_s(\omega, \kappa) \rightarrow C_s(\omega_r^{(s)}, \vec{\kappa}), \quad (4.22)$$

in the expressions for the propagators in Eqs. (4.17) and (4.19). That is, approximate the propagator by evaluating the numerators at the dispersion relation. Thus, for example,

$$C_+ \rightarrow C_+(\omega_r^{(+)}, \kappa) = \frac{1}{\sigma} \begin{pmatrix} \sigma + \sigma_- - m_V^2 & -m_V \gamma_+ \\ -m_V \gamma_+ & \sigma_+ - m_\rho^2 - \gamma_+^2 \end{pmatrix}, \quad (4.23)$$

and similarly for $C_-(\omega_r^{(-)}, \kappa)$, where $\gamma_\pm$ are given in Eq. (3.30). It is straightforward to confirm that

$$C_+(\omega_r^{(+)}, \kappa) = \frac{1}{\Lambda \sigma} u_+ u_+^\dagger,$$

$$C_-(\omega_r^{(-)}, \kappa) = \frac{1}{(\Lambda + \gamma_-^2) \sigma} u_- u_-^\dagger, \quad (4.24)$$

where $u_\pm$ are the (flavor) spinors given in Eq. (3.29). In other words, in the one-particle approximation, the zero-temperature flavor propagators are given by

$$(\Delta_{F0}^{(\xi)}(k))_{1p} = \frac{U_+ U_+^\dagger}{k^2 - \sigma_+ + i\epsilon} + \frac{U_- U_-^\dagger}{k^2 - \sigma_- + i\epsilon}, \quad (4.25)$$

where

$$U_+ = \sqrt{\frac{1}{\Lambda \sigma}} u_+,$$

$$U_- = \sqrt{\frac{1}{(\Lambda + \gamma_-^2) \sigma}} u_-. \quad (4.26)$$

The $U_\pm$ factors defined above are analogous in the present context to the normalization factors for plasmons propagating in a plasma or fermions propagating in a thermal background [28,29]. They in turn determine the properly normalized transformation matrix between the ρ, V_L fields and the fields of the eigenmodes. If we denote the latter by $\phi_\pm$, then

$$\begin{pmatrix} \rho \\ V_L \end{pmatrix} = U_+ \phi_+ + U_- \phi_-. \quad (4.27)$$

The thermal propagators given in Eq. (4.19) can be expressed as

$$\Delta_{ab}^{(\xi)}(k) = \sum_s \Delta_{ab}^{(s)}(k) U_s U_s^\dagger, \quad (4.28)$$

in particular,

$$\Delta_{12}^{(\xi)}(k) = \sum_s \Delta_{12}^{(s)}(k) U_s U_s^\dagger, \quad (4.29) \quad \text{and}$$

and similarly for the (21) components, where the $\Delta_{12}^{(\pm)}(k)$ and $\Delta_{21}^{(\pm)}(k)$ propagators are given explicitly in Eq. (4.5). The spinors $U_\pm$ that we have identified above allow us to calculate the rate of processes such as

$$\phi_+ \rightarrow \phi_- + \phi_- \quad (4.30)$$

in a straightforward way once we know the vertices to use. In turn, the thermal propagators allow us to calculate the thermal corrections to such processes. The proper vertices for such purposes are determined in Sec. V B.

V. IMAGINARY PART OF THE SELF-ENERGY: FORMULATION OF THE PROBLEM

A. General formulas for π_i

Denoting by $\pi_r^{(\pm)}$ the real part of the self-energy of each propagating mode, then from Eq. (4.1) they are simply

$$\pi_r^{(\pm)} = \sigma_\pm. \quad (5.1)$$

In general, including the loop corrections, the self-energy will develop an imaginary part due to the interactions with the background. Thus we write the total self-energy in the form

$$\pi_{\text{eff}}^{(\pm)}(k) = \pi_r^{(\pm)}(k) + i\pi_i^{(\pm)}(k). \quad (5.2)$$

In the case of one scalar field, a convenient formula to determine π_i is

$$\pi_i(k) = \frac{i\pi_{12}(k)}{2n_B(x)}, \quad (5.3)$$

where $\pi_{12}(k)$ is the 12 component of the thermal self-energy matrix. The dispersion relation is the solution of

$$k^2 - (\pi_r + i\pi_i) = 0. \quad (5.4)$$

Writing

$$\omega = \omega_r + i\omega_i \quad (5.5)$$

to the lowest order

$$\omega_i(\kappa) = \frac{\pi_i(\omega_r, \kappa)}{2\omega_r N} \equiv -\frac{\Gamma(\kappa)}{2}, \quad (5.6)$$

where ω_r is the solution of

$$\omega_r^2 - \pi(\omega_r, \kappa) = 0, \quad (5.7)$$

$$N = 1 - \frac{\pi'_r}{2\omega_r}, \quad (5.8)$$

with

$$\pi'_r \equiv \left. \frac{\partial \pi_r}{\partial \omega} \right|_{\omega=\omega_r}. \quad (5.9)$$

The quantity Γ defined in Eq. (5.6) has the interpretation of the total decay rate; the evolution amplitude of the propagating mode has the factor

$$e^{-i\omega t} = e^{-i\omega_r t} e^{-\frac{\Gamma}{2}t}, \quad (5.10)$$

and therefore the flux is damped by $e^{-\Gamma t}$.

In the spirit of the one-particle approximation, similar formulas apply also to our case, for each propagating ($s = \pm$) mode. At this point it is useful to recall the fact that $\sigma_\pm$, and whence $\pi_r^{(\pm)}$ by Eq. (5.1), are independent of ω [Eq. (3.24)], and therefore the normalization factors analogous to N above are 1. Thus we write their dispersion relation in the form

$$\omega_s = \omega_r^{(s)} - i\frac{\Gamma_s}{2}, \quad (5.11)$$

with

$$\Gamma_s \equiv -\frac{\pi_i^{(s)}(\omega_r^{(s)}, \kappa)}{\omega_r^{(s)}}. \quad (5.12)$$

The strategy now is to consider the cubic interaction terms in the Lagrangian and write them in terms of the $\phi_\pm$ fields by means of Eq. (4.27). Then it becomes straightforward to calculate the rate for processes such as $\phi_+ \rightarrow \phi_- + \phi_-$, and related ones, as well as the thermal corrections from the one-loop diagrams for the imaginary part of the self-energy of each mode.

B. Vertices

The relevant interaction terms are obtained from Eq. (3.8) of Ref. [17],

$$L_{\phi A} = \frac{1}{2}(\partial\rho)^2 + \frac{1}{2}(q\phi_0)^2 V^2 + \frac{1}{2}q^2 V^2 \rho^2 + q^2 \phi_0 \rho V^2 - 2q\mu(u \cdot V)(\phi_0 + \rho)^2 - U_\phi. \quad (5.13)$$

To write down the $V_L - \rho$ interaction terms we must eliminate the component V_k in the cubic and quartic terms, as we did it for the bilinear terms, using Eq. (3.15). Since that formula depends on k , it will produce vertices that are momentum dependent. We consider the relevant interactions

terms that follow from Eq. (5.13) one by one. The terms that must be looked at are

$$L' = \frac{1}{2} q^2 V^2 \rho^2 + q^2 \phi_0 \rho V^2 - 2q\mu(u \cdot V)\rho^2, \quad (5.14)$$

where

$$\begin{aligned} V^2 &= V_T \cdot V_T - V_L^2 - V_k^2 \rightarrow V_T \cdot V_T - V_L^2 - \frac{4\mu^2(k \cdot u)^2}{m_V^2 k^2} \rho^2, \\ u \cdot V &= -\sqrt{-\tilde{u}^2} V_L + \frac{(k \cdot u) V_k}{\sqrt{k^2}} \rightarrow -\sqrt{-\tilde{u}^2} V_L + \frac{2\mu(k \cdot u)^2}{m_V k^2} \rho. \end{aligned} \quad (5.15)$$

Therefore, the $V_L - \rho$ interaction terms contained in Eq. (5.14) are the following quartic and cubic terms:

$$\begin{aligned} L^{(4)} &= -\frac{1}{2} q^2 V_L^2 \rho^2, \\ L^{(3)} &= -qm_V \rho V_L^2 + 2q\mu\sqrt{-\tilde{u}^2} V_L \rho^2, \\ &= -qm_V \rho V_L^2 + q\gamma(k) V_L \rho^2, \end{aligned} \quad (5.16)$$

where we have used

$$m_V = q\phi_0, \quad (5.17)$$

and defined

$$\gamma(k) = \frac{2\mu\kappa}{\sqrt{k^2}}. \quad (5.18)$$

In Fig. 1, we have shown the $V_L \rho^2$ vertex.

C. $L^{(3)}$ in terms of the eigenmode fields $\phi_{\pm}$

For definiteness, we consider the calculation of $\pi_i^{(+)}$, that is, the imaginary part of the self-energy of the (+) mode, from which we determine the corresponding damping as already explained. As we know, the imaginary part of the self-energy is related to the rates of various processes, and our focus on $\pi_i^{(+)}$ is based on choosing the common case in which the dominant processes that contribute are

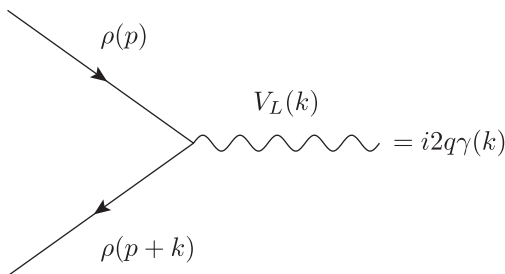


FIG. 1. Feynman rule for the $V_L \rho^2$ vertex that appears in Eq. (5.16), and $\gamma(k)$ defined in Eq. (5.18).

$\phi_+ \leftrightarrow \phi_- + \phi_-$. The quartic term $V_L^2 \rho^2$ can contribute to the diagonal (11 and 22) elements of the self-energy but not to the 12 (or 21) element; therefore, we do not consider them further. In short, the problem is to start from

$$L^{(3)} = -qm_V \rho V_L^2 + q\gamma(k) V_L \rho^2, \quad (5.19)$$

and write it in terms of $\phi_{\pm}$ using

$$\begin{pmatrix} \rho \\ V_L \end{pmatrix} = \sum_{i=\pm} U_i \phi_i. \quad (5.20)$$

Thus writing

$$\begin{aligned} \rho &= \sum_i U_{1i} \phi_i, \\ V_L &= \sum_i U_{2i} \phi_i, \end{aligned} \quad (5.21)$$

we have

$$\begin{aligned} L^{(3)} &= -qm_V \sum_{ijk} U_{1i} U_{2j} U_{2k} \phi_i \phi_j \phi_k \\ &\quad + q\gamma(k) \sum_{ijk} U_{2i} U_{1j} U_{1k} \phi_i \phi_j \phi_k. \end{aligned} \quad (5.22)$$

Now we want to single out the term of the form $\phi_+ \phi_-^2$. This is,

$$L'^{(3)} = A \phi_+ \phi_-^2, \quad (5.23)$$

where

$$\begin{aligned} A &= -qm_V (U_{1+} U_{2-} U_{2-} + U_{1-} U_{2+} U_{2-} + U_{1-} U_{2-} U_{2+}) \\ &\quad + q\gamma(k) (U_{2+} U_{1-} U_{1-} + U_{2-} U_{1+} U_{1-} + U_{2-} U_{1-} U_{1+}). \end{aligned} \quad (5.24)$$

To finish this step we have to use the explicit formulas for the components U_{ai} to determine the coupling A . The U_{ai} are given explicitly by

$$\begin{aligned} U_{1+} &= \sqrt{\frac{1}{\Lambda\sigma}} \Lambda = \sqrt{\frac{\Lambda}{\sigma}}, \\ U_{2+} &= \sqrt{\frac{1}{\Lambda\sigma}} (-m_V \gamma_+), \\ U_{1-} &= \sqrt{\frac{1}{(\Lambda + \gamma_-^2)\sigma}} (m_V \gamma_-), \\ U_{2-} &= \sqrt{\frac{1}{(\Lambda + \gamma_-^2)\sigma}} (\Lambda + \gamma_-^2) = \sqrt{\frac{\Lambda + \gamma_-^2}{\sigma}}. \end{aligned} \quad (5.25)$$

Using the vertex given in Eq. (5.23), we can calculate $(\pi_{12}^{(+)})$ by the diagram given in Fig. 2, and then

$$\pi_i^{(+)}(k) \equiv \frac{i\pi_{12}^{(+)}(k)}{2n_B(x)}, \quad (5.26)$$

from which we obtain the damping rate of the (+) mode by means of Eq. (5.12). Moreover, from the result for $\pi_i^{(+)}$, the corresponding contribution to the imaginary part of the V_L self-energy is

$$\pi_i^{(L)}(k) = |U_{2+}|^2 \pi_i^{(+)}(k), \quad (5.27)$$

and therefore, remembering Eq. (3.41), we obtain for the imaginary part of the *longitudinal* dielectric constant

$$\text{Im } \epsilon_\ell(\omega, \kappa) = -\frac{|U_{2+}|^2 \pi_i^{(+)}(k)}{k^2}. \quad (5.28)$$

VI. IMAGINARY PART OF THE SELF-ENERGY: CALCULATION OF $\pi_i^{(+)}$

A. Expression for $\pi_i^{(+)}$

From Fig. 2,

$$-i(\pi_{12}^{(+)}(k)) = (iA)(-iA^*) \int \frac{d^4 p}{(2\pi)^4} i\Delta_{21}^{(-)}(p) i\Delta_{12}^{(-)}(k+p), \quad (6.1)$$

where $\Delta_{21}^{(-)}(q)$ and $\Delta_{12}^{(-)}(q)$ are given in Eq. (4.5). Using the identities

$$\begin{aligned} \eta_B(q) + \theta(q) &= e^{x_q} n_B(x_q) \epsilon(q), \\ \eta_B(q) + \theta(-q) &= n_B(x_q) \epsilon(q), \end{aligned} \quad (6.2)$$

where

$$\epsilon(q) = \theta(q) - \theta(-q), \quad (6.3)$$

the propagators can be written in the form

$$\begin{aligned} \Delta_{21}^{(s)}(q) &= -2\pi i \delta(q^2 - \hat{s}_s(q)) e^{x_q} n_B(x_q) \epsilon(q), \\ \Delta_{12}^{(s)}(q) &= -2\pi i \delta(q^2 - \hat{s}_s(q)) n_B(x_q) \epsilon(q). \end{aligned} \quad (6.4)$$

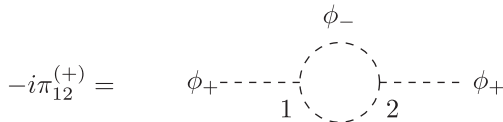


FIG. 2. Diagrams for the 12 element of the self-energy of the (+) mode due to the interaction term given in Eq. (5.23).

Then substituting Eq. (6.4) in Eq. (6.1), after some straightforward algebra (see Appendix A) we arrive at

$$\begin{aligned} \pi_i^{(+)}(k) &= -\frac{1}{2} |A|^2 \int \frac{d^3 P}{(2\pi)^3 2\omega_r^{(-)}(P)} \frac{d^3 Q}{(2\pi)^3 2\omega_r^{(-)}(Q)} (2\pi)^4 \\ &\times \{ \delta^{(4)}(k+p-q)(n_-(p) - n_-(q)) \\ &+ \delta^{(4)}(k+q-p)(-n_-(p) + n_-(q)) \\ &- \delta^{(4)}(k+p+q)(n_-(p) + n_-(q) + 1) \\ &- \delta^{(4)}(k-p-q)(-n_-(p) - n_-(q) - 1) \}. \end{aligned} \quad (6.5)$$

In this formula,

$$\begin{aligned} p^\mu &= (\omega_r^{(-)}(P), \vec{P}), \\ q^\mu &= (\omega_r^{(-)}(Q), \vec{Q}), \\ k^\mu &= (\omega, \vec{\kappa}). \end{aligned} \quad (6.6)$$

$$n_s(p) \equiv \frac{1}{e^{\beta\omega_r^{(s)}(P)} - 1}, \quad (6.7)$$

and similarly for $n_s(q)$ with $P \rightarrow Q$. The damping of the propagating (+) mode is then obtained by Eq. (5.12).

This expression for $\pi_i^{(+)}$ in Eq. (6.5) can be interpreted in terms of the transition probabilities for various processes,

$$\begin{aligned} \phi_+(k) + \phi_-(p) &\leftrightarrow \phi_-(q), \\ \phi_+(k) + \phi_-(q) &\leftrightarrow \phi_-(p), \\ \phi_+(k) + \phi_-(q) + \phi_-(p) &\leftrightarrow 0, \\ \phi_+(k) &\leftrightarrow \phi_-(q) + \phi_-(p). \end{aligned} \quad (6.8)$$

Consider, for example, the first term. Using

$$n_-(p) - n_-(q) = n_-(p)(1 + n_-(q)) - n_-(q)(1 + n_-(p)), \quad (6.9)$$

and remembering that the delta function fixes $k+p=q$, the first term corresponds to the processes

$$\phi_+(k) + \phi_-(p) \leftrightarrow \phi_-(q). \quad (6.10)$$

Some processes are forbidden by energy conservation while others suppressed by our specific assumptions about the background conditions.

The dominant process in the case we consider is the decay process $\phi_+(k) \rightarrow \phi_-(q) + \phi_-(p)$. For completeness we also include its inverse, although it will be suppressed under the background conditions that we assume. The corresponding contribution to $\pi_i^{(+)}$ is

$$\pi_i^{(+)}(k) = -\omega [\Gamma(\phi_+ \rightarrow \phi_- \phi_-) - \Gamma(\phi_- \phi_- \rightarrow \phi_+)], \quad (6.11)$$

where

$$\begin{aligned}\Gamma(\phi_+ \rightarrow \phi_- \phi_-) &= \frac{1}{2\omega} |A|^2 \int \frac{d^3 P}{(2\pi)^3 2\omega_r^{(-)}(P)} \frac{d^3 Q}{(2\pi)^3 2\omega_r^{(-)}(Q)} (2\pi)^4 \delta^{(4)}(k - p - q) (1 + n_-(p))(1 + n_-(q)), \\ \Gamma(\phi_- \phi_- \rightarrow \phi_+) &= \frac{1}{2\omega} |A|^2 \int \frac{d^3 P}{(2\pi)^3 2\omega_r^{(-)}(P)} \frac{d^3 Q}{(2\pi)^3 2\omega_r^{(-)}(Q)} (2\pi)^4 \delta^{(4)}(k - p - q) n_-(p) n_-(q).\end{aligned}\quad (6.12)$$

In the limit that the n_- distribution is negligible, the dominant contribution to $\pi_i^{(+)}$ is due to the decay rate

$$\Gamma^{(0)}(\phi_+ \rightarrow \phi_- \phi_-) = \frac{1}{2\omega} |A|^2 \int \frac{d^3 P}{(2\pi)^3 2\omega_r^{(-)}(P)} \frac{d^3 Q}{(2\pi)^3 2\omega_r^{(-)}(Q)} (2\pi)^4 \delta^{(4)}(k - p - q). \quad (6.13)$$

In any case, once we have computed $\pi_i^{(+)}$, we obtain the damping of the (+) mode from Eq. (5.12).

B. Long wavelength limit

Here we will consider the calculation of $\pi_i^{(+)}$ in the long wavelength limit ($\kappa \rightarrow 0$), which is a useful case for many applications.

1. Evaluation of the integral

We must evaluate the integral in Eq. (6.13) for $\kappa = 0$, that is

$$I = \int \frac{d^3 P}{(2\pi)^3 2\omega_r^{(-)}(P)} \frac{d^3 Q}{(2\pi)^3 2\omega_r^{(-)}(Q)} (2\pi)^4 \delta^{(4)}(k - p - q), \quad (6.14)$$

with

$$k^\mu = (\omega, \vec{0}), \quad (6.15)$$

but using the exact dispersion relation for $\omega_r^{(-)}$. In this case we can use the results of Appendix B, with the notation adapted to the present case. As indicated at the end of Appendix B, the results simplify in the present case because $m_1 = m_2$. Thus, the result is

$$I = \sum_{P^*} \left(\frac{1}{2\pi} \right)^2 \frac{2\pi P^{*2}}{\omega^2} \frac{1}{|G_0|}, \quad (6.16)$$

where P^* is the solution of

$$\omega_r^{(-)}(P^*) = \frac{1}{2}\omega, \quad (6.17)$$

and

$$G_0 \equiv \left(\frac{\partial \omega_r^{(-)}}{\partial P} \right) \Big|_{P=P^*}. \quad (6.18)$$

The $\sum_{P^*}$ symbol in Eq. (6.16) is meant to indicate a sum over each valid solution of Eq. (6.17). The identification of the valid solutions is carried out below.

2. Solution for P^*

The full dispersion relation can be written in the form

$$\omega_r^{(-)} = \sqrt{P^2 + M^2 - \sqrt{\Delta^2 + 4\mu^2 P^2}}, \quad (6.19)$$

where we are defining

$$\begin{aligned}M^2 &\equiv \frac{1}{2}(m_\rho^2 + m_V^2), \\ \Delta &\equiv \frac{1}{2}(m_\rho^2 - m_V^2).\end{aligned}\quad (6.20)$$

We will solve Eq. (6.19) for $P(\omega_r^{(-)})$ and then

$$P^*(\omega) = P(\omega/2). \quad (6.21)$$

Solving Eq. (6.19) for P is tedious but straightforward since it is a quadratic equation. The equation can be written in the form

$$P^4 - 2BP^2 - C = 0, \quad (6.22)$$

where

$$\begin{aligned}B &= 2\mu^2 - (M^2 - \omega_r^{(-)2}), \\ C &= \Delta^2 - (M^2 - \omega_r^{(-)2})^2 \\ &= (\omega_r^{(-)2} - m_V^2)(m_\rho^2 - \omega_r^{(-)2}).\end{aligned}\quad (6.23)$$

Therefore the solutions are

$$P^2 = B \pm \sqrt{D}, \quad (6.24)$$

with

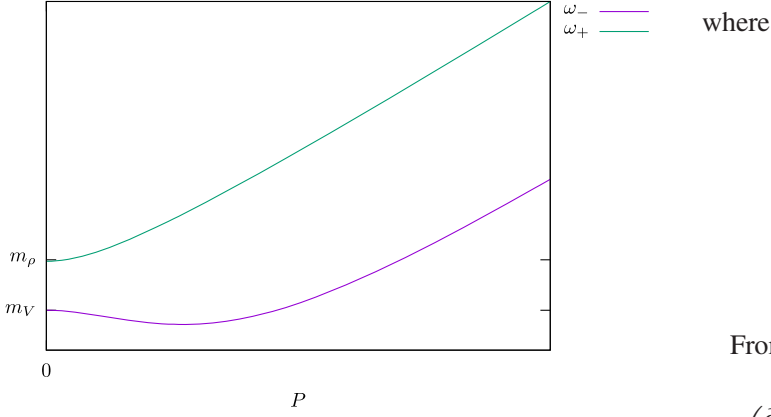


FIG. 3. Sample sketches of the dispersions $\omega_{\pm}$ that illustrate the features discussed in the text and used in arriving at Eq. (6.27).

$$\begin{aligned} D &= B^2 + C \\ &= \Delta^2 + 4\mu^2[\mu^2 - M^2 + \omega_r^{(-)2}]. \end{aligned} \quad (6.25)$$

For some values of $\omega_r^{(-)}$ both solutions in Eq. (6.24) are valid, but for others only $B + \sqrt{D}$ is valid. As an aid to determine the ranges of $\omega_r^{(-)}$, we show a sketch of the dispersion relations $\omega_r^{(\pm)}$ in Fig. 3.

As the figure shows, the dispersion relation $\omega_r^{(-)}$ has a minimum value, which we denote by $\omega_{r0}^{(-)}$. In that point the two solutions for P coincide, that is $D = 0$. Thus from Eq. (6.25),

$$\omega_{r0}^{(-)2} = M^2 - \mu^2 - \frac{\Delta^2}{4\mu^2}. \quad (6.26)$$

For $\omega_{r0}^{(-)} < \omega_r^{(-)} < m_V$, it follows from Eq. (6.23) that $C < 0$. This implies that $\sqrt{D} < B$ and therefore both solutions in Eq. (6.24) are valid, as can be deduced also by a glance at Fig. 3 in that region. In the region $m_V < \omega_r^{(-)} < m_\rho$, we see from Eq. (6.23) that $C > 0$, and therefore only the solution $B + \sqrt{D}$ is valid. Finally, for $\omega_r^{(-)} > m_\rho$, it follows from Eq. (6.23) that $C < 0$ and therefore both solutions exist. However, as seen from Fig. 3, the solution with the smaller (larger) value of P corresponds the $\omega_r^{(+)}$ ($\omega_r^{(-)}$) dispersion relation (which is due to the fact that $\omega_r^{(+)}$ increases faster than $\omega_r^{(-)}$ as P increases.) Thus in that range the valid solution is $B + \sqrt{D}$.

In summary,

$$P^{*2}(\omega) = \begin{cases} B_\omega \pm \sqrt{D_\omega} & (\omega_{r0}^{(-)} < \omega/2 < m_V) \\ B_\omega + \sqrt{D_\omega} & (\omega/2 > m_V) \end{cases} \quad (6.27)$$

where

$$\begin{aligned} B_\omega &= 2\mu^2 - \left(M^2 - \frac{1}{4}\omega^2\right), \\ D_\omega &= \Delta^2 + 4\mu^2\left[\mu^2 - M^2 + \frac{1}{4}\omega^2\right]. \end{aligned} \quad (6.28)$$

3. Expression for $\Gamma^{(0)}(\phi_+ \rightarrow \phi_- \phi_-)$

From Eq. (6.19), it follows that

$$\left(\frac{\partial \omega_r^{(-)}}{\partial P}\right)\bigg|_{P=P^*} = \frac{2P^*}{\omega} \left[1 - \frac{2\mu^2}{\sqrt{\Delta^2 + 4\mu^2 P^{*2}}}\right], \quad (6.29)$$

and therefore from Eq. (6.16)

$$I = \sum_{P^*} \frac{1}{4\pi} \left(\frac{P^*}{\omega}\right) \left[1 - \frac{2\mu^2}{\sqrt{\Delta^2 + 4\mu^2 P^{*2}}}\right]^{-1}. \quad (6.30)$$

We now evaluate the parameters U_{ab} and A , evaluated at $\kappa = 0$. In the limit $\kappa \rightarrow 0$, the mixing parameters defined in Eq. (5.25) become [remembering Eq. (3.26)],

$$\begin{aligned} U_{1+}(\kappa \rightarrow 0) &= U_{2-}(\kappa \rightarrow 0) = 1, \\ U_{1-}(\kappa \rightarrow 0) &= \frac{2\mu\kappa}{m_\rho^2 - m_V^2}, \\ U_{2+}(\kappa \rightarrow 0) &= -\frac{m_V}{m_\rho} U_{1-}(\kappa \rightarrow 0), \end{aligned} \quad (6.31)$$

which from Eq. (5.24) in turn gives

$$A(\kappa \rightarrow 0) = A_1 + A_2, \quad (6.32)$$

where

$$\begin{aligned} A_1 &= -qm_V[1 - 8rx^2], \\ A_2 &= 8q\mu\left(\frac{\kappa}{\omega}\right)x[1 - 2rx^2], \end{aligned} \quad (6.33)$$

with

$$\begin{aligned} r &\equiv \frac{m_V}{m_\rho}, \\ x &\equiv \frac{\mu\kappa}{m_\rho^2 - m_V^2}. \end{aligned} \quad (6.34)$$

Thus, setting strictly $\kappa = 0$,

$$A(0) = -qm_V. \quad (6.35)$$

From Eq. (6.13), we then finally obtain

$$\Gamma^{(0)}(\phi_+ \rightarrow \phi_- \phi_-)|_{\kappa \rightarrow 0} = \sum_{P^*} \frac{q^2 P^*}{8\pi} \frac{m_V^2}{\omega^2} \left[1 - \frac{2\mu^2}{\sqrt{\Delta^2 + 4\mu^2 P^{*2}}} \right]^{-1}, \quad (6.36)$$

which gives the leading term in Eq. (6.11) in the long wavelength ($\kappa \rightarrow 0$) limit. The corrections due to the distribution functions can be readily included using Eq. (6.12), thus

$$\pi_i^{(+)}(\omega, \kappa \rightarrow 0) = -\omega \Gamma^{(0)}(\phi_+ \rightarrow \phi_- \phi_-)|_{\kappa \rightarrow 0} (n_-(p) + n_-(q) + 1), \quad (6.37)$$

remembering that in this limit

$$n_-(p) = n_-(q) = \frac{1}{e^{\frac{1}{2}\beta\omega} - 1}. \quad (6.38)$$

C. Damping in the long wavelength limit

The damping of the (+) mode is given by Eq. (5.12). Explicitly, in this limit

$$\Gamma_+ = \frac{q^2 P_0^*}{8\pi} \frac{m_V^2}{m_\rho^2} \left[1 - \frac{2\mu^2}{\sqrt{\Delta^2 + 4\mu^2 P_0^{*2}}} \right]^{-1} \left[\frac{2}{e^{\frac{1}{2}\beta m_\rho} - 1} + 1 \right], \quad (6.39)$$

where we have used the fact that in this limit $\omega_r^{(+)}(\kappa \rightarrow 0) = m_\rho$, and we have defined

$$P_0^* \equiv P^*(m_\rho). \quad (6.40)$$

From Eqs. (6.27) and (6.28), using Eq. (6.20),

$$P_0^{*2} = 2\mu^2 - \frac{1}{4}(m_\rho^2 + 2m_V^2) + \sqrt{4\mu^4 + \frac{1}{4}(m_\rho^2 - m_V^2)^2 - \mu^2(m_\rho^2 + 2m_V^2)}, \quad (6.41)$$

provided that

$$m_\rho > 2m_V. \quad (6.42)$$

D. Longitudinal dielectric constant

The corresponding contribution to the longitudinal dielectric constant is given by Eq. (5.28). Therefore, using Eqs. (6.36) and (6.37), together with Eq. (6.31),

$$\text{Im } \epsilon_\ell = \sum_{P^*} \frac{q^2}{8\pi} \frac{P^*(\omega) \kappa^2}{\omega^3} \frac{m_V^2}{m_\rho^2} \left(\frac{2\mu m_V}{m_\rho^2 - m_V^2} \right)^2 \left[1 - \frac{2\mu^2}{\sqrt{\Delta^2 + 4\mu^2 P^{*2}(\omega)}} \right]^{-1} \left[\frac{2}{e^{\frac{1}{2}\beta\omega} - 1} + 1 \right], \quad (6.43)$$

which complements Eq. (3.40).

VII. DISCUSSION

For guidance and illustrative purposes, let us consider the large m_ρ limit. There are three independent parameters,

$$\mu, m_\rho, m_V. \quad (7.1)$$

Remember that

$$\begin{aligned} m_\rho^2 &= 4\mu^2 + 2(\mu^2 - m^2), \\ m_V^2 &= \frac{q^2}{\lambda}(\mu^2 - m^2). \end{aligned} \quad (7.2)$$

So we can think also in terms of

$$\mu, m, \frac{q^2}{\lambda}. \quad (7.3)$$

For the purposes of this discussion we will take

$$\frac{q^2}{\lambda} \sim 1, \quad \mu \sim m, \quad (7.4)$$

so that, in particular,

$$m_\rho \sim 2\mu \sim 2m \gg m_V. \quad (7.5)$$

A. Damping

From Eq. (6.41), neglecting m_V ,

$$P_0^{*2} \simeq \frac{3}{4} m_\rho^2, \quad (7.6)$$

and therefore we will put

$$P_0^* \sim m_\rho. \quad (7.7)$$

Then, under the conditions of Eq. (7.5), in the $m_V \rightarrow 0$ limit Eq. (6.39) gives

$$\Gamma_+ \simeq \frac{q^2 m_V^2}{4\pi m_\rho}, \quad (7.8)$$

neglecting the factor involving the distribution function.

B. Imaginary part of the longitudinal dielectric constant

We consider Eq. (6.43) in the same limit discussed above. Since we are considering $m_V \rightarrow 0$, the appropriate formula to use for $P^{*2}(\omega)$ is the second one in Eq. (6.27) (i.e., $\omega/2 > m_V$), which in the $m_V \rightarrow 0$ limit reduces to

$$P^{*2}(\omega) = \frac{1}{4} \omega^2 + \frac{1}{2} m_\rho \omega. \quad (7.9)$$

In the limit stated in Eq. (7.5), the formula in Eq. (6.43) then gives

$$\text{Im} \epsilon_\ell = \frac{q^2 P^*(\omega) \kappa^2}{8\pi \omega^3} \left(\frac{m_V}{m_\rho} \right)^4 \left[1 - \frac{1}{\sqrt{1 + \frac{4P^{*2}(\omega)}{m_\rho^2}}} \right]^{-1}, \quad (7.10)$$

neglecting the correction due to the distribution functions. Further insight can be obtained by considering the cases $\omega \gg m_\rho$ or $\omega \ll m_\rho$. From Eq. (7.9),

$$\frac{P^{*2}(\omega)}{m_\rho^2} = \begin{cases} \frac{\omega^2}{4m_\rho^2} & (\omega \gg m_\rho), \\ \frac{\omega}{2m_\rho} & (\omega \ll m_\rho), \end{cases} \quad (7.11)$$

and correspondingly

$$\left[1 - \frac{1}{\sqrt{1 + \frac{4P^{*2}(\omega)}{m_\rho^2}}} \right]^{-1} = \begin{cases} 1 + O\left(\frac{m_\rho}{\omega}\right) & (\omega \gg m_\rho), \\ \frac{m_\rho}{\omega} & (\omega \ll m_\rho). \end{cases} \quad (7.12)$$

Therefore,

$$\text{Im} \epsilon_\ell = \frac{q^2}{8\pi} \left(\frac{m_V}{m_\rho} \right)^4 \left(\frac{\kappa}{m_\rho} \right)^2 \times \begin{cases} \frac{m_\rho^2}{2\omega^2} & (\omega \gg m_\rho), \\ \frac{1}{\sqrt{2}} \left(\frac{m_\rho}{\omega} \right)^{7/2} & (\omega \ll m_\rho). \end{cases} \quad (7.13)$$

C. Real part of the longitudinal dielectric constant

In the same limit we have considered above, more specifically

$$m_\rho \sim 2\mu \sim 2m \gg m_V, \omega, \kappa, \quad (7.14)$$

it follows from Eq. (3.40) that

$$\text{Re} \epsilon_\ell = 1 - \frac{m_V^2}{\omega^2 - \frac{\kappa^4}{m_\rho^2}}. \quad (7.15)$$

The details leading to Eq. (7.15) are given in Appendix C. Here we note that the dispersion relation that follows from

$$\text{Re} \epsilon_\ell = 0 \quad (7.16)$$

(neglecting the damping term) is simply

$$\omega_V^2 = m_V^2 + \frac{\kappa^4}{m_\rho^2}. \quad (7.17)$$

This is the same result that is obtained from Eq. (3.23) for $\omega_r^{(-)}$ in the same limit [17]. Furthermore, it is reassuring that, putting $m_\rho \sim 2m$, as indicated in Eqs. (7.14), (7.15), and (7.17) reproduce the results obtained for the dielectric constant and dispersion relation in nonrelativistic models of the BE condensation of charged scalars [23].

However it must be kept in mind that the results given in Eqs. (7.15) and (7.17) are limited by the fact that they are valid and hold in the limit of large m_ρ . In this respect, Eq. (3.40) gives the appropriate formula to use for the longitudinal dielectric constant when this limiting case is not appropriate, and correspondingly Eq. (3.23) for the dispersion relation $\omega_r^{(-)}$.

For illustrative purposes and guidance, Fig. 4 shows the plots of the dispersion relations $\omega_r^{(-)}$ and ω_V , given in Eqs. (3.23) and (7.17), respectively. For the purpose of the plots we have set $2\mu/m_\rho = 1$ [see Eq. (7.14)]. The dispersion

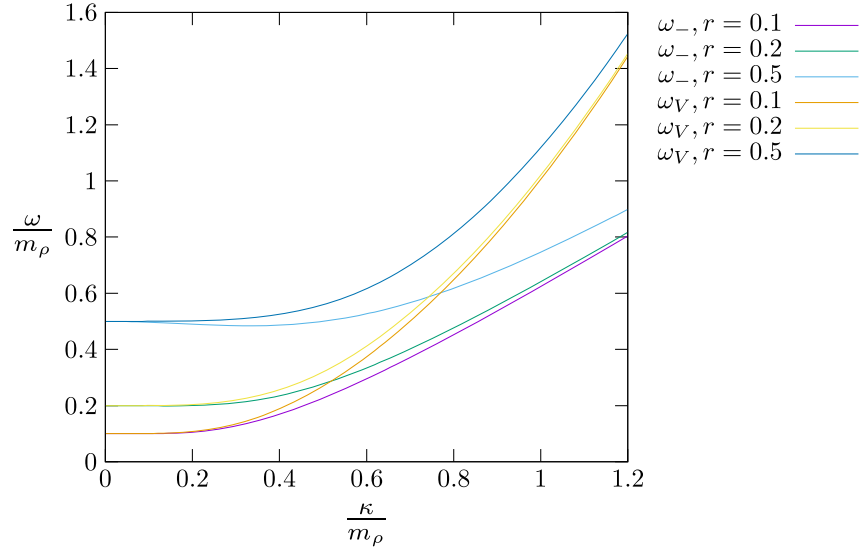


FIG. 4. Plots of the dispersion relations $\omega_r^{(-)}$ and ω_V , given in Eqs. (3.23) and (7.17), respectively. For the purpose of the plots we have set $2\mu/m_\rho = 1$ [see Eq. (7.14)]. The dispersion relations are plotted for the values of the parameter $r \equiv \frac{m_V}{m_\rho}$, as indicated.

relations are plotted for various values of the parameter $r \equiv \frac{m_V}{m_\rho}$ indicated in the figure.

VIII. CONCLUSIONS AND OUTLOOK

The subject of the present work is the imaginary part of the dispersion relations of the propagating modes in a model of a charged scalar Bose-Einstein condensate. In a previous work [17] we considered the real part of the dispersion relations. In that model, two modes correspond to the transverse photon polarizations, while the other two modes are combinations of the longitudinal photon and the massive scalar field, which we denote as the $(\pm)$ modes. While the dispersion relations of the transverse modes have the usual form for transverse photons in a plasma, the dispersion relations of the $(\pm)$ modes reveal some unique features. In our previous work cited above, we determined and analyzed the real part of the dispersion relations of the $(\pm)$ modes, as well as the real part of the contribution to the longitudinal component of polarization tensor and the corresponding contribution to the dielectric constant. The focus in the present work is the calculation, in the same model, of the imaginary part of the dispersion relations of the $(\pm)$ modes, as well as the corresponding contributions to the imaginary part of the longitudinal component of polarization tensor and the dielectric constant.

The main result of the present work is an explicit formula for the imaginary part of the dispersion relations, as well as the corresponding contributions to the imaginary part of the longitudinal component of polarization tensor and the dielectric constant, obtained by the one-loop calculations of the self-energy of the propagating $(\pm)$ modes in the context of TFT. To illustrate some features of the results, we considered in some detail the long wavelength limit,

including specifically the high and low frequency limits, and obtained explicit simple explicit expressions for the damping rate as well as the background contribution to the longitudinal polarization tensor and dielectric constant in those cases. This work and the results obtained pave the way for further exploration of the properties of the model, which can be useful for applications related to the optical properties of this kind of system in various physical contexts, such as those mentioned in the references already cited, and experiments related to dark-matter searches using materials with special optical properties [25].

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DETAILS OF EQ. (6.5)

After substituting Eq. (6.4) in Eq. (6.1) we use

$$e^{x_p} n_B(x_p) n_B(x_{p+k}) = n_B(x_k) (n_B(x_p) - n_B(x_{p+k})). \quad (A1)$$

Then from Eq. (5.26), we obtain

$$\begin{aligned} \pi_i^{(+)}(k) = & -2\pi^2 |A|^2 \int \frac{d^4 p}{(2\pi)^4} \delta(p^2 - \hat{\sigma}_-(p)) \delta((p+k)^2 \\ & - \hat{\sigma}_-(p+k)) (n_B(x_p) - n_B(x_{p+k})) \epsilon(p) \epsilon(p+k). \end{aligned} \quad (A2)$$

We now put

$$p + k = q, \quad (A3)$$

and insert

$$(2\pi)^4 \int \frac{d^4 q}{(2\pi)^4} \delta^{(4)}(q - p - k). \quad (\text{A4})$$

Thus,

$$\pi_i^{(+)}(k) = -\frac{|A|^2}{2} \int \frac{d^4 p}{(2\pi)^3} \int \frac{d^4 q}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(q - p - k) \delta(p^2 - \hat{\sigma}_-(p)) \delta(q^2 - \hat{\sigma}_-(q)) (n_B(x_p) - n_B(x_q)) \epsilon(p) \epsilon(q). \quad (\text{A5})$$

Using

$$\epsilon(p) \epsilon(q) = \theta(p) \theta(q) + \theta(-p) \theta(-q) - \theta(p) \theta(-q) - \theta(-p) \theta(q), \quad (\text{A6})$$

and

$$n_B(x) = -1 - n_B(-x), \quad (\text{A7})$$

we have

$$\begin{aligned} (n_B(x_p) - n_B(x_q)) \epsilon(p) \epsilon(q) &= \theta(p) \theta(q) (n_B(x_p) - n_B(x_q)) + \theta(-p) \theta(-q) (-n_B(-x_p) + n_B(-x_q)) \\ &\quad - \theta(p) \theta(-q) (n_B(x_p) + 1 + n_B(-x_q)) - \theta(-p) \theta(q) (-1 - n_B(-x_p) - n_B(x_q)). \end{aligned} \quad (\text{A8})$$

We will then use

$$\begin{aligned} \int \frac{d^4 p}{(2\pi)^3} \delta(p^2 - \hat{\sigma}_s(p)) \theta(p) n_B(x_p) &= \int \frac{d^3 P}{(2\pi)^3 2\omega_r^{(s)}(P)} n_s(P), \\ \int \frac{d^4 p}{(2\pi)^3} \delta(p^2 - \hat{\sigma}_s(p)) \theta(-p) n_B(-x_p) &= \int \frac{d^3 P}{(2\pi)^3 2\omega_r^{(s)}(P)} n_s(P), \end{aligned} \quad (\text{A9})$$

where

$$n_s(P) \equiv \frac{1}{e^{\beta \omega_r^{(s)}(P)} - 1}, \quad (\text{A10})$$

and with the understanding that, after the integration over p^0 ,

$$p^\mu = (\omega_r^{(s)}(P), \vec{P}). \quad (\text{A11})$$

Similarly with the integrals over q , with $q^\mu = (\omega_r^{(s)}(Q), \vec{Q})$. The functions $\omega_r^{(s)}(P)$ and $\omega_r^{(s)}(Q)$ are the dispersion relation defined in Eq. (3.23).

In this way we arrive at

$$\begin{aligned} \pi_i^{(+)}(k) &= -\frac{1}{2} |A|^2 \int \frac{d^3 P}{(2\pi)^3 2\omega_r^{(-)}(P)} \frac{d^3 Q}{(2\pi)^3 2\omega_r^{(-)}(Q)} (2\pi)^4 \{ \delta^{(4)}(k + p - q) (n_-(p) - n_-(q)) \\ &\quad + \delta^{(4)}(k + q - p) (-n_-(p) + n_-(q)) - \delta^{(4)}(k + p + q) (n_-(p) + n_-(q) + 1) \\ &\quad - \delta^{(4)}(k - p - q) (-n_-(p) - n_-(q) - 1) \}, \end{aligned} \quad (\text{A12})$$

with

$$\begin{aligned} p^\mu &= (\omega_r^{(-)}(P), \vec{P}), \\ q^\mu &= (\omega_r^{(-)}(Q), \vec{Q}), \\ k^\mu &= (\omega, \vec{k}), \end{aligned} \quad (\text{A13})$$

which is Eq. (6.5).

APPENDIX B: PHASE SPACE INTEGRAL FOR $\phi_+ \rightarrow \phi_- \phi_-$ IN THE LONG WAVELENGTH LIMIT

We consider the phase space integral of the type we have to do for the $\phi_+ \rightarrow \phi_- \phi_-$ decay, that is

$$I \equiv \int \frac{d^3 P_1}{2E_1} \int \frac{d^3 P_2}{2E_2} \delta^{(4)}(k - p_1 - p_2), \quad (\text{B1})$$

but with the dispersion relations of the form

$$E_i(P) = \sqrt{P^2 + m_i^2(P)}. \quad (\text{B2})$$

In this section we consider the long wavelength limit, which corresponds to setting $\kappa \rightarrow 0$,

$$k^\mu = (\omega, \vec{0}). \quad (\text{B3})$$

We start from

$$I = \int \frac{d^3 P_1}{2E_1} \delta((k - p_1)^2 - m_2^2) \theta(\omega - E_1), \quad (\text{B4})$$

where now

$$(k - p_1)^2 = \omega^2 + m_1^2 - 2\omega E_1, \quad (\text{B5})$$

and

$$p_2 = k - p_1 \Rightarrow P_2 = P_1. \quad (\text{B6})$$

Then putting

$$\frac{d^3 P_1}{2E_1} \rightarrow 2\pi \frac{P_1^2}{E_1} dP_1, \quad (\text{B7})$$

we obtain

$$\begin{aligned} dI &= 2\pi dP_1 \frac{P_1^2}{E_1} \delta[\omega^2 + m_1^2(P_1) - m_2^2(P_1) - 2\omega E_1] \theta(\omega - E_1) \\ &= \frac{\pi}{\omega} dP_1 \frac{P_1^2}{E_1} \delta[\omega^2 + m_1^2(P_1) - m_2^2(P_1) - 2\omega E_1] \theta(\omega - E_1). \end{aligned} \quad (\text{B8})$$

The delta function then fixes $P_1 = P_1^*$, where P_1^* is such that

$$E_1(P_1^*) = \frac{\omega^2 + m_1^2(P_1^*) - m_2^2(P_1^*)}{2\omega}, \quad (\text{B9})$$

with $E_1(P_1)$ given in Eq. (B2). This is an implicit equation that must be solved for P_1^* . The theta function $\theta(\omega - E_1)$ requires that ω and the parameters that enter in the definition of the $m_i(P)$ be such that

$$E_1(P_1^*) < \omega \quad (\text{B10})$$

otherwise the integral is zero (this is the analog of the condition $m > m_1 + m_2$ in the standard case). Assuming this is satisfied, then

$$I = \frac{\pi P_1^{*2}}{\omega E_1^*} \frac{1}{|G_0|}, \quad (\text{B11})$$

where we have defined E_1^* ,

$$E_1^* \equiv E_1(P_1^*) = \frac{\omega^2 + m_1^2(P_1^*) - m_2^2(P_1^*)}{2\omega}, \quad (\text{B12})$$

and

$$G_0 \equiv \left(\frac{\partial G}{\partial P_1} \right) \Big|_{P_1=P_1^*}, \quad (\text{B13})$$

with

$$G = E_1(P_1) + \frac{m_1^2(P_1) - m_2^2(P_1)}{2\omega}. \quad (\text{B14})$$

1. Special case: $m_1 = m_2$

In this case the formulas simplify. The equation for P_1^* is simply

$$E_1(P_1^*) = \frac{1}{2}\omega. \quad (\text{B15})$$

Assuming that ω and the parameters that enter in the definition of $m_1(P)$ are such that the solution exists, the condition $E_1(P_1^*) < \omega$ is automatically satisfied. G reduces to $E_1(P)$. Therefore,

$$I = 2\pi \frac{P_1^{*2}}{\omega^2} \frac{1}{|G_0|}, \quad (\text{B16})$$

with

$$G_0 = \left(\frac{\partial E_1}{\partial P_1} \right) \Big|_{P_1=P_1^*}. \quad (\text{B17})$$

APPENDIX C: DERIVATION OF EQ. (7.15)

Rewriting the term in the square brackets in Eq. (7.15),

$$\begin{aligned} \text{Re } \epsilon_\ell &= 1 - \frac{m_V^2(k^2 - m_\rho^2)}{k^2(k^2 - m_\rho^2) - 4\mu^2\kappa^2} \\ &= 1 - \frac{m_V^2}{X}, \end{aligned} \quad (\text{C1})$$

where

$$X \equiv k^2 + \frac{4\mu^2\kappa^2}{m_\rho^2(1 - k^2/m_\rho^2)}. \quad (\text{C2})$$

Expanding the denominator,

$$X = \omega^2 \left(1 + \frac{\eta\kappa^2}{m_\rho^2} \right) + \kappa^2(\eta - 1) - \frac{\eta\kappa^4}{m_\rho^2}, \quad (\text{C3})$$

where we have put

$$\eta = \frac{4\mu^2}{m_\rho^2}. \quad (\text{C4})$$

In the limit stated in Eq. (7.5), $\eta \rightarrow 1$, and

$$X \rightarrow \omega^2 - \frac{\kappa^4}{m_\rho^2}, \quad (\text{C5})$$

which leads to Eq. (7.15).

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