

Unitarity of 4D lattice theory of gravity

S. N. Vergeles^{*}

*Landau Institute for Theoretical Physics, Russian Academy of Sciences,
Chernogolovka, Moscow region, 142432 Russia
and Moscow Institute of Physics and Technology, Department of Theoretical Physics,
Dolgoprudny, Moscow region, 141707 Russia*



(Received 30 May 2025; accepted 2 September 2025; published 22 September 2025)

The unitarity of the 4D lattice theory of gravity in the case of the Minkowski signature is proved. The proof is valid only for lattices that conserve the number of degrees of freedom during time evolution. The Euclidean signature and the Minkowski signature are related by the deformation of the integration contours of dynamic variables in a discrete lattice functional integral. It is important that the result is obtained directly on the lattice. Since the studied lattice theory of gravity in the long-wave limit transforms into the well-known Einstein-Cartan-Palatini theory, the obtained result means that this lattice theory of gravity has the right to be considered as a discrete regularization of the generally accepted continuous physical theory of gravity.

DOI: [10.1103/yw8k-fcd2](https://doi.org/10.1103/yw8k-fcd2)

I. INTRODUCTION

The idea of discretizing space-time to describe the theory of gravity at a fundamental level was first formulated by Regge in [1]. In this work, the discretization of space-time was carried out using a simplicial complex. Each 1-simplex was assigned its length, so that all the sizes of each 2-simplex (triangle) were fixed. The lengths of the three sides of each triangle satisfied the triangle inequality. In this way, the geometry of the entire complex was fixed. An approach to discrete geometry similar to Regge calculus can also be found in [2,3].

Despite the obvious elegance of the Regge calculus, this theory turns out to be very inconvenient when moving to quantum theory. Indeed, the independent variables determining the Regge action are the lengths of one-dimensional simplices, subject to a large number of constraints, namely, triangle inequalities. Moreover, the introduction of Dirac fields into the theory creates a new difficulty, consisting in the absence of explicit orthonormal bases. Perhaps this is why the version of discrete gravity based on the so-called B-F theory soon gained more intensive development. A detailed description of the discrete quantum theory of gravity based on the B-F formalism can be found in [4–7]. The B-F theory is developed on the basis of the action of the

theory of gravity in the form of Palatini [see below (2.27)]. The most characteristic property of B-F theory is that the curvature tensor of the 2-form is zero. However, while B-F theory does describe gravity in three-dimensional space, it does not do so in higher-dimensional spaces. For example, in a four-dimensional space with action (2.27), the curvature tensor is not equal to zero. But this is not the only difficulty of the theory: The introduction of matter is also inconvenient.

In contrast to the multidimensional case, significant computational progress has been made in two-dimensional discrete quantum gravity [8,9].

Later, more quantizable models of discrete gravity emerged [10–20]. These models are characterized by the fact that in the naive long-wave limit they are transformed into the well-known Einstein-Cartan-Palatini continuum theory. By Einstein-Cartan-Palatini theory, we mean a theory with the Hilbert-Einstein action, plus the action of the Dirac field minimally coupled to gravity, plus a contribution to the action from the cosmological constant, all of which are considered in Cartan-Palatini form.

The Einstein-Cartan-Palatini action is invariant under diffeomorphisms. The following fact is interesting here. In the lattice theory of gravity [10–20], local coordinates are completely absent; they appear only when passing to the long-wave limit, and they can be introduced arbitrarily. The long-wave limit is nothing more than the expansion of the original action in terms of a small parameter $\xi \sim l_P/\lambda \ll 1$ that is independent of local coordinates. Here, l_P is the Planck scale, and λ is the physical scale. It is natural to assume that l_P is the characteristic scale of one 1-simplex of the lattice, and $\lambda \sim nl_P$, where $n \gg 1$ is the number

^{*}Contact author: vergeles@itp.ac.ru

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

of 1-simplices of the graph that fits into the wavelength. From the above, it is clear that the parameter ξ does not depend on the local coordinates, which are arbitrary. Therefore, when expanding the lattice action in the parameter ξ , each term of a given degree relative to ξ is invariant under diffeomorphisms. In this expansion, the main term is the Einstein-Cartan-Palatini action whose invariance with respect to diffeomorphisms is established independently. The remaining terms, which quickly die out according to a power law in the long-wave limit, are also invariants with respect to diffeomorphisms (see Sec. II).

The following problem of quantization of generally covariant theories should be pointed out. Such theories represent degenerate dynamical systems with constraints. To construct a unitary and covariant S matrix, it is necessary to prove (i) the gauge independence of the covariant S matrix and (ii) its unitarity at least in one (nonrelativistic) gauge. If both requirements can be fulfilled, then the unitarity is guaranteed also in a relativistic gauge, and the suggested Feynman rules are correct. To construct a unitary S matrix, the Hamiltonian and all constraints are expressed in terms of the chosen independent physical canonical variables, the gauge is fixed, the functional measure is determined, and the transition amplitude is written out in the form of a functional integral [21]. Thus, the second requirement is fulfilled. To satisfy the first requirement, many theories (for example, Yang-Mills theory) use the Faddeev-Popov trick [22]. However, in the case of theories that are invariant under diffeomorphisms (for example, the theory of gravity), the Faddeev-Popov technology is fundamentally unsuitable. The reason for this is that, under general coordinate transformations in chronologically ordered averages, the order of the operators, generally speaking, changes. These difficulties were overcome in the works of Fradkin and Fradkin-Vilkovisky [23]. Although Fradkin-Vilkovisky technology is extremely complex, it fundamentally solves the problem.

From the above, it follows that the Einstein-Cartan-Palatini quantum theory allows the construction of a unitary and gauge-invariant S matrix. However, this model is only the main term in the expansion of the lattice theory under study in the parameter ξ . An important question arises: Do the following terms of this expansion violate the unitarity of the theory? If this were so, then the lattice model of gravity that we are studying could not be considered as a model that even slightly describes the real world.

We now formulate the result of the present work: The evolution of the quantum model of discrete gravity formulated in Sec. II is a unitary transformation. Let us emphasize that this result was obtained directly in lattice theory and, of course, for the Minkowski signature. Note also that, unlike the continuum theory of gravity, the lattice theory of gravity has no local coordinates at all, since neither Cartesian spaces nor manifolds are used in defining abstract simplicial complexes [24]. We will see that this

circumstance radically simplifies the proof of unitarity of lattice theory.

The work is organized as follows.

In Sec. II, we define the lattice gravity model we study in two related versions: the Euclidean signature and the Minkowski signature. These two versions of the theory transform into each other by deformation in the complex plane of the integration contours over the dynamic variables in the functional integral, representing either the partition function or the transition amplitude.

In Sec. III, a proof of the unitarity of the transition amplitude in lattice theory is given.

The Appendix provides some information from the field of combinatorial topology necessary for proving unitarity.

II. DEFINITION OF LATTICE THEORY OF GRAVITY

In the works of the author [10–17], the lattice theory of gravity coupled to Dirac fields was studied. Let us give a brief definition of it.

Consider the orientable four-dimensional simplicial complex $\mathfrak{K}$. We recommend the book [24], Sec. II.4, for an introduction to the definition of abstract simplicial complexes. Suppose that each of its 4-simplices belongs to a subcomplex of $\mathfrak{K}' \in \mathfrak{K}$ that has a geometric realization in $\mathbb{R}^4$ without cavities. The vertices are denoted by $a_{\mathcal{V}}$, and the indices $\mathcal{V}$ and $\mathcal{W}$ numerate the vertices and 4-simplices $s_{\mathcal{W}}^4$, respectively. It is necessary to use local enumeration of vertices belonging to a given 4-simplex: All five vertices of the 4-simplex $s_{\mathcal{W}}^4$ are numbered as $a_{\mathcal{V}_{(\mathcal{W})i}}$, $i = 1, 2, 3, 4, 5$. In what follows, notations with an additional subscript ($\mathcal{W}$) indicate that the corresponding object belongs to the 4-simplex $s_{\mathcal{W}}^4$. Let us denote $\epsilon_{\mathcal{V}_{(\mathcal{W})1}\mathcal{V}_{(\mathcal{W})2}\mathcal{V}_{(\mathcal{W})3}\mathcal{V}_{(\mathcal{W})4}\mathcal{V}_{(\mathcal{W})5}} = \pm 1$ the Levi-Civita symbol. The upper (lower) sign depends on the orientation of the 4-simplex $s_{\mathcal{W}}^4 = a_{\mathcal{V}_{(\mathcal{W})1}}a_{\mathcal{V}_{(\mathcal{W})2}}a_{\mathcal{V}_{(\mathcal{W})3}}a_{\mathcal{V}_{(\mathcal{W})4}}a_{\mathcal{V}_{(\mathcal{W})5}}$.

A. Euclidean signature

Element of the compact group Spin(4) and element of the Clifford algebra

$$\begin{aligned}
 \Omega_{\mathcal{V}_1\mathcal{V}_2} &= \Omega_{\mathcal{V}_2\mathcal{V}_1}^{-1} = \exp(\omega_{\mathcal{V}_1\mathcal{V}_2}) \\
 &= \exp\left(\frac{1}{2}\sigma^{ab}\omega_{\mathcal{V}_1\mathcal{V}_2}^{ab}\right) \in \text{Spin}(4), \\
 \sigma^{ab} &\equiv \frac{1}{4}[\gamma^a\gamma^b], \\
 \gamma^a\gamma^b + \gamma^b\gamma^a &= 2\delta^{ab}, \quad a = 1, 2, 3, 4, \quad \gamma^5 \equiv \gamma^1\gamma^2\gamma^3\gamma^4, \\
 \hat{e}_{\mathcal{V}_1\mathcal{V}_2} &\equiv e_{\mathcal{V}_1\mathcal{V}_2}^a\gamma^a \equiv -\Omega_{\mathcal{V}_1\mathcal{V}_2}\hat{e}_{\mathcal{V}_2\mathcal{V}_1}\Omega_{\mathcal{V}_1\mathcal{V}_2}^{-1}, \\
 |e_{\mathcal{V}_1\mathcal{V}_2}| &< 1, \quad |e_{\mathcal{V}_1\mathcal{V}_2}| \equiv \sqrt{\sum_a (e_{\mathcal{V}_1\mathcal{V}_2}^a)^2}, \quad (2.1)
 \end{aligned}$$

are defined on each oriented 1-simplex $a_{\nu_1}a_{\nu_2}$. By assumption, the set of variables $\{\Omega, \hat{e}\}$ is a set of independent bosonic dynamical variables. Fermionic degrees of freedom (Dirac spinors) are defined at the vertices of the complex:

$$\Psi_{\nu}^{\dagger}, \Psi_{\nu}. \quad (2.2)$$

The set of variables $\{\Psi^{\dagger}, \Psi\}$ are mutually independent, and the spinors $\Psi_{\nu}^{\dagger}$ and Ψ_{ν} are in mutual involution (or anti-involution) with respect to the operation of Hermitian conjugation.

Let us consider a model with action

$$\mathfrak{A} = \mathfrak{A}_g + \mathfrak{A}_{\Psi} + \mathfrak{A}_{\Lambda_0}. \quad (2.3)$$

Here, $\mathfrak{A}_g$ and $\mathfrak{A}_{\Psi}$ are the actions of pure gravity and the Dirac field, respectively:

$$\begin{aligned} \mathfrak{A}_g = & -\frac{1}{5! \cdot 2 \cdot l_P^2} \sum_{\mathcal{W}} \sum_{\sigma} \varepsilon_{\sigma(\mathcal{V}_{(\mathcal{W})1})\sigma(\mathcal{V}_{(\mathcal{W})2})\sigma(\mathcal{V}_{(\mathcal{W})3})\sigma(\mathcal{V}_{(\mathcal{W})4})\sigma(\mathcal{V}_{(\mathcal{W})5})} \\ & \times \text{tr} \gamma^5 \{ \Omega_{\sigma(\mathcal{V}_{(\mathcal{W})5})\sigma(\mathcal{V}_{(\mathcal{W})1})} \Omega_{\sigma(\mathcal{V}_{(\mathcal{W})1})\sigma(\mathcal{V}_{(\mathcal{W})2})} \Omega_{\sigma(\mathcal{V}_{(\mathcal{W})2})\sigma(\mathcal{V}_{(\mathcal{W})3})} \hat{e}_{\sigma(\mathcal{V}_{(\mathcal{W})3})\sigma(\mathcal{V}_{(\mathcal{W})4})} \hat{e}_{\sigma(\mathcal{V}_{(\mathcal{W})4})\sigma(\mathcal{V}_{(\mathcal{W})5})} \}, \end{aligned} \quad (2.4)$$

$$\mathfrak{A}_{\Psi} = \frac{1}{5 \cdot 24^2} \sum_{\mathcal{W}} \sum_{\sigma} \varepsilon_{\sigma(\mathcal{V}_{(\mathcal{W})1})\sigma(\mathcal{V}_{(\mathcal{W})2})\sigma(\mathcal{V}_{(\mathcal{W})3})\sigma(\mathcal{V}_{(\mathcal{W})4})\sigma(\mathcal{V}_{(\mathcal{W})5})} \text{tr} \gamma^5 \{ \hat{\Theta}_{\sigma(\mathcal{V}_{(\mathcal{W})5})\sigma(\mathcal{V}_{(\mathcal{W})1})} \hat{e}_{\sigma(\mathcal{V}_{(\mathcal{W})1})\sigma(\mathcal{V}_{(\mathcal{W})2})} \hat{e}_{\sigma(\mathcal{V}_{(\mathcal{W})2})\sigma(\mathcal{V}_{(\mathcal{W})3})} \hat{e}_{\sigma(\mathcal{V}_{(\mathcal{W})3})\sigma(\mathcal{V}_{(\mathcal{W})4})} \hat{e}_{\sigma(\mathcal{V}_{(\mathcal{W})4})\sigma(\mathcal{V}_{(\mathcal{W})5})} \}. \quad (2.5)$$

Each σ is one of 5! permutations of vertices $\mathcal{V}_{(\mathcal{W})i} \rightarrow \sigma(\mathcal{V}_{(\mathcal{W})i})$.

$$\hat{\Theta}_{\nu_1\nu_2} \equiv \Theta_{\nu_1\nu_2}^a \gamma^a = \hat{\Theta}_{\nu_1\nu_2}^{\dagger}, \quad \Theta_{\nu_1\nu_2}^a = \frac{i}{2} (\Psi_{\nu_1}^{\dagger} \gamma^a \Omega_{\nu_1\nu_2} \Psi_{\nu_2} - \Psi_{\nu_2}^{\dagger} \Omega_{\nu_2\nu_1} \gamma^a \Psi_{\nu_1}). \quad (2.6)$$

It can be verified that [compare with (2.1)]

$$\hat{\Theta}_{\nu_1\nu_2} \equiv -\Omega_{\nu_1\nu_2} \hat{\Theta}_{\nu_2\nu_1} \Omega_{\nu_1\nu_2}^{-1}. \quad (2.7)$$

The contribution to the lattice action from the cosmological constant is

$$\begin{aligned} \mathfrak{A}_{\Lambda_0} = & -\frac{1}{5! \cdot 12} \cdot \Lambda_0 \cdot \varepsilon_{abcd} \sum_{\mathcal{W}} \sum_{\sigma} \varepsilon_{\sigma(\mathcal{V}_{(\mathcal{W})1})\sigma(\mathcal{V}_{(\mathcal{W})2})\sigma(\mathcal{V}_{(\mathcal{W})3})\sigma(\mathcal{V}_{(\mathcal{W})4})\sigma(\mathcal{V}_{(\mathcal{W})5})} \\ & \times e_{\sigma(\mathcal{V}_{(\mathcal{W})5})\sigma(\mathcal{V}_{(\mathcal{W})1})}^a e_{\sigma(\mathcal{V}_{(\mathcal{W})1})\sigma(\mathcal{V}_{(\mathcal{W})2})}^b e_{\sigma(\mathcal{V}_{(\mathcal{W})2})\sigma(\mathcal{V}_{(\mathcal{W})3})}^c e_{\sigma(\mathcal{V}_{(\mathcal{W})3})\sigma(\mathcal{V}_{(\mathcal{W})4})}^d. \end{aligned} \quad (2.8)$$

The partition function is represented by an integral

$$Z = \prod_{1\text{-simplices}} \int_{|e_{\nu_1\nu_2}| < 1} \prod_a d e_{\nu_1\nu_2}^a \int d\mu \{ \Omega_{\nu_1\nu_2} \} \times \prod_{\nu} \int d\Psi_{\nu}^{\dagger} d\Psi_{\nu} \exp(\mathfrak{A}). \quad (2.9)$$

The action (2.3), as well as the integral (2.9), is invariant under gauge transformations

$$\begin{aligned} \tilde{\Omega}_{\nu_1\nu_2} &= S_{\nu_1} \Omega_{\nu_1\nu_2} S_{\nu_2}^{-1}, & \tilde{e}_{\nu_1\nu_2} &= S_{\nu_1} \hat{e}_{\nu_1\nu_2} S_{\nu_1}^{-1}, \\ \tilde{\Psi}_{\nu} &= S_{\nu} \Psi_{\nu}, & \tilde{\Psi}_{\nu}^{\dagger} &= \Psi_{\nu}^{\dagger} S_{\nu}^{-1}, \quad S_{\nu} \in \text{Spin}(4). \end{aligned} \quad (2.10)$$

Verification of this fact is made easier by using the relation [compare with the relation for $\hat{e}_{\nu_1\nu_2}$ in (2.10)]

$$\tilde{\Theta}_{\nu_1\nu_2} = S_{\nu_1} \hat{\Theta}_{\nu_1\nu_2} S_{\nu_1}^{-1}, \quad (2.11)$$

which follows directly from (2.10).

The lattice model under consideration is invariant with respect to the global discrete $\mathbb{Z}_2$ symmetry, which is an analog of the combined PT symmetry. Let $\hat{\mathcal{U}}_{PT}$ denote the operator of this transformation. Then, the transformed dynamic variables are expressed through the original variables as follows:

$$\begin{aligned} \hat{\mathcal{U}}_{PT}^{-1} \Psi_{\nu} \hat{\mathcal{U}}_{PT} &= U_{PT} (\Psi_{\nu}^{\dagger})^t, \\ \hat{\mathcal{U}}_{PT}^{-1} \Psi_{\nu}^{\dagger} \hat{\mathcal{U}}_{PT} &= -(\Psi_{\nu})^t U_{PT}^{-1}, \quad U_{PT} = i\gamma^1 \gamma^3 \\ \hat{\mathcal{U}}_{PT}^{-1} e_{\nu_1\nu_2}^a \hat{\mathcal{U}}_{PT} &= -e_{\nu_1\nu_2}^a, \quad \hat{\mathcal{U}}_{PT}^{-1} \omega_{\nu_1\nu_2}^{ab} \hat{\mathcal{U}}_{PT} = \omega_{\nu_1\nu_2}^{ab}. \end{aligned} \quad (2.12)$$

Here, the superscript “ t ” denotes the transposition of the Dirac matrices and spinors. We have

$$U_{PT}^{-1} \gamma^a U_{PT} = (\gamma^a)^t, \quad U_{PT}^{-1} \sigma^{ab} U_{PT} = -(\sigma^{ab})^t. \quad (2.13)$$

From (2.12) and (2.13), it follows that

$$U_{PT}^{-1} \Omega_{\mathcal{V}_1 \mathcal{V}_2} U_{PT} = (\Omega_{\mathcal{V}_2 \mathcal{V}_1})^t, \quad (2.14)$$

$$\hat{\mathcal{U}}_{PT}^{-1} \Theta_{\mathcal{V}_1 \mathcal{V}_2}^a \hat{\mathcal{U}}_{PT} = -\Theta_{\mathcal{V}_1 \mathcal{V}_2}^a. \quad (2.15)$$

We see that the action (2.3) is invariant under the transformations (2.12)–(2.15).

B. Minkowski signature

Let us pass on to the Minkowski signature. Further in this section, all lattice variables in the case of the Euclidean signature are provided with a prime. For field variables in the case of the Minkowski signature, the old notations are used.

For the specified transformation of the action, the following deformations of the integration contours in the integral (2.9) are necessary:

$$\begin{aligned} \omega_{\mathcal{V}_1 \mathcal{V}_2}^{4\alpha} &= i\omega_{\mathcal{V}_1 \mathcal{V}_2}^{0\alpha}, & \omega_{\mathcal{V}_1 \mathcal{V}_2}^{a\beta} &= -\omega_{\mathcal{V}_1 \mathcal{V}_2}^{a\beta}, \\ e_{\mathcal{V}_1 \mathcal{V}_2}^{4\alpha} &= e_{\mathcal{V}_1 \mathcal{V}_2}^{0\alpha}, & e_{\mathcal{V}_1 \mathcal{V}_2}^{a\alpha} &= ie_{\mathcal{V}_1 \mathcal{V}_2}^{a\alpha}. \end{aligned} \quad (2.16)$$

The variables $\omega_{\mathcal{V}ij}^{ab}$ and $e_{\mathcal{V}ij}^a$ in the Minkowski signature are real, and their indices take the values

$$a, b, \dots = 0, 1, 2, 3, \quad \alpha, \beta, \dots = 1, 2, 3. \quad (2.17)$$

In the orthonormal basis (ONB), the metric tensor $\eta^{ab} = \text{diag}(1, -1, -1, -1)$. Dirac matrices are transformed as

$$\gamma'^4 = \gamma^0, \quad \gamma'^a = i\gamma^a, \quad \gamma'^5 = \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3. \quad (2.18)$$

Thus, for spin matrices $\sigma^{ab} = (1/4)[\gamma^a, \gamma^b]$ we have

$$\sigma'^{4\alpha} = i\sigma^{0\alpha}, \quad \sigma'^{a\beta} = -\sigma^{a\beta}. \quad (2.19)$$

When passing to the Minkowski signature, the Dirac spinors are transformed as follows:

$$\Psi'_{\mathcal{V}} = \Psi_{\mathcal{V}}, \quad \Psi'^{\dagger}_{\mathcal{V}} = \Psi^{\dagger}_{\mathcal{V}} \gamma^0 = \bar{\Psi}_{\mathcal{V}}. \quad (2.20)$$

Using (2.16)–(2.20), we find

$$\begin{aligned} \omega'_{\mathcal{V}_1 \mathcal{V}_2} &= \frac{1}{2} \omega_{\mathcal{V}_1 \mathcal{V}_2}^{ab} \sigma_{ab} \equiv \omega_{\mathcal{V}_1 \mathcal{V}_2}, \\ \hat{e}'_{\mathcal{V}_1 \mathcal{V}_2} &= \gamma_a e_{\mathcal{V}_1 \mathcal{V}_2}^a \equiv \hat{e}_{\mathcal{V}_1 \mathcal{V}_2}, \\ \Omega'_{\mathcal{V}_1 \mathcal{V}_2} &= \exp\left(\frac{1}{2} \omega_{\mathcal{V}_1 \mathcal{V}_2}^{ab} \sigma'^{ab}\right) \\ &= \exp\left(\frac{1}{2} \omega_{\mathcal{V}_1 \mathcal{V}_2}^{ab} \sigma_{ab}\right) \equiv \Omega_{\mathcal{V}_1 \mathcal{V}_2} \in \text{Spin}(3, 1), \\ \hat{\Theta}'_{\mathcal{V}_1 \mathcal{V}_2} &= \gamma_a \Theta_{\mathcal{V}_1 \mathcal{V}_2}^a \equiv \hat{\Theta}_{\mathcal{V}_1 \mathcal{V}_2}, \\ \Theta_{\mathcal{V}_1 \mathcal{V}_2}^a &= \frac{i}{2} (\bar{\Psi}_{\mathcal{V}_1} \gamma^a \Omega_{\mathcal{V}_1 \mathcal{V}_2} \Psi_{\mathcal{V}_2} - \bar{\Psi}_{\mathcal{V}_2} \Omega_{\mathcal{V}_2 \mathcal{V}_1} \gamma^a \Psi_{\mathcal{V}_1}) \\ &= (\Theta_{\mathcal{V}_1 \mathcal{V}_2}^a)^{\dagger}. \end{aligned} \quad (2.21)$$

In the Minkowski signature, the action still has the form (2.3), but it is a functional of the unprimed variables from the right-hand sides of the equations (2.21) and is purely imaginary. This is a result of the deformation of the integration contours according to (2.16) and the substitutions (2.20). Let us prove this fact.

Let us write out the elementary term from the action (2.4) with an accuracy of up to a real factor, and let us represent the Euclidean primed variables in the form (2.16). Thus, we go to the Minkowski signature:

$$\begin{aligned} \varepsilon_{\mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \mathcal{V}_4 \mathcal{V}_5} \text{tr} \gamma^5 \Omega'_{\mathcal{V}_5 \mathcal{V}_1} \Omega'_{\mathcal{V}_1 \mathcal{V}_2} \Omega'_{\mathcal{V}_2 \mathcal{V}_3} \hat{e}'_{\mathcal{V}_3 \mathcal{V}_4} \hat{e}'_{\mathcal{V}_5 \mathcal{V}_4} \\ = \varepsilon_{\mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \mathcal{V}_4 \mathcal{V}_5} \text{tr} \gamma^5 \Omega_{\mathcal{V}_5 \mathcal{V}_1} \Omega_{\mathcal{V}_1 \mathcal{V}_2} \Omega_{\mathcal{V}_2 \mathcal{V}_3} \hat{e}_{\mathcal{V}_3 \mathcal{V}_4} \hat{e}_{\mathcal{V}_5 \mathcal{V}_4} \equiv i\mathcal{A}. \end{aligned} \quad (2.22)$$

The first equality in (2.22) holds due to the equalities (2.21). Let us show that $\mathcal{A}$ is a real quantity. To do this, we perform the Hermitian conjugation operation of the quantity (2.22):

$$-i\mathcal{A}^{\dagger} = \varepsilon_{\mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \mathcal{V}_4 \mathcal{V}_5} \text{tr} \gamma^5 \hat{e}_{\mathcal{V}_5 \mathcal{V}_4}^{\dagger} \hat{e}_{\mathcal{V}_3 \mathcal{V}_4}^{\dagger} \Omega_{\mathcal{V}_2 \mathcal{V}_3}^{\dagger} \Omega_{\mathcal{V}_1 \mathcal{V}_2}^{\dagger} \Omega_{\mathcal{V}_5 \mathcal{V}_1}^{\dagger}. \quad (2.23)$$

Now, let us take into account the equalities

$$\begin{aligned} \hat{e}_{\mathcal{V}_i \mathcal{V}_j}^{\dagger} &= \gamma^0 \hat{e}_{\mathcal{V}_i \mathcal{V}_j} \gamma^0, & \Omega_{\mathcal{V}_i \mathcal{V}_j}^{\dagger} &= \gamma^0 \Omega_{\mathcal{V}_i \mathcal{V}_j} \gamma^0, \\ \gamma^0 \gamma^5 \gamma^0 &= -\gamma^5 \end{aligned} \quad (2.24)$$

and use them to transform the value (2.23):

$$\begin{aligned} -i\mathcal{A}^{\dagger} &= -\varepsilon_{\mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \mathcal{V}_4 \mathcal{V}_5} \text{tr} \gamma^5 \hat{e}_{\mathcal{V}_5 \mathcal{V}_4} \hat{e}_{\mathcal{V}_3 \mathcal{V}_4} \Omega_{\mathcal{V}_5 \mathcal{V}_2} \Omega_{\mathcal{V}_2 \mathcal{V}_1} \Omega_{\mathcal{V}_1 \mathcal{V}_5} \\ &= -\varepsilon_{\mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \mathcal{V}_4 \mathcal{V}_5} \text{tr} \gamma^5 \Omega_{\mathcal{V}_5 \mathcal{V}_1} \Omega_{\mathcal{V}_1 \mathcal{V}_2} \Omega_{\mathcal{V}_2 \mathcal{V}_3} \hat{e}_{\mathcal{V}_3 \mathcal{V}_4} \hat{e}_{\mathcal{V}_5 \mathcal{V}_4} \\ &\equiv -i\mathcal{A}. \end{aligned} \quad (2.25)$$

Here, we have taken advantage of the fact that simultaneous permutations $\mathcal{V}_1 \leftrightarrow \mathcal{V}_2$ and $\mathcal{V}_3 \leftrightarrow \mathcal{V}_4$ leave the symbol $\varepsilon_{\mathcal{V}_1 \mathcal{V}_2 \mathcal{V}_3 \mathcal{V}_4 \mathcal{V}_5}$ invariant. From (2.25), it is clear that the

quantity $\mathcal{A}$ is Hermitian, and, therefore, the contribution to the action (2.4) is an anti-Hermitian quantity in the Minkowski signature.

The fact that the contributions to the action (2.5) and (2.8) become anti-Hermitian in the Minkowski signature is already evident from the equality $\text{tr } \gamma^5 \gamma_a \gamma_b \gamma_c \gamma_d = 4i\epsilon_{abcd}$.

We see that the full action (2.3) becomes anti-Hermitian in the Minkowski signature:

$$e^{\mathfrak{A}'} = e^{i\mathfrak{A}}, \quad \mathfrak{A}^\dagger = \mathfrak{A}. \quad (2.26)$$

It should be noted that the Hermitian property of the expression $\mathfrak{A}$ in (2.26) is formal, reflecting only the formal application of the operator $\dagger$ to all terms in the action $\mathfrak{A}$. To actually determine whether an operator is Hermitian, it is necessary to have a definition of the scalar product in the vector space where the operator acts. This problem is solved further.

The transition to the long-wave limit is possible for such field configurations that change slowly enough during transitions from simplex to simplex, i.e., during small or significant displacements along the lattice. In our theory, it is at the stage of transition to the long-wave limit that the need arises to introduce local coordinates. Local coordinates x^μ ($\mu = 0, 1, 2, 3$) are markers of the lattice vertices. Without going into details, we write out the long-wave limit of action (2.3) in the Minkowski signature [10–17]:

$$\begin{aligned} \mathfrak{A}'_g &\longrightarrow i\mathfrak{A}_g, \\ \mathfrak{A}_g &= -\frac{1}{4l_P^2} \epsilon_{abcd} \int \mathfrak{R}^{ab} \wedge e^c \wedge e^d, \\ \frac{1}{2} \sigma_{ab} \mathfrak{R}^{ab} &= \frac{1}{2} \sigma_{ab} \mathfrak{R}_{\mu\nu}^{ab} dx^\mu \wedge dx^\nu \\ &= (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu]) dx^\mu \wedge dx^\nu, \end{aligned} \quad (2.27)$$

$$\begin{aligned} \mathfrak{A}'_\Psi &\longrightarrow i\mathfrak{A}_\Psi, \\ \mathfrak{A}_\Psi &= \frac{1}{6} \epsilon_{abcd} \int \Theta^a \wedge e^b \wedge e^c \wedge e^d, \\ \Theta^a &= \frac{i}{2} [\bar{\Psi} \gamma^a \mathcal{D}_\mu \Psi - (\overline{\mathcal{D}_\mu \Psi}) \gamma^a \Psi] dx^\mu, \\ \mathcal{D}_\mu &= (\partial_\mu + \omega_\mu), \end{aligned} \quad (2.28)$$

$$\begin{aligned} \mathfrak{A}'_{\Lambda_0} &\longrightarrow i\mathfrak{A}_{\Lambda_0}, \\ \mathfrak{A}_{\Lambda_0} &= -2\Lambda_0 \int e^0 \wedge e^1 \wedge e^2 \wedge e^3. \end{aligned} \quad (2.29)$$

Here, 1-forms

$$\omega_\mu^{ab} dx^\mu, \quad e_\mu^a dx^\mu \quad (2.30)$$

are the long-wave limits of the lattice variables (2.16).

All other terms in such a transition will contain additional factor $\xi \sim (l_P/\lambda) \longrightarrow 0$ to the positive power, and,

therefore, they are omitted (λ is the characteristic wavelength of the physical subsystem).

The action (2.27)–(2.29) is the Einstein-Cartan-Palatini action minimally coupled to the Dirac field.

III. TIME EVOLUTION OF STATES IN THE LATTICE THEORY OF GRAVITY

Next, we study lattice theory in Minkowski signature.

Let Σ denote the 3D simplicial complex on which the wave function (WF) of lattice gravity is defined. The WF is a functional of bosonic variables (2.1) defined on 1-simplices and fermionic variables (2.2) defined at the vertices, and it is invariant under gauge transformations (2.10), but with $S_V \in \text{Spin}(3, 1)$. It is assumed that the 3D complex Σ models a spatial lattice. Therefore, bosonic variables (2.1) defined on the complex Σ are dynamic variables.

Note that, in the long-wave limit, the 3D complex Σ transforms into a spatially similar curved hypersurface Σ in a 4D manifold $\mathfrak{K}$. Let x^μ be local coordinates in $\mathfrak{K}$. We can assume that the hypersurface Σ is given by the equation $x^0 = 0$. Then, the coordinates x^i , $i = 1, 2, 3$, are local coordinates on the hypersurface Σ . It is known that the set of bosonic dynamic variables of the system (2.27)–(2.29) is determined by the restriction of 1-forms (2.30) on Σ . Therefore, the bosonic dynamical variables are the fields ω_i^{ab} and e_i^a , and the fields ω_0^{ab} and e_0^a play the role of Lagrange multipliers for the constraints imposed on the fields ω_i^{ab} and e_i^a . In the functional integral for the transition amplitude, the effective action and the functional measure are very complicated [23]: The effective action contains, in addition to the original contribution (2.27)–(2.29), also contributions fixing all gauges and the contribution of the additional fermion fields (ghosts). The latter contribution is nonlinear relative to the ghost fields.

From the above, it follows that in lattice theory the WF is defined on physical boson degrees of freedom (2.1), which are analogous to fields ω_i^{ab} and e_i^a in continuum theory.

A. Local time evolution operator: First example

The mechanism of the evolution of the WF in the lattice theory, described below, is radically different from the corresponding mechanism in the continuum theory. These differences boil down to two points.

- (i) In the continuum theory, the functional integral determines the transition amplitude immediately for the entire WF of the system defined on the hypersurface Σ . On the contrary, in lattice theory, there is the possibility of defining a local transition amplitude or transfer matrix that affects the evolution of degrees of freedom only on those lattice simplices that contain one (arbitrary) vertex. This method is implemented here.
- (ii) The second difference is that in the lattice theory there are no local coordinates. This fact radically

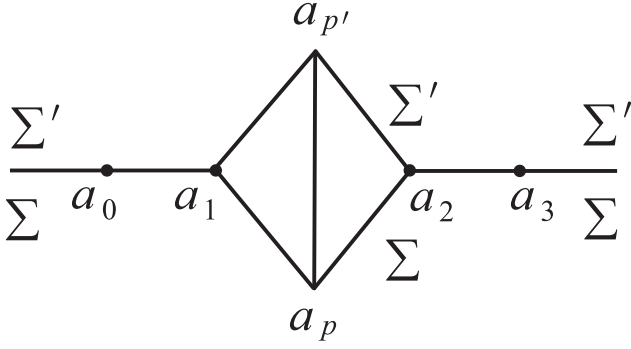


FIG. 1. The 1-complex Σ is the lower broken line, and near the vertex a_p it contains 1-simplices $(a_1 a_p)$ and $(a_p a_2)$. The 1-complex Σ' differs from Σ only in that it contains 1-simplices $(a_1 a_{p'})$ and $(a_{p'} a_2)$ instead of 1-simplices $(a_1 a_p)$ and $(a_p a_2)$. Here, the 2-complex $K_{pp'}^2$ is the union of two 2-simplices: $K_{pp'}^2 = (a_1 a_p a_{p'}) \cup (a_2 a_p a_{p'})$.

simplifies the construction of the integration measure and secondary constraints. The secondary constraints are reduced to fixing the gauge, which is done without changing the original gauge-invariant measure. The effective action remains unchanged; it coincides with the original action in the Minkowski signature.

Let us proceed to constructing the local transfer matrix and proving its unitarity.

Let us begin by describing the 4-complex that defines the local transfer matrix that translates the WF in time locally near the vertex $a_p \in \Sigma$. As a result, the vertex a_p and its nearest simplices advance in time to the point $a_{p'}$, which is outside the complex Σ ahead in time. The 3D complex Σ is transformed into a 3D complex $\Sigma' \supset a_{p'}$, on which a new WF is defined. Let us denote the bounded 4D complex that transforms Σ into Σ' as $K_{pp'}^4$. The described action is schematically depicted in Fig. 1 in the case of 1D complexes Σ and Σ' .

Let an arbitrary vertex $a_p \in \Sigma$, and denote by $K_p^3 \subset \Sigma$ the subcomplex consisting of all 3-simplices $s_{1p}^3, \dots, s_{np}^3$ containing a_p : $a_p \in s_{ip}^3, i = 1, \dots, n$. Thus, $s_{ip}^3 = (a_{i1} a_{i2} a_{i3} a_p)$, where a_{i1}, a_{i2}, a_{i3} , and a_p are the four vertices of the simplex s_{ip}^3 , $K_p^3 = \cup_i^n s_{ip}^3$. Of course, each of the vertices $a_{ip} \in s_{ip}^3$, $\rho = 1, 2, 3$, also belongs to the adjacent simplices $s_{i'p}^3$. The boundary of a subcomplex K_p^3 is as follows: $\partial K_p^3 = \cup_i^n (a_{i1} a_{i2} a_{i3})$.

Let the 0-simplex $a_{p'}$ not belong to the complex Σ : $a_{p'} \notin \Sigma$. Let us consider the set of 3-simplices $s_{ip'}^3 = (a_{i1} a_{i2} a_{i3} a_{p'})$, as well as the 3-complex $K_{p'}^3 = \cup_i^n s_{ip'}^3$. We have

$$\partial K_{p'}^3 = \partial K_p^3 = S^2. \quad (3.1)$$

Here, S^2 is a two-dimensional sphere.

Let us introduce into consideration 4-simplices

$$s_{ip'p'}^4 = (a_{i1} a_{i2} a_{i3} a_p a_{p'}), \quad i = 1, \dots, n, \quad (3.2)$$

and the union of these 4-simplices:

$$K_{pp'}^4 = \cup_i^n s_{ip'p'}^4, \quad \partial K_{pp'}^4 = K_p^3 \cup K_{p'}^3. \quad (3.3)$$

It was indicated above that equalities $a_{i_1 \rho_1} = a_{i_2 \rho_2}$ hold for some different values of indices $(i_1 \rho_1)$ and $(i_2 \rho_2)$. Therefore, from the set of vertices $\{a_{ip}\}$ we select r different vertices $a_\alpha \in \partial K_p^3$, $\alpha = 1, \dots, r$, that exhaust this set.

Let us denote the part of the action defined on $K_{pp'}^4$ as $\mathfrak{A}_{pp'}$. To translate a WF defined on a 3D complex Σ from a vertex a_p to a vertex $a_{p'}$ in the immediate vicinity of these vertices, the following operations must be performed: (i) The WF must be multiplied by the factor $\exp(i\mathfrak{A}_{pp'})$; (ii) it is necessary to integrate over all variables defined at the vertex a_p and 1-simplices $(a_p a_{p'})$ and $(a_\alpha a_p)$, $\alpha = 1, \dots, r$. As a result, a WF is generated, defined on the 3D complex Σ' . Formally, the complex Σ' differs from the complex Σ by replacing vertex a_p with vertex $a_{p'}$ everywhere. This means that all simplices $(a_{i1} a_{i2} a_{i3} a_p) \subset \Sigma$ are replaced by simplices $(a_{i1} a_{i2} a_{i3} a_{p'}) \subset \Sigma'$. Since there is a one-to-one correspondence between all simplices of the complexes Σ and Σ' , as a result of the described quantum evolution the number of degrees of freedom of the system is conserved. It seems to us that this is a very important property of the theory, and only such a theory can be unitary. The last statement is a hypothesis.

B. Gauge fixing

The gauge group $\text{Spin}(3,1)$ is noncompact, and the described integral for the time evolution of the WF admits a finite number of gauge transformations that preserve both the measure and the integrand. Therefore, it is necessary to fix the gauge in the Minkowski signature. To solve this problem, some mathematical results concerning the homology theory of 1D simplicial complexes are needed. These results and the necessary terminology are given in the Appendix.

Let us consider a subcomplex of some abstract complex consisting of all its 1-simplices and 0-simplices and call this 1D subcomplex the one-dimensional skeleton or graph of the complex. Let us choose a basis in the one-dimensional homology group of this graph or, more precisely, one representative for each basis element. Thus, the cycles $c_1, \dots, c_C$ can be considered as a basis of the one-dimensional homology group of the graph. Each of these cycles is a closed broken contour consisting of 1-simplices connected in series. Of course, the set of cycles $c_1, \dots, c_C$ representing the basis of a one-dimensional homology group is not unique.

Statement 1. To fix the gauge, it is necessary and sufficient to have on each basic element $\mathfrak{c}$ of the graph at least one independent variable $\Omega_{\nu_1\nu_2}$ related to a certain 1-simplex $a_1a_2 \subset \mathfrak{c}$. The system contains the same number of independent variables Ω as the dimension of the one-dimensional homology group of the graph. All other variables Ω should be fixed (for example, are set equal to one).

Note that the choice of 1-simplices $(a_1a_2) \subset \mathfrak{c}$ on which the independent variables $\Omega_{\nu_1\nu_2}$ are defined is also ambiguous.

Bringing the set of variables $\{\Omega\}$ to the form specified in Statement 1 is achieved by means of gauge transformations. Suppose that the graph contains 1-simplices that are not contained in any cycle. We will call the set of such 1-simplices a tree and denote it by $\mathfrak{t}$. Obviously, moving sequentially along the tree (or its parts in the case of a disconnected tree), one can convert all elements Ω into units using gauge transformations. Similarly, moving along each cycle, using gauge transformations, we can convert all elements Ω except one on each cycle to units. Thus, Statement 1 is proved.

Let n_0 and n_1 denote the numbers of vertices and 1-simplices of the graph, respectively. Equation (A9) provides the following formula for the number of cycles of the graph:

$$C = n_1 - n_0 + 1. \quad (3.4)$$

Let us calculate the number of cycles of the graph of the complex $K_{pp'}^4$ (3.3). We have the number of vertices of this graph or complex:

$$n_0 = r + 2. \quad (3.5)$$

Indeed, all vertices of the complex $K_{pp'}^4$ are exhausted by the set of vertices $\{a_{ip}\}$ plus two vertices a_p and $a_{p'}$. The set of vertices $\{a_{ip}\}$ contains r independent vertices. The number of 1-simplices of the complex $K_{pp'}^4$ is equal to

$$n_{1K^4} = 5(r - 1). \quad (3.6)$$

This number consists of $3(r - 2)$ 1-simplices on $\partial K_p^3 = S^2$ (3.1), plus $2r$ simplices $(a_\alpha a_p)$ and $(a_\alpha a_{p'})$, $\alpha = 1, \dots, r$, plus the simplex $(a_p a_{p'})$. The number of 1-simplices on ∂K_p^3 is calculated as follows. Let $\tilde{n}_0 = r$, $\tilde{n}_1$, and $\tilde{n}_2$ be the numbers of 0-, 1-, and 2-simplices on S^2 , respectively. The Euler characteristic for S^2 gives the relation $\chi = 2 = \tilde{n}_0 - \tilde{n}_1 + \tilde{n}_2 = r - \tilde{n}_1 + \tilde{n}_2$. On the other hand, on S^2 there is the relation $2\tilde{n}_1 = 3\tilde{n}_2$. Eliminating $\tilde{n}_2$ from the last two equalities, we arrive at the equality of 1-simplices on S^2 : $\tilde{n}_1 = 3(r - 2)$. Using equalities (3.4)–(3.6), we find the number of cycles on the graph of complex $K_{pp'}^4$:

$$C_{K^4} = 4r - 6. \quad (3.7)$$

The number of 1-simplices of the complex $K_{pp'}^4$ on which the variables $\Omega = 1$, according to (3.6) and (3.7), is equal to

$$n_{\Omega=1} = n_{1K^4} - C_{K^4} = r + 1. \quad (3.8)$$

Taking advantage of the freedom to choose a detailed gauge fixation, we choose the 1-simplices $(a_\alpha a_p)$ and $(a_p a_{p'})$ as the $(r + 1)$ simplices on which $\Omega = 1$.

Let us denote the WF on the 3D complex Σ (Σ') as Φ_Σ ($\Phi_{\Sigma'}$). As stated above, they are related by the following equation:

$$\begin{aligned} \Phi_{\Sigma'} &= \int_{-\infty}^{+\infty} \prod_{\alpha=1}^r d e_{a_\alpha a_p} d e_{a_p a_{p'}} \int d \bar{\Psi}_{a_p} d \Psi_{a_p} \\ &\times \exp(i \mathfrak{A}_{pp'}) \Phi_\Sigma \equiv \exp(i \mathfrak{A}_{pp'}) |\Phi_\Sigma\rangle. \end{aligned} \quad (3.9)$$

The last identity is simply a designation. Here, the integral over the variables Ω is absent, since on 1-simplices $(a_\alpha a_p)$ and $(a_p a_{p'})$ we have $\Omega = 1$. In the Minkowski signature, the integrand is oscillating, and, therefore, the integrals over tetrads in (3.9) converge. This is why the integration limits over tetrads are extended to infinity [compare with integral in (2.9)].

Statement 2. If the WF Φ_Σ is gauge invariant, then the WF $\Phi_{\Sigma'}$ is also gauge invariant.

Statement 2 is fairly clear: Both the integrand and the measure of integration in (3.9) are gauge invariant. Therefore, the result of integration is also gauge invariant.

IV. UNITARITY ON A LATTICE

A. Unitarity of the local time evolution operator: First example

Any WF defined on the complex Σ is also defined on the complex Σ' , since these two complexes are identical. Therefore, we will not indicate further on which of these complexes the WF is defined.

Let us introduce some notations:

$$\begin{aligned} \langle \Phi_{\lambda_1} | \Phi_{\lambda_2} \rangle &= \langle \Phi_{\lambda_2} | \Phi_{\lambda_1} \rangle^\dagger \\ &\equiv \int_{-\infty}^{+\infty} D e \int D \mu \{ \Omega \}_{\text{Gauge Fix}} \int D \bar{\Psi} D \Psi \cdot \Phi_{\lambda_1}^\dagger \Phi_{\lambda_2}. \end{aligned} \quad (4.1)$$

Here, all integrals are taken over variables defined on the complex Σ (or Σ'), and the gauge is fixed, which is indicated in the notation of the measure of variables Ω . This operation is necessary, since all WFs are gauge invariant. We will need the following integral:

$$\mathfrak{A}_{pp'} |\Phi_\lambda\rangle \equiv \int_{-\infty}^{+\infty} \prod_{\alpha=1}^r d e_{a_\alpha a_p} d e_{a_p a_{p'}} \int d \bar{\Psi}_{a_p} d \Psi_{a_p} \mathfrak{A}_{pp'} \Phi_\lambda. \quad (4.2)$$

It is useful to compare the right-hand sides of Eqs. (3.9) and (4.2).

We will assume that the set of WF $\{|\Phi_\lambda\rangle\}$ is an ONB:

$$\langle\Phi_{\lambda_1}|\Phi_{\lambda_2}\rangle = \delta_{\lambda_1\lambda_2}. \quad (4.3)$$

Here, the nature of the delta function on the right-hand side does not matter.

Furthermore, we consider only the normalizable WF $|\Phi\rangle = \sum_\lambda A_\lambda |\Phi_\lambda\rangle$:

$$\langle\Phi|\Phi\rangle = \sum_\lambda |A_\lambda|^2 < \infty. \quad (4.4)$$

Let us write out the second condition imposed on all normalizable WF:

$$|\langle\Phi_1|\mathfrak{A}_{pp'}|\Phi_2\rangle| < \infty. \quad (4.5)$$

We have

$$\langle\Phi_1|\mathfrak{A}_{pp'}|\Phi_2\rangle^\dagger = \langle\Phi_2|\mathfrak{A}_{p'p}|\Phi_1\rangle. \quad (4.6)$$

Here, as a result of conjugation, the complexes Σ and Σ' change places, which is taken into account in the designation of the operator $\mathfrak{A}_{pp'}^\dagger = \mathfrak{A}_{p'p}$. However, using gauge transformations that do not change the integrals under consideration, the operator $\mathfrak{A}_{p'p}$ can be brought back to its previous form $\mathfrak{A}_{pp'}$, so that

$$\mathfrak{A}_{pp'}^\dagger = \mathfrak{A}_{pp'}. \quad (4.7)$$

Since the operator $\mathfrak{A}_{pp'}$ is Hermitian on the considered space of functions, we will assume the ONB $\{|\Phi_\lambda\rangle\}$ to be a complete set of eigenfunctions of this operator:

$$\mathfrak{A}_{pp'}|\Phi_\lambda\rangle = E_\lambda|\Phi_\lambda\rangle, \quad |E_\lambda| < \infty, \quad \bar{E}_\lambda = E_\lambda. \quad (4.8)$$

Let $|\Phi_1\rangle = \sum_\lambda A_{1\lambda}|\Phi_\lambda\rangle$ and $|\Phi_2\rangle = \sum_\lambda A_{2\lambda}|\Phi_\lambda\rangle$. Then, according to (4.3) and (4.8) we have

$$\langle\Phi_1|\mathfrak{A}_{pp'}|\Phi_2\rangle = \sum_\lambda \bar{A}_{1\lambda} A_{2\lambda} E_\lambda, \quad (4.9)$$

$$\exp(i\mathfrak{A}_{pp'}) \sum_\lambda A_\lambda |\Phi_\lambda\rangle = \sum_\lambda A_\lambda e^{iE_\lambda} |\Phi_\lambda\rangle. \quad (4.10)$$

From the last equality, we see that as a result of the described time evolution (3.9) the coefficients A_λ of the expansion of the WF in terms of the ONB $\{|\Phi_\lambda\rangle\}$ acquire phases: $A_\lambda \rightarrow A_\lambda e^{iE_\lambda}$. From this, in turn, it follows that the time evolution preserves both the norm (4.4) of the vectors and the values (4.5) and (4.9).

The above means the truth of the following statement.

Statement 3. The operation of time evolution (3.9) is unitary, and it preserves the class of functions on which unitarity is proven, as well as the number of degrees of freedom of the system.

B. Further examples

1. Second example

Let us consider the second example of local transformation of the WF in time. As in the first example, in this case a certain vertex $a_p \in \Sigma$ transforms into a vertex $a_{p'}$ that does not belong to the complex Σ . Here, the 4D complex transforming the complex Σ into the complex Σ' is denoted by $K_{2pp'}^4$. The rest of the designations remain the same. We believe that in this example $n = r = 4$. Four 3-simplices $(a_1 a_2 a_3 a_p), (a_2 a_3 a_4 a_p), (a_1 a_3 a_4 a_p), (a_1 a_2 a_4 a_p) \subset \Sigma$, and $(a_1 a_2 a_3 a_{p'}), (a_2 a_3 a_4 a_{p'}), (a_1 a_3 a_4 a_{p'}), (a_1 a_2 a_4 a_{p'}) \subset \Sigma'$. By definition $K_{2pp'}^4 = s_p^4 \cup s_{p'}^4$, where 4-simplices $s_p^4 = a_1 a_2 a_3 a_4 a_p$ and $s_{p'}^4 = a_1 a_2 a_3 a_4 a_{p'}$.

The graph of the complex $K_{2pp'}^4$ contains $n_1 = 14$ 1-simplices and $n_0 = 6$ vertices. Therefore, according to Eq. (A9), this graph has nine independent cycles, and there remain five 1-simplices on which the variables Ω must be fixed. The corresponding connected tree subgraph containing all the original vertices of the graph may contain, for example, the following 1-simplices: $a_p a_1, a_p a_2, a_p a_3, a_3 a_{p'}$, and $a_{p'} a_4$.

The proof of the unitarity of the time evolution operator constructed here repeats verbatim the analogous proof given in the first example. The number of degrees of freedom is also conserved in the second example.

2. Third example

Let us consider a 4D complex $K_{3pp'}^4$ with the following properties: $\partial K_{3pp'}^4 = K_{3p}^3 \cup K_{3p'}^3$, $K_{3p}^3 \subset \Sigma$, $K_{3p'}^3 \subset \Sigma'$, and the vertex a_p ($a_{p'}$) is internal to the complex K_{3p}^3 ($K_{3p'}^3$). Moreover, the complex $K_{3pp'}^4$ is symmetric with respect to the inversion (reflection) operation $\hat{P}$ such that $\hat{P}K_{3p}^3 = K_{3p'}^3$, $\hat{P}K_{3p'}^3 = K_{3p}^3$, $\hat{P}a_p = a_{p'}$, $\hat{P}a_{p'} = a_p$, $\hat{P}K_{3pp'}^4 = K_{3pp'}^4$, and $\hat{P}^2 = 1$. The $K_{3pp'}^4$ and $K_{2pp'}^4$ complexes from the first two examples have these properties.

The evolution operator constructed on the complex $K_{3pp'}^4$ is also unitary, and the number of degrees of freedom of the system is reproduced.

3. Fourth example

Let, as above, Σ and Σ' be isomorphic 3D complexes. Let us consider a 4D complex $\mathfrak{K}$ with the following properties: (i) $\partial\mathfrak{K} = \Sigma \cup \Sigma'$ and (ii) in a finite or infinite number of steps on the complex $\mathfrak{K}$ such as those described in

examples one, two, and three, the complex Σ is transformed into a complex Σ' . Since at each step the evolution of the wave function is unitary, and the evolution also conserves the number of degrees of freedom, then these same properties take place during the complete evolution from complex Σ to complex Σ' . Of course, for this to be possible, the conditions (4.5) must be met at each vertex of the complex Σ . Note also that the result of the described evolution is determined uniquely up to a gauge transformation. The validity of the last statement follows from the fact that the value of the integral does not depend, generally speaking, on the order of integration. But the methods of fixing the gauge, although reflected in the final result, are not significant: The final WF is gauge invariant.

V. DISCUSSION

Let us ask ourselves the first question that arises when studying the problem of unitarity: Will the theory be unitary when the number of degrees of freedom changes in the process of evolution? Let us try to present some (sketchy and shaky) arguments in favor of the existing unitarity of such evolution. Here, the argumentation will be fundamentally different from the one given above.

By definition, we will consider a linear operator $\mathcal{U}$, acting in a complex linear space with a scalar product, to be unitary if, as a result of its action on any vector, the scalar product of the resulting vector does not change:

$$|B\rangle = \mathcal{U}|A\rangle, \quad \langle B|B\rangle = \langle A|A\rangle. \quad (5.1)$$

Here, we admit that the vector $|B\rangle$ belongs to a space with a changed dimension. This is an expansive interpretation of the concept of unitarity, but it is precisely this interpretation that is needed here.

In our case, we denote the set of bosonic and fermionic variables on 3D complexes Σ and Σ' as V and V' , respectively, and the transfer matrix transforming the WF $\Phi_A\{V\}$ on Σ into WF $\Phi_B\{V'\}$ on Σ' as $\exp i\mathfrak{A}\{V', V\}$. The transfer matrix is defined on one 4D lattice layer. Then,

$$\Phi_B\{V'\} = \int_V \exp(i\mathfrak{A}\{V', V\}) \Phi_A\{V\} \quad (5.2)$$

and

$$\begin{aligned} \langle B|B\rangle &= \int_{V_1} \int_{V_2} \int_{V'} \exp(i\mathfrak{A}\{V', V_1\} - \mathfrak{A}\{V', V_2\}) \\ &\times \Phi_A^\dagger\{V_2\} \Phi_A\{V_1\}. \end{aligned} \quad (5.3)$$

Since the lattice actions are Hermitian, the inner integral over the variables V' is an alternating integral for $V_1 \neq V_2$. The simplest model of an alternating integral is the integral

$$q = \left| \int_0^a e^{i\lambda x} dx \right| = \left| \frac{2}{\lambda} \sin \frac{\lambda a}{2} \right| < a. \quad (5.4)$$

Only for $\lambda = 0$ do we have $q/a = 1$. From this, we can conclude that the inner integral (5.3) saturates at $V_1 = V_2$. In other words, as a result of integration over variables V' , as the number of these integrations tends to infinity, the integral tends to the integral

$$\begin{aligned} \langle B|B\rangle &= \int_{V_1} \int_{V_2} \Phi_A^\dagger\{V_1\} \Phi_A\{V_1\} \mathcal{F}\{V_1\} \delta_{V_1 V_2}, \\ \delta_{V_1 V_2} &= 0 \quad \text{for } V_1 \neq V_2, \quad \int_{V_2} \delta_{V_1 V_2} = 1. \end{aligned} \quad (5.5)$$

Indeed, let us denote by $q_{\mathcal{W}}$ the integral over the variables V' of one 4-simplex $s_{\mathcal{W}}^4$ at $V_1 \neq V_2$. According to (5.4), we have $q_{\mathcal{W}}/(\text{measure volume}) < 1$. Therefore,

$$\prod_{\mathcal{W}} \left(\frac{q_{\mathcal{W}}}{\text{measure volume}_{\mathcal{W}}} \right) \rightarrow 0 \quad (5.6)$$

for an increasing number of integrations, that is, for an infinite transfer matrix in space. If we could prove that the functional

$$\mathcal{F}\{V_1\} = \text{const}, \quad (5.7)$$

then the unitarity of the lattice theory would be proven for the case with a nonfixed number of degrees of freedom. Unfortunately, we do not know how to prove Eq. (5.7) at present. Moreover, it is not clear to us whether this problem is correct.

The fermionic part of the actions in (5.3) would not cause any significant difficulties in proving unitarity. The problem is in proving the equality (5.7).

Let us briefly dwell on the problem of convergence of integrals when integrating over a noncompact subgroup of the gauge group $\text{Spin}(3,1)$. We will denote the elements from this subgroup as $\mathcal{L}$:

$$\begin{aligned} \mathcal{L} &= e^{\vec{\sigma} \vec{\chi}} = \cosh \chi + \sinh \chi (\vec{\sigma} \vec{n}), \\ \vec{\chi} &= \chi \vec{n}, \quad \vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \phi), \\ 0 &\leq \chi < \infty, \end{aligned} \quad (5.8)$$

and $\vec{\sigma}$ are three Pauli matrices. The gauge invariant measure has the form

$$\begin{aligned} d\mu(\mathcal{L}) &= \frac{1}{12} \text{tr} \left\{ (\mathcal{L}^{-1} d\mathcal{L}) \wedge (\mathcal{L}^{-1} d\mathcal{L}) \wedge (\mathcal{L}^{-1} d\mathcal{L}) \right\} \\ &= \sinh^2 \chi \sin \theta d\chi \wedge d\theta \wedge d\phi. \end{aligned} \quad (5.9)$$

The oscillating exponential in the functional integral depends linearly on the value $\mathcal{L}$ in (5.8). Integrating over

the angles θ and ϕ eliminates the factor $(\sinh \chi \sin \theta)$ from the measure (5.9). Thus, the effective measure of integration over the variable χ becomes

$$d\mu_{\text{eff}} = \sinh \chi d\chi = d \cosh \chi. \quad (5.10)$$

Since the exponential contains a term of the form $(i\lambda \cosh \chi)$, the integral over variable χ turns out to be similar to the integral $\int dx \exp(i\lambda x)$. The result of such integration leads to generalized functions that must be integrated during further integration.

The question arises about numerical modeling of the theory under study. It seems to us that the Monte Carlo method could be adequate here.

VI. CONCLUSION

The main conclusion of the work: If space-time has the property of granularity, then it is possible to construct a quantum theory that models this granularity and has the property of unitarity.

The obtained result means that the lattice theory of gravity under consideration has the right to be considered a discrete regularization of the generally accepted continuous physical theory of gravity.

It should be noted that the lattice local transfer matrix constructed here has an analog in the continuum theory of gravity. In the latter case, the Hamiltonian is an infinite linear combination of first-order constraints: $\mathcal{H}_T = \int c^\Lambda(x) \hat{\phi}_\Lambda(x) d^3x$. Here, $\hat{\phi}_\Lambda(x)$ are local first-class constraint operators, which are analogous to local transfer matrices in lattice theory. Physical states satisfy the conditions $\hat{\phi}_\Lambda(x)|\Phi\rangle = 0$ (see [25], Lecture 3). The difference between the lattice and the continuum theory is that our consideration is carried out in the Schrödinger representation, while in the lectures [25] in the Heisenberg representation.

In conclusion, we point out a modified model of discrete gravity, for which all the conclusions of this work remain valid. The modified model differs from the one considered only in that in the modified model the bosonic variables $e_{\nu_1\nu_2}^a$ are everywhere replaced by bilinear forms with respect to Dirac fields [see (2.21)]:

$$e_{\nu_1\nu_2}^a \longrightarrow \Theta_{\nu_1\nu_2}^a. \quad (6.1)$$

This is possible because under the action of all symmetry transformations, both continuous and discrete, these quantities are transformed identically.

Note that the idea of representing a tetrad as a bilinear form of Dirac fields (quark-antiquark fields) has long roots (see, for example, Refs. [18–20, 26–29]). In addition to the above-mentioned common property that unites the cited works, there are also significant differences between them. In particular, in Refs. [26–28], continuum theories are

studied and fundamental Lagrangians describing pregeometry are determined. The achievement of this approach is that it leads to Einstein's theory of gravity in the long-wave limit. In Refs. [18–20], as in the present paper, the lattice regularization of the continuum theory of gravity is studied, and the lattice models coincide. In Ref. [29], the ideas of Refs. [18–20] are developed but within the framework of the continuum theory.

The replacement according to (6.1) simplifies the integration in the transition amplitude of type (3.9), since integrals over fermions, unlike integrals over bosons, always converge by definition. This approach was used in Refs. [18, 20]. However, the problem of fixing the gauge in the Minkowski metric, as well as the problem of integration over the divergent volume of the gauge group $\text{Spin}(3,1)$, still remains in the indicated modification of the theory.

ACKNOWLEDGMENTS

We are grateful to one of our referees, who insisted on the need to study the unitarity problem directly in lattice theory. We are grateful to A. Ya. Maltsev for significant assistance in solving mathematical issues, especially in the field of combinatorial topology. This work was carried out as part of the State Program FFWR-2024-0011.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: SOME FORMULAS FROM THE BASICS OF COMBINATORIAL TOPOLOGY

Let us consider a one-dimensional abstract connected simplicial complex, which we will call a graph and denote by $\mathfrak{R}^1$. On each oriented 1-simplex $a_{\nu_1}a_{\nu_2}$ an element of the group $\Omega_{\nu_1\nu_2} = \Omega_{\nu_2\nu_1}^{-1} \in \text{Spin}(3,1)$ is defined up to gauge transformations of the form (2.10) and $S_\nu \in \text{Spin}(3,1)$. By definition, any two configurations of quantities $\{\Omega_{\nu_1\nu_2}\}_1$ and $\{\Omega_{\nu_1\nu_2}\}_2$ are equivalent if they are transformed into each other by gauge transformations alone. Next, the set of equivalent configurations of quantities $\{\Omega_{\nu_1\nu_2}\}$ is called an equivalence class. Here, the question of the number of equivalence classes on the graph is decided. The solution to this problem is necessary to fixing the gauge.

Let us call a closed contour

$$(a_{\nu_1}a_{\nu_2})(a_{\nu_2}a_{\nu_3})\dots(a_{\nu_i}a_{\nu_1}) \quad (\text{A1})$$

without self-intersections a cycle. Here, 1-simplices $a_{\nu_i}a_{\nu_{i+1}} \in \mathfrak{R}^1$.

A graph is called tree if it does not contain cycles.

Let us show that on a tree graph the number of equivalence classes is equal to zero. Indeed, let us make

the following gauge transformation: On the 1-simplex $a_{\nu_1}a_{\nu_2}$, we take $S_{\nu_1} = 1$ and $S_{\nu_2} = \Omega_{\nu_1\nu_2}$. As a result of this gauge transformation, according to (2.10) we get $\tilde{\Omega}_{\nu_1\nu_2} = 1$. In the next step on the 1-simplex $a_{\nu_2}a_{\nu_3}$, putting $S_{\nu_2} = 1$ and $S_{\nu_3} = \Omega_{\nu_2\nu_3}$, we get $\tilde{\Omega}_{\nu_2\nu_3} = 1$. Similarly, we can move in all directions from the initial 1-simplex $a_{\nu_1}a_{\nu_2}$. Since there are no cycles, there is no obstacle to achieving the result $\Omega = 1$ on all 1-simplices. This means that on the tree graph there are no real degrees of freedom of the gauge field.

If the graph contains a cycle (A1), then the described procedure is impossible. Indeed, as a result of any gauge transformations, the value

$$\text{tr } \Omega_{\nu_1\nu_2}\Omega_{\nu_2\nu_3}\dots\Omega_{\nu_i\nu_1} \quad (\text{A2})$$

is conserved. Thus, the problem of fixing the gauge is reduced to calculating the number of independent cycles of the graph. The definition of the set of independent cycles is given below. Although this set is not uniquely chosen, the number of independent cycles is a topological invariant.

Before we proceed to a theoretical consideration, let us demonstrate what has been said using a simple example. Consider a graph consisting of six tetrahedron edges a_1a_2 , a_2a_3 , a_3a_1 , a_1a_4 , a_2a_4 , and a_3a_4 . Using the method described above, we can, for example, make $\Omega_{12} = \Omega_{23} = \Omega_{34} = 1$. With the specified gauge fixing, three quantities Ω_{13} , Ω_{14} , and Ω_{24} remain independent variables. The stated choice of gauge fixing is ambiguous. In this example, the number of elementary cycles is four:

$$\begin{aligned} (a_1a_2)(a_2a_3)(a_3a_1), & \quad (a_1a_2)(a_2a_4)(a_4a_1), \\ (a_2a_3)(a_3a_4)(a_4a_2), & \quad (a_3a_1)(a_1a_4)(a_4a_3). \end{aligned} \quad (\text{A3})$$

But the number of independent cycles is three.

Let us move on to a general solution to the problem. For a more in-depth study of the problem, we recommend to the reader Refs. [24,30].

In general, the number of independent degrees of freedom $\{\Omega\}$ is equal to the dimension (rank) of the one-dimensional homology group H_1 of the graph. By definition, this same number is the number of independent cycles on the graph. Let a base consisting of C elements be selected in the group H_1 . Let us select one cycle c in the graph representing each basis element in the group H_1 . Thus, the cycles $c_1, \dots, c_C$ can be considered as a basis of the one-dimensional homology group of the graph. Generally speaking, the choice of base cycles is not straightforward. Each of these loops contains one independent variable Ω , which is defined on some 1-simplex belonging to this loop.

Let us define a group H_1 . Let V_0 denote the linear space of zero-dimensional chains. A zero-dimensional chain is a formal linear combination

$$\sum_{\nu} x^{\nu} a_{\nu}. \quad (\text{A4})$$

Here, x^{ν} are real numbers, and a_{ν} are graph vertices. Chains (A4) can be multiplied by real numbers and added term by term. The linear space of 1-chains V_1 is defined similarly:

$$\sum_{\text{1-simplices}} x^{\nu_1\nu_2} (a_{\nu_1}a_{\nu_2}). \quad (\text{A5})$$

Here, the coefficients of 1-simplices $a_{\nu_1}a_{\nu_2}$ are real numbers. The boundary operator on chains $d: V_1 \rightarrow V_0, V_0 \rightarrow 0$ is linear and acts according to the rule

$$d_1(a_{\nu_1}a_{\nu_2}) = a_{\nu_2} - a_{\nu_1}. \quad (\text{A6})$$

The homology group H_1 is the kernel of the operator d_1 :

$$H_1 = \text{Ker } d_1, \quad \text{rank } H_1 = \dim \text{Ker } d_1, \quad (\text{A7})$$

“rank” means the dimension of linear spaces H_1 or H_0 . We have for the Euler characteristic of the graph [24,30]

$$\chi = n_0 - n_1 = \text{rank } H_0 - \text{rank } H_1. \quad (\text{A8})$$

Since by definition the number of independent cycles $C = \text{rank } H_1$ and for a connected graph we have $\text{rank } H_0 = 1$, then from here we get

$$C = n_1 - n_0 + 1. \quad (\text{A9})$$

To fix the gauge on the graph, you can do the following. We will sequentially throw out (mentally) one 1-simplex from each independent cycle. After each such act, the number of independent cycles of the original graph decreases by one. As a result of C steps, the original graph becomes treelike. On the remaining 1-simplices of the tree subgraph, we fix the variables Ω , setting them equal to one or giving them other fixed values. On all mentally thrown out 1-simplices, the variables Ω are independent, and they take any values. Thus, the gauge is fixed. The gauge fixation procedure is not unique.

Let us consider the first example from this point of view. Let us (mentally) throw out from the complex $K_{pp'}^4$ all 1-simplices, with the exception of the 1-simplices $(a_{\alpha}a_p)$ and $(a_p a_{p'})$. The remaining graph becomes a connected tree graph with the number of 1-simplices $n_1 = (r + 1)$ and the original number of vertices $n_0 = (r + 2)$. According to Eq. (A9), this graph has no cycles: $C = 0$.

Let us go back to the example (A3). Since the tetrahedron graph has $n_1 = 6$ and $n_0 = 4$, then according to Eq. (A9) the number of independent cycles is three.

- [1] T. Regge, *Nuovo Cimento* (1955–1965) **19**, 558 (1961).
- [2] J. Cheeger, W. Muller, and R. Schrader, *Comm. Math. Phys.* **92**, 405 (1984).
- [3] U. Pinkall and A. Bobenko, *J. Reine Angew. Math.* **1996**, 187 (1996).
- [4] T. Regge and R. M. Williams, *J. Math. Phys. (N.Y.)* **41**, 3964 (2000).
- [5] J. C. Baez, *Lect. Notes Phys.* **543**, 25 (2000); *38th Internationale Universitatswochen fuer Kernphysik und Teilchenphysik (International University School of Nuclear and Particle Physics): Geometry and Quantum Physics (IUKT)*, [arXiv:gr-qc/9905087](#).
- [6] M. P. Reisenberger, [arXiv:gr-qc/9711052](#).
- [7] J. Iwasaki, [arXiv:gr-qc/9903112](#).
- [8] F. David, *Nucl. Phys.* **B257**, 543 (1985).
- [9] D. Boulatov, V. Kazakov, I. Kostov, and A. A. Migdal, *Nucl. Phys.* **B275**, 641 (1986).
- [10] S. Vergeles, *JETP Lett.* **120**, 461 (2024).
- [11] S. Vergeles, *Nucl. Phys.* **B735**, 172 (2006).
- [12] S. Vergeles, *Phys. Rev. D* **92**, 025053 (2015).
- [13] S. Vergeles, *J. High Energy Phys.* 02 2017 044.
- [14] S. Vergeles, *Phys. Rev. D* **96**, 054512 (2017).
- [15] S. Vergeles, *Classical Quantum Gravity* **38**, 085022 (2021).
- [16] S. Vergeles, *Classical Quantum Gravity* **39**, 038001 (2021).
- [17] S. N. Vergeles, *J. Exp. Theor. Phys.* **166**, 721 (2024).
- [18] D. Diakonov, [arXiv:1109.0091](#).
- [19] A. A. Vladimirov and D. Diakonov, *Phys. Rev. D* **86**, 104019 (2012).
- [20] A. A. Vladimirov and D. Diakonov, *Phys. Part. Nucl.* **45**, 800 (2014).
- [21] P. A. M. Dirac, *Can. J. Math.* **2**, 129 (1950).
- [22] L. D. Faddeev and V. N. Popov, *Sov. Phys. Usp.* **16**, 777 (1974).
- [23] E. S. Fradkin and G. Vilkovisky, *Phys. Lett.* **55B**, 224 (1975).
- [24] L. Pontryagin, *Basics of Combinatorial Topology* (Science, Moscow, 1976).
- [25] P. A. Dirac, *Lectures on Quantum Mechanics* (Courier Corporation, New York, 2013).
- [26] K. Akama, *Prog. Theor. Phys.* **60**, 1900 (1978).
- [27] C. Wetterich, *Phys. Rev. D* **70**, 105004 (2004).
- [28] C. Wetterich, *J. High Energy Phys.* 06 2022 069.
- [29] Y. N. Obukhov and F. W. Hehl, *Phys. Lett. B* **713**, 321 (2012).
- [30] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern Geometry—Methods and Applications. Part III. Introduction to Homology Theory* (Springer-Verlag New York Inc., 1990).