Vacuum polarization corrections to the $\mu^+\mu^-$ spectrum

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(Received 16 May 2025; accepted 23 June 2025; published 28 July 2025)

The energy levels of the $n = 1$ and $n = 2$ bound states of the $\mu^+\mu^-$ atom (true muonium) are calculated starting from a previously derived potential that correctly describes positronium to order α^5 supplemented by the known α^6 contributions. All electron vacuum polarization corrections on the true muonium levels are computed to order α^6. The second-order perturbative contributions from all of the α^4 and α^5 terms in the potential are also included. Additional order α^6 contributions from the two-photon and three-photon annihilation processes are included in the evaluation of the ground state ($1^3S_1 - 1^1S_0$) hyperfine splitting.

DOI: 10.1103/yqyn-v8pn

I. INTRODUCTION

Since the discussion of the $\mu^+\mu^-$ bound system (true muonium) by Hughes and Maglic [1] in 1971 and a calculation of the ground state hyperfine splitting due to electron vacuum polarization shortly thereafter [2], the topic remained relatively dormant until the late 1990s [3]. Recently, with the increasing prospect of experimental searches for these bound states, there has been a renewed theoretical interest in the $\mu^+\mu^-$ atom [4–8] and numerous discussions about the best method to discover true muonium [9–13]. The calculations that follow amount to another take on the spectrum of the $\mu^+\mu^-$ bound system. They start from a nonrelativistic positronium potential that contains all nonrelativistic and QED one-loop corrections. As usual, this potential is supplemented by numerous electron vacuum polarization contributions to the $\mu^+\mu^-$ spectrum. Brief sketches of the electron vacuum polarization calculations are given below and the individual results are contained in Table I. A more detailed description of these calculations can be found in [14]. The net result may add something to the theoretical discussion of true muonium particularly in the values of the $2P \to 2S$ transitions.

II. POSITRONIUM POTENTIAL

A convenient starting point for calculation of the true muonium energy level spectrum is the perturbative potential that describes the spectrum of positronium to order α^5. The history of the derivation of this potential is summarized in [16]. It consists of a fine structure (α^4) term and a one-loop correction (α^5). The latter is derived in Ref. [17] from the tree and one-loop scattering amplitudes and in Ref. [18] using nonrelativistic QED and includes all QED corrections to order α^5. It can be expressed as a direct potential $V'(\vec{r})$ and an annihilation potential $V''(\vec{r})$ in a form given by

$$V'(\vec{r}) = -\alpha\left[\frac{1}{r} + \frac{\pi}{m^2}\delta(\vec{r}) + \frac{1}{2m^2}p_i\left(\frac{\delta_{ij}}{r} + \hat{x}_i\frac{1}{r}\hat{x}_j\right)p_j - \frac{4\pi}{3m^2}\vec{S}^2\delta(\vec{r}) - \frac{1}{4m^2r^3}S_{12} - \frac{3}{2m^2r^3}\vec{L}\cdot\vec{S}\right]$$
$$+\frac{\alpha^2}{m^2}\left[\left(\frac{26}{15} - \frac{14}{3}\ln(2) + \frac{16}{3}\ln(m/\lambda)\right)\delta(\vec{r}) + \frac{7}{6\pi}\nabla^2\left(\frac{\ln(mr/2)+\gamma}{r}\right) - \frac{2}{3}\vec{S}^2\delta(\vec{r}) + \frac{1}{4\pi r^3}S_{12} + \frac{1}{\pi r^3}\vec{L}\cdot\vec{S}\right] - \frac{(\vec{p}^2)^2}{4m^3} \tag{1}$$

and

$$V''(\vec{r}) = \frac{\alpha\pi}{m^2}\vec{S}^2\delta(\vec{r}) - \frac{\alpha^2}{m^2}\left[\left(\frac{26}{9} + 2\ln(2)\right)\vec{S}^2 + 4(1 - \ln(2))\right]\delta(\vec{r}), \tag{2}$$

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TABLE I. The entries summarize the various corrections (in meV) to the $n = 1$ and $n = 2$ states of true muonium. The pion one-loop contributions are included in the spin and spin annih columns. These columns contain all contributions proportional to $\vec{S}^2$ in the direct or annihilation channels, respectively [15]. The third column contains the contribution from the remaining spin independent terms in the potential.

	E_0	Coulomb	Spin ind	Spin	LS	Tensor	Spin annih	Two-loop	Kinetic energy	E
1^3S_1	$-1.4065\,10^6$	−489.3156	−0.0075	0.5440			0.58320	−0.0008	−0.03172	$-1.40701\,10^6$
1^1S_0	$-1.4065\,10^6$	−489.3156	−0.0075				−0.0009	−0.0011	−0.03172	$-1.40719\,10^6$
2^3S_1	$-3.5165\,10^5$	−58.7263	−0.0095	0.0616			0.0729		−0.00327	$-3.51705\,10^5$
2^1S_0	$-3.5165\,10^5$	−58.7263	−0.0095				−0.0001		−0.00327	$-3.51727\,10^5$
2^3P_2	$-3.5165\,10^5$	−1.4451	−0.0003		0.0005	−0.00001			−0.00003	-3.5165510^5
2^3P_1	$-3.5165\,10^5$	−1.4451	−0.0003		−0.0005	0.0002			−0.00003	$-3.51659\,10^5$
2^3P_0	$-3.5165\,10^5$	−1.4451	−0.0003		−0.0010	−0.0005			−0.00003	$-3.51664\,10^5$
2^1P_1	$-3.5165\,10^5$	−1.4451	−0.0003						−0.00003	$-3.51657\,10^5$

where $\vec{S}$ is the total spin, γ is Euler's constant, $S_{12} = 3\vec{S}\cdot\hat{r}\,\vec{S}\cdot\hat{r} - \vec{S}^2$ and λ is the infrared cutoff that is removed using the Bethe sum. The fine structure term is equivalent to the form in [16] written in a Hermitian form. The α^4 and α^5 contributions to the energy levels from these terms supplemented by the α^6 terms from [16,19] give the energies obtained by replacing the electron mass with the muon mass. The result is found in the E_0 column of Table I. To obtain the vacuum polarization corrections it is essential to have an explicit form for the potential accurate to order α^5.

III. VACUUM POLARIZATION CORRECTIONS

The large electron vacuum polarization correction to the Coulomb interaction can be included by using the dispersion representation for the photon propagator [20–22]

$$D(k^2) = -\frac{1}{k^2} + \int_0^\infty \frac{d\lambda}{\lambda}\frac{\Delta(\lambda)}{\lambda - k^2 - i\varepsilon}, \tag{3}$$

where $\Delta(q^2)$ is

$$\Delta(q^2) = -\frac{(2\pi)^3}{3q^2}\sum_n \delta^{(4)}(q - q_n)\langle 0|j_\mu(0)|n\rangle\langle n|j^\mu(0)|0\rangle. \tag{4}$$

For the $e\bar{e}$ intermediate state, the leading contribution, $\Delta^{(2)}(\lambda)$, is

$$\Delta^{(2)}(\lambda) = \frac{\alpha}{3\pi}(1 + 2m_e^2/\lambda)\sqrt{1 - 4m_e^2/\lambda}\;\theta(\lambda - 4m_e^2). \tag{5}$$

If we take k^2 to be spacelike, then the modified Coulomb interaction is

$$V(\vec{k}^2) = -\frac{e^2}{\vec{k}^2} - e^2\int_{4m_e^2}^\infty \frac{d\lambda}{\lambda}\frac{\Delta^{(2)}(\lambda)}{\vec{k}^2 + \lambda} \tag{6}$$

$$= V_C(\vec{k}^2) + V_{VP}(\vec{k}^2). \tag{7}$$

Transforming to coordinate space,

$$V_{VP}(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3k V_{VP}(\vec{k})e^{i\vec{k}\cdot\vec{r}}, \tag{8}$$

$$V_{VP}(\vec{r}) = -\frac{\alpha}{r}\int_{4m_e^2}^\infty \frac{d\lambda}{\lambda}\Delta^{(2)}(\lambda)e^{-\sqrt{\lambda}r}. \tag{9}$$

In general, the leading e^+e^- vacuum polarization correction to the $1S$, $2S$ and $2P$ states is

$$\begin{aligned}\langle n,l|V_{VP}|n,l\rangle &= \int_0^\infty dr r^2 R_{nl}^2(r)V_{VP}(r)\\ &= -\frac{m_\mu\alpha^3}{6\pi}\left(\frac{n}{2}\right)^2\int_4^\infty \frac{dx}{x}(1 + 2/x)\sqrt{1 - 4/x}\\ &\quad\times\int_0^\infty dy y f(n,l,y)^2 e^{-(n\beta\sqrt{x}/2+1)y},\end{aligned} \tag{10}$$

where $\beta = 2m_e/m_\mu\alpha$, $R_{nl}(r)$ is the radial wave function and

$$f(1,0,y) = 2, \qquad f(2,0,y) = \frac{(2-y)}{2\sqrt{2}},$$
$$\text{and} \quad f(2,1,y) = \frac{y}{2\sqrt{6}}. \tag{11}$$

These factors arise when the integration variable in $\langle n,l|V_{VP}|n,l\rangle$ is changed to $r \to nay/2$. Then, for example,

$$R_{20}(ay) = \frac{1}{a^{3/2}}\frac{(2-y)e^{-y/2}}{2\sqrt{2}}. \tag{12}$$

This separates a common normalization factor and exponential behavior from any radial wave function.

In particular, the ground state contribution is

$$\langle 1,0|V_{VP}|1,0\rangle = -485.143\ \text{meV}. \tag{13}$$

Because the effective coupling $m_\mu\alpha/m_e$ appears in the y integral, this result is of order $m_\mu\alpha^5/m_e$. In the equal mass case, the leading term reproduces the Ühling α^5 effect

contained in the potential as well as higher-order corrections.

There is also a higher-order contribution to $\Delta(\lambda)$, $\Delta^{(4)}(\lambda)$, due to Källen and Sabry [22]. $\Delta^{(4)}(\lambda)$ is the result of including the intermediate state $e^+e^-\gamma$ in Eq. (4). It results in a potential $V_{KS}(r)$. This contribution has the form that resembles Eq. (10):

$$\langle n,l|V_{KS}|n,l\rangle = \frac{m_\mu\alpha^2}{2\pi}\left(\frac{n}{2}\right)^2\int_4^\infty\frac{dx}{x}\Delta^{(4)}(\sqrt{x}/2)\int_0^\infty dyyf(n,l,y)^2e^{-(n\beta\sqrt{x}/2+1)y} \tag{14}$$

with

$$\begin{aligned}\Delta^{(4)}(x) = -\frac{\alpha^2}{2\pi^2}\Bigg\{&\left(\frac{13}{54x}+\frac{7}{108x^3}+\frac{2}{9x^5}\right)\sqrt{x^2-1}+\left(\frac{4}{3x}+\frac{2}{3x^3}\right)\sqrt{x^2-1}\log\left(8x(x^2-1)\right)\\&+\left(-\frac{44}{9}+\frac{2}{3x^2}+\frac{5}{4x^4}+\frac{2}{9x^6}\right)\operatorname{arccosh}(x)+\left(-\frac{8}{3}+\frac{2}{3x^4}\right)\Bigg[\frac{2\pi^2}{3}-\operatorname{arccosh}^2(x)-\log\left(8x(x^2-1)\right)\operatorname{arccosh}(x)\\&-2\mathrm{Re}\left[\mathrm{Li}_2\left(\left(x+\sqrt{x^2-1}\right)^2\right)\right]+\mathrm{Li}_2\left(-\left(x-\sqrt{x^2-1}\right)^2\right)\Bigg]\Bigg\}.\end{aligned} \tag{15}$$

In this case, the contribution ground state contribution is

$$\langle 1,0|V_{KS}|1,0\rangle = -7.6999\ \text{meV}. \tag{16}$$

The potential $V_{VP}(\vec{r})$ can also contribute in the second order of nonrelativistic perturbation theory. For the ground state, this contribution is obtained from

$$E_{VPVP}(1S) = \frac{m_\mu\alpha^4}{72\pi^2}\int_4^\infty\frac{d\lambda}{\lambda}\Delta^{(2)}(\lambda)\int_4^\infty\frac{d\rho}{\rho}\Delta^{(2)}(\rho)\int_0^\infty dxxe^{-(\beta\sqrt{\lambda}+1)x/2}\int_0^\infty dx'x'g_{10}(x,x')e^{-(\beta\sqrt{\rho}+1)x'/2}, \tag{17}$$

where $g_{10}(x,x')$ is the Coulomb radial Green's function and $\beta = m_e/\mu\alpha$:

$$g_{10}(x,x') = 4e^{-x/2}e^{-x'/2}\left[\ln(x)+\ln(x')+\frac{x}{2}+\frac{x'}{2}+2\gamma-\frac{7}{2}-\frac{1}{x}-\frac{1}{x'}+\frac{e^{x_<}}{x_<}-Ei(x_<)\right]\frac{1}{4\pi}. \tag{18}$$

Here $Ei(x)$ is the exponential integral and $x_<$ is the smaller of x and x'. The x and x' integrals can be evaluated analytically using *Mathematica* and the remaining integrals over $\Delta^{(2)}(\lambda)$ are evaluated using NIntegrate. With this, the ground contribution is -0.32260 meV. The sum of these corrections for all states is listed in the Coulomb column of Table I.

In addition to vacuum polarization corrections to propagator, each of the fine structure terms in Eq. (1) receives a similar correction. Omitting the integral over λ, the modified terms are

$$-\frac{\pi\alpha}{m_\mu^2}\delta(\vec{r}) \to -\frac{\pi\alpha}{m_\mu^2}\left(\delta(\vec{r})-\frac{\lambda}{4\pi r}e^{-\sqrt{\lambda}r}\right), \tag{19a}$$

$$-\frac{\alpha}{2m^2}p_i\left(\delta_{ij}\frac{1}{r}+\hat{x}_i\frac{1}{r}\hat{x}_j\right)p_j \to -\frac{\alpha}{2m^2}p_i\left(\delta_{ij}\frac{e^{-\sqrt{\lambda}r}}{r}+\hat{x}_i\frac{(1+\sqrt{\lambda}r)e^{-\sqrt{\lambda}r}}{r}\hat{x}_j\right)p_j, \tag{19b}$$

$$\frac{4\pi\alpha}{3m_\mu^2}\vec{S}^2\delta(\vec{r}) \to \frac{4\pi\alpha}{3m_\mu^2}\left(\delta(\vec{r})-\frac{\lambda}{4\pi r}e^{-\sqrt{\lambda}r}\right)\vec{S}^2, \tag{19c}$$

$$\frac{3\alpha}{2m^2}\frac{1}{r^3}\vec{L}\cdot\vec{S} \to \frac{3\alpha}{2m^2}\frac{(1+\sqrt{\lambda}r)}{r^3}e^{-\sqrt{\lambda}r}\vec{L}\cdot\vec{S}, \tag{19d}$$

$$\frac{\alpha}{4m^2}\frac{1}{r^3}S_{12} \to \frac{\alpha}{4m^2}\frac{(1+\sqrt{\lambda}r+\lambda r^2)}{r^3}e^{-\sqrt{\lambda}r}S_{12}. \tag{19e}$$

TABLE II. The transition frequencies between various levels are shown. Included are $\pi^+\pi^-$ vacuum polarization corrections. The α^6 (modified) singlet annihilation contribution [23,24] and the α^6 triplet annihilation contribution [25–27] to the ground state are included. The results are compared with those of Ref. [28].

Transition	Theory (MHz)	Reference [28]
$\Delta(1^3S_1 \to 1^1S_0)$	4.23281 10^7	4.2328355(51) 10^7
$\Delta(2^3S_1 \to 2^1S_0)$	5.28916 10^6	
$\Delta(2^3S_1 \to 1^3S_1)$	2.55172 10^{11}	2.550014(16) 10^{11}
$\Delta(2^3P_2 \to 2^3S_1)$	1.20372 10^7	1.206(3) 10^7
$\Delta(2^3P_1 \to 2^3S_1)$	1.11302 10^7	1.115(3) 10^7
$\Delta(2^3P_0 \to 2^3S_1)$	9.99479 10^6	1.002(3) 10^7
$\Delta(2^1P_1 \to 2^3S_1)$	1.15080 10^7	1.153(3) 10^7

The energy corrections for all these terms can be calculated by substituting one of the modified operators into the integral in Eq. (10) after taking into account the change in its coupling.

Since the $\mu^+\mu^-$ bound system [1] has an annihilation channel, there are corrections associated with this process. Some of them are contained in Eq. (2), but the largest effect is the electron vacuum polarization correction to the fine structure annihilation term, which depends on $\vec{S}^2$ [2]. The energy associated with the annihilation process is

$$E_A(nS) = -\frac{m_\mu \alpha^4}{8n^3}\langle \vec{S}^2\rangle 4m_\mu^2 \int_{4m_e^2}^{\infty}\frac{d\lambda}{\lambda}\frac{\Delta(\lambda)}{(\lambda - 4m_\mu^2 - i\varepsilon)}. \quad (20)$$

For $\Delta^{(2)}(\lambda)$, the integral can be evaluated analytically [2] and the result is $-3.4609\alpha/\pi$. The analogous calculation using $\Delta^{(4)}(\lambda)$ was performed numerically and the total annihilation contribution is given in the spin annih column of Table I.

The columns in Table I contain sum of the electron vacuum polarization corrections to the terms in Eqs. (1) and (2) as well as the second-order contributions using V_{VP} with all of the terms in Eq. (19). The second-order perturbative corrections are a straightforward calculation using the appropriate Green's functions [14].

The pion vacuum polarization correction is included in spin and spin annih columns. In addition, the α^6 modified singlet annihilation contribution [23,24] and the α^6 triplet annihilation contribution [25–27] are included in two-loop column. The evaluation of these terms is contained in the Appendix. The frequencies of various transitions are shown in Table II. These results are compared with Ref. [28].

IV. CONCLUSIONS

Comparing the results in Table II with Ref. [28], the ground state hyperfine splitting and the $2^3S_1 - 1^3S_1$ splitting agree within the quoted errors. The $2P - 2S$ frequencies are consistent those in Ref. [28]. All the known α^6 contributions from the underlying positronium potential are included [16]. The vacuum polarization corrections to the terms in Eq. (19) also include their α^6 corrections. The spin-orbit and tensor α^6 terms are negligible. Second-order perturbative corrections include the one-loop terms and, again, the p-state spin-orbit and tensor contributions can be neglected.

ACKNOWLEDGMENTS

The author thanks Stan Brodsky and Richard Lebed for conversations and email communications regarding this work.

DATA AVAILABILITY

The data supporting this study's findings are available within the article.

APPENDIX: GROUND STATE HYPERFINE SPLITTING CORRECTIONS

In order to generalize the positronium result for the α^6 two-photon annihilation correction, given in the paper of Adkins *et al.* [23], to the $\mu^+\mu^-$ atom, one can simply scale the vertex, self-energy and box contributions by replacing the electron mass with the muon mass. However, the vacuum polarization contribution contains an additional contribution due to the electron in the loop. To include this contribution, the calculation in Ref. [23] must be generalized.

According to Ref. [23], the vacuum polarization correction for $\mu^+\mu^-$ can be expressed as

$$\Delta E_{VP} = \frac{2m_\mu^3\alpha^5}{\pi}\int\frac{d^4k}{i\pi^2}\frac{\vec{k}^2\Pi_f(k^2)}{k^2(k-K)^2(k^2 - k\cdot K)^2}, \quad (A1)$$

where $K = (2m_\mu, \vec{0})$ and $\Pi_f(k^2)$ is the renormalized vacuum polarization factor with an arbitrary mass in the loop. To proceed, we can use the dispersion representation

$$\frac{\Pi_f(k^2)}{k^2} = \int_{4m^2}^{\infty}\frac{d\lambda}{\lambda}\frac{\Delta(\lambda)}{\lambda - k^2 - i\varepsilon}. \quad (A2)$$

For the leading order, $\Delta(\lambda)$ is

$$\Delta(\lambda) = \frac{\alpha}{3\pi}\left(1 + \frac{2m^2}{\lambda}\right)\sqrt{1 - \frac{4m^2}{\lambda}}\theta(\lambda - 4m^2). \quad (A3)$$

Expressing $\vec{k}^2$ as

$$\vec{k}^2 = \frac{(k\cdot K)^2}{K^2} - k^2, \quad (A4)$$

ΔE_{VP} is

$$\Delta E_{VP} = -\frac{2m_\mu^3\alpha^5}{\pi}\int_{4m^2}^{\infty}\frac{d\lambda}{\lambda}\Delta(\lambda)\int\frac{d^4k}{i\pi^2}\frac{(k\cdot K)^2/K^2-k^2}{(k-K)^2(k^2-\lambda)(k^2-k\cdot K)^2}. \tag{A5}$$

Shifting k to $k+K$ and expressing k in units of m_μ gives

$$\Delta E_{VP} = -\frac{2m_\mu\alpha^5}{\pi}\int_{4m^2}^{\infty}\frac{d\lambda}{\lambda}\Delta(\lambda)\int\frac{d^4k}{i\pi^2}\frac{(k\cdot N)^2/N^2-k^2}{k^2(k^2+2k\cdot N+4-\bar{\lambda})(k^2+k\cdot N)^2}, \tag{A6}$$

with $N=(2,\vec{0})$ and $\bar{\lambda}=\lambda/m_\mu^2$. Replacing λ by $4m^2z$ and evaluating the d^4k integral in the usual way results in the expression

$$\Delta E_{VP} = -\frac{m_\mu\alpha^6}{\pi^2}I(r), \tag{A7}$$

with $r=m^2/m_\mu^2$ and

$$I(r) = \int_r^{\infty}\frac{dz(2z+r)\sqrt{z-r}F(z)}{2z^{5/2}}. \tag{A8}$$

The function $F(z)$ is

$$F(z) = \frac{1}{6}\left[2-2\,\log(2)-4z\,\log(2)+(z-1)\log(z-1)-3z\,\log(z)+4z\sqrt{\frac{z}{z-1}}\log\left(\sqrt{z}+\sqrt{z-1}\right)\right]. \tag{A9}$$

If the muon is in the loop, $r=1$ and *Mathematica* gives $I(1)=\pi^2/36$, leading to the usual result

$$\Delta E_{VP}^{\mu\bar{\mu}} = -0.27416\frac{m_\mu\alpha^6}{\pi^2}. \tag{A10}$$

For $r=m_e^2/m_\mu^2$, numerically integrating using NIntegrate gives

$$I(r) = 1.0501-4.9129i, \tag{A11}$$

which results in an additional contribution

$$\Delta E_{VP}^{e\bar{e}} = -1.0501\frac{m_\mu\alpha^6}{\pi^2}. \tag{A12}$$

The imaginary part is proportional to an additional contribution to the decay width.

Combining Eq. (A12) with the rescaled positronium result gives a frequency shift for the $\mu^+\mu^-$ atom of

$$\Delta\nu^S_{\mu^+\mu^-} = -285.18\ \text{MHz} \tag{A13}$$

for the α^6 singlet annihilation process.

There is a corresponding two-loop annihilation contribution to the triplet hyperfine interval [25–27], which can be rescaled to give

$$\Delta\nu^T_{\mu^+\mu^-} = -200.38\ \text{MHz}. \tag{A14}$$

The net α^6 annihilation contribution for the ground state hyperfine splitting is then

$$\Delta\nu_A(\alpha^6) = 84.80\ \text{MHz}. \tag{A15}$$

This is quite small in comparison to the $\alpha^6\log(\alpha^{-1})$ ground state contribution [27], which rescales to

$$\Delta\nu(\alpha^6\log(\alpha^{-1})) = \frac{5m_\mu\alpha^6\log(\alpha^{-1})}{24} = 3.95454\times10^3\ \text{MHz}. \tag{A16}$$

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