

All-loop soft-photon theorems and higher spin currents on the Celestial sphere

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Soft factorization theorems can be reinterpreted as Ward identities for (asymptotic) symmetries of scattering amplitudes in asymptotically flat space-time. In this letter we study the symmetries implied by the all-loop soft-photon theorems when all the charged particles are highly energetic and the relation $\omega \ll m \ll E$ holds where E is the typical energy of a charged particle, m is the typical mass and ω is the soft photon energy. Loop-level soft theorems are qualitatively different from the tree-level soft theorems because loop-level soft factors contain multiparticle sums. If we want to interpret them as Ward identities or define Celestial operator product expansion between soft and hard operators then we need to introduce additional fields which live on the Celestial sphere but do not appear as asymptotic states in any scattering experiment. For example, if we want to interpret the one-loop exact $\mathcal{O}(\ln \omega)$ soft theorem for a positive helicity soft photon (with energy ω) as a Ward identity then we need to introduce a pair of antiholomorphic currents on the Celestial sphere which transform as a doublet under the $SL(2, \mathbb{R})_R$. We call them dipole currents because the corresponding charges measure the monopole and the dipole moment of an electrically charged particle on the Celestial sphere. More generally, the soft photon theorem at $\mathcal{O}(\omega^{2j-1}(\ln \omega)^{2j})$ for every $j \in \frac{1}{2}\mathbb{Z}_+$ gives rise to $(2j+1)$ antiholomorphic currents which transform in the spin- j representation of the $SL(2, \mathbb{R})_R$. These currents exist in the quantum theory because they follow from loop-level soft theorems. We argue that under certain circumstances the (classical) algebra of the higher-spin currents is the wedge subalgebra of the $w_{1+\infty}$.

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Introduction. Soft factorization theorems can be reinterpreted as Ward identities for symmetries of scattering amplitudes in asymptotically flat space-time. These symmetries are known as the soft symmetries. This is a central point in our current understanding of flat space holography and the framework in which such reinterpretation becomes natural is known as the Celestial holography [1–3]. Celestial holography posits that the holographic dual theory of the quantum theory of gravity in asymptotically flat four dimensional space time is a CFT_2 living on the celestial two-sphere. The correlation functions of the Celestial CFT_2 are related to the bulk scattering amplitudes by an integral transformation known as the Mellin transformation. As a

result, the soft symmetries of the scattering amplitudes become symmetries of the Celestial CFT_2 and in many cases it is infinite dimensional [4–11]. In this letter we study the soft symmetries of the scattering amplitudes implied by the soft photon theorems in QED which remain valid even at the loop level. So we start by briefly reviewing various soft photon theorems in QED.

It is well known that the leading soft photon theorem [12] is all-loop exact [13]. However, since four-dimensional S -matrices are infrared divergent due to the long-range interactions mediated by massless particles, deriving finite subleading corrections has long remained a challenge [15]. This program was successfully carried out in [16], with a proof of universality subsequently established in [17]. The resulting subleading soft photon theorem appears at $\mathcal{O}(\ln \omega)$ in the soft photon energy ω and is one-loop exact. The $\mathcal{O}(\ln \omega)$ soft factor consists of two distinct contributions, termed the “classical” and “quantum” parts in [16]. The classical contribution encodes information about the classical electromagnetic waveform in the low-frequency expansion, as demonstrated through independent classical analyses in [18,19]. In contrast, a clear physical

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interpretation of the quantum contribution remains elusive to date. We fill up this gap in this letter.

From a four-dimensional bulk perspective, the *classical* part of the subleading soft photon theorem at $\mathcal{O}(\ln \omega)$ has already been understood as a Ward identity associated with a large gauge symmetry. This symmetry acts on charged matter fields as a local enhancement of global phase rotations, referred to as “superphaserotations” in [20–22]. In the present work we provide an interpretation of the *quantum* contribution to the $\mathcal{O}(\ln \omega)$ soft photon theorem in QED as a symmetry of the dual Celestial CFT₂ in the *high-energy limit*. Furthermore, we analyze higher subleading corrections to the soft photon theorem at orders $\omega^n (\ln \omega)^{n+1}$ as conjectured and explicitly derived up to subsubleading order in [23,24], and argued to be $(n+1)$ -loop exact.

Loop-level soft theorems are qualitatively different from the tree-level ones [25] because they involve multiparticle sums. It turns out that if we want to interpret them as Ward identities of symmetries of a Celestial CFT₂ or define a Celestial operator product expansion (OPE) between a hard and a soft operator then we need to introduce chiral currents of all spins on the Celestial sphere which do *not* appear as asymptotic states in any scattering experiment. We show that depending on the value of a real parameter k the algebra of the higher-spin currents can be Abelian ($k=0$) or the wedge subalgebra of $w_{1+\infty}$ [28–31] in the semi-classical limit $k \rightarrow 0$.

Appearance of higher-spin currents is not new in the context of Celestial holography. Both (tree-level) soft gluons and gravitons generate infinite dimensional higher-spin algebras known as the S -algebra and the wedge subalgebra of $w_{1+\infty}$ [8–11,32–34], respectively. However, the infinite number of higher-spin currents we get in the high-energy limit and the symmetries they generate are robust against quantum corrections.

All-loop soft theorems at high-energy limit. Consider the scattering of massive charged particles and photons in a generic theory of quantum electrodynamics in four space-time dimensions, where, apart from electromagnetism, no other long-range interaction is present. Now let $\mathcal{A}_{N+1}^{(\text{em})}$ denotes the scattering amplitude of N finite-energy charged particles with charges $\{e_a\}$ and momenta $\{p_a\}$ together with one outgoing photon, while $\mathcal{A}_N^{(\text{em})}$ denotes the scattering amplitude for the same process without the external photon. The photon carries polarization ϵ_μ^h , helicity h , and momentum $k^\mu = -\omega \mathbf{n}^\mu$, where ω is the photon energy and $\mathbf{n}^\mu = (1, \hat{n})$ is a null vector with $\hat{n}$ being the unit spatial vector specifying the photon emission direction. Given this the soft photon theorem, valid to all orders in perturbation theory, relates the scattering amplitude $\mathcal{A}_{N+1}^{(\text{em})}$ to the amplitude $\mathcal{A}_N^{(\text{em})}$ [35] and has the following structure:

$$\mathcal{A}_{N+1}^{(\text{em}) \omega \rightarrow 0} = S_{\text{em}}(\{e_a, p_a\}, (\epsilon^h, k)) \mathcal{A}_N^{(\text{em})} + \mathcal{R}_{N+1}^{(\text{em})}, \quad (1)$$

where S_{em} represents the universal soft photon factor determined only in terms of the scattering data specified in its argument and $\mathcal{R}_{N+1}^{(\text{em})}$ represents the nonfactorized part of $\mathcal{A}_{N+1}^{(\text{em})}$ in the soft limit [37]. They have the following expansion in the soft limit:

$$S_{\text{em}}(\{e_a, p_a\}, (\epsilon^h, k)) \stackrel{\omega \rightarrow 0}{=} \sum_{n=-1}^{\infty} S_{\text{em}}^{(n)} \times \omega^n (\ln \omega)^{n+1},$$

$$\mathcal{R}_{N+1}^{(\text{em}) \omega \rightarrow 0} \stackrel{\omega \rightarrow 0}{=} \sum_{n=0}^{\infty} \sum_{t=0}^n \mathbb{R}_{\text{em}, N+1}^{(n,t)} \omega^n (\ln \omega)^t. \quad (2)$$

The soft factor $S_{\text{em}}^{(-1)}$ was derived in [12], $S_{\text{em}}^{(0)}$ in [16], and $S_{\text{em}}^{(1)}$ in [23], for the scattering of massive charged particles. The universality of $S_{\text{em}}^{(0)}$ was established in [17]. For general n , $S_{\text{em}}^{(n)}$ was conjectured in [23], up to an undetermined function, to be $(n+1)$ -loop exact, and its full classical contribution was evaluated in [24] as a low-frequency expansion of the electromagnetic waveform.

From the soft expansion (1) and (2), the soft factor coefficient $S_{\text{em}}^{(n)}$ can be extracted by using the projection operator

$$S_{\text{em}}^{(n)} = \frac{1}{(n+1)!} \lim_{\omega \rightarrow 0} \frac{d^{n+1}}{d\omega^{n+1}} \omega \prod_{\ell=0}^{n+1} \frac{1}{\ell!} \times \left(\omega \frac{d}{d\omega} - (\ell-1) \right)^\ell \left[\frac{\mathcal{A}_{N+1}^{(\text{em})}}{\mathcal{A}_N^{(\text{em})}} \right], \quad (3)$$

where the projection operator for $n=0$ was already found in [20].

High-energy limit: In this letter we want to study the soft expansion (2) in the high-energy limit,

$$\omega^2 \ll m^2 \ll |p_a \cdot p_b| \sim E^2, \quad \forall a \neq b, \quad (4)$$

where m represents the typical mass of a charged particle and E is the characteristic energy scale of the hard scattering process.

In the high-energy limit (4) the expressions for the soft factors simplify considerably. For example, the $\mathcal{O}(\ln \omega)$ soft factor derived in [16] admits an expansion in the limit (4), yielding

$$S_{\text{em}}^{(0)} = -\frac{1}{4\pi^2} \sum_{\substack{a,b=1 \\ b \neq a}}^N \frac{e_a^2 e_b}{p_a \cdot p_b} \frac{\epsilon_\mu \mathbf{n}_\nu}{p_a \cdot \mathbf{n}} (p_a^\mu p_b^\nu - p_a^\nu p_b^\mu) + \mathcal{O}\left(\frac{m^4}{E^5} \log \frac{m^4}{E^4}, \frac{m^4}{E^5}\right). \quad (5)$$

We observe that $S_{\text{em}}^{(0)}$ contains a leading $\mathcal{O}(1/E)$ term and subleading power-law and logarithmic corrections. In this letter, we focus on the *leading contribution* only [39].

Motivated by the analysis presented in the Supplemental Material [40] and inspired by Refs. [23,24], we can write down the expressions for the *leading terms* of the logarithmic soft factors in the high-energy limit (4) as

$$\begin{aligned} \mathbb{S}_{\text{em}}^{(-1)} &= -\sum_{a=1}^N e_a \frac{\epsilon_\mu^h p_a^\mu}{p_a \cdot \mathbf{n}}, \quad (6) \\ \mathbb{S}_{\text{em}}^{(n)} &= -\frac{1}{(n+1)!(4\pi^2)^{n+1}} \sum_{\substack{a,b=1 \\ b \neq a}}^N \frac{e_a^2 e_b}{p_a \cdot p_b} \frac{\epsilon_\mu^h \mathbf{n}_\nu}{p_a \cdot \mathbf{n}} \\ &\quad \times (p_a^\mu p_b^\nu - p_a^\nu p_b^\mu) \left(\sum_{\substack{c=1 \\ c \neq a}}^N \frac{e_a e_c}{p_a \cdot p_c} (p_c \cdot \mathbf{n}) \right)^n, \quad (7) \end{aligned}$$

where $n \geq 0$. The $\mathbb{S}_{\text{em}}^{(n)}$ for $n \geq 0$ presented above are intrinsically quantum contributions to the soft photon theorems since the classical contributions are *subleading* in the high-energy limit.

Soft theorems in conformal primary basis. In the high-energy limit (4), hard particles are effectively massless at leading order, since finite-mass corrections are suppressed by powers of m/E . We can therefore parametrize the soft momentum k^μ , the positive-helicity photon polarization vector $\epsilon^{+\mu}(k)$, and the momentum p^μ of a hard particle as

$$\begin{aligned} k^\mu &= \eta \omega \mathbf{n}^\mu(w, \bar{w}), \quad \omega \rightarrow 0+, \\ \mathbf{n}^\mu(w, \bar{w}) &= (1 + w\bar{w}, w + \bar{w}, -i(w - \bar{w}), 1 - w\bar{w}), \\ \epsilon^{+\mu}(k) &= \frac{1}{\sqrt{2}}(\bar{w}, 1, -i, -\bar{w}), \\ p^\mu &= \eta \omega \mathbf{n}^\mu(z, \bar{z}), \end{aligned} \quad (8)$$

where $\eta = \pm 1$ for an incoming (outgoing) particle and $(w, \bar{w})((z, \bar{z}))$ represent the (stereographic) coordinates of the Celestial sphere. We also define celestial primary field $\phi_{h,\bar{h}}^{(\eta)}(z, \bar{z}, e)$ by Mellin transforming the creation and annihilation operators $b^\dagger$ and b as [2]

$$\begin{aligned} \phi_{h,\bar{h}}^{(+1)}(z, \bar{z}, e) &= \int_0^\infty d\omega \omega^{\Delta-1} b^\dagger(p, \sigma, e), \\ \phi_{h,\bar{h}}^{(-1)}(z, \bar{z}, e) &= \int_0^\infty d\omega \omega^{\Delta-1} b(p, \sigma, e), \end{aligned} \quad (9)$$

where Δ is in general a complex number, $\sigma \in \frac{1}{2}\mathbb{Z}$ is the helicity of the particle and e is the electric charge. One can show that under Lorentz transformation which acts on the Celestial sphere as the group $[SL(2, \mathbb{C})]$ of global conformal transformations, the field $\phi_{h,\bar{h}}^{(\eta)}(z, \bar{z}, e)$ transforms as a conformal primary with left and right conformal dimensions given by $h = \frac{1}{2}(\Delta + \sigma)$ and $\bar{h} = \frac{1}{2}(\Delta - \sigma)$, respectively.

In the conformal primary basis, the soft photon theorems associated with the soft factors in (6) and (7) can be recast as relations among celestial correlation functions as given by

$$\left\langle \mathcal{S}^1(w, \bar{w}) \prod_{i=1}^N \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle = \sum_{a=1}^N \frac{e_a}{w - z_a} \left\langle \prod_{i=1}^N \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle, \quad (10)$$

$$\begin{aligned} \left\langle \mathcal{S}^{1-2j}(w, \bar{w}) \prod_{i=1}^N \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle &= \frac{1}{(2j)!} \sum_{a=1}^N \frac{e_a^{2j+1} \eta_a^{2j}}{w - z_a} \left[\sum_{\substack{b=1 \\ b \neq a}}^N e_b \frac{\bar{w} - \bar{z}_b}{\bar{z}_a - \bar{z}_b} \right] \\ &\quad \times \left[\sum_{\substack{c=1 \\ c \neq a}}^N e_c \frac{(w - z_c)(\bar{w} - \bar{z}_c)}{(z_a - z_c)(\bar{z}_a - \bar{z}_c)} \right]^{2j-1} \left\langle \phi_{\Delta_a-2j}^{(\eta_a)}(z_a, \bar{z}_a, e_a) \prod_{\substack{i=1 \\ i \neq a}}^N \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle, \end{aligned} \quad (11)$$

for $j = 1/2, 1, 3/2, \dots$. In the above equations we have kept the helicity and all other internal quantum numbers of hard particles implicit because the soft factors depend only on the electric charges and the momenta of them. The conformally soft [26,41] photon $\mathcal{S}^\Delta(w, \bar{w})$ is defined as

$$\begin{aligned} \mathcal{S}^{1-2j}(w, \bar{w}) &= -\sqrt{2}(4\pi^2)^{2j} \frac{(-1)^{2j}}{(2j)!} \\ &\quad \times \lim_{\Delta \rightarrow 1-2j} (\Delta + 2j - 1)^{2j+1} \Gamma_\Delta^+(w, \bar{w}), \\ \Gamma_\Delta^+(w, \bar{w}) &= \int_0^\infty d\omega \omega^{\Delta-1} b(-\omega \mathbf{n}(w, \bar{w}), \sigma = +1), \end{aligned} \quad (12)$$

for $j = 0, 1/2, 1, 3/2, \dots$. The left and right conformal dimensions of $\mathcal{S}^{1-2j}$ are $h = 1 - j$, $\bar{h} = -j$ as follows from the (Lorentz) conformal invariance of (10) and (11).

The leading soft theorem (10) is known to be equivalent to the Ward identity of a $U(1)$ Kac-Moody algebra at level zero [1]. In the rest of the letter we focus on the subleading soft theorems (11).

$\mathcal{O}(\ln \omega)$ Soft theorem as Ward identity and the dipole current operator. In the conformal primary basis, the $\mathcal{O}(\ln \omega)$ soft theorem follows from (11) upon setting $j = \frac{1}{2}$. The expression (11) for $j = \frac{1}{2}$, as it stands, does

not allow us to define the Celestial OPE between soft and hard operators because of the two particle summation. For example, if we send $w \rightarrow z_a$ then the most singular term in $(w - z_a)$

$$\begin{aligned} & \mathcal{S}^0(w, \bar{w}) \phi_{\Delta_a}^{(\eta_a)}(z_a, \bar{z}_a, e_a) \\ & \sim \frac{\eta_a e_a^2}{w - z_a} \left(\sum_{\substack{b=1 \\ b \neq a}}^N e_b \frac{\bar{w} - \bar{z}_b}{\bar{z}_a - \bar{z}_b} \right) \phi_{\Delta_{a-1}}^{(\eta_a)}(z_a, \bar{z}_a, e_a), \end{aligned} \quad (13)$$

does not have the standard form of an OPE. To cure this problem we introduce the ‘‘dipole current operator’’ $\bar{d}(\bar{w}, \bar{z})$ [42], whose correlation function with the charged operators is given by

$$\begin{aligned} & \left\langle \bar{d}(\bar{w}, \bar{z}) \prod_i \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle \\ & = \sum_a e_a \frac{\bar{w} - \bar{z}_a}{\bar{z} - \bar{z}_a} \left\langle \prod_i \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle. \end{aligned} \quad (14)$$

Using the dipole current operator (13) can be rewritten as an OPE,

$$\begin{aligned} & \mathcal{S}^0(w, \bar{w}) \phi_{\Delta_a}^{(\eta_a)}(z_a, \bar{z}_a, e_a) \\ & \sim \frac{\eta_a e_a^2}{w - z_a} : \bar{d}\phi_{\Delta_{a-1}}^{(\eta_a)} : (\bar{w}, z_a, \bar{z}_a, e_a). \end{aligned} \quad (15)$$

and the $\mathcal{O}(\ln \omega)$ soft theorem can be recast as a Ward identity,

$$\begin{aligned} & \left\langle \mathcal{S}^0(w, \bar{w}) \prod_{i=1}^N \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle = \sum_{a=1}^N \frac{\eta_a e_a^2}{w - z_a} \\ & \times \left\langle : \bar{d}\phi_{\Delta_{a-1}}^{(\eta_a)} : (\bar{w}, z_a, \bar{z}_a, e_a) \prod_{\substack{i=1 \\ i \neq a}}^N \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle. \end{aligned} \quad (16)$$

Here the composite operator [43] $: \bar{d}\phi_{\Delta}^{(\eta)} :$ is defined as

$$\begin{aligned} & : \bar{d}\phi_{\Delta}^{(\eta)} : (\bar{w}, z, \bar{z}, e) \\ & \equiv \lim_{\bar{\xi} \rightarrow \bar{z}} \left(\bar{d}(\bar{w}, \bar{\xi}) \phi_{\Delta}^{(\eta)}(z, \bar{z}, e) - e \frac{\bar{w} - \bar{z}}{\bar{\xi} - \bar{z}} \phi_{\Delta}^{(\eta)}(z, \bar{z}, e) \right). \end{aligned} \quad (17)$$

We emphasize that the existence of the dipole current follows from the locality of the Celestial CFT₂ in the presence of loop corrections in the bulk.

Conformal transformation of the dipole operator: Under an $SL(2, \mathbb{R})_R$ transformation $\bar{z} \rightarrow \bar{z}' = \frac{a\bar{z}+b}{c\bar{z}+d}$ with $a, b, c, d \in \mathbb{R}$

and $ad - bc = 1$, the invariance of the correlation function (14) requires the dipole current $\bar{d}(\bar{w}, \bar{z})$ to transform as

$$\bar{d}(\bar{w}, \bar{z}) \rightarrow \bar{d}'(\bar{w}, \bar{z}) = \frac{c\bar{w} + d}{c\bar{z} + d} \times \bar{d}(\bar{w}', \bar{z}'). \quad (18)$$

Now the rhs of (14) is a polynomial of degree one in $\bar{w}$. So following [6–8] we Taylor expand $\bar{d}(\bar{w}, \bar{z})$ in $\bar{w}$ coordinate and write,

$$\bar{d}(\bar{w}, \bar{z}) = \bar{d}_{-\frac{1}{2}}^{\frac{1}{2}}(\bar{z}) + \bar{w} \bar{d}_{\frac{1}{2}}^{\frac{1}{2}}(\bar{z}). \quad (19)$$

Then the transformation law (18) shows that the doublet $(\bar{d}_{\frac{1}{2}}^{\frac{1}{2}}(\bar{z}), \bar{d}_{-\frac{1}{2}}^{\frac{1}{2}}(\bar{z}))$ transforms in the tensor product of two $SL(2, \mathbb{R})_R$ representations: a conformal highest-weight representation with weight $\bar{h} = \frac{1}{2}$ and the two-dimensional spin- $\frac{1}{2}$ representation. Under the latter, the fields $\bar{d}_{\pm\frac{1}{2}}^{\frac{1}{2}}(\bar{z})$ carry weights $\pm\frac{1}{2}$ (see Supplementary Material [40] for details).

To make this manifest we use the two component $SL(2, \mathbb{R})$ spinor notation [46] and rewrite (19) and the soft photon $\mathcal{S}^0(w, \bar{w})$ as

$$\bar{d}(\bar{w}, \bar{z}) = \bar{W}^{\alpha} \bar{d}_{\alpha}^{\frac{1}{2}}(\bar{z}), \quad \mathcal{S}^0(w, \bar{w}) = \bar{W}^{\alpha} \mathcal{S}_{\alpha}^0(w), \quad (20)$$

for $\alpha = \pm\frac{1}{2}$. In the spinor notation the dipole Ward identity (14) and the $\mathcal{O}(\ln \omega)$ soft theorem (16) become

$$\begin{aligned} & \left\langle \bar{d}_{\alpha}^{\frac{1}{2}}(\bar{z}) \prod_i \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle \\ & = \sum_a e_a \frac{\bar{Z}_{a\alpha}}{\bar{z} - \bar{z}_a} \left\langle \prod_i \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle, \end{aligned} \quad (21)$$

and

$$\begin{aligned} & \left\langle \mathcal{S}_{\alpha}^0(w) \prod_{i=1}^N \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle = \sum_{a=1}^N \frac{\eta_a e_a^2}{w - z_a} \\ & \times \left\langle : \bar{d}_{\alpha}^{\frac{1}{2}} \phi_{\Delta_{a-1}}^{(\eta_a)} : (z_a, \bar{z}_a, e_a) \prod_{\substack{i=1 \\ i \neq a}}^N \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle, \end{aligned} \quad (22)$$

respectively.

$\mathcal{O}(\omega^{2j-1}(\ln \omega)^{2j})$ Soft theorem and the Celestial OPE. In this section, we show how to write down the OPE between a conformally soft photon for general values of j and a hard charged operator. As $w \rightarrow z_a$ the singular parts of the leading (10) and the subleading soft theorems (11) can be combined to give

$$\begin{aligned}
& \left\langle \mathcal{S}^{1-2j}(w, \bar{w}) \prod_{i=1}^N \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle_{w \rightarrow z_a} \\
&= \frac{1}{(2j)!} \frac{\eta_a^{2j} e_a^{2j+1}}{w - z_a} \left[\sum_{\substack{b=1 \\ b \neq a}}^N e_b \frac{\bar{w} - \bar{z}_b}{\bar{z}_a - \bar{z}_b} \right]^{2j} \\
&\times \left\langle \phi_{\Delta_a-2j}^{(\eta_a)}(z_a, \bar{z}_a, e_a) \prod_{\substack{i=1 \\ i \neq a}}^N \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle \\
&+ \text{nonsingular terms}, \tag{23}
\end{aligned}$$

where $j = 0, \frac{1}{2}, 1, \dots$. The singular terms (23) contain multiparticle sums for $j \geq 1/2$. Therefore to make the OPE interpretation possible we show that for general values

of j we need to introduce higher-spin currents built from the normal-ordered products of the dipole current. It is not difficult to see this. First let us note that the conformal invariance determines the two point function of the dipole current to be [47]

$$\langle \bar{d}_\alpha^\dagger(\bar{z}_1) \bar{d}_\beta^\dagger(\bar{z}_2) \rangle = k \frac{\epsilon_{\alpha\beta}}{\bar{z}_1 - \bar{z}_2}, \tag{24}$$

where k is a real parameter which can be zero also. This together with the boundary condition that in the absence of any operator insertion at infinity $\bar{d}_\alpha^\dagger(\bar{z}) \rightarrow 1/\bar{z}$ as $\bar{z} \rightarrow \infty$ determine the correlation function with multiple insertion of the dipole currents to be

$$\begin{aligned}
\left\langle \bar{d}_{a_1}^\dagger(\bar{\xi}_1) \prod_{p=2}^n \bar{d}_{a_p}^\dagger(\bar{\xi}_p) \prod_i \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle &= \left(\sum_a e_a \frac{\bar{Z}_{aa_1}}{\bar{\xi}_1 - \bar{z}_a} \right) \left\langle \prod_{p=2}^n \bar{d}_{a_p}^\dagger(\bar{\xi}_p) \prod_i \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle \\
&+ \sum_{a=2}^n k \frac{\epsilon_{a_1 a_a}}{\bar{\xi}_1 - \bar{\xi}_a} \left\langle \bar{d}_{a_2}^\dagger(\bar{\xi}_2) \cdots \bar{d}_{a_a}^\dagger(\bar{\xi}_a) \cdots \bar{d}_{a_n}^\dagger(\bar{\xi}_n) \prod_i \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle. \tag{25}
\end{aligned}$$

Using (25) we can define a spin- j current $\bar{d}_{a_1 \dots a_{2j}}^j$ as the normal ordered product of $2j$ dipole currents [48], i.e.,

$$\bar{d}_{a_1 \dots a_{2j}}^j = : \bar{d}_{a_1}^\dagger \bar{d}_{a_2}^\dagger \cdots \bar{d}_{a_{2j}}^\dagger :, \tag{26}$$

whose correlation function with the charged operators is given by

$$\begin{aligned}
& \left\langle \bar{d}_{a_1 a_2 \dots a_{2j}}^j(\bar{\xi}) \prod_i \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle \\
&= \prod_{k=1}^{2j} \left(\sum_a e_a \frac{\bar{Z}_{aa_k}}{\bar{\xi} - \bar{z}_a} \right) \left\langle \prod_i \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle. \tag{27}
\end{aligned}$$

The current $\bar{d}_{a_1 a_2 \dots a_{2j}}^j$ is completely symmetric in the spinor indices and transforms in the tensor product of two $SL(2, \mathbb{R})$ representations one of which is a conformal highest-weight representation of weight $\bar{h} = j$ and the other one is the $(2j+1)$ -dimensional spin- j representation [40].

In order to extract the OPE between the conformally soft photon and a hard operator we write the soft operator in spinor notation,

$$\mathcal{S}^{1-2j}(w, \bar{w}) = \frac{1}{(2j)!} \bar{W}^{\alpha_1} \bar{W}^{\alpha_2} \cdots \bar{W}^{\alpha_{2j}} \mathcal{S}_{a_1 a_2 \dots a_{2j}}^{1-2j}(w). \tag{28}$$

Now using (28) and comparing (23) and (27), it is easy to find the OPE between the soft operator $\mathcal{S}_{a_1 a_2 \dots a_{2j}}^{1-2j}(w)$ and a hard operator $\phi_{\Delta_a}^{(\eta_a)}(z_a, \bar{z}_a, e_a)$

$$\begin{aligned}
& \mathcal{S}_{a_1 a_2 \dots a_{2j}}^{1-2j}(w) \phi_{\Delta_a}^{(\eta_a)}(z_a, \bar{z}_a, e_a) \\
& \sim \frac{\eta_a^{2j} e_a^{2j+1}}{w - z_a} : \bar{d}_{a_1 a_2 \dots a_{2j}}^j \phi_{\Delta_a-2j}^{(\eta_a)}(z_a, \bar{z}_a, e_a) :, \tag{29}
\end{aligned}$$

for $j \in \frac{1}{2} \mathbb{Z}_+$. Now unlike the $j = 0$ and $j = \frac{1}{2}$ cases, the higher j soft theorems cannot be written as Ward identities using the higher-spin currents alone. However, as Eq. (29) shows the singular term in the OPE is completely determined in terms of the higher-spin current. To write the soft theorems as Ward identities for $j \geq 1$ we need to understand the part of the soft theorem which does not have a pole in w . We leave this interesting question for future investigation.

Algebra of higher-spin currents. To summarize we have shown that if we want the Celestial CFT₂ to be local, in particular, if we want the Celestial OPE to exist then we need to introduce an infinite number of higher-spin currents which live on the Celestial sphere. So a natural question is what is the algebra generated by the higher-spin currents? To answer this we first note that the contraction between two dipole currents $\bar{d}^\dagger$ is a c -number (24) and so we can compute the singular terms of the OPE between the currents defined in (26) using Wick's theorem. The result can be summarized as follows. The currents $\bar{d}_{\alpha\beta}^1$ generate a $SL(2, \mathbb{R})$ Kac-Moody algebra and the currents $\bar{d}_a^{1/2}$ and $\bar{d}_{\alpha\beta}^1$ form a closed subalgebra. However, starting from $j = 3/2$ the OPE of the currents do not close among themselves as discussed in detail in the Supplementary Material [40].

However, in the semiclassical limit $k \rightarrow 0$, the problematic terms which are all of $\mathcal{O}(k^2)$ and higher are subleading. If we keep only the leading terms of $\mathcal{O}(k)$ then we get the closed form OPE,

$$\begin{aligned} & \bar{d}_{\alpha_1 \dots \alpha_{2j_1}}^{j_1}(\bar{z}_1) \bar{d}_{\beta_1 \dots \beta_{2j_2}}^{j_2}(\bar{z}_2) \\ & \sim k \frac{\epsilon_{\alpha_1 \beta_1} \bar{d}_{\alpha_2 \dots \alpha_{2j_1} \beta_2 \dots \beta_{2j_2}}^{j_1+j_2-1}(\bar{z}_2) + \text{Contractions}}{\bar{z}_1 - \bar{z}_2} + \mathcal{O}(k^2). \end{aligned} \quad (30)$$

Now we show that the global part of the algebra generated by the higher-spin currents in the semiclassical limit $k \rightarrow 0$ is the wedge subalgebra of the $w_{1+\infty}$ algebra. We denote the wedge subalgebra by $\wedge w_{1+\infty}$.

We define the global modes of the spin- j current by

$$w_n^p \equiv \frac{1}{2} \oint \frac{d\bar{z}}{2\pi i} \bar{d}_{\alpha_1 \alpha_2 \dots \alpha_{2j}}^j(\bar{z}), \quad (31)$$

where

$$p = j + 1, \quad n = \sum_{i=1}^{2j} \alpha_i, \alpha_i = \pm \frac{1}{2}, \quad j \in \frac{1}{2}\mathbb{Z}_+, \quad (32)$$

with the restriction $1 - p \leq n \leq p - 1$. It is easy to see that in terms of the generators w_n^p the algebra following from the OPE (30) is

$$[w_m^p, w_n^q] = k[m(q-1) - n(p-1)]w_{m+n}^{p+q-2}, \quad (33)$$

with $p, q = \frac{3}{2}, 2, \frac{5}{2}, 3, \dots$ and the value for $w_0^1 = \frac{1}{2}$ appears when $p = q = \frac{3}{2}$. The algebra (33) subject to the restriction $1 - p \leq n \leq p - 1$ is well known to be the wedge subalgebra $\wedge w_{1+\infty}$ [28–31]. If $k = 0$ then of course this reduces to an infinite dimensional Abelian algebra. However, we emphasize that the existence of the higher-spin currents and the symmetries of the scattering amplitudes they generate via the Ward identities (14) and (27) do not depend on the value of k .

Discussion. In this letter we have shown that in the high-energy QED when quantum corrections in the bulk are taken into account the locality of the dual Celestial CFT₂ requires the existence of an infinite number of higher-spin currents. These currents generate higher-spin symmetries of the S -matrix of QED. The algebra of the higher-spin currents depends on a real parameter k and a nonvanishing k is essential for the emergence of $\wedge w_{1+\infty}$ in the

semiclassical ($k \rightarrow 0$) limit. To determine the value of k we need to calculate celestial correlators or S -matrix elements of the form,

$$\left\langle \bar{d}^{j_1}(\bar{w}_1) \bar{d}^{j_2}(\bar{w}_2) \prod_{i=1}^N \phi_{\Delta_i}^{(\eta_i)}(z_i, \bar{z}_i, e_i) \right\rangle. \quad (34)$$

in QED. However, we do not know the Feynman diagram representation of (34) because *the higher-spin currents do not appear as asymptotic states in a scattering experiment in an obvious way*. Understanding this better is one of the most important problems posed by our finding. Another important problem is to understand the breaking of the higher-spin symmetry at not so high energy. Recently $\wedge w_{1+\infty}$ was shown [32,49] to be the symmetry of any 4D CFT on Minkowski space. It will be very interesting to explore the relation between this and our work. And last but not the least it will be very interesting to understand how one can constrain the QED amplitudes at high energy using the higher-spin symmetry along the lines of [6,50].

To summarize, it turns out that higher-spin symmetry is ubiquitous in flat space holography [8–11,32–34] and a proper understanding of this will shed light on the nature of quantum gravity and the high-energy symmetries of string theory. We hope to return to this very interesting topic in future.

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- $$[Q_{\vec{d}}(\vec{w}), \phi_{\Delta}^{(\eta)}(z, \bar{z}, e)] = -e(\vec{z} - \vec{w})\phi_{\Delta}^{(\eta)}(z, \bar{z}, e),$$
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- $$\bar{Z}^{\alpha} = \begin{pmatrix} \bar{z} \\ 1 \end{pmatrix}, \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
- $$\bar{Z}_{\alpha} = \epsilon_{\alpha\beta} \bar{Z}^{\beta} = \begin{pmatrix} 1 \\ -\bar{z} \end{pmatrix}, \quad \alpha, \beta = \pm 1/2.$$
- [47] In terms of the original field $\vec{d}(\vec{w}, \vec{z})$ this is nothing but $\langle \vec{d}(\vec{w}_1, \vec{z}_1) \vec{d}(\vec{w}_2, \vec{z}_2) \rangle = k \frac{\vec{w}_1 - \vec{w}_2}{\vec{z}_1 - \vec{z}_2}$. This form of the two point function is uniquely determined by the $SL(2, \mathbb{R})_R$ transformation law (18).
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- $$\vec{d}_{\alpha\beta}^1(\vec{z}) = \vec{d}_{\beta\alpha}^1(\vec{z}) = \lim_{\vec{z}' \rightarrow \vec{z}} \left(\vec{d}_{\alpha}^1(\vec{z}') \vec{d}_{\beta}^1(\vec{z}) - k \frac{\epsilon_{\alpha\beta}}{\vec{z}' - \vec{z}} \right)$$
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