

Multiplicity distribution of produced gluons in deep inelastic scattering: Main equations and their homotopy solutions for heavy nuclei

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In this paper we discuss the multiplicity distribution in the deep inelastic processes in the frame work of high energy QCD. We obtained three results. First, we get the new derivation of the equations for the cross sections of productions of n -cut Pomerons in the final states (σ_n). These equations coincide with the equations that have been derived using the Abramovsky-Gribov-Kancheli cutting rules but based on the dipole approach to QCD. Second, we developed the homotopy approach for finding the solutions to these equations. It consists with the analytic solution for the first iteration and the converge procedure of calculating the next iterations using computing. Third, we found the analytical solution for σ_n at large $n \gtrsim N(z) = 2N_0 z \exp(z^2/(2\kappa))$ with $z = \ln(r^2 Q_s^2)$. Using this solution we calculate the entropy of the produced gluons at large z : $S_E = \ln(N(z))$, where the saturation momentum Q_s and all constants are discussed in the text.

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I. INTRODUCTION

We continue [1,2] to develop the approach for the solution of the nonlinear evolution equations that govern the deep inelastic scattering (DIS) processes in QCD. In this paper we are going to discuss the multiplicity distribution of the produced gluons in QCD. This distribution has become a hot subject during the past several years [3–21] especially because of the new view on the entropy in DIS. In Ref. [7] it was proposed that the entropy in DIS is closely related to the entropy of entanglement between the spatial region probed by deep inelastic scattering and the rest of the proton.

Our approach is based on two main ideas: the homotopy approach to the solution of nonlinear equations [22,23] and Abramovsky-Gribov-Kancheli (AGK) cutting rules [24]. The AGK cutting rules allow us [25] to relate the gluons that come to detectors ($t = \infty$ in Fig. 1) with the dipoles that have been in the wave function of the fast hadron

$\Psi(r, b, \{r_i, b_i\})$. The coherence of this wave function is destroyed at time $t = 0$ in Fig. 1. We will discuss these ideas in the next section, while in this Introduction we summarize our theoretical understanding of DIS processes (see Ref. [26] for the review).

- (1) The scattering amplitude of the colorless dipole with the size x_{01} , which determines the DIS cross section, satisfies the Balitsky-Kovchegov (BK) nonlinear equation [27]

$$\frac{\partial}{\partial Y} N_{01}(Y) = \frac{\bar{\alpha}_S}{2\pi} \int d^2 x_2 \frac{x_{01}^2}{x_{02}^2 x_{12}^2} \left[N_{02}(Y) + N_{12}(Y) - N_{01}(Y) - N_{02}(Y) N_{12}(Y) \right], \quad (1.1)$$

where $N_{ik}(Y) = N(Y, \mathbf{x}_{ik}, \mathbf{b})$ is the scattering amplitude of the dipoles with size $x_{ik} = |\mathbf{x}_{ik}| = |\mathbf{x}_i - \mathbf{x}_k|$ and with rapidity Y at the impact parameter $\mathbf{b} = (\mathbf{x}_0 + \mathbf{x}_1)/2$.

- (2) In Refs. [28–30] it has been shown that Eq. (1.1) leads to a new dimensional scale: saturation momentum [30], which has the following Y dependence:

$$Q_s^2(Y, \mathbf{b}) = Q_s^2(Y = Y_0, \mathbf{b}) e^{\bar{\alpha}_S \kappa Y - \frac{3}{2(1-\gamma_{cr})} \ln(\bar{\alpha}_S Y)}, \quad (1.2)$$

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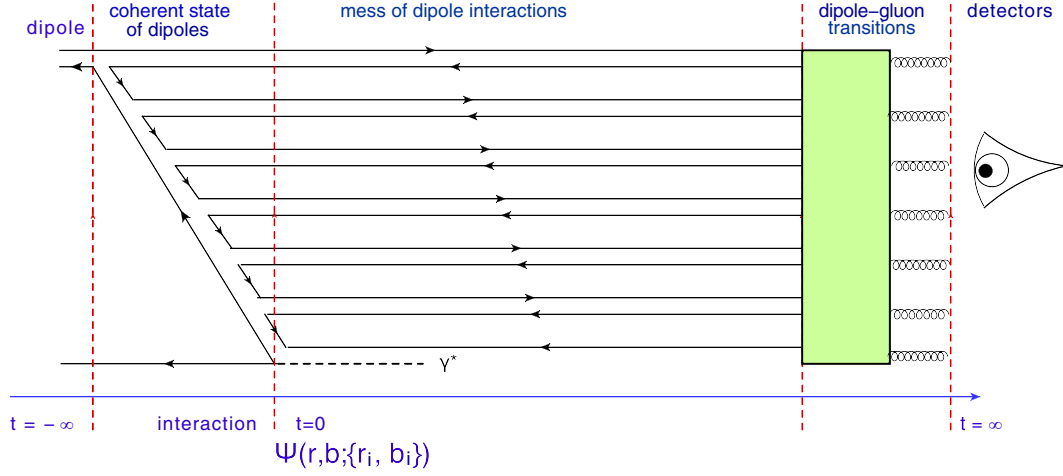


FIG. 1. The interaction of fast hadron (dipole) with the virtual photon (γ^*). The coherence of the partonic wave function of the fast hadron is destroyed at $t = 0$, while the gluons can be measured at $t = \infty$.

where Y_0 is the initial value of rapidity and κ and γ_{cr} are determined by the following equations¹:

$$\kappa \equiv \frac{\chi(\gamma_{\text{cr}})}{1 - \gamma_{\text{cr}}} = -\frac{d\chi(\gamma_{\text{cr}})}{d\gamma_{\text{cr}}}$$

and $\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma)$. (1.3)

- (3) It is shown [28] (see also Ref. [29]) that in the vicinity of the saturation scale the scattering amplitude takes the following form:

$$N(z) = \text{Const}(x_{01}^2 Q_s^2(Y, \mathbf{b}))^{\bar{\gamma}} = \text{Const} e^{\bar{\gamma} z}, \quad (1.4)$$

with $\bar{\gamma} = 1 - \gamma_{\text{cr}}$. In Eq. (1.4) we introduce a new variable z , which is equal to

$$z = \ln(x_{01}^2 Q_s^2(Y, \mathbf{b})) = \bar{\alpha}_s \kappa (Y - Y_A) + \xi, \quad (1.5)$$

where $\xi = \ln(x_{01}^2 Q_s^2(Y = Y_0, \mathbf{b}))$.

- (4) Inside the saturation region: $x_{01}^2 Q_s^2(Y) > 1$ the scattering amplitude $N_{01}(Y, x_{01}, b)$ is a function of one variable $x_{01}^2 Q_s^2(Y)$ (geometric scaling behavior), viz.

$$N(Y, x_{01}, \mathbf{b}) = N(x_{01}^2 Q_s^2(Y, \mathbf{b})). \quad (1.6)$$

It is important to note that this behavior has been proven on general theoretical grounds [33] and has been seen in the experimental data on DIS [34].

- (5) Finally, in Ref. [35] it is shown that the solution to Eq. (1.1) deep into the saturation region for $z \gg 1$ has the following form:

¹ $\chi(\gamma)$ is the Balitsky-Fadin-Kuraev-Lipatov (BFKL) kernel [31] in anomalous dimension (γ) representation. ψ is the Euler ψ function [see Ref. [32] formula (8.36)].

$$N(z) = 1 - C(z) \exp\left(-\frac{z^2}{2\kappa}\right), \quad (1.7)$$

where $C(z)$ is a smooth function of z .

- (6) The saturation region has been defined as $x_{01}^2 Q_s^2(Y, b) > 1$ (see Fig. 2). However, in Refs. [36,37] it has been noted that actually for very large impact parameters the nonlinear corrections become small and we have to solve the linear BFKL equation. This feature can be seen directly from the eigenfunction of this equation. Indeed, the eigenfunction has the following form for any kernel that satisfies the conformal symmetry [38,39]:

$$\phi_\gamma(\mathbf{r}, \mathbf{R}, \mathbf{b}) = \left(\frac{r^2 R^2}{(b + \frac{1}{2}(\mathbf{r} - \mathbf{R}))^2 (b - \frac{1}{2}(\mathbf{r} - \mathbf{R}))^2} \right)^\gamma \xrightarrow{b \gg r, R} \left(\frac{r^2 R^2}{b^4} \right)^\gamma \equiv e^{\gamma \xi} \text{ with } 0 < \text{Re } \gamma < 1, \quad (1.8)$$

where $\xi = \ln(\frac{r^2 R^2}{b^4})$. In Eq. (1.8), R is the size of the initial dipole at $Y = 0$, while $r \equiv x_{01}$ is the size of the dipole with rapidity Y . However, we consider the DIS with nuclei and for such target in Ref. [1] it has been shown that we can absorb all dependence on the impact parameter in the b dependence of the saturation scale.

- (7) Figure 2 shows the typical kinematic regions in DIS. As has been mentioned, we are dealing with the saturation region in this paper. However, this region for the elastic amplitude has to be divided in two parts: region I and region II (shown in Fig. 2). In region I, we expect the geometric scaling behavior of the scattering amplitude. This amplitude in the vicinity of the saturation scale (red curve in Fig. 2) has the behavior of Eq. (1.4). We use this behavior to

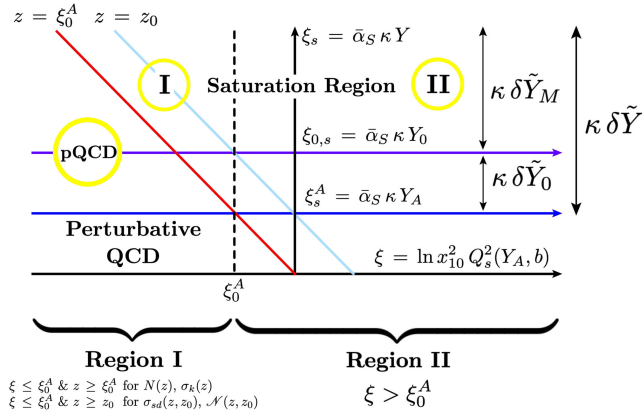


FIG. 2. Saturation region of QCD for elastic amplitude. The critical line ($z = 0$) is shown in red. The initial condition for scattering with the dilute system of partons (with proton) is given at $\xi_s = 0$. For heavy nuclei, the initial conditions are placed at $Y_A = (1/3) \ln A \gg 1$, where A is the number of nucleon in a nucleus. Two blue lines show the kinematic regions for the initial conditions: the upper one for nuclei and the lower one for proton.

put the boundary condition to the saturation region on the red line. They have the form

$$N(z = \xi_0^A) = N_0; \quad (1.9a)$$

$$\left. \frac{d \ln N(z)}{dz} \right|_{z=\xi_0^A} = \bar{\gamma} = 1 - \gamma_{\text{cr}}; \quad (1.9b)$$

where $Y_A = \ln A^{1/3}$ and A is the number of the nucleons in a nucleus.

For $\xi > \xi_0^A$ we use the McLerran-Venugopalan formula for the initial condition at $Y = Y_A$ for DIS with nuclei² (see Fig. 2),

$$\begin{aligned} N(Y = Y_A, \mathbf{x}_{01}, \mathbf{b}) &= 1 - \exp(-r^2 Q_s^2(Y = Y_A, \mathbf{b})/4) \\ &= 1 - \exp\left(-\frac{1}{4} e^{\xi}\right), \end{aligned} \quad (1.10)$$

where $r = x_{01}$ and we will use this notation for the size of the dipole below.

In the next section, we will discuss our approach. It is based on two main ideas. The first is the derivation of the amplitude given by the BK equation using only the wave function of the dipoles $\Psi(r, b, \{r_i, b_i\})$. The second is the AGK [24] approach to the gluon production. In these approaches we are dealing with the cross sections of production of n -cut Pomerons (σ_n^{AGK}) which we introduce in this section. We derive the evolution equation for σ_n^{AGK} .

²We use the simplified form of the McLerran-Venugopalan formula given in Ref. [40]. The exact form of this formula can be found in Ref. [41].

Our derivation does not use the AGK cutting rules in the explicit form, as it was done in Ref. [25], based on the main features of the BFKL cascade. We generalized the approach for the process of diffraction production of Refs. [26,42] to the general case and show that this approach leads to the same set of equations as the AGK cutting rules. In particular, in this section we discuss how to calculate the multiplicity distributions of the produced gluons using σ_n^{AGK} and the Poisson distributions for the produced gluons for the BFKL Pomeron.

In Sec. III we develop the homotopy approach for solutions of these equations. This method was discussed in our previous papers (see Refs. [1,2]). It is based on the idea [22,23] to divide the general nonlinear equation in two parts,

$$\mathcal{L}[u] + \mathcal{N}_{\mathcal{L}}[u] = 0, \quad (1.11)$$

where $\mathcal{L}[u]$ include the linear evolution and part of the nonlinear corrections, which can be treated analytically (or almost analytically). The nonlinear part $\mathcal{N}_{\mathcal{L}}[u]$ has an arbitrary form. As a solution, we introduce the following equation for the homotopy function $\mathcal{H}(p, u)$:

$$\mathcal{H}(p, u) = \mathcal{L}[u_p] + p \mathcal{N}_{\mathcal{L}}[u_p] = 0. \quad (1.12)$$

Solving Eq. (1.12) we reconstruct the function

$$\begin{aligned} u_p(Y, \mathbf{x}_{10}, \mathbf{b}) &= u_0(Y, \mathbf{x}_{10}, \mathbf{b}) + p u_1(Y, \mathbf{x}_{10}, \mathbf{b}) \\ &+ p^2 u_2(Y, \mathbf{x}_{10}, \mathbf{b}) + \dots \end{aligned} \quad (1.13)$$

with $\mathcal{L}[u_0] = 0$. Equation (1.13) gives the solution to the nonlinear equation at $p = 1$. The hope is that several terms in the series of Eq. (1.13) will give a good approximation in the solution of the nonlinear equation. In Sec. III we discuss the particular form of this approach and show that we can choose $\mathcal{L}[u]$ as the BK equation with the simplified leading twist BFKL kernel. It should be emphasized that we change the homotopy approach of Refs. [22,23] taking for $\mathcal{L}[u]$ the simplified nonlinear equation, which has the analytic solution. In Sec. IV we solve the equations for σ_n at large n : $n \gtrsim N(z) = 2N_0 z \exp(z^2/(2\kappa))$. We show that these cross sections have the Koba-Nielsen-Olesen (KNO) [43–45] scaling behavior. We calculate the entropy (S_E) of the produced gluons and show that $S_E = \ln N_{\text{DIS}}$ where N_{DIS} is the multiplicity in the deep inelastic scattering process, confirming the main ideas of Ref. [7]. In the Conclusion, we discuss our main results.

II. MAIN EQUATIONS

A. Setting the problem

As has been mentioned in the Introduction, we discuss the multiplicity distributions of produced gluons at $t = +\infty$ in Fig. 1. The distribution of dipoles at $t = 0$ is given by the

dipole wave function of the fast dipole with size r and rapidity Y . From $t = 0$ and $t = \infty$, the dipoles can interact. We use the AGK cutting rules, which give us an economical way to treat a mess of dipole interactions during propagation of the dipole cascade (see Fig. 1) from the moment of interaction of the dipole with the target ($t = 0$ in Fig. 1) until it reaches the detectors ($t = \infty$ in Fig. 1).

In our treatment of the multiplicity distributions, we are going to explore that the scattering amplitudes at high energy in QCD, which can be found as the solution to the BK equation [see Eq. (1.1)], on the other hand, are the sum of the “fan” diagrams of the BFKL Pomeron calculus [31]. Generally, the scattering amplitude can be represented as the sum of multipomeron exchanges,

$$N(Y, \mathbf{r}, \mathbf{b}) = \sum_{k=1}^{\infty} (-1)^{k+1} \underbrace{C_k(\mathbf{r}, \mathbf{b}) (\text{Im } G_P(Y, \mathbf{r}, \mathbf{b}))^k}_{F_k(Y, \mathbf{r}, \mathbf{b})} \quad (2.1)$$

where F_k is the contribution of the exchange of k Pomerons to the cross section. The total cross section σ_{tot} is therefore

$$\sigma_{\text{tot}}(Y, \mathbf{r}, \mathbf{b}) = 2 \sum_{k=1}^{\infty} (-1)^{k+1} F_k(Y, \mathbf{r}, \mathbf{b}). \quad (2.2)$$

Our approach is based on two main ideas. First, we recall that it is proven in Refs. [31,46] that the s -channel unitarity for the BFKL Pomeron has the form

$$2 \text{Im } G_P(Y, \mathbf{r}, \mathbf{b}) = \sigma_{\text{in}}^{\text{BFKL}}(Y, \mathbf{r}, \mathbf{b}), \quad (2.3)$$

where G_P is the Green’s function for the BFKL Pomeron and $\sigma_{\text{in}}^{\text{BFKL}}$ is the inelastic cross sections of produced gluons with mean multiplicity $\bar{n} = \Delta_P Y$, where $\Delta_P = 4\bar{\alpha}_S \ln 2$ is the intercept of the BFKL Pomeron. We also know that produced gluons have the Poisson distribution with this mean multiplicity (see Ref. [46] and Appendix A in Ref. [47]). Equation (2.3) gives us connection between the Pomeron Green’s function and the cross section of produced gluons and determines the so-called cut Pomeron.

The next problem is to rewrite the scattering amplitude as the sum of contributions with n -cut Pomeron (σ_n^{AGK}). The AGK cutting rules [24] solve this problem and allow us to calculate the contributions of n -cut Pomerons if we know F_k : the contribution of the exchange of k Pomerons to the cross section. They take the form

$$\begin{aligned} \sigma_n^{\text{AGK}}(Y, \mathbf{r}, \mathbf{b}) &= \sum_{k=n}^{\infty} \sigma_n^k(Y, \mathbf{r}, \mathbf{b}); \quad \text{with} \\ \sigma_n^k(Y, \mathbf{r}, \mathbf{b}) &= \begin{cases} (-1)^k (2^k - 2) F_k(Y, \mathbf{r}, \mathbf{b}) & \text{for } n = 0 \\ (-1)^{k-n} \frac{k!}{(k-n)! n!} 2^k F_k(Y, \mathbf{r}, \mathbf{b}) & \text{for } n \geq 1, \end{cases} \end{aligned} \quad (2.4)$$

where σ_0 denotes the cross section with the multiplicity of produced dipoles, which is much less than $\Delta_P Y$. In other words, it is the cross section of the diffraction production.

Finally, we can discuss the multiplicity distributions of produced gluons, calculating the cross section of productions of n gluons in the final state ($\tilde{\sigma}_n$). Our master formula for ($\tilde{\sigma}_n$) takes the form of convolution for the cross section of produced k -cut Pomerons with the Poisson distributions of gluons in these k Pomerons,

$$\underbrace{\sum_{n_i} \prod_{i=1}^k \mathcal{P}_{n_i}^P(\Delta_P Y)}_{\sum_i n_i = n} = \underbrace{\frac{(k \Delta_P Y)^n}{n!} e^{-k \Delta_P Y}}_{\text{Poisson distribution}} = \mathcal{P}_n^P(k \Delta_P Y). \quad (2.5)$$

Finally,

$$\begin{aligned} \tilde{\sigma}_n(Y, \mathbf{r}, \mathbf{b}) &= \sum_{k=1}^{\infty} \underbrace{\sigma_k^{\text{AGK}}(Y, \mathbf{r}, \mathbf{b})}_{\propto (\sigma_{\text{in}}^{\text{BFKL}}(Y, \mathbf{r}, \mathbf{b}))^k} \\ \mathcal{P}_n^P(k \Delta_P Y) &\xrightarrow{Y \gg 1} \sigma_{k=n/(\Delta_P Y)}^{\text{AGK}}(Y, \mathbf{r}, \mathbf{b}). \end{aligned} \quad (2.6)$$

In the last equation we used that the Poisson distribution with average number of gluons $k \Delta_P Y$ is a much steeper function than σ_k^{AGK} with typical k of the order of G_P .

Let us illustrate the procedure by obtaining the initial condition for σ_n^{AGK} . The initial conditions for the scattering amplitude of Eq. (1.10) can be rewritten as

$$\sigma_{\text{tot}}(Y = Y_A, \mathbf{r}, \mathbf{b}) = 2 \sum_{k=1}^{\infty} (-1)^{k+1} \underbrace{\frac{1}{k!} \left(\frac{e^\xi}{4} \right)^k}_{F_k(Y_A, \mathbf{r}, \mathbf{b})}, \quad (2.7)$$

where $F_k(Y_A, \mathbf{r}, \mathbf{b}) = \frac{1}{k!} (\text{Im } G_P(Y = Y_A, \mathbf{r}, \mathbf{b}))^k$ and $G_P(Y = Y_A, \mathbf{r}, \mathbf{b})$ are the initial conditions for the BFKL Pomeron, which is the scattering amplitude of the dipoles with exchange of two gluons. Using Eq. (2.4) we calculate the initial conditions for the cross section of n -cut Pomerons,

$$\begin{aligned} \sigma_n^{\text{AGK}}(Y = Y_A, \mathbf{r}, \mathbf{b}) &= \sum_{k=n}^{\infty} (-1)^{k-n} \frac{k!}{(k-n)! n!} 2^k F_k(Y_A, \mathbf{r}, \mathbf{b}) \\ &= \frac{(\frac{1}{2} e^\xi)^n}{n!} \exp\left(-\frac{1}{2} e^\xi\right). \end{aligned} \quad (2.8)$$

Note that, in Eq. (2.8), $\frac{1}{2} e^\xi$ is $2\sigma^{\text{BA}}$ where σ^{BA} is the cross section of two dipoles scattering in the Born approximation of perturbative QCD. In the equation, $Y_A = \ln A^{1/3}$ and A is the number of the nucleons in a nucleus. For nuclei, the b dependence of $Q_s(Y_A, \mathbf{b})$ is determined by the optical width of the nucleus $T_A(\mathbf{b})$, which gives the number of nucleons

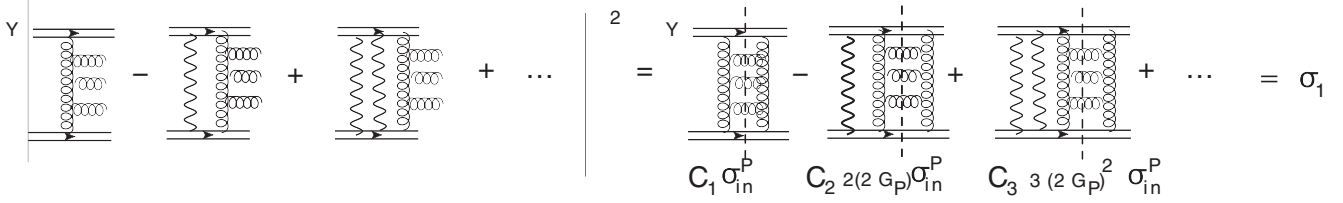


FIG. 3. σ_1^{AGK} for the scattering amplitude of Eq. (2.1). The vertical dashed lines show cut Pomerons.

at a given value of impact parameter b : $Q_s(Y_A, \mathbf{b}) = Q_0 T_A(\mathbf{b})$.

The amplitude in the form of Eq. (2.1) as well as σ_n^{AGK} have been found for the BK cascade in Ref. [25].

Therefore, in this paper we calculate³ $\sigma_n \equiv \sigma_n^{\text{AGK}}$ or, in other words, the cross section of n -cut Pomerons in the framework of the AGK approach. For the sake of illustration, we show in Fig. 3 σ_1^{AGK} for the scattering amplitude of Eq. (2.1).

B. AGK equations without AGK cutting rules

The main equation for σ_n has been written in Ref. [25], where they were derived from the AGK cutting rules. In this section, we are going to rewrite them in the dipole approach to QCD [48] without addressing the AGK cutting rules in the explicit form. We have two indications that we can do this. First, it was shown in Refs. [26,42] that this can be done for the processes of diffraction production. The equations in Ref. [25] include these processes and this indicates that we can rewrite the result of Ref. [25] in a such

way. Second, in the simple but instructive one-dimensional models for σ_n , the evolution equation can be written for σ_n , which does not address the AGK cutting rules in an explicit way [47].

As has been discussed in Refs. [26,42] the derivation of the evolution equation for the diffractive production is based on two principal ingredients. The first one is that only dipoles in the parton wave function of the fast hadron at $t \rightarrow -\infty$ in Fig. 1 can be produced at $t \rightarrow +\infty$ and measured by our detectors. This point was proven in Refs. [26,42] and the proof used the results of Ref. [49]. In these papers, it was shown that neither production of the new partons nor annihilation of the partons in the hadronic wave function contribute to process of diffractive production. Actually, the proof is general for any final state and does not depend on specific interaction between dipoles. It has been demonstrated [42] that the AGK cutting rules provide these features in the framework of the dipole approach for the diffractive production.

The equation for the S -matrix in the BFKL cascade has the following form [48] in the dipole approach to QCD:

$$\frac{\partial}{\partial Y} S(Y, \mathbf{r}, \mathbf{b}) = \frac{\bar{\alpha}_S}{2\pi} \int d^2 r' \underbrace{\frac{r^2}{r'^2 (\mathbf{r} - \mathbf{r}')^2}}_{K(\mathbf{r}, \mathbf{r}')} \left[S\left(Y, \mathbf{r}', \mathbf{b} - \frac{1}{2}(\mathbf{r} - \mathbf{r}')\right) S\left(Y, \mathbf{r} - \mathbf{r}', \mathbf{b} - \frac{1}{2}\mathbf{r}'\right) - S(Y, \mathbf{r}, \mathbf{b}) \right]. \quad (2.9)$$

The equation for the cross section of the diffractive production can be written in the same form as Eq. (2.9) but for [42]

$$S^D(Y, \mathbf{r}; \mathbf{b}) = 1 - 2N(Y, \mathbf{r}; \mathbf{b}) + \sigma_{sd}(Y, \mathbf{r}; \mathbf{b}). \quad (2.10)$$

Indeed, plugging in Eq. (2.9) S^D and rewriting the equation for $\sigma_{sd}(Y, \mathbf{r}; \mathbf{b}) = 1 - S^D(Y, \mathbf{r}; \mathbf{b})$ we obtain

$$\begin{aligned} \frac{\partial}{\partial Y} \sigma_{sd}(Y, \mathbf{r}_{01}, \mathbf{b}) &= \frac{\bar{\alpha}_S}{2\pi} \int d^2 r_2 K(\mathbf{r}_{01} | \mathbf{r}_{12}, \mathbf{r}_{02}) [\sigma_{sd}(Y, \mathbf{r}_{12}, \mathbf{b}) + \sigma_{sd}(Y, \mathbf{r}_{02}, \mathbf{b}) - \sigma_{sd}(Y, \mathbf{r}_{01}, \mathbf{b}) \\ &\quad + \sigma_{sd}(Y, \mathbf{r}_{12}, \mathbf{b}) \sigma_{sd}(Y, \mathbf{r}_{02}, \mathbf{b}) - 2\sigma_{sd}(Y, \mathbf{r}_{12}, \mathbf{b}) N(Y, \mathbf{r}_{02}, \mathbf{b}) \\ &\quad - 2N(Y, \mathbf{r}_{12}, \mathbf{b}) \sigma_{sd}(Y, \mathbf{r}_{02}, \mathbf{b}) + 2N(Y, \mathbf{r}_{12}, \mathbf{b}) N(Y, \mathbf{r}_{02}, \mathbf{b})]. \end{aligned} \quad (2.11)$$

Equation (2.11) is the equation of Ref. [42] for the cross section of the diffractive production with all possible rapidity gaps for the same scattering process. $N(Y, \mathbf{r}_{01}, \mathbf{b}) = 1 - S(Y, \mathbf{r}_{01}, \mathbf{b})$ is the imaginary part of the elastic scattering amplitude of the dipole with the size $\mathbf{r}_{01} \equiv \mathbf{r}$ and rapidity Y at the impact parameter $\mathbf{b}$. We use notations $\mathbf{r}_{02} \equiv \mathbf{r}'$ and $\mathbf{r}_{12} \equiv \mathbf{r} - \mathbf{r}'$.

³We apologize for any inconvenience that these notations may cause.

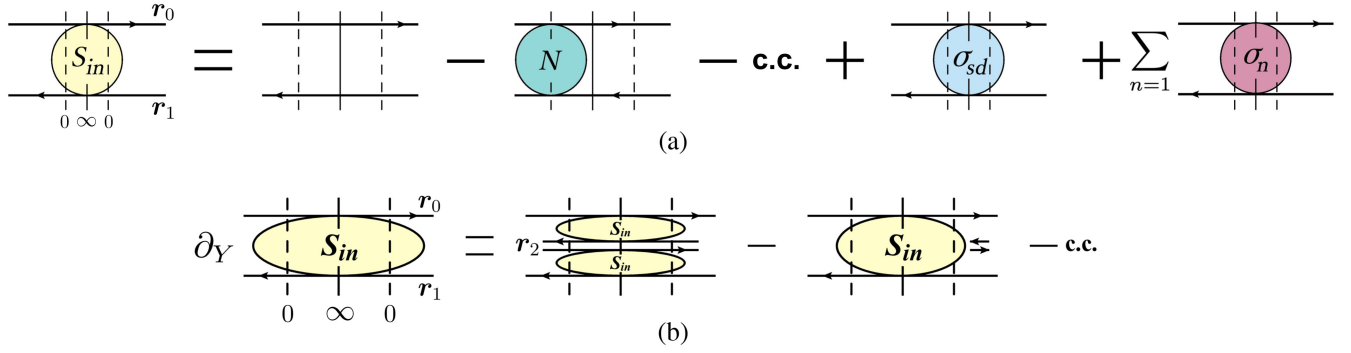


FIG. 4. (a) The graphic form of Eq. (2.12) for the interaction of one dipole which is denoted by two parallel horizontal lines with opposite directions of arrows. All notations in the caption of Fig. 5. (b) The evolution equation for S_{in} .

We propose Eq. (2.9) for $S_{in}(Y, \mathbf{r}; \mathbf{b})$ which is equal to

$$S_{in}(Y, \mathbf{r}, \mathbf{b}) = 1 - 2N(Y, \mathbf{r}, \mathbf{b}) + \sigma_{sd}(Y, \mathbf{r}, \mathbf{b}) + \sum_{n=1} \sigma_n(Y, \mathbf{r}, \mathbf{b}) \quad (2.12)$$

with additional selection for finding σ_n : the fixed multiplicity, which is equal to $n\Delta_P Y$, for each term.

The equation has the form

$$\underbrace{\frac{\partial}{\partial Y} S_{in}(Y, \mathbf{r}, \mathbf{b})}_{\text{select fixed multiplicity terms}} = \frac{\bar{\alpha}_S}{2\pi} \int d^2 r' K(\mathbf{r}, \mathbf{r}') \underbrace{\left[S_{in}\left(Y, \mathbf{r}', \mathbf{b} - \frac{1}{2}(\mathbf{r} - \mathbf{r}')\right) S_{in}\left(Y, \mathbf{r} - \mathbf{r}', \mathbf{b} - \frac{1}{2}\mathbf{r}'\right) - S_{in}(Y, \mathbf{r}, \mathbf{b}) \right]}_{\text{select fixed multiplicity terms}}. \quad (2.13)$$

In Figs. 4(a) and 4(b), one can see the graphic form of Eqs. (2.12) and (2.13), respectively.

The equation for σ_n takes the following form from Eq. (2.13):

$$\begin{aligned} \frac{\partial}{\partial Y} \sigma_n(Y, \mathbf{r}_{01}, \mathbf{b}) &= \frac{\bar{\alpha}_S}{2\pi} \int d^2 r_2 K(\mathbf{r}_{01} | \mathbf{r}_{12}, \mathbf{r}_{02}) \left[\sigma_n(Y, \mathbf{r}_{12}, \mathbf{b}) + \sigma_n(Y, \mathbf{r}_{02}, \mathbf{b}) - \sigma_n(Y, \mathbf{r}_{01}, \mathbf{b}) \right. \\ &\quad + \sigma_n(Y, \mathbf{r}_{12}, \mathbf{b}) \sigma_{sd}(Y, \mathbf{r}_{02}, \mathbf{b}) + \sigma_n(Y, \mathbf{r}_{02}, \mathbf{b}) \sigma_{sd}(Y, \mathbf{r}_{12}, \mathbf{b}) + \sum_{k=1}^{n-1} \sigma_{n-k}(Y, \mathbf{r}_{02}, \mathbf{b}) \sigma_k(Y, \mathbf{r}_{12}, \mathbf{b}) \\ &\quad \left. - 2\sigma_n(Y, \mathbf{r}_{12}, \mathbf{b}) N(Y, \mathbf{r}_{02}, \mathbf{b}) - 2\sigma_n(Y, \mathbf{r}_{02}, \mathbf{b}) N(Y, \mathbf{r}_{12}, \mathbf{b}) \right], \end{aligned} \quad (2.14)$$

where σ_{sd} is the cross section for the diffraction dissociation.

Equation (2.14) coincides with the equations that have been derived in the BFKL Pomeron calculus in QCD from the AGK cutting rules [25] (see also Ref. [50]). It is instructive to note that all these equations have a natural coefficient for each term. To illustrate this point, we present all these equations in Fig. 5. Note that all these coefficients reproduce the AGK cutting rules.

The main point is that these coefficients are natural if you account for the dipoles in the initial wave function and only they go to the detectors. Let us consider the equation of σ_2 [see Fig. 5 and Eq. (2.14) with $n = 2$]. The first stage is the decay of the dipole with the size $r_{01} \equiv r$ into two dipoles $r_{02} \equiv r'$ and $r_{12} \equiv r - r'$ which is described by $K(\mathbf{r}, \mathbf{r}')$. The first terms describe the linear evolution of σ_2 . It should be noted that we put coefficient 1 in front of this term.⁴ Actually, this term has the form (Fig. 6)

$$\frac{\bar{\alpha}_S}{2\pi} \int d^2 r_2 K(\mathbf{r}_{01} | \mathbf{r}_{12}, \mathbf{r}_{02}) [\sigma_n(Y, \mathbf{r}_{02}, \mathbf{b}) + \sigma_n(Y, \mathbf{r}_{12}, \mathbf{b}) - \sigma_n(Y, \mathbf{r}_{01}, \mathbf{b})] \quad (2.15)$$

⁴This coefficient is correct in the one-dimensional model, which could be viewed as QCD but with fixed dipole size [47].

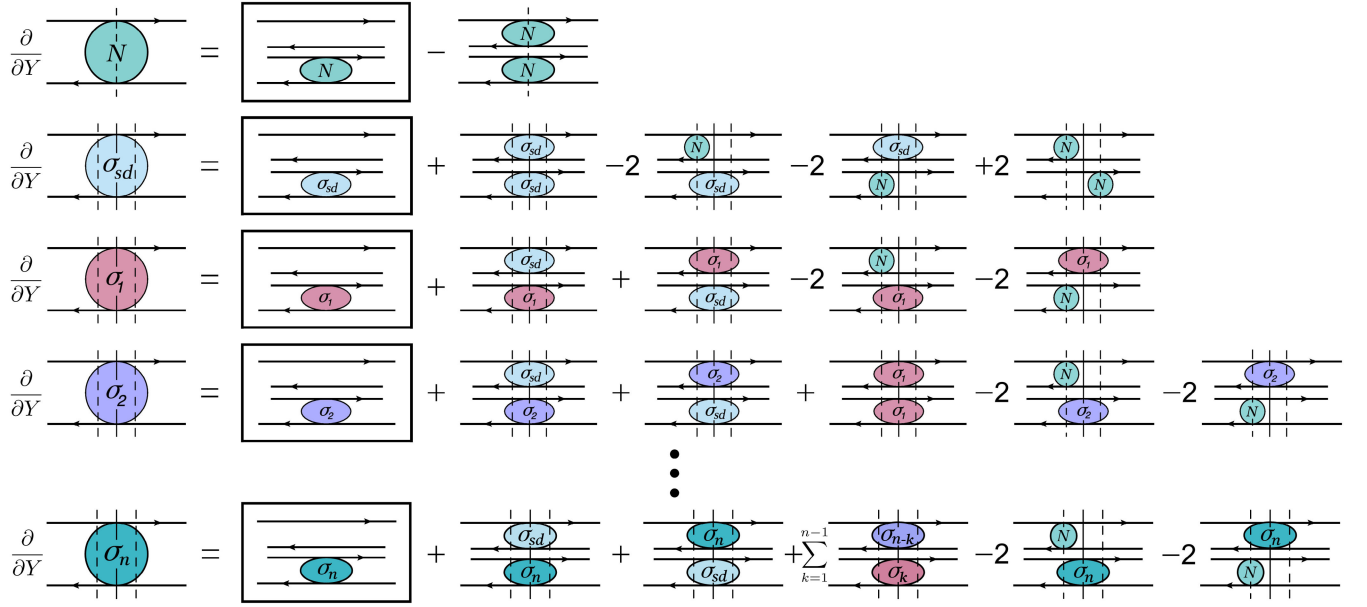


FIG. 5. The graphic form of Eq. (2.14) for the interaction of one dipole, which is denoted by two parallel horizontal lines with opposite directions of arrows. The first stage of the interaction is the decay of one dipole to two, which is described by the kernel $K(\mathbf{r}, \mathbf{r}')$. The vertical dashed lines denote time $t = 0$ in Fig. 1 where the interaction with the target occurs in the amplitude and complex conjugated amplitude. The vertical line denotes $t = +\infty$, where our detectors are placed. The Pomeron intercept is absorbed in the definition of Y . The coefficients reproduce the AGK cutting rules. Note that the linear term has the form of Eq. (2.15).

for any value of n including elastic amplitude and the cross section of diffraction production. In the second and third term, one dipole interacts with the target with σ_{sd} , while the second dipole interaction is still σ_2 . Note, that the multiplicity of the final stage is $2\Delta_P Y$ which is the same as for σ_2 . Δ_{BFKL} is the intercept of the BFKL Pomeron. We have two such diagrams, since any of two dipoles can interact with σ_{sd} . The fourth term is the interaction of both dipoles with σ_1 , which again gives us the multiplicity $2\Delta_P Y$. The

fifth term is the shadowing in the initial state induced by the elastic amplitude. Factor 2 stems from the fact that any of two dipoles can interact in the initial state. The last term is the same shadowing but in the final state. Hence, you see that all these coefficients have a simple meaning.

It is instructive to note that the equation for the inelastic cross section, $\sigma_{\text{in}}(Y, \mathbf{r}_{01}, \mathbf{b}) = \sum_{n=1}^{\infty} \sigma_n(Y, \mathbf{r}_{01}, \mathbf{b})$, which we obtain by summing $\sigma_n(Y, \mathbf{r}_{01}, \mathbf{b})$ of Eq. (2.14), has the following form:

$$\frac{\partial}{\partial Y} \sigma_{\text{in}}(Y, \mathbf{r}_{01}, \mathbf{b}) = \frac{\bar{\alpha}_S}{2\pi} \int d^2 r_2 K(\mathbf{r}_{01} | \mathbf{r}_{12}, \mathbf{r}_{02}) [\sigma_{\text{in}}(Y, \mathbf{r}_{12}, \mathbf{b}) + \sigma_{\text{in}}(Y, \mathbf{r}_{02}, \mathbf{b}) - \sigma_{\text{in}}(Y, \mathbf{r}_{01}, \mathbf{b}) + \sigma_{\text{in}}(Y, \mathbf{r}_{12}, \mathbf{b}) \sigma_{\text{in}}(Y, \mathbf{r}_{02}, \mathbf{b}) - \sigma_{\text{in}}(Y, \mathbf{r}_{12}, \mathbf{b}) (2N(Y, \mathbf{r}_{02}, \mathbf{b}) - \sigma_{sd}(Y, \mathbf{r}_{02}, \mathbf{b})) - \sigma_{\text{in}}(Y, \mathbf{r}_{02}, \mathbf{b}) (2N(Y, \mathbf{r}_{12}, \mathbf{b}) - \sigma_{sd}(Y, \mathbf{r}_{12}, \mathbf{b}))]. \quad (2.16)$$

From the s -channel unitarity constraints $2N(Y, \mathbf{r}_{ik}, \mathbf{b}) = \sigma_{sd}(Y, \mathbf{r}_{ik}, \mathbf{b}) + \sigma_{\text{in}}(Y, \mathbf{r}_{ik}, \mathbf{b})$, one can see that Eq. (2.16) takes the form

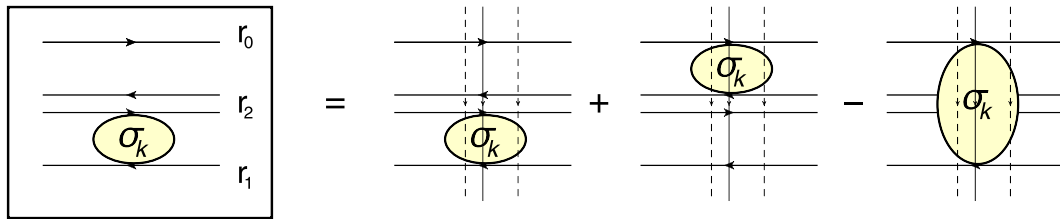


FIG. 6. The graphic form of the first term in each equation of Fig. 5 [see Eqs. (2.13) and (2.14)] for the interaction of one dipole, which is denoted by two parallel horizontal lines with opposite directions of arrows. The first stage of the interaction is the decay of one dipole to two, which is described by kernel $K(\mathbf{r}_{01} | \mathbf{r}_{12}, \mathbf{r}_{02})$. The vertical dashed lines denote time $t = 0$ in Fig. 1 where the interaction with the target occurs. The vertical line denotes $t = +\infty$, where our detectors are placed.

$$\frac{\partial}{\partial Y} \sigma_{\text{in}}(Y, \mathbf{r}_{01}, \mathbf{b}) = \frac{\bar{\alpha}_S}{2\pi} \int d^2 r_2 K(\mathbf{r}_{01} | \mathbf{r}_{12}, \mathbf{r}_{02}) [\sigma_{\text{in}}(Y, \mathbf{r}_{12}, \mathbf{b}) + \sigma_{\text{in}}(Y, \mathbf{r}_{02}, \mathbf{b}) - \sigma_{\text{in}}(Y, \mathbf{r}_{01}, \mathbf{b}) - \sigma_{\text{in}}(Y, \mathbf{r}_{12}, \mathbf{b}) \sigma_{\text{in}}(Y, \mathbf{r}_{02}, \mathbf{b})].$$

Therefore, we obtain the BK equation for σ_{in} as it is expected from the unitarity constraint.

Concluding this section, we state that we found the evolution equations for σ_n^{AGK} which reproduce the AGK cutting rules. However, it is well known that AGK cutting rules are violated in QCD [51–57]. We wish to recall that all these violations are related to the emission of the measured gluon from the triple Pomeron vertex. Calculating $\sigma_n \equiv \sigma_n^{\text{AGK}}$ we integrated over all momenta of emitted gluons and for such processes AGK cutting rules are correct.

III. LEADING TWIST BFKL KERNEL: HOMOTOPY APPROACH FOR THE SOLUTIONS

All equations for σ_n depend on the amplitude $\mathcal{N}(Y, \mathbf{r}_{ik}, \mathbf{b}) = 2N(Y, \mathbf{r}_{ik}, \mathbf{b}) - \sigma_{\text{sd}}(Y, \mathbf{r}_{ik}, \mathbf{b})$. Introducing $\Delta(Y, \mathbf{r}_{ik}, \mathbf{b}) = 1 - \mathcal{N}(Y, \mathbf{r}_{ik}, \mathbf{b})$, we rewrite Eq. (2.14) as follows:

$$\begin{aligned} \frac{\partial}{\partial Y} \sigma_n(Y, \mathbf{r}_{01}, \mathbf{b}) = \frac{\bar{\alpha}_S}{2\pi} \int d^2 r_2 K(\mathbf{r}_{01} | \mathbf{r}_{12}, \mathbf{r}_{02}) & \left[-\sigma_n(Y, \mathbf{r}_{01}, \mathbf{b}) + \sigma_n(Y, \mathbf{r}_{12}, \mathbf{b}) \Delta(Y, \mathbf{r}_{02}, \mathbf{b}) + \Delta(Y, \mathbf{r}_{12}, \mathbf{b}) \sigma_n(Y, \mathbf{r}_{02}, \mathbf{b}) \right. \\ & \left. + \sum_{k=1}^{n-1} \sigma_{n-k}(Y, \mathbf{r}_{02}, \mathbf{b}) \sigma_k(Y, \mathbf{r}_{12}, \mathbf{b}) \right]. \end{aligned} \quad (3.1)$$

The homotopy approach for finding $\Delta(Y, \mathbf{r}_{ik}, \mathbf{b})$ has been developed in our previous paper (see Ref. [2]). We wish only to point out that, for the equation of σ_n , which we consider in this paper, we find σ_{sd} with rapidity gaps larger than z_0 (see Ref. [2]). We fix $z_0 = 2$ since we can consider the Pomeron exchange only at $Y \geq 2$.

To find solutions of Eq. (3.1), we use the simplified BFKL kernel suggested in Ref. [35]. This kernel describes the high energy asymptotic solution of the nonlinear BK equation and leads to the geometric scaling behavior,

$$\chi(\gamma) = \begin{cases} \frac{1}{1-\gamma} & \text{for } z \geq \xi_0^A, \text{ summing}(z)^n; \\ \frac{1}{\gamma} & \text{for } z < \xi_0^A, \text{ summing}(-\xi)^n, \end{cases} \quad (3.2)$$

where ξ has been introduced in Eq. (1.8).

Since this kernel sums log contributions, it corresponds to the leading twist term of the full BFKL kernel. It has a very simple form in the coordinate representation [35],

$$\frac{\bar{\alpha}_S}{2\pi} \int d^2 r' K(\mathbf{r}, \mathbf{r}') \rightarrow \frac{\bar{\alpha}_S}{2} \int_{Q_s^{-2}(Y, \mathbf{b})}^{r^2} \frac{dr'^2}{r'^2} + \frac{\bar{\alpha}_S}{2} \int_{Q_s^{-2}(Y, \mathbf{b})}^{r^2} \frac{d|\mathbf{r} - \mathbf{r}'|}{|\mathbf{r} - \mathbf{r}'|^2} = \frac{\bar{\alpha}_S}{2} \int_{-\xi_s}^{\xi} d\xi_{02} + \frac{\bar{\alpha}_S}{2} \int_{-\xi_s}^{\xi} d\xi_{12} \equiv \int_{-\xi_s}^{\xi} K_{LT}(\xi, \xi') d\xi', \quad (3.3)$$

where $\xi_{ik} = \ln(r_{ik}^2 Q_s^2(Y = Y_A, \mathbf{b}))$ and $\xi_s = \bar{\alpha}_S \kappa(Y - Y_A)$. Note that the logarithms originate from the decay of a large size dipole into one small size dipole and one large size dipole [35]. However, the size of the small dipole is still larger than $1/Q_s^2 = \frac{b^4}{R^2} e^{-\xi_s}$.

The equations σ_n of Eq. (3.1) for the leading twist BFKL kernel, Eq. (3.3), take the form

$$\begin{aligned} \frac{\partial}{\partial Y} \sigma_n(Y, \mathbf{r}_{01}, \mathbf{b}) = -2\bar{\alpha}_S \ln(r_{01} Q_s(Y, \mathbf{b})) \sigma_n(Y, \mathbf{r}_{01}, \mathbf{b}) \\ + \bar{\alpha}_S \sigma_n(Y, \mathbf{r}_{01}, \mathbf{b}) \int_{\ln(Q_s^{-2}(Y, \mathbf{b}))}^{\ln(r_{01}^2)} d\ln(r_{02}^2) \Delta(Y, \mathbf{r}_{02}, \mathbf{b}) + \bar{\alpha}_S \Delta(Y, \mathbf{r}_{01}, \mathbf{b}) \int_{\ln(Q_s^{-2}(Y, \mathbf{b}))}^{\ln(r_{01}^2)} d\ln(r_{02}^2) \sigma_n(Y, \mathbf{r}_{02}, \mathbf{b}) \\ + \bar{\alpha}_S \sum_{k=1}^{n-1} \int_{\ln(Q_s^{-2}(Y, \mathbf{b}))}^{\ln(r_{01}^2)} d\ln(r_{02}^2) \sigma_{n-k}(Y, \mathbf{r}_{02}, \mathbf{b}) \sigma_k(Y, \mathbf{r}_{01}, \mathbf{b}). \end{aligned} \quad (3.4)$$

Introducing $z = \ln(r_{01}^2 Q_s^2(Y, \mathbf{b}))$ ($z' = \ln(r_{02}^2 Q_s^2(Y, \mathbf{b}))$) and $\delta\tilde{Y} = \bar{\alpha}_S(Y - Y_A)$ we rewrite Eq. (3.4) as follows:

$$\begin{aligned} \frac{\partial}{\partial \tilde{Y}} \sigma_n(\delta \tilde{Y}, z) + \kappa \frac{\partial}{\partial z} \sigma_n(\delta \tilde{Y}, z) = & -z \sigma_n(\delta \tilde{Y}, z) + \sigma_n(\delta \tilde{Y}, z) \int_{\xi_0^A}^z dz' \Delta(\delta \tilde{Y}, z') + \Delta(\delta \tilde{Y}, z) \int_{\xi_0^A}^z dz' \sigma_n(\delta \tilde{Y}, z') \\ & + \sum_{k=1}^{n-1} \int_{\xi_0^A}^z dz' \sigma_{n-k}(\delta \tilde{Y}, z') \sigma_k(\delta \tilde{Y}, z). \end{aligned} \quad (3.5)$$

In the following subsections, we develop the homotopy approach [1,22,23] for solving Eq. (3.5) for small n .

A. $\sigma_1(Y, r, b)$

1. Region I

For $z > \xi_0^A$ and $\xi < \xi_0^A$ we are in region I (see Fig. 2). For this region, we anticipate the geometric scaling (GS) behavior of σ_n and Eq. (3.5) for $n = 1$ takes the form

$$\kappa \frac{d\sigma_1(z)}{dz} = -z \sigma_1(z) + \sigma_1(z) \int_{\xi_0^A}^z dz' \Delta(z') + \Delta(z) \int_{\xi_0^A}^z dz' \sigma_1(z'). \quad (3.6)$$

Splitting the integrals into two parts, we have

$$\int_{\xi_0^A}^z dz' \Delta(z') = \underbrace{\int_{\xi_0^A}^{\infty} dz' \Delta(z')}_{=z_\Delta} - \underbrace{\int_z^{\infty} dz' \Delta(z')}_{=\tilde{\Sigma}_\Delta(z)}; \quad \int_{\xi_0^A}^z dz' \sigma_1(z') = \underbrace{\int_{\xi_0^A}^{\infty} dz' \sigma_1(z')}_{=\sigma_{0,1}} - \underbrace{\int_z^{\infty} dz' \sigma_1(z')}_{=\tilde{\Sigma}_1(z)}. \quad (3.7)$$

Using Eq. (3.7) we can rewrite Eq. (3.6) as

$$\kappa \frac{d\sigma_1(z)}{dz} = -\underbrace{(z - z_\Delta + \tilde{\Sigma}_\Delta(z))}_{=T(z)} \sigma_1(z) + \sigma_{0,1} \Delta(z) - \Delta(z) \tilde{\Sigma}_1(z). \quad (3.8)$$

In the homotopy approach, we rewrite Eq. (3.8) in the form of Eq. (1.11) with the following choice of $\mathcal{L}[\sigma_1]$ and $\mathcal{N}_\mathcal{L}[\sigma_1]$:

$$\mathcal{L}[\sigma_1] = \kappa \frac{d\sigma_1(z)}{dz} + T(z) \sigma_1(z) - \sigma_{0,1} \Delta(z), \quad (3.9a)$$

$$\mathcal{N}_\mathcal{L}[\sigma_1] = \Delta(z) \tilde{\Sigma}_1(z). \quad (3.9b)$$

Following the procedure that we have discussed in the Introduction [see Eqs. (1.12) and (1.13)] with $\mathcal{L}[\sigma_1^{(0)}] = 0$, Eq. (1.13) gives the solution to the nonlinear equation at $p = 1$. The hope is that several terms in series of Eq. (1.13) will give a good approximation to the solution of the nonlinear equation. This method has been applied to the solution of the BK and KL equations in Refs. [1,2,58] with encouraging results.

First consider $p = 0$. Equation $\mathcal{L}[\sigma_1^{(0)}] = 0$ has the form

$$\kappa \frac{d\sigma_1^{(0,I)}(z)}{dz} = -T(z) \sigma_1^{(0,I)}(z) + \sigma_{0,1} \Delta(z). \quad (3.10)$$

The solution of Eq. (3.10) consists of a homogeneous equation plus a particular solution, viz. $\sigma_1^{(0,I)}(z) = \phi_1(z) + \Delta_{\sigma_1}(z)$. The solution of the homogenous equation is given by

$$\kappa \frac{d\phi_1(z)}{dz} = -T(z) \phi_1(z); \quad \phi_1(z) = C_\phi^{(1)} \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \quad (3.11)$$

and the particular solution of the nonhomogenous equation reads

$$\Delta_{\sigma_1}(z) = \frac{\sigma_{0,1}}{\kappa} \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \int_{\xi_0^A}^z dz' \Delta(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right). \quad (3.12)$$

Finally, we write the full solution for the zeroth iteration as

$$\sigma_1^{(0,I)}(z) = \phi_1(z) + \Delta_{\sigma_1}(z) = \underbrace{\exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right)}_{\tilde{\sigma}_1^{(0)}} \underbrace{\left\{ \frac{\sigma_{0,1}}{\kappa} \int_{\xi_0^A}^z dz' \Delta(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right) + C_\phi^{(1)} \right\}}_{\tilde{\sigma}_1^{(0,I)}}. \quad (3.13)$$

From the initial condition at $z = \xi_0^A$ [see Eq. (2.8)], we find $C_\phi^{(1)} = \frac{1}{2} e^{\xi_0^A} \exp(-\frac{1}{2} e^{\xi_0^A})$. We draw your attention to the asymptotic behavior of the solution of Eq. (3.11): $\phi_1 \xrightarrow{z \gg 1} C_\phi^{(1)} \exp(-\frac{(z-z_\Delta)^2}{2\kappa} + \frac{(\xi_0^A - z_\Delta)^2}{2\kappa})$. This asymptotic behavior coincides with the solution with the general BFKL kernel [35].

For the next homotopy iteration, we need to account for the linear terms in p in Eq. (1.12),

$$\mathcal{L}[\sigma_1^{(0)} + p\sigma_1^{(1)}] + p\mathcal{N}_\mathcal{L}[\sigma_1^{(0)}] = 0. \quad (3.14)$$

The equation takes the form

$$\kappa \frac{d\sigma_1^{(1,I)}(z)}{dz} = -T(z)\sigma_1^{(1,I)}(z) - \Delta(z)\tilde{\Sigma}_1^{(0)}(z), \quad (3.15)$$

where $\tilde{\Sigma}_1^{(0)}(z) = \int_z^\infty dz' \sigma_1^{(0)}(z')$. The solution of Eq. (3.15) takes the form

$$\sigma_1^{(1,I)}(z) = -\frac{1}{\kappa} \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \int_{\xi_0^A}^z dz' \Delta(z') \tilde{\Sigma}_1^{(0)}(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right). \quad (3.16)$$

In general, the equation for the p iteration ($p \geq 1$) in region I takes the form

$$\kappa \frac{d\sigma_1^{(p,I)}(z)}{dz} = -T(z)\sigma_1^{(p,I)}(z) - \Delta(z)\tilde{\Sigma}_1^{(p-1)}(z) \quad (3.17)$$

with solution

$$\sigma_1^{(p,I)}(z) = -\frac{1}{\kappa} \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \int_{\xi_0^A}^z dz' \Delta(z') \tilde{\Sigma}_1^{(p-1)}(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right). \quad (3.18)$$

2. Region II

In region II for $\xi > \xi_0^A$ (see Fig. 2) there is no GS behavior of σ_n and we need to find a function that depends both on $\delta\tilde{Y}$ and ξ . In this region, Eq. (3.5) for $n = 1$ takes the form

$$\frac{\partial \sigma_1(\delta\tilde{Y}, z)}{\partial \delta\tilde{Y}} + \kappa \frac{\partial \sigma_1(\delta\tilde{Y}, z)}{\partial z} = -z\sigma_1(\delta\tilde{Y}, z) + \sigma_1(\delta\tilde{Y}, z) \int_{\xi_0^A}^z dz' \Delta(\delta\tilde{Y}, z') + \Delta(\delta\tilde{Y}, z) \int_{\xi_0^A}^z dz' \sigma_1(\delta\tilde{Y}, z'). \quad (3.19)$$

Splitting the integrals in the following form:

$$\int_{\xi_0^A}^z dz' \Delta(\delta\tilde{Y}, z') = \underbrace{\int_{\xi_0^A}^{\infty} dz' \Delta(\delta\tilde{Y}, z')}_{=z_{\Delta}(\delta\tilde{Y})} - \underbrace{\int_z^{\infty} dz' \Delta(\delta\tilde{Y}, z')}_{=\tilde{\Sigma}_{\Delta}(\delta\tilde{Y}, z)}; \quad (3.20a)$$

$$\int_{\xi_0^A}^z dz' \sigma_1(\delta\tilde{Y}, z') = \underbrace{\int_{\xi_0^A}^{\infty} dz' \sigma_1(z')}_{=\sigma_{0,1}} + \underbrace{\int_{\xi_0^A}^{\infty} dz' (\sigma_1(\delta\tilde{Y}, z') - \sigma_1(z'))}_{=\delta\Sigma_1(\delta\tilde{Y})} - \underbrace{\int_z^{\infty} dz' \sigma_1(\delta\tilde{Y}, z')}_{=\tilde{\Sigma}_1(\delta\tilde{Y}, z)} \quad (3.20b)$$

we can rewrite Eq. (3.19) as

$$\begin{aligned} \frac{\partial \sigma_1(\delta\tilde{Y}, z)}{\partial \delta\tilde{Y}} + \kappa \frac{\partial \sigma_1(\delta\tilde{Y}, z)}{\partial z} = & - \underbrace{(z - z_{\Delta}(\delta\tilde{Y}) + \tilde{\Sigma}_{\Delta}(\delta\tilde{Y}, z))}_{=T(\delta\tilde{Y}, z)} \sigma_1(\delta\tilde{Y}, z) + \sigma_{0,1} \Delta(\delta\tilde{Y}, z) \\ & - \Delta(\delta\tilde{Y}, z) \delta\Sigma_1(\delta\tilde{Y}) + \Delta(\delta\tilde{Y}, z) \tilde{\Sigma}_1(\delta\tilde{Y}, z). \end{aligned} \quad (3.21)$$

Developing the homotopy approach in this region in a similar way as in region I, we choose

$$\mathcal{L}[\sigma_1] = \frac{\partial \sigma_1(\delta\tilde{Y}, z)}{\partial \delta\tilde{Y}} + \kappa \frac{\partial \sigma_1(\delta\tilde{Y}, z)}{\partial z} + T(\delta\tilde{Y}, z) \sigma_1(\delta\tilde{Y}, z) - \sigma_{0,1} \Delta(\delta\tilde{Y}, z), \quad (3.22a)$$

$$\mathcal{N}_{\mathcal{L}}[\sigma_1] = \Delta(\delta\tilde{Y}, z) \delta\Sigma_1(\delta\tilde{Y}) - \Delta(\delta\tilde{Y}, z) \tilde{\Sigma}_1(\delta\tilde{Y}, z). \quad (3.22b)$$

For the zeroth iteration (i.e., for the solution of the equation $\mathcal{L}[\sigma_1^{(0)}] = 0$), we search a solution in the form $\sigma_1^{(0,II)}(\delta\tilde{Y}, z) = \phi_1(\delta\tilde{Y}, z) + \Delta_{\sigma_1}(\delta\tilde{Y}, z)$. The homogeneous equation for $\phi_1(\delta\tilde{Y}, z)$ looks as follows:

$$\frac{\partial \phi_1(\delta\tilde{Y}, z)}{\partial \delta\tilde{Y}} + \kappa \frac{\partial \phi_1(\delta\tilde{Y}, z)}{\partial z} = -T(\delta\tilde{Y}, z) \phi_1(\delta\tilde{Y}, z) \quad (3.23)$$

with the general solution

$$\phi_1(\delta\tilde{Y}, z) = \Phi_1(-\kappa\delta\tilde{Y} + z) \exp \left(\underbrace{- \int_0^{\delta\tilde{Y}} d\delta\tilde{Y}' T(\delta\tilde{Y}', -\kappa(\delta\tilde{Y} - \delta\tilde{Y}') + z)}_{=\sigma_1^{0'}(\delta\tilde{Y}, z)} \right) \quad (3.24)$$

with arbitrary function Φ_1 . For the particular solution, searching in the form $\Delta_{\sigma_1}(\delta\tilde{Y}, z) = \sigma_1^{0'}(\delta\tilde{Y}, z) \tilde{\sigma}_1^{0'}(\delta\tilde{Y}, z)$, we have the following equation for $\tilde{\sigma}_1^{0'}(\delta\tilde{Y}, z)$:

$$\frac{\partial \tilde{\sigma}_1^{0'}(\delta\tilde{Y}, z)}{\partial \delta\tilde{Y}} + \kappa \frac{\partial \tilde{\sigma}_1^{0'}(\delta\tilde{Y}, z)}{\partial z} = \underbrace{\frac{\sigma_{0,1}}{\sigma_1^{0'}(\delta\tilde{Y}, z)} \Delta(\delta\tilde{Y}, z)}_{=R_1(\delta\tilde{Y}, z)}. \quad (3.25)$$

Using the ω representation for $\tilde{\sigma}_1^{0'}$,

$$\tilde{\sigma}_1^{0'}(\delta\tilde{Y}, z) = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} e^{\omega\delta\tilde{Y}} \hat{\sigma}_1^{0'}(\omega, z) \quad (3.26)$$

we obtain

$$\omega \hat{\sigma}_1^{0'}(\omega, z) + \kappa \frac{\partial \hat{\sigma}_1^{0'}(\omega, z)}{\partial z} = \underbrace{\frac{\sigma_{0,1}}{\sigma_1^{0'}(\omega, z)} \Delta(\omega, z)}_{=R_1(\omega, z)}. \quad (3.27)$$

As function of $\delta\tilde{Y}$ Eq. (3.27) gives

$$\hat{\sigma}_1^{0'}(\delta\tilde{Y}, z) = \frac{1}{\kappa} \int_{\xi_0^A}^z dz' R_1\left(\delta\tilde{Y} - \frac{z - z'}{\kappa}, z'\right) + C_{\sigma_1}. \quad (3.28)$$

Therefore, the general solution to Eq. (3.11) takes the form

$$\sigma_1^{(0,II)}(\delta\tilde{Y}, z) = \Phi_1(-\kappa\delta\tilde{Y} + z) \sigma_1^{0'}(\delta\tilde{Y}, z) + \sigma_1^{0'}(\delta\tilde{Y}, z) \hat{\sigma}_1^{0'}(\delta\tilde{Y}, z). \quad (3.29)$$

The solution of Eq. (3.29) has to satisfy the initial condition of Eq. (2.8) at $\delta\tilde{Y} = 0$ and boundary conditions that have the following form:

$$\sigma_1^{(0,I)}(z = \kappa\delta\tilde{Y} + \xi_0^A) = \sigma_1^{(0,II)}(\delta\tilde{Y}, z = \kappa\delta\tilde{Y} + \xi_0^A). \quad (3.30)$$

The initial conditions can determine the function $\Phi_1(-\kappa\delta\tilde{Y} + z) = \Phi_1(\xi)$, viz.

$$\sigma_1^{0'}(\delta\tilde{Y} = 0, z = \xi) \Phi_1(\xi) = \frac{1}{2} e^\xi \exp\left(-\frac{1}{2} e^\xi\right) - \sigma_1^{0'}(\delta\tilde{Y} = 0, z = \xi) \hat{\sigma}_1^{0'}(\delta\tilde{Y} = 0, z = \xi). \quad (3.31)$$

The boundary condition at $\xi = \xi_0^A$ allows us to determine C_{σ_1} . However, before doing this, we need to consider the integration over z' in Eq. (3.28). First, the argument $\delta\tilde{Y} - \frac{z - z'}{\kappa}$ has to be larger than 0. Indeed, only for such z' we are in region II. For $\delta\tilde{Y} - \frac{z - z'}{\kappa} < 0$, we are in region I for Δ and Δ does not depend on $\delta\tilde{Y}$. Therefore, we can replace Eq. (3.28) by the following expression:

$$\hat{\sigma}_1^{0'}(\delta\tilde{Y}, z) = \frac{1}{\kappa} \int_{\kappa\delta\tilde{Y} + \xi_0^A}^z dz' \frac{\sigma_{0,1}}{\sigma_1^{0'}(\delta\tilde{Y}, z)} \Delta\left(\delta\tilde{Y} - \frac{z - z'}{\kappa}, z'\right) + \underbrace{\frac{1}{\kappa} \int_{\xi_0^A}^{\kappa\delta\tilde{Y} + \xi_0^A} dz' \frac{\sigma_{0,1}}{\tilde{\sigma}_1^{(0)}(z')} \Delta(z')}_{=\tilde{\sigma}_1^{(0,I)}(\kappa\delta\tilde{Y} + \xi_0^A)} + C_{\sigma_1}. \quad (3.32)$$

$\sigma(0)_{n(z)}$ and $\sigma(p)_{n(z)}$ are taken from Eq. (3.62) and (3.69), respectively.

Indeed, from Eq. (3.30) we have

$$\begin{aligned} \sigma_1^{(0,I)}(z = \kappa\delta\tilde{Y} + \xi_0^A) &= \tilde{\sigma}_1^{(0)}(\kappa\delta\tilde{Y} + \xi_0^A) \left(\tilde{\sigma}_1^{(0,I)}(\kappa\delta\tilde{Y} + \xi_0^A) + C_\phi^{(1)} \right) \\ &= \sigma_1^{(0,II)}(\delta\tilde{Y}, z = \kappa\delta\tilde{Y} + \xi_0^A) = \sigma_1^{0'}(\delta\tilde{Y}, \kappa\delta\tilde{Y} + \xi_0^A) \left\{ \Phi_1(\xi_0^A) + \tilde{\sigma}_1^{(0,I)}(\kappa\delta\tilde{Y} + \xi_0^A) + C_{\sigma_1} \right\}. \end{aligned} \quad (3.33)$$

Constant C_{σ_1} can be determined from Eq. (3.33) with solution $C_{\sigma_1} = C_\phi^{(1)} - \Phi_1(\xi_0^A)$.

The equation for the first iteration in the homotopy approach takes the form

$$\frac{\partial \sigma_1^{(1,II)}(\delta\tilde{Y}, z)}{\partial \delta\tilde{Y}} + \kappa \frac{\partial \sigma_1^{(1,II)}(\delta\tilde{Y}, z)}{\partial z} = -T(\delta\tilde{Y}, z) \sigma_1^{(1,II)}(\delta\tilde{Y}, z) - \Delta(\delta\tilde{Y}, z) (\delta\Sigma_1^{(0)}(\delta\tilde{Y}) - \tilde{\Sigma}_1^{(0)}(\delta\tilde{Y}, z)). \quad (3.34)$$

Searching for the solution in the form: $\sigma_1^{(1,II)}(Y, r_{01}, b) = \sigma_1^{0'}(\delta\tilde{Y}, z) \sigma_1^{1'}(\delta\tilde{Y}, z)$, we obtain

$$\frac{\partial \sigma_1^{1'}(\delta\tilde{Y}, z)}{\partial \delta\tilde{Y}} + \kappa \frac{\partial \sigma_1^{1'}(\delta\tilde{Y}, z)}{\partial z} = - \underbrace{\frac{\Delta(\delta\tilde{Y}, z) (\delta\Sigma_1^{(0)}(\delta\tilde{Y}) - \tilde{\Sigma}_1^{(0)}(\delta\tilde{Y}, z))}{\sigma_1^{0'}(\delta\tilde{Y}, z)}}_{=R_1^{(1)}(\delta\tilde{Y}, z)}. \quad (3.35)$$

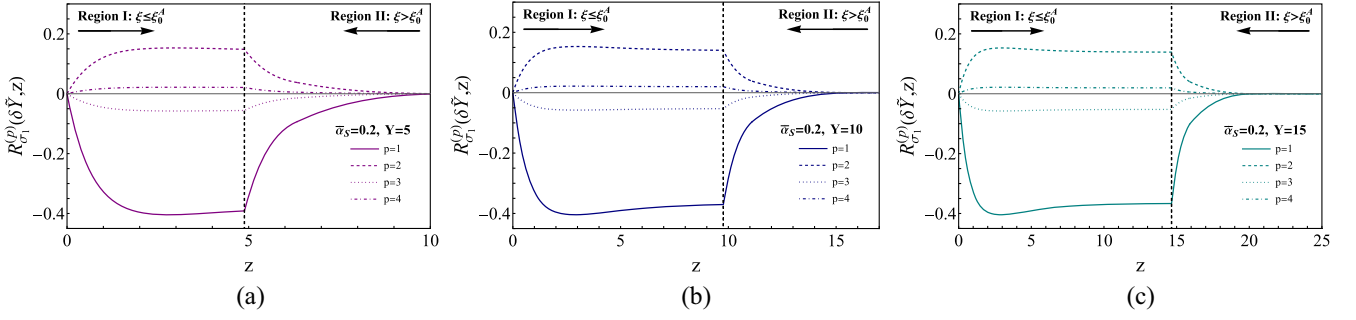


FIG. 7. The ratio of $R_{\sigma_1^{(p)}}(\delta\tilde{Y}, z)$ versus z at different values of $\delta\tilde{Y}$. $N_0 = 0.22$, $\bar{\gamma} = 0.63$, $\xi_0^A = 0$, $z_0 = 2$, $z_\Delta = 0.87$. (a)–(c) show the ratio $R_{\sigma_1^{(p)}}$ as a function of Y .

Using the ω representation for $\sigma_1^{1'}$,

$$\sigma_1^{1'}(\delta\tilde{Y}, z) = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} e^{\omega\delta\tilde{Y}} \tilde{\sigma}_1^{1'}(\omega, z) \quad (3.36)$$

we obtain

$$\omega \tilde{\sigma}_1^{1'}(\omega, z) + \kappa \frac{\partial \tilde{\sigma}_1^{1'}(\omega, z)}{\partial z} = R_1^{(1)}(\omega, z) \quad (3.37)$$

with solution

$$\sigma_1^{1'}(\delta\tilde{Y}, z) = \frac{1}{\kappa} \int_0^z dz' R_1^{(1)}\left(\delta\tilde{Y} - \frac{z-z'}{\kappa}, z'\right) + C_{\sigma_1}^{(1)}. \quad (3.38)$$

In general, for the p iteration ($p \geq 1$) in region II we have

$$\begin{aligned} & \frac{\partial \sigma_1^{(p,II)}(\delta\tilde{Y}, z)}{\partial \delta\tilde{Y}} + \kappa \frac{\partial \sigma_1^{(p,II)}(\delta\tilde{Y}, z)}{\partial z} \\ &= -T(\delta\tilde{Y}, z) \sigma_1^{(p,II)}(\delta\tilde{Y}, z) - \Delta(\delta\tilde{Y}, z) (\delta\Sigma_1^{(p-1)}(\delta\tilde{Y}) \\ & \quad - \tilde{\Sigma}_1^{(p-1)}(\delta\tilde{Y}, z)). \end{aligned} \quad (3.39)$$

Searching in the form $\sigma_1^{(p,II)}(\delta\tilde{Y}, z) = \sigma_1^{(p,I)}(\delta\tilde{Y}, z) \sigma_1^{p'}(\delta\tilde{Y}, z)$, we obtain

$$\sigma_1^{p'}(\delta\tilde{Y}, z) = \frac{1}{\kappa} \int_0^z dz' R_1^{(p)}(\delta\tilde{Y}, z') + C_{\sigma_1}^{(p)}, \quad (3.40)$$

where $C_{\sigma_1}^{(p)}$ is found by requiring that $\sigma_1^{(p,II)}(\delta\tilde{Y} = 0, z = \xi) = 0$.

In Fig. 7, we plot $R_{\sigma_1^{(p)}}(\delta\tilde{Y}, z) = \frac{\sigma_1^{(p,II)}(\delta\tilde{Y}, z)}{\sigma_1^{(0,II)}(\delta\tilde{Y}, z)} \Theta(\xi_0^A - \xi) + \frac{\sigma_1^{(p,II)}(\delta\tilde{Y}, z)}{\sigma_1^{(0,II)}(\delta\tilde{Y}, z)} \Theta(\xi - \xi_0^A)$ up to fourth iteration. One can see that the fourth iteration gives an accuracy less than 2%. Therefore, our choice of $\mathcal{L}$ and $\mathcal{N}_{\mathcal{L}}$ looks rather good, making the homotopy approach a regular procedure of getting the solution.

3. Corrections for the general BFKL kernel

For the general BFKL kernel, we need to add corrections for the linear equation in the first iteration. They lead to the following contribution to the linear part of the equation for σ_1 :

$$\Delta I = \int d^2 r' \{K(\mathbf{r}, \mathbf{r}') - K_{LT}(\mathbf{r}, \mathbf{r}')\} \sigma_1^{(0)}(\delta\tilde{Y}, \mathbf{r}; \mathbf{b}). \quad (3.41)$$

In Fig. 8, we plot the ratio of Eq. (3.41) to the contribution in the leading twist kernel approach, viz.

$$R(\delta\tilde{Y}, \mathbf{r}, \mathbf{b}) = \frac{\Delta I(\delta\tilde{Y}, \mathbf{r}, \mathbf{b})}{\int d^2 r' K(\mathbf{r}, \mathbf{r}') \sigma_1^{(0)}(\delta\tilde{Y}, \mathbf{r}', \mathbf{b})}. \quad (3.42)$$

One can see that these corrections are small, especially at large values of z . It does not look surprising since the solution $\propto \exp(-\frac{z^2}{2\kappa})$ stems from the general BFKL kernel.

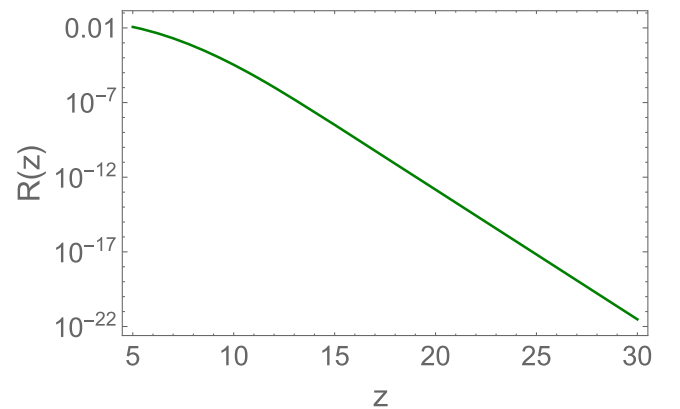


FIG. 8. The ratio R of Eq. (3.42) versus z in region I of Fig. 2.

B. $\sigma_2(Y, r, b)$

1. Region I

From Eq. (3.5) we obtain the following equation for σ_2 :

$$\kappa \frac{d\sigma_2(z)}{dz} = -z\sigma_2(z) + \sigma_2(z) \int_{\xi_0^A}^z dz' \Delta(z') + \Delta(z) \int_{\xi_0^A}^z dz' \sigma_2(z') + \int_{\xi_0^A}^z dz' \sigma_1(z') \sigma_1(z). \quad (3.43)$$

As discussed in the previous section, $\sigma_1(Y_A, r_{12}, b) \sigma_1(Y_A, r_{02}, b) \sim \sigma_2(Y_A r_{01}, b)$. Bearing this in mind, we suggest to add the contribution of this term to the definition of the operator $\mathcal{L}$,

$$\mathcal{L}[\sigma_2] = \kappa \frac{d\sigma_2(z)}{dz} + T(z)\sigma_2(z) - U_2(z), \quad (3.44a)$$

$$\mathcal{N}_{\mathcal{L}}[\sigma_2] = \Delta(z) \tilde{\Sigma}_2(z), \quad (3.44b)$$

where we have defined $U_2(z) = \sigma_{0,2} \Delta(z) + \int_{\xi_0^A}^z dz' \sigma_1(z') \sigma_1(z)$ and $\tilde{\Sigma}_2(z) = \int_z^\infty dz' \sigma_2(z')$. The solution to $\mathcal{L}[\sigma_2] = 0$ has the form $\sigma_2^{(0,I)}(z) = \phi_2(z) + \Delta_{\sigma_2}(z)$, where

$$\kappa \frac{d\phi_2(z)}{dz} = -T(z)\phi_2(z); \quad \phi_2(z) = C_\phi^{(2)} \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \quad (3.45)$$

and the particular solution is given by

$$\Delta_{\sigma_2}(z) = \frac{1}{\kappa} \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \int_{\xi_0^A}^z dz' U_2(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right). \quad (3.46)$$

Finally,

$$\sigma_2^{(0,I)}(z) = \phi_2(z) + \Delta_{\sigma_2}(z) = \underbrace{\exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right)}_{\tilde{\sigma}_2^{(0)}} \underbrace{\left\{ \frac{1}{\kappa} \int_{\xi_0^A}^z dz' U_2(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right) + C_\phi^{(2)} \right\}}_{\tilde{\sigma}_2^{(0,I)}}. \quad (3.47)$$

Using the initial condition of Eq. (2.8) at $\xi = \xi_0^A$, we find $C_\phi^{(2)} = \frac{1}{8} e^{2\xi_0^A} \exp(-\frac{1}{2} e^{\xi_0^A})$.

Performing the following iterations in a similar way as done for σ_1 , we obtain ($p \geq 1$)

$$\sigma_2^{(p,I)}(z) = -\frac{1}{\kappa} \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \int_{\xi_0^A}^z dz' U_2(z') \tilde{\Sigma}_2^{(p-1)}(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right). \quad (3.48)$$

2. Region II

In region II, Eq. (3.44a) takes the form

$$\frac{\partial \sigma_2^{(0,II)}(\delta \tilde{Y}, z)}{\partial \delta \tilde{Y}} + \kappa \frac{\partial \sigma_2^{(0,II)}(\delta \tilde{Y}, z)}{\partial z} = -T(\delta \tilde{Y}, z) \sigma_2^{(0,II)}(\delta \tilde{Y}, z) + U_2(\delta \tilde{Y}, z). \quad (3.49)$$

The solution to the homogeneous part of this equation has the same form as Eq. (3.24),

$$\Phi_2(-\kappa \delta \tilde{Y} + z) \sigma_2^{0'}(\delta \tilde{Y}, z) = \Phi_2(-\kappa \delta \tilde{Y} + z) \underbrace{\exp\left(-\int_0^{\delta \tilde{Y}} d\delta \tilde{Y}' T(\delta \tilde{Y}', -\kappa(\delta \tilde{Y} - \delta \tilde{Y}') + z)\right)}_{=\sigma_2^{0'}(\delta \tilde{Y}, z)} \quad (3.50)$$

with arbitrary function Φ_2 .

The general solution to Eq. (3.49) has the form $\sigma_2^{0'}(\delta\tilde{Y}, z)\tilde{\sigma}_2^{0'}(\delta\tilde{Y}, z)$ with the following equation for $\tilde{\sigma}_2^{0'}(\delta\tilde{Y}, z)$:

$$\frac{\partial \tilde{\sigma}_2^{0'}(\delta\tilde{Y}, z)}{\partial \delta\tilde{Y}} + \kappa \frac{\partial \tilde{\sigma}_2^{0'}(\delta\tilde{Y}, z)}{\partial z} = \underbrace{\frac{U_2(\delta\tilde{Y}, z)}{\sigma_2^{0'}(\delta\tilde{Y}, z)}}_{R_2(\delta\tilde{Y}, z)}. \quad (3.51)$$

The solution to Eq. (3.51) can be obtained following the steps of Eqs. (3.26) and (3.27). It has the form

$$\tilde{\sigma}_2^{0'}(\delta\tilde{Y}, z) = \frac{1}{\kappa} \int_{\xi_0^A}^z dz' R_2\left(\delta\tilde{Y} - \frac{z-z'}{\kappa}, z'\right) + C_{\sigma_2}, \quad (3.52)$$

which leads to the general solution to Eq. (3.49),

$$\sigma_2^{(0,II)}(\delta\tilde{Y}, z) = \Phi_2(-\kappa\delta\tilde{Y} + z)\sigma_2^{0'}(\delta\tilde{Y}, z) + \sigma_2^{0'}(\delta\tilde{Y}, z)\tilde{\sigma}_2^{0'}(\delta\tilde{Y}, z). \quad (3.53)$$

Equation (3.53) has to satisfy the initial condition of Eq. (2.8) at $\delta\tilde{Y} = 0$ and boundary conditions that have the following form:

$$\sigma_2^{(0,I)}(z = \kappa\delta\tilde{Y} + \xi_0^A) = \sigma_2^{(0,II)}(\delta\tilde{Y}, z = \kappa\delta\tilde{Y} + \xi_0^A). \quad (3.54)$$

The initial conditions determine the function $\Phi_2(-\kappa\delta\tilde{Y} + z) = \Phi_2(\xi)$, viz.

$$\sigma_2^{0'}(\delta\tilde{Y} = 0, z = \xi)\Phi_2(\xi) = \frac{1}{8}e^{2\xi}\exp\left(-\frac{1}{2}e^\xi\right) - \sigma_2^{0'}(\delta\tilde{Y} = 0, z = \xi)\tilde{\sigma}_2^{0'}(\delta\tilde{Y} = 0, z = \xi). \quad (3.55)$$

We need to use Eq. (3.32), which looks as follows for σ_2 :

$$\tilde{\sigma}_2^{0'}(\delta\tilde{Y}, z) = \frac{1}{\kappa} \int_{\kappa\delta\tilde{Y} + \xi_0^A}^z dz' R_2\left(\delta\tilde{Y} - \frac{z-z'}{\kappa}, z'\right) + \frac{1}{\kappa} \int_{\xi_0^A}^{\kappa\delta\tilde{Y} + \xi_0^A} dz' \frac{U_2(z')}{\tilde{\sigma}_2^{(0)}(z')} + C_{\sigma_2}. \quad (3.56)$$

From Eq. (3.56), our solution to Eq. (3.54) is $C_{\sigma_2} = C_\phi^{(2)} - \Phi_2(\xi_0^A)$.

For iterations $p \geq 1$, one finds a similar expression to that obtained for $\sigma_1^{(p,II)}$. Searching in the form $\sigma_2^{(p,II)}(\delta\tilde{Y}, z) = \sigma_2^{0'}(\delta\tilde{Y}, z)\sigma_2^{p'}(\delta\tilde{Y}, z)$, we obtain

$$\sigma_2^{p'}(\delta\tilde{Y}, z) = \frac{1}{\kappa} \int_0^z dz' R_2^{(p)}(\delta\tilde{Y}, z') + C_{\sigma_2}^{(p)}, \quad (3.57)$$

where $R_2^{(p)}(\delta\tilde{Y}, z) = -\frac{\mathcal{N}_c[\sigma_2(\delta\tilde{Y}, z)]}{\sigma_2^{0'}(\delta\tilde{Y}, z)}$ and $C_{\sigma_2}^{(p)}$ is found by requiring that $\sigma_2^{(p,II)}(\delta\tilde{Y} = 0, z = \xi) = 0$.

The numerical calculations of $R_2^{(p)}(\delta\tilde{Y}, z) = \frac{\sigma_2^{(p,I)}(\delta\tilde{Y}, z)}{\sigma_2^{(0,I)}(\delta\tilde{Y}, z)}\Theta(\xi_0^A - \xi) + \frac{\sigma_2^{(p,II)}(\delta\tilde{Y}, z)}{\sigma_2^{(0,II)}(\delta\tilde{Y}, z)}\Theta(\xi - \xi_0^A)$ are shown in Fig. 9. One can see that they show the same pattern as the calculation for σ_1 (see Fig. 7): The first iteration is about 50% of the first at low values of z but decreases faster than the corrections of σ_1 . Also, we see the fourth iteration gives an accuracy of less than 3%.

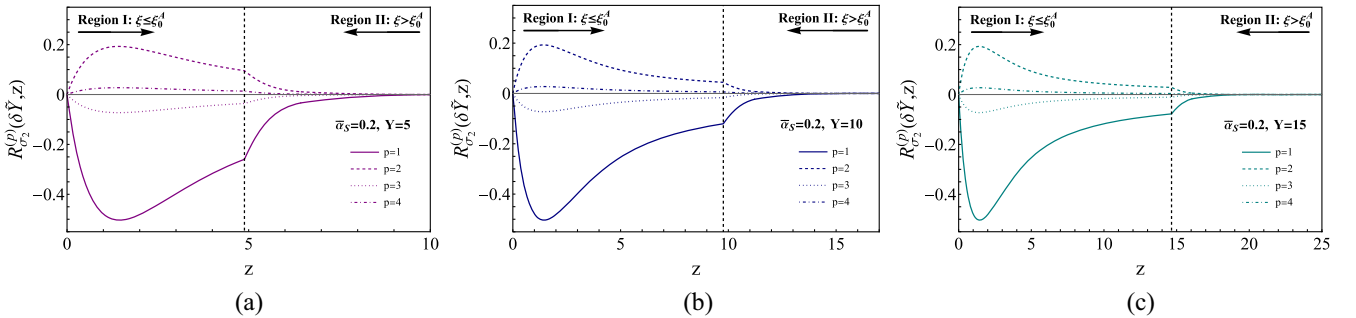


FIG. 9. The ratio of $R_2^{(p)}(\delta\tilde{Y}, z)$ versus z at different values of $\delta\tilde{Y}$. $N_0 = 0.22$, $\bar{\gamma} = 0.63$, $\xi_0^A = 0$, $z_0 = 2$, $z_\Delta = 0.87$. (a)–(c) show the ratio $R_2^{(p)}$ as a function of Y .

C. $\sigma_3(Y, r, b)$ and $\sigma_n(Y, r, b)$

1. Region I

Equation (2.14) for $\sigma_3(Y, r_{01}; b)$ has the form

$$\begin{aligned} \frac{\partial \sigma_3(Y, r_{01}, b)}{\partial Y} = & \frac{\bar{\alpha}_S}{2\pi} \int_{\substack{r_{02} \gg 1/Q_s(Y) \\ r_{12} \gg 1/Q_s(Y)}} d^2 r_2 K(r_{01}|r_{12}, r_{02}) \{ \sigma_3(Y, r_{12}, b) + \sigma_3(Y, r_{02}, b) - \sigma_3(Y, r_{01}, b) \\ & + \sigma_3(Y, r_{12}, b) \sigma_{sd}(Y, r_{02}, b) + \sigma_3(Y, r_{02}, b) \sigma_{sd}(Y, r_{12}, b) \\ & + \sigma_2(Y, r_{12}, b) \sigma_1(Y, r_{02}, b) + \sigma_2(Y, r_{02}, b) \sigma_1(Y, r_{12}, b) \\ & - 2\sigma_3(Y, r_{12}, b) N(Y, r_{02}, b) - 2\sigma_3(Y, r_{02}, b) N(Y, r_{12}, b) \}. \end{aligned} \quad (3.58)$$

Our choice of $\mathcal{L}(\sigma_3)$ follows the same reasoning as in the previous sections, but with the inclusion of an additional term $\propto 2\sigma_2(Y, r; b)\sigma_1(Y, r; b)$. The equation in region I takes the form

$$\kappa \frac{\partial \sigma_3^{(0,I)}(z)}{\partial z} = -T(z) \sigma_3^{(0,I)}(z) + \underbrace{\sigma_{0,3} \Delta(z) + \sigma_1(z) \int_{\xi_0^A}^z dz' \sigma_2(z') + \sigma_2(z) \int_{\xi_0^A}^z dz' \sigma_1(z')}_{U_3(z)} \quad (3.59)$$

with solution

$$\sigma_3^{(0,I)}(z) = \phi_3(z) + \Delta_{\sigma_3}(z) = \underbrace{\exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right)}_{\tilde{\sigma}_3^{(0)}} \left\{ \underbrace{\frac{1}{\kappa} \int_{\xi_0^A}^z dz' U_3(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right)}_{\tilde{\sigma}_3^{(0,I)}} + C_\phi^{(3)} \right\}. \quad (3.60)$$

Using the initial condition of Eq. (2.8) at $\xi = \xi_0^A$, we find $C_\phi^{(3)} = \frac{1}{48} e^{3\xi_0^A} \exp(-\frac{1}{2} e^{\xi_0^A})$.

Equation (3.60) can be easily generalized for $\sigma_n^{(0,I)}(Y, r; b)$ and it takes the form

$$\kappa \frac{\partial \sigma_n^{(0,I)}(z)}{\partial z} = -T(z) \sigma_n^{(0,I)}(z) + \underbrace{\sigma_{0,n} \Delta(z) + \sum_{k=1}^{n-1} \int_{\xi_0^A}^z dz' \sigma_{n-k}(z') \sigma_k(z')}_{U_n(z)} \quad (3.61)$$

with the following solution:

$$\sigma_n^{(0,I)}(z) = \phi_n(z) + \Delta_{\sigma_n}(z) = \underbrace{\exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right)}_{\tilde{\sigma}_n^{(0)}} \left\{ \underbrace{\frac{1}{\kappa} \int_{\xi_0^A}^z dz' U_n(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right)}_{\tilde{\sigma}_n^{(0,I)}} + C_\phi^{(n)} \right\}, \quad (3.62)$$

where the constant $C_\phi^{(n)}$ is fixed from the initial condition of Eq. (2.8),

$$C_\phi^{(n)} = \frac{\left(\frac{1}{2} e^{\xi_0^A}\right)^n}{n!} e^{-\frac{1}{2} e^{\xi_0^A}}. \quad (3.63)$$

For $p \geq 1$, it takes the form

$$\sigma_n^{(p,I)}(z) = \exp\left(-\frac{1}{\kappa} \int_{\xi_0^A}^z dz' T(z')\right) \left\{ \frac{1}{\kappa} \int_{\xi_0^A}^z dz' U_n(z') \tilde{\Sigma}_n^{(p-1)}(z') \exp\left(\frac{1}{\kappa} \int_{\xi_0^A}^{z'} dz'' T(z'')\right) + C_\phi^{(n)} \right\}. \quad (3.64)$$

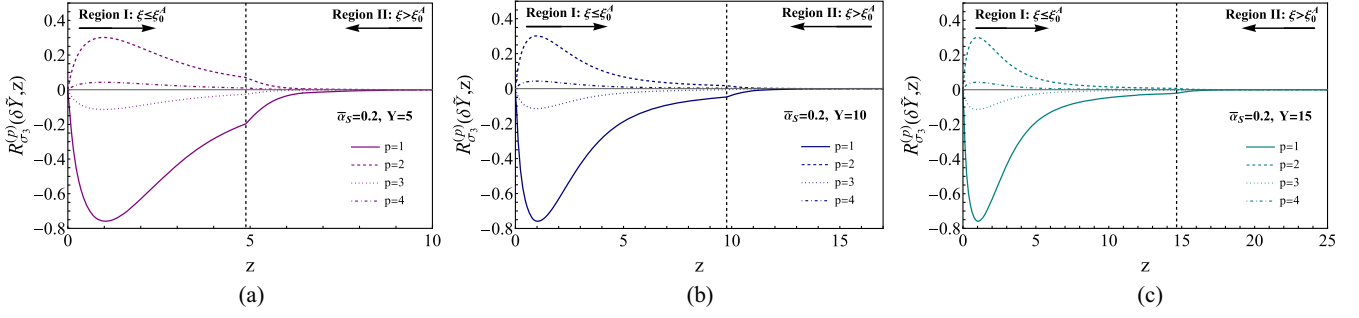


FIG. 10. The ratio of $R_{\sigma_3}^{(p)}(\delta\tilde{Y}, z)$ versus z at different values of $\delta\tilde{Y}$. $N_0 = 0.22$, $\bar{\gamma} = 0.63$, $\xi_0^A = 0$, $z_0 = 2$, $z_\Delta = 0.87$. (a)–(c) show the ratio $R_{\sigma_3}^{(p)}$ as a function of Y .

One can see that Eq. (3.62) leads to the cross section that decreases as $e^{-\frac{(z-z_\Delta)^2}{2\kappa}}$. We need to find the solution to σ_n at large n to see how the summation over n will result in the inelastic cross section of about a constant. We will do this in Sec. IV.

2. Region II

In region II, Eq. (3.58) takes the form

$$\begin{aligned} \frac{\partial \sigma_3^{(0,II)}(\delta\tilde{Y}, z)}{\partial \delta\tilde{Y}} + \kappa \frac{\partial \sigma_3^{(0,II)}(\delta\tilde{Y}, z)}{\partial z} = & -T(\delta\tilde{Y}, z) \sigma_3^{(0,II)}(z) \\ & + \underbrace{\sigma_{0,3} \Delta(\delta\tilde{Y}, z) + \sigma_1(\delta\tilde{Y}, z) \int_{\xi_0^A}^z dz' \sigma_2(\delta\tilde{Y}, z') + \sigma_2(\delta\tilde{Y}, z) \int_{\xi_0^A}^z dz' \sigma_1(\delta\tilde{Y}, z')}_{U_3(\delta\tilde{Y}, z)} \end{aligned} \quad (3.65)$$

with the general solution

$$\sigma_3^{(0,II)}(\delta\tilde{Y}, z) = \Phi_3(-\kappa\delta\tilde{Y} + z) \sigma_3^{0'}(\delta\tilde{Y}, z) + \sigma_3^{0'}(\delta\tilde{Y}, z) \tilde{\sigma}_3^{0'}(\delta\tilde{Y}, z), \quad (3.66)$$

where

$$\tilde{\sigma}_3^{0'}(\delta\tilde{Y}, z) = \frac{1}{\kappa} \int_{\xi_0^A}^z dz' R_3 \left(\delta\tilde{Y} - \frac{z-z'}{\kappa}, z' \right) + C_{\sigma_3}. \quad (3.67)$$

We need to use the same procedure as was described for $\sigma_2^{(0,II)}(\delta\tilde{Y}, z)$ to find all unknown functions. For the $p \geq 1$ iteration, we search in the form $\sigma_3^{(p,II)}(\delta\tilde{Y}, z) = \sigma_3^{0'}(\delta\tilde{Y}, z) \sigma_3^{p'}(\delta\tilde{Y}, z)$ with solution

$$\sigma_3^{p'}(\delta\tilde{Y}, z) = \frac{1}{\kappa} \int_0^z dz' R_3^{(p)}(\delta\tilde{Y}, z') + C_{\sigma_3}^{(p)}. \quad (3.68)$$

The numerical calculations of $R_{\sigma_3}^{(p)}(\delta\tilde{Y}, z) = \frac{\sigma_3^{(p,II)}(\delta\tilde{Y}, z)}{\sigma_3^{(0,II)}(\delta\tilde{Y}, z)} \times \Theta(\xi_0^A - \xi) + \frac{\sigma_3^{(p,II)}(\delta\tilde{Y}, z)}{\sigma_3^{(0,II)}(\delta\tilde{Y}, z)} \Theta(\xi - \xi_0^A)$ are shown in Fig. 10.

One can see that they show the same pattern as the calculation for σ_1 (see Fig. 7) and for σ_2 : The second iteration is about 30% of the first, but the fourth iteration gives an accuracy of less than 3%.

The common feature of σ_1 , σ_2 , and σ_3 calculations (see Figs. 7–10) is that corrections at large z are very small. The equation can be generalized to arbitrary n as follows:

$$\begin{aligned} \frac{\partial \sigma_n^{(0,II)}(\delta\tilde{Y}, z)}{\partial \delta\tilde{Y}} + \kappa \frac{\partial \sigma_n^{(0,II)}(\delta\tilde{Y}, z)}{\partial z} = & -T(\delta\tilde{Y}, z) \sigma_n^{(0,II)}(\delta\tilde{Y}, z) \\ & + \underbrace{\sigma_{0,n} \Delta(\delta\tilde{Y}, z) + \sum_{k=1}^{n-1} \int_{\xi_0^A}^z dz' \sigma_{n-k}(\delta\tilde{Y}, z') \sigma_k(\delta\tilde{Y}, z)}_{U_n(\delta\tilde{Y}, z)} \end{aligned} \quad (3.69)$$

with the solution

$$\begin{aligned} \sigma_n^{(0,II)}(\delta\tilde{Y}, z) = & \Phi_n(-\kappa\delta\tilde{Y} + z) \sigma_n^{0'}(\delta\tilde{Y}, z) \\ & + \sigma_n^{0'}(\delta\tilde{Y}, z) \tilde{\sigma}_n^{0'}(\delta\tilde{Y}, z), \end{aligned} \quad (3.70)$$

where

$$\bar{\sigma}_n^{0'}(\delta\tilde{Y}, z) = \frac{1}{\kappa} \int_{\xi_0^A}^z dz' R_n \left(\delta\tilde{Y} - \frac{z - z'}{\kappa}, z' \right) + C_{\sigma_n}, \quad (3.71)$$

with $R_n = U_n/\sigma_n^{0'}$ and the functions C_{σ_n} and Φ_n are determined to satisfy the initial condition given by Eq. (2.8). For the $p \geq 1$ iteration, the solution takes the form $\sigma_n^{(p, II)}(\delta\tilde{Y}, z) = \sigma_n^{0'}(\delta\tilde{Y}, z) \sigma_n^{p'}(\delta\tilde{Y}, z)$, where

$$\sigma_n^{p'}(\delta\tilde{Y}, z) = \frac{1}{\kappa} \int_0^z dz' R_n^{(p)}(\delta\tilde{Y}, z') + C_{\sigma_n}^{(p)}, \quad (3.72)$$

where $R_n^{(p)}(\delta\tilde{Y}, z) = -\frac{\mathcal{N}_L[\sigma_n(\delta\tilde{Y}, z)]}{\sigma_n^{0'}(\delta\tilde{Y}, z)}$ and $C_{\sigma_n}^{(p)}$ is found by requiring that $\sigma_n^{(p, II)}(\delta\tilde{Y} = 0, z = \xi) = 0$.

As an application for the results of general n , we plot the quantity (see Fig. 11)

$$R_{\sigma_5}^{(p)}(\delta\tilde{Y}, z) = \frac{\sigma_5^{(p, I)}(\delta\tilde{Y}, z)}{\sigma_5^{(0, I)}(\delta\tilde{Y}, z)} \Theta(\xi_0^A - \xi) + \frac{\sigma_5^{(p, II)}(\delta\tilde{Y}, z)}{\sigma_5^{(0, II)}(\delta\tilde{Y}, z)} \Theta(\xi - \xi_0^A). \quad (3.73)$$

One can see that the first iteration turns out to be rather large in the small- z region which corresponds to region I in Fig. 2. Note, that the behavior in region II of the same figure gives a small contribution. Despite the fact that, after an appropriate number of iterations the corrections become small, the approach may break down for a sufficiently large n . For this reason, we use this method only for small values of n . In the next section, we solve the equation in the large- n limit and study the possible matching with this solution. We will show that actually at small values of z the solution at large n is prevailed, and therefore, the large corrections in this region for σ_5 , for example, do not look important.

IV. MULTIPLICITY DISTRIBUTION AT LARGE n

A. Solutions to the equations

In this section, we solve the equation for σ_n at large n : $n \gtrsim N(z)$, where the typical multiplicity $N(z)$ we have to find from the solution. The equation takes the form

$$\kappa \frac{d\sigma_n(z)}{dz} = -z\sigma_n + \Delta(z)\Sigma_n(z) + \int_0^z dz' \Delta(z')\sigma_n(z) + \sum_{k=1}^{n-1} \Sigma_{n-k}(z)\sigma_k(z), \quad (4.1)$$

where $\Sigma_n(z) = \int_0^z dz' \sigma_n(z')$.

We suggest that the solution has the following form:

$$\Sigma_n(z) = \phi(z) \exp(-n\Phi(z)). \quad (4.2)$$

Functions $\Phi(z)$ and $\phi(z)$ we will find from Eq. (4.1). First, we note that

$$\sum_{k=1}^{n-1} \Sigma_{n-k}(z)\sigma_k(z) = \frac{1}{2} \frac{d}{dz} \sum_{k=1}^{n-1} \Sigma_{n-k}(z)\Sigma_k(z) = \frac{n-1}{2} \frac{d}{dz} \phi^2(z) \exp(-n\Phi(z)). \quad (4.3)$$

Equation (4.1) can be rewritten as

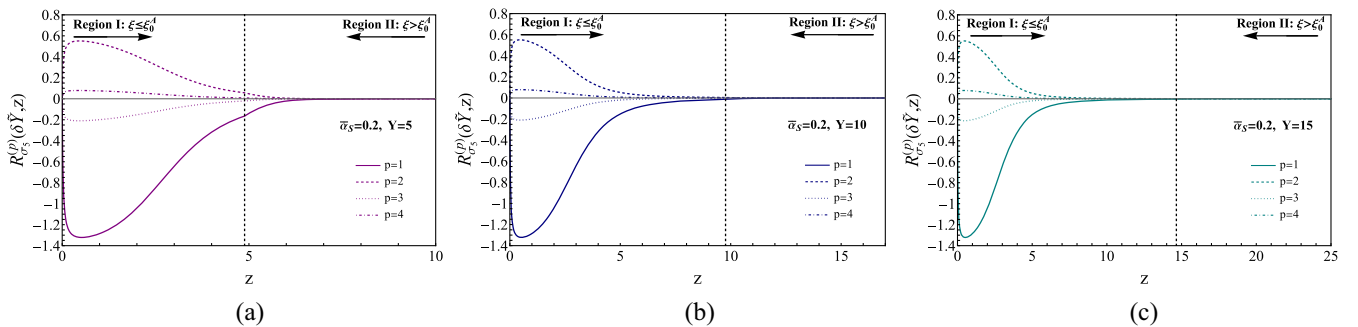


FIG. 11. The ratio of $R_{\sigma_5}^{(p)}(\delta\tilde{Y}, z)$ versus z at different values of $\delta\tilde{Y}$. $N_0 = 0.22$, $\bar{\gamma} = 0.63$, $\xi_0^A = 0$, $z_0 = 2$, $z_\Delta = 0.87$. (a)–(c) show the ratio $R_{\sigma_5}^{(p)}$ as a function of Y .

$$\kappa \frac{d^2 \Sigma_n}{dz^2} = -\frac{d}{dz}(z \Sigma_n(z)) + \Sigma_n(z) + \frac{d}{dz}(\Sigma_n(z) \Sigma_\Delta(z)) + \frac{n-1}{2} \frac{d}{dz}(\phi(z) \Sigma_n(z)) \quad (4.4)$$

with $\Sigma_\Delta(z) = \int^z dz' \Delta(z')$. Introducing $S_n(z) = \int^z dz' \Sigma_n(z')$, we reduce Eq. (4.4) to the following equation:

$$\kappa \frac{d^2 \Sigma_n}{dz^2} = -\frac{d}{dz}(z \Sigma_n(z)) + \frac{d}{dz} S_n(z) + \frac{d}{dz}(\Sigma_n(z) \Sigma_\Delta(z)) + \frac{n-1}{2} \frac{d}{dz}(\phi(z) \Sigma_n(z)). \quad (4.5)$$

Assuming that all functions decrease at large z , we have

$$\begin{aligned} \kappa \frac{d \Sigma_n}{dz} &= -z \Sigma_n(z) + S_n(z) + \Sigma_n(z) \Sigma_\Delta(z) + \frac{n-1}{2} \phi(z) \Sigma_n(z) \\ &= -(z - z_\Delta) \Sigma_n(z) + S_n(z) - \Sigma_n(z) \tilde{\Sigma}_\Delta(z) + \frac{n-1}{2} \phi(z) \Sigma_n(z) \end{aligned} \quad (4.6)$$

with $\tilde{\Sigma}_\Delta(z) = \int_z^\infty dz' \Delta(z')$.

Using that $\frac{d \Sigma_n}{dz} = (\phi'(z) - n \phi(z) \Phi'(z)) \exp(-n \Phi(z))$ and equating terms that are proportional to n , we obtain

$$\Phi'(z) = -\frac{1}{2\kappa} \phi(z); \quad \Phi(z) = -\frac{1}{2\kappa} \int^z dz' \phi(z') + C. \quad (4.7)$$

From this equation, $S_n(z) = \frac{2\kappa}{n} \exp(-n \Phi(z))$ and Eq. (4.6) takes the form

$$\kappa \phi'(z) = -(z - z_\Delta) \phi(z) - \tilde{\Sigma}_\Delta(z) \phi(z) + \frac{2\kappa}{n} - \frac{1}{2} \phi^2(z). \quad (4.8)$$

Neglecting the term $\frac{2\kappa}{n}$ at large n , we solve the homogenous equation

$$\kappa \phi'(z) = -\underbrace{(z - z_\Delta + \tilde{\Sigma}_\Delta(z))}_{T(z)} \phi(z) - \frac{1}{2} \phi^2(z). \quad (4.9)$$

Introducing $\phi(z) = 2\kappa \frac{U'(z)}{U(z)}$, we obtain for function U the following equation:

$$\kappa U''(z) = -T(z) U'(z); U'(z) = C_\phi^{(1)} \exp\left(-\frac{1}{\kappa} \int^z dz' T(z')\right); U(z) = C_\phi^{(1)} \int^z dz' \exp\left(-\frac{1}{\kappa} \int^{z'} dz'' T(z'')\right) + C_\phi^{(2)}. \quad (4.10)$$

Therefore, for the solution of the homogenous equation $\phi_h(z)$, we have

$$\begin{aligned} \phi(z) &= 2\kappa \frac{\exp\left(-\frac{1}{\kappa} \int^z dz' T(z')\right)}{\int^z dz' \exp\left(-\frac{1}{\kappa} \int^{z'} dz'' T(z'')\right) + C_\phi^{(2)}/C_\phi^{(1)}} \\ &\xrightarrow{z \gg 1} 2\kappa \frac{e^{-\frac{(z-z_\Delta)^2}{2\kappa}}}{\int_{z_\Delta}^z dz' e^{-\frac{(z'-z_\Delta)^2}{2\kappa}} + C_\phi^{(2)}/C_\phi^{(1)}} = 2\kappa \frac{e^{-\frac{(z-z_\Delta)^2}{2\kappa}}}{\sqrt{\frac{\pi\kappa}{2}} \operatorname{erf}\left(\frac{z-z_\Delta}{\sqrt{2\kappa}}\right) + C_\phi^{(2)}/C_\phi^{(1)}} \xrightarrow{z \gg 1} C_\phi e^{-\frac{(z-z_\Delta)^2}{2\kappa}} \equiv \phi_{\text{asym}}(z). \end{aligned} \quad (4.11)$$

In Eq. (4.11) the arbitrary constants C_ϕ , $C_\phi^{(1)}$, and $C_\phi^{(2)}$ have to be chosen from the matching of our solution with the initial conditions of Eq. (2.8).

Plugging Eqs. (4.7) and (4.11) into Eq. (4.2) we obtain Σ_n , which is equal to

$$\begin{aligned}\Sigma_n(z) &= \phi(z) \exp(-n\Phi(z)) \xrightarrow{z \gg 1} C_\phi e^{-\frac{(z-z_\Delta)^2}{2\kappa}} \exp\left(-n \frac{C_\phi}{2\kappa} \int_z^\infty dz' e^{-\frac{(z'-z_\Delta)^2}{2\kappa}}\right) \\ &= C_\phi e^{-\frac{(z-z_\Delta)^2}{2\kappa}} \exp\left(-n \frac{C_\phi}{2} \sqrt{\frac{\pi}{2\kappa}} \operatorname{erfc}\left(\frac{z-z_\Delta}{\sqrt{2\kappa}}\right)\right) \xrightarrow{z \gg 1} C_\phi e^{-\frac{(z-z_\Delta)^2}{2\kappa}} \exp\left(-nC_\phi \frac{1}{2z} e^{-\frac{(z-z_\Delta)^2}{2\kappa}}\right).\end{aligned}\quad (4.12)$$

From this equation, we can calculate the value of σ_n at large n , which is equal to

$$\sigma_n(z) = -\phi(z)n\Phi'(z) \exp(-n\Phi(z)) \xrightarrow{z \gg 1} \phi_{\text{asym}}(-n\Phi'_{\text{asym}}(z)) \exp\{-n\Phi_{\text{asym}}(z)\}.\quad (4.13)$$

From Eq. (4.12), one can see that $\Phi_{\text{asym}}(z) = \frac{\phi_{\text{asym}}}{2z}$ and, therefore, at large z the multiplicity distribution follows the almost KNO [43–45] scaling behavior,

$$\sigma_n(z) = (\phi'(z) - n\phi(z)\Phi'(z)) \exp(-n\Phi(z)) \xrightarrow{n \geq N(z)} \frac{2}{\kappa} \frac{z^2}{N(z)} \Psi\left(\xi = \frac{n}{N(z)}\right); \quad \Psi(\xi) = \xi e^{-\xi};\quad (4.14)$$

with

$$N(z) = 1/\Phi_{\text{asym}}(z) = \frac{2z}{C_\phi} e^{\frac{(z-z_\Delta)^2}{2\kappa}}.\quad (4.15)$$

B. Entropy of produced gluons

Since $\sigma_{\text{in}} = \sum_{n=1}^\infty \sigma_n$ at large z tends to 1 [see Eq. (2.17) and the next subsection], the probability to have n -cut Pomerons in the final state from Eq. (4.14) is equal to

$$\mathcal{P}_n^{\text{AGK}}(z) \equiv \frac{\sigma_n^{\text{AGK}}(z)}{\sigma_{\text{in}}^{\text{AGK}}(z)} = \frac{2}{\kappa} \frac{z^2}{N(z)} \Psi\left(\frac{n}{N(z)}\right).\quad (4.16)$$

The entropy content of multiplicity distributions is defined by

$$S_E(z) = -\sum_n \mathcal{P}_n^{\text{AGK}}(z) \ln(\mathcal{P}_n^{\text{AGK}}(z)),\quad (4.17)$$

which we interpret as the entropy for the produced gluons. Using Eq. (4.16) in Eq. (4.17) we find that the von Neumann entropy at large z is equal to

$$\begin{aligned}S_E(z) &= \underbrace{\sum_{n=n_0}^\infty \frac{2}{\kappa} \frac{z^2}{N(z)} \Psi\left(\frac{n}{N(z)}\right) \ln\left(\frac{\kappa N(z)}{2} \frac{z^2}{z^2}\right)}_1 - \underbrace{\sum_{n=n_0}^\infty \frac{2}{\kappa} \frac{z^2}{N(z)} \Psi\left(\frac{n}{N(z)}\right) \ln\left(\Psi\left(\frac{n}{N(z)}\right)\right)}_{\text{Const}} \\ &= \ln(N(z)) - \ln\left(\frac{z^2}{2\kappa}\right) - 2 \ln 2 + \frac{2}{\kappa} \frac{n_0}{N(z)} \underbrace{2 \exp\left(-\frac{n_0}{N(z)}\right)}_{\substack{1 \text{ see Eq. (4.31)}}} \\ &= \frac{(z-z_\Delta)^2}{2\kappa} - \ln z - \ln\left(\frac{C_\phi}{\kappa}\right) + \frac{2}{\kappa} \ln 2 \xrightarrow{z \gg 1} \frac{z^2}{2\kappa}.\end{aligned}\quad (4.18)$$

In this equation, we introduce $n_0 \sim N(z)$, which we will explain below [see Eq. (4.31)]. Choosing the value of n_0 from the condition that $\sum_{n=n_0}^\infty \sigma_n = \text{Const}$, we get the second term in Eq. (4.18) is equal to $\ln 2$. Therefore, in QCD $S_E(z) = \ln N(z) = \frac{z^2}{2\kappa}$ in accordance with the result of Refs. [7,59,60]. It should be noted that we assume that Eq. (4.16) described all n having $\sum_{n=1}^\infty \mathcal{P}_n^{\text{AGK}}(z) = 1$. As we will discuss below in Sec. IV E the main contribution to the total inelastic cross section comes from $n \geq N(z)$. Based on this fact, we can safely consider Eq. (4.16) for calculation of S_E .

C. Inelastic cross section

The inelastic cross section can be found as

$$\sigma_{\text{in}}(z) = \sum_{n=1}^{\infty} \sigma_n(z) = \frac{d}{dz} \sum_{n=1}^{\infty} \Sigma_n(z). \quad (4.19)$$

Using Eq. (4.12), we obtain

$$\Sigma_{\text{in}}(z) = \sum_{n=1}^{\infty} \Sigma_n(z) = 2z \sum_{n=1}^{\infty} \frac{\phi_{\text{asym}}(z)}{2z} \exp\left(-n \frac{\phi_{\text{asym}}(z)}{2z}\right) = \frac{\phi_{\text{asym}}(z)}{\exp\left(\frac{\phi_{\text{asym}}(z)}{2z}\right) - 1} \xrightarrow{z \gg 1} 2z - \frac{1}{2} \phi_{\text{asym}}(z). \quad (4.20)$$

Finally, from Eq. (4.20), $\sigma_{\text{in}}(z) = \Sigma'_{\text{in}}(z) = 2$ at large z . It is instructive to note that this cannot be obtained using Eq. (4.14) at large n . We need to use the first equation [see Eq. (4.14)] which is correct at any value of n for our solution and expand $\Phi(z)$ at large z within accuracy $1/z^3$.

The limit $\sigma_{\text{in}} = 2$ at large z contradicts the unitarity constraints that lead to $\sigma_{\text{in}}(z) \rightarrow 1$ at large z . The unitarity limit can be found from Eq. (2.17). Indeed, the inelastic cross section $\sigma_{\text{in}} = \sum_{n=1}^{\infty} \sigma_n$ satisfies the BK equation, as has been demonstrated in Eq. (2.17). For the leading twist BFKL kernel, it takes the form

$$\kappa \frac{d\sigma_{\text{in}}(z)}{dz} = -z\sigma_{\text{in}} + \Delta(z)\Sigma_{\text{in}}(z) + \int_0^z dz' \Delta(z')\sigma_{\text{in}}(z) + \Sigma_{\text{in}}(z)\sigma_{\text{in}}(z), \quad (4.21)$$

where $\Sigma_{\text{in}}(z) = \int_0^z dz' \sigma_{\text{in}}(z')$.

Introducing $\sigma_{\text{in}}(z) = 1 - \Delta_{\sigma}(z)$ and $\Sigma_{\text{in}}(z) = z - \Delta_{\Sigma}(z)$ with $\Delta_{\Sigma}(z) = \int_0^z dz' \Delta_{\sigma}(z')$, we reduce Eq. (4.21) to the following form:

$$\kappa \frac{d\Delta_{\sigma}(z)}{dz} = \Delta_{\Sigma}(z)(1 - \Delta_{\sigma}(z)) - z\Delta(z) + \Delta(z)\Delta_{\Sigma}(z) - \int_0^z dz' \Delta(z')(1 - \Delta_{\sigma}(z)). \quad (4.22)$$

The solution of this equation is $\Delta_{\sigma}(z) = \Delta(z)$. Indeed, for this solution, Eq. (4.22) reduces to

$$\kappa \frac{d\Delta_{\sigma}(z)}{dz} = -z\Delta_{\sigma}(z) + \Delta_{\sigma}(z)\Delta_{\Sigma}(z), \quad (4.23)$$

which is the BK equation for our kernel.

Finally,

$$\sigma_{\text{in}}(z) \xrightarrow{z \gg 1} 1 - \Delta(z). \quad (4.24)$$

Comparing this expression with Eq. (4.2) we can see that solution of Eq. (4.2) deciphers what 1 means in Eq. (4.24) from the point of production processes.

D. Corrections $1/n$

Plugging solutions of Eqs. (4.12) and (4.13) into Eq. (4.5), we can see that our assumption that $\mathcal{S}(z)$ decreases at large z is not correct: $\mathcal{S}(z) \xrightarrow{z \gg 1} \frac{2\kappa}{n}$. Hence, Eq. (4.6) takes the form

$$\begin{aligned} \kappa \frac{d\Sigma_n(z)}{dz} = & -(z - z_{\Delta})\Sigma_n(z) + \left(\mathcal{S}_n(z) - \frac{2\kappa}{n}\right) - \Sigma_n(z)\tilde{\Sigma}_{\Delta}(z) \\ & + \frac{n-1}{2}\phi(z)\Sigma_n(z). \end{aligned} \quad (4.25)$$

For solving Eq. (4.25), we are searching the solution in the form $\phi(z) = \phi_h(z) + \Delta_{\phi}(z)$ and neglecting the $\Delta_{\phi}^2(z)$ contribution. The equation for $\Delta_{\phi}(z)$ takes the form

$$\kappa \Delta'_{\phi}(z) = -(T(z) + \phi_h(z))\Delta_{\phi}(z) + \frac{2\kappa}{n}(\exp(n\Phi(z)) - 1). \quad (4.26)$$

For large z we can rewrite this equation as

$$\kappa \Delta'_{\phi}(z) = -(T(z) + \phi_h(z))\Delta_{\phi}(z) + \int_z^{\infty} dz' \phi_h(z'). \quad (4.27)$$

Hence, the particular solution of the nonhomogenous equation is equal to

$$\Delta_\phi(z) = \frac{1}{\kappa} \exp \left(-\frac{1}{\kappa} \int_{z_\Delta}^z dz' (T(z') + \phi_h(z')) \right) \int_{z_\Delta}^z dz' \left(\int_{z'}^\infty d\bar{z} \phi_h(\bar{z}) \right) \exp \left(\frac{1}{\kappa} \int_{z_\Delta}^{z'} dz'' (T(z'') + \phi_h(z'')) \right) \\ \xrightarrow{z \gg 1} C_\phi \ln(z) \exp \left(-\frac{(z - z_\Delta)^2}{2\kappa} \right). \quad (4.28)$$

Finally, the solution to Eq. (4.25) is

$$\phi_{n.h.}(z) = \phi_h(z) + \Delta_\phi(z) \xrightarrow{z \gg 1} \phi_{\text{asym}}(z)(1 + \ln z). \quad (4.29)$$

Plugging Eq. (4.29) into Eq. (4.7) we see that

$$\Phi(z) = \frac{1}{2\kappa} \int_z^\infty dz' \phi_{n.h.}(z') \xrightarrow{z \gg 1} \frac{C_\phi}{2\kappa} \int_z^\infty dz' e^{-\frac{(z'-z_\Delta)^2}{2\kappa}} (1 + \ln(z')) = \Phi_{\text{asym}}(z)(1 + \ln z). \quad (4.30)$$

Therefore, these corrections do not change our solution, adding only weak $\ln z$ dependence of $N(z)$.

E. Discussion of the solution

Concluding this section, we infer that each σ_n of our solution decreases as $\exp(-z^2/(2\kappa))$ as it follows from the solutions of Sec. III. We consider this property as an attractive feature of the solution. On the other hand, the inelastic cross section is equal to 2 at large z instead of 1 as expected from the unitarity constraints.

This shortcoming is closely related to the fact that we do not know from which value of n we can trust our solution. Coming back to Eq. (4.20), let us introduce $\sigma_{\text{in}}^{n_0} = \sum_{n=n_0}^\infty \sigma_n$. For $\sigma_{\text{in}}^{n_0}$, Eq. (4.20) takes the form

$$\sigma_{\text{in}}^{n_0}(z) \xrightarrow{z \gg 1} 2 \exp \left(-\frac{n_0}{N(z)} \right). \quad (4.31)$$

From this equation, it follows that we can trust our solution for $n_0 \sim N(z)$. Indeed, for such n_0 sum of all σ_n with $n > n_0$ will give the cross section smaller than 1.

Hence, we conclude that we need to solve a difficult problem of matching our solution with the solution for $n \leq N(z)$, which we have not found. To illustrate the difficulties we could face in finding such a matching, we note that our solution at large n leads to σ_n at $z \rightarrow 0$, which are constants with quite different n dependence than the initial conditions of Eq. (2.8). Indeed, n dependence of our solution at $z = 0$ is $\sigma_n \propto n \exp(-\text{Const } n)$ instead of $\propto \frac{\text{Const}^n}{n!}$.

The question arises whether $\sum_{n=1}^{n_0} \sigma_n$ could give the substantial part of the inelastic cross section. As has been discussed in Sec. III, σ_n at small n in the homotopy approach takes the following form for large z :

$$\sigma_n(z) = \sigma_n^{(0)}(z) \left(1 + \sum_{p=1}^{p_{\text{max}}} \frac{\sigma_n^{(p)}(z)}{\sigma_n^{(0)}(z)} \right) \\ \xrightarrow{z \gg 1} \sigma_n^{(0)}(z), \quad (4.32)$$

where p_{max} is the maximum number of iterations taken in the homotopy approach. Using this equation, we compute $\sum_{n=1}^{n_0} \sigma_n(z)$ in order to see if it is much less than 1. As can be seen in Fig. 12, it approaches 0 for sufficiently large z . Therefore, it is expected that the sum over $n \leq n_0$ leads to the negligible contribution to the inelastic cross section. Hence, in spite of the fact that we do not know solutions for $n < N(z)$, we obtain correct estimates for the entropy of the produced gluons since this result is based on the expectation that the main contributions come from $n \sim N(z)$.

Coming back to the problem of matching of our solution at large n with the solution of Eq. (4.13), we could estimate the value of $n = n_m$ at which we expect the matching occurs. This value is the solution to the following equation:

$$\sigma_{\text{small } n} = \sigma_{n_m}^{(0)}(z) \left(1 + \sum_{p=1}^{p_{\text{max}}} \frac{\sigma_{n_m}^{(p)}(z)}{\sigma_{n_m}^{(0)}(z)} \right) \\ = \sigma_{\text{large } n} = \frac{2}{\kappa} \frac{z^2}{N(z)} \Psi \left(\frac{n_m}{N(z)} \right). \quad (4.33)$$

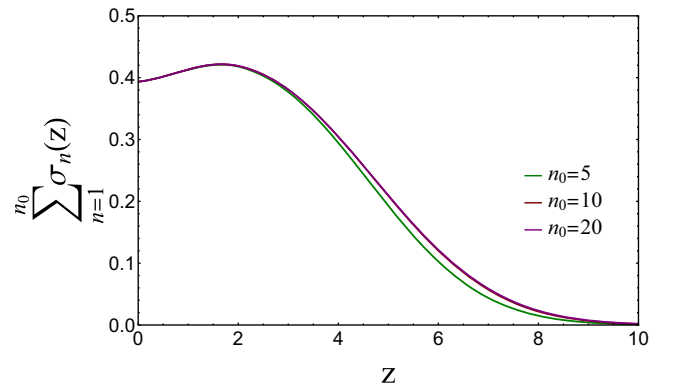


FIG. 12. Plot of $\sum_{n=1}^{n_0} \sigma_n(z)$ for different values of n_0 .

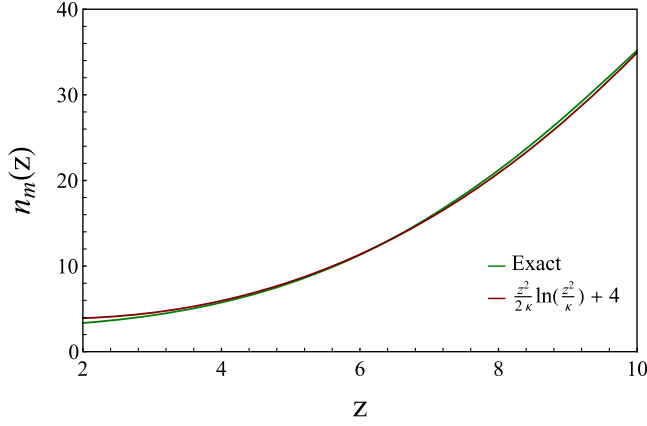


FIG. 13. Solution of Eq. (4.33) for $\xi_0^A = 0$ and $z_\Delta = 0.87$ (blue line), together with the approximating function (purple line).

Figure 13 shows the value of $n_m(z)$, which is the solution to Eq. (4.33) for $\xi_0^A = 0$ and $z_\Delta = 0.87$ (exact in Fig. 13). This solution is compared in this figure with the approximate analytic solution $\frac{z^2}{2\kappa} \ln(\frac{z^2}{\kappa}) + 4$.

Therefore, we see that we can hope for finding a matching procedure at rather large n but much smaller than $N(z)$.

Considering

$$\sigma_n(z) = \sigma_{\text{large } n}(z)\Theta(z_m - z) + \sigma_{\text{small } n}(z)\Theta(z - z_m) \quad (4.34)$$

in Fig. 14, we plot σ_{10} , σ_{20} to show explicitly the transition. One can see that we have not solved the problem of matching the solutions at large and small n . Note that $z_m(n)$ in Eq. (4.34) is the inverse function of $n = n_m(z)$.

V. CONCLUSIONS

In this paper we discuss the multiplicity distribution in the deep inelastic processes in the framework of high energy QCD. We obtained three results. First, we give a new derivation of the equations for the cross sections of productions of n -cut BFKL Pomerons in the final states [σ_n , see Eq. (2.14) and Fig. 5]. This derivation is carried out in the framework of the dipole approach to high energy QCD and it is not based on the Abramovsky-Gribov-Kancheli cutting rules but reproduces them.

Second, we developed the homotopy approach for finding the solutions to these equations. It consists of the analytic solution for the first iteration and the convergent procedure of calculating the next iterations using computing. We demonstrated that we need three iterations to obtain the solution within 0.2% accuracy. Using this approach, we can calculate several σ_n^{AGK} with small $n = 1, 2, \dots$. Having these cross sections, we

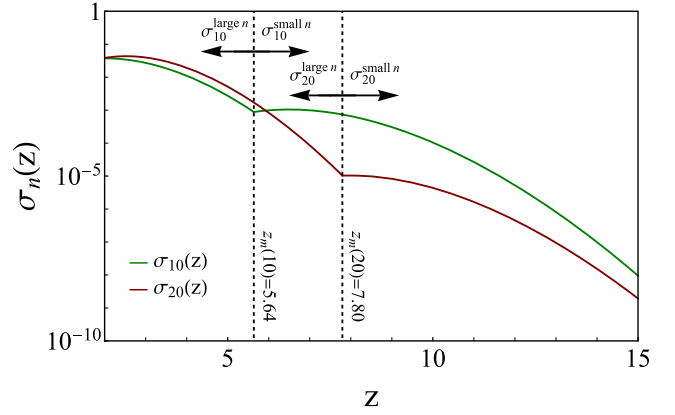


FIG. 14. The cross sections σ_{10} , σ_{20} showing the transition at $n = n_m$ [see Eq. (4.34)].

can calculate the moments of our multiplicity distributions [61,62] and obtain the multiplicity distribution $P_n(z)$ since

$$P_n(z) = \oint \frac{d\lambda}{2\pi i} \frac{e^{f(\lambda)}}{\lambda^{n+1}}, \quad (5.1)$$

where the contour of integration is the circle around the point $\lambda = 0$ and $f(\lambda)$ is the cumulant generating function, which is defined as

$$f(\lambda) = \sum_{n=1}^{\infty} \frac{C_n}{n!} (\lambda - 1)^n, \quad (5.2)$$

where C_n are cumulants [63]. The first cumulant $C_1 = \bar{n}$ is the average number of produced gluons, the second is the variance, and the third is the third central moment of our distributions (see Refs. [62,63] for more information). All cumulants can be calculated in the BFKL Pomeron approach, using that $\sigma_k^{\text{AGK}}(z)$ ⁵ gives the k th moment of the multiplicity distribution [$\langle n^k \rangle = \sigma_k^{\text{AGK}}(z) n_P^k$]⁶ [24,61]. The distributions with the first two cumulants are given in Ref. [61]. Hence, using the procedure suggested in this paper, we can describe the experimental data using several σ_n^{AGK} . This approach will be the subject of our future publications.

Third, we found an analytical solution for σ_n at large $n \gtrsim N(z) = \frac{2z}{C_\phi} e^{\frac{(z-z_\Delta)^2}{2\kappa}}$ [see Eq. (4.16)]. This solution has two remarkable features: (i) it decreases as $\exp(-\frac{(z-z_\Delta)^2}{2\kappa})$ at large z ; and (ii) it gives $\sigma_{\text{in}} \rightarrow \text{Const}$ at large z . In spite of

⁵Recall that σ_k is given at fixed b which enters the definition of z .

⁶ n_P is the multiplicity of the produced gluons in the BFKL Pomeron.

this, $Const$ turns out to be larger than 1 that comes from the s -channel unitarity constraints; we argue that we can trust this solution for $n \geq N(z)$. The first property we observed in Sec. III from the homotopy approach to finding the solution to the main equations.

The problem that has not been solved in the paper is the matching of our solution with the initial condition of Eq. (2.8). σ_n in our solution tends to the value at $z \rightarrow 0$ which has a different n dependence than the initial conditions. In Sec. IV E we argued that the matching occurs at large multiplicities proportional to $z^2/(2\kappa)$, which, however, are much smaller than $N(z)$. Therefore, we need to find a solution for small n , which has been discussed in Sec. III, at rather large n . We expect that this will be a difficult problem.

Finally, our solution leads to the value of the entropy of the produced gluons $S_E = \ln(N(z))$ which confirms the result of Refs. [7,59,60].

We hope that this paper will contribute to the gradual but inevitable understanding of the main features of the Pomeron calculus approach to high energy QCD.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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