

# Jet quenching in holographic QCD as an indicator of phase transitions in anisotropic regimes

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In this paper, we employ the gauge-gravity duality to study jet quenching (JQ) phenomena in the quark-gluon plasma. For this purpose, we implement holographic quantum chromodynamics (QCD) models constructed from an Einstein-Maxwell-dilaton gravity at finite temperature and finite chemical potential for light and heavy quarks. The models capture both the confinement and deconfinement phases of QCD and the first-order phase transitions. We calculate the JQ parameter in different models and compare them with the experimental data obtained in heavy-ions studies. In particular, we investigate how JQ, as a function of temperature  $T$ , chemical potential  $\mu$ , and magnetic field  $c_B$ , serves as a probe for identifying first-order phase transitions within the  $(T, \mu, c_B)$  parameter space of holographic QCD. Particular attention is paid to the dependence of JQ on the parameter  $\nu$ , which characterizes longitudinal versus transverse anisotropy relative to the heavy-ion collision axis. By analyzing the dependence of the JQ parameters on these thermodynamic variables, we map critical regions associated with phase boundaries. We compare our findings to earlier studies of the running coupling constant's behavior within the gauge-gravity duality framework. This approach provides new insights into the interplay between nonperturbative dynamics and phase structure in strongly coupled systems.

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## I. INTRODUCTION

It is well known that collisions of high-energy particles produce jets of elementary particles. In particular, collisions of ultrarelativistic heavy-ion beams create a hot, dense medium comparable to conditions in the early Universe. The resulting jets interact strongly with this medium, leading to a significant energy decrease known as jet quenching (JQ). JQ phenomenon is studied in high-energy heavy-ion collisions, particularly in the context of quark-gluon plasma (QGP) formation. It refers to the energy loss of high-energy partons (quarks and gluons) as they traverse the QGP, a hot and dense medium created in such collisions. The JQ parameter, denoted as  $\hat{q}$ , quantifies the average transverse momentum squared transferred from the parton to the medium per unit path length. It is a key observable for understanding the properties of the QGP.

The concept of the JQ parameter emerged from studies of parton energy loss in dense media by Baier *et al.* [1,2] (see also [3,4]). Quantitatively, the JQ parameter is characterized by the parameter  $\hat{q}$ , defined as the average squared transverse momentum transfer per unit path length,  $\hat{q} = \langle p_T^2 \rangle / L$ . This parameter measures how energetic partons lose energy via medium-induced gluon radiation. Crucially,  $\hat{q}$  links microscopic parton-medium interactions to observable jet suppression. For a quark with energy  $E$ , the average energy loss scales as  $\langle \Delta E \rangle \propto \alpha_s \hat{q} L^2$ , where  $L$  is the medium length. Experimental results from Relativistic Heavy Ion Collider (RHIC) and Large Hadron Collider (LHC) show significant jet suppression through the nuclear modification factor  $R_{AA} < 1$  and dijet asymmetry, yielding key values:  $\hat{q} \approx 1.2 \pm 0.3 \text{ GeV}^2/\text{fm}$  for  $\sqrt{s_{NN}} = 200 \text{ GeV}$  Au + Au collisions (RHIC) and  $\hat{q} \approx 1.5 \pm 0.4 \text{ GeV}^2/\text{fm}$  for  $\sqrt{s_{NN}} = 2.76 \text{ TeV}$  Pb + Pb collisions (LHC) [5–7].

A nonperturbative calculation scheme for  $\hat{q}$  using gauge-gravity duality was proposed by Liu *et al.* [8]. They related  $\hat{q}$  to a lightlike Wilson loop in the adjoint representation (denoted  $W^A$ ) of  $\mathcal{N} = 4$  SYM theory,

$$\langle W^A(C) \rangle \sim e^{-\frac{1}{4\sqrt{2}} \hat{q} L^- L_\perp^2}, \quad (1.1)$$

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where  $\mathcal{C}$  is a rectangular loop with light-cone ( $L^-$ ) and transverse ( $L_\perp$ ) extensions. Subsequent dual calculations using the Nambu-Goto action in AdS-Schwarzschild spacetime established  $\hat{q}$ 's explicit dependence on the 't Hooft coupling  $\lambda$  and horizon position  $z_h$  [for general holographic applications to HIC (see [9–11])]. Later work incorporated chemical potential  $\mu$ , magnetic fields  $B$ , and other parameters [12–44], revealing  $\hat{q}$ 's thermodynamic dependence in holographic quantum chromodynamics (QCD). Nonperturbative calculations have also been performed on the lattice [45,46].

In this paper, we study  $\hat{q}$ 's dependence on temperature, chemical potential, external magnetic field, and anisotropy parameter  $\nu$  across multiple holographic models—including those sensitive to quark masses. Special attention is paid to  $\hat{q}$ 's behavior near phase transitions: it varies smoothly across second-order transitions but exhibits discontinuities at first-order transitions. We compare our results with those in [39–41].

The paper is organized as follows. In Sec. II, we first briefly remind the five-dimensional anisotropic holographic models in the presence of a nonzero magnetic field for light and heavy quarks, as shown in Sec. II A, and then in Sec. II B, we present the analytical expressions for the JQ parameters for these models. In Sec. III, we present the numerical results of studies of the JQ parameter for light quarks. In Sec. III A, we show the results for the zero magnetic fields and  $\nu = 1, 1.5, 3$ , and  $4.5$  and in Sec. III B for the nonzero magnetic fields. In Sec. IV, we present numerical results of studies of the JQ parameter for heavy quarks. In Sec. IV A, we show results for the zero magnetic fields and  $\nu = 1, 1.5, 4.5$ , and in Sec. IV B for the nonzero magnetic fields. In Sec. V, we summarize our numerical results obtained for the LQ (light quark) and HQ (heavy quark) models by discussing their dependence of the anisotropic parameter  $\nu$  and parameter  $c_B$ , specifying the magnetic field. The paper is also complemented by an Appendix. Appendix A describes how the equations of motion (EOMs) were solved and which boundary conditions were applied. In Appendices B and C, some

complimentary plots are presented for LQ and HQ models, respectively.

## II. SETUP

### A. Anisotropic holographic models in the external magnetic field

In this paper, we deal with the Lagrangian in the Einstein frame used in previous papers [47–49],

$$\mathcal{L} = \sqrt{-g} \left[ R - \frac{f_0(\phi)}{4} F_0^2 - \frac{f_1(\phi)}{4} F_1^2 - \frac{f_3(\phi)}{4} F_3^2 - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right], \quad (2.1)$$

where  $g$  is the determinant of the metric tensor,  $R$  is the Ricci scalar,  $\phi = \phi(z)$  is the dilaton field,  $f_0(\phi)$ ,  $f_1(\phi)$ , and  $f_3(\phi)$  are the coupling functions associated with stresses  $F_0$ ,  $F_1$ , and  $F_3$  of the Maxwell fields, and  $F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho$ . The indexes  $\rho$  and  $\sigma$  numerate the spacetime coordinates  $(t, x_1, x_2, x_3, z)$ , with  $z$  being the holographic radial coordinate.  $V(\phi)$  is the potential of the dilaton field. In this paper, we considered  $F_0$ ,  $F_1$ , and  $F_3$  as the first, second, and third Maxwell fields, respectively.

In these cases, the metric is

$$ds^2 = \frac{L^2 \mathfrak{b}(z)}{z^2} \left[ -g(z) dt^2 + dx_1^2 + \left( \frac{z}{L} \right)^{2-\frac{2}{\nu}} (dx_2^2 + e^{c_B z^2} dx_3^2) + \frac{dz^2}{g(z)} \right], \quad (2.2)$$

$$\mathfrak{b}(z) = e^{2\mathcal{A}(z)}, \quad \mathcal{A}_s(z) = \mathcal{A}(z) + \sqrt{\frac{1}{6}} \phi(z), \quad (2.3)$$

and the matter fields<sup>1</sup> are

$$\phi = \phi(z), \quad (2.4)$$

$$\begin{aligned} F_0 - \text{electric ansatz}, \quad A_0 &= A_t(z), \quad A_i = 0, \quad i = 1, 2, 3, 4, \\ F_k - \text{magnetic ansatz}, \quad F_1 &= q_1 dx^2 \wedge dx^3, \quad F_3 = q_3 dx^1 \wedge dx^2. \end{aligned} \quad (2.5)$$

In (2.2),  $L$  is the length parameter,  $\mathfrak{b}(z)$  is the warp factor set by  $\mathcal{A}(z)$ ,  $g(z)$  is the blackening function,  $\nu$  is the parameter of the primary anisotropy caused by the non-symmetry of the heavy-ion collision (HIC), and  $c_B$  is the coefficient of the secondary anisotropy related to the magnetic field  $F_3$ . The choice of  $\mathcal{A}(z)$  determines the light/heavy quark description of the model. In previous

works, we considered  $\mathcal{A}_{LQ}(z) = -a \ln(\mathfrak{b} z^2 + 1)$  for light quarks [47,51], with  $a = 4.046$  and  $b = 0.01613 \text{ GeV}^2$ . In this paper, we also accept the same values for parameters  $a$

<sup>1</sup>Also, we can add a new Maxwell field  $F_2$  with magnetic ansatz  $F_2 = q_2 dx^1 \wedge dx^3$  to our model [50].

and  $b^2 \mathcal{A}_{HQ}(z) = -cz^2/4$  for heavy quarks [49,53,54] and  $\mathcal{A}_{HQ}(z) = -\frac{s}{3}z^2 - (p - c_B q_3)z^4$  for heavy quarks with magnetic catalysis [48].  $s$  and  $p$  are parameters that can be fitted to the experimental data such as  $s = 1.16 \text{ GeV}^2$  and  $p = 0.273 \text{ GeV}^4$ . We take  $f_1$  in the case of light quarks as [47]  $f_1 = (1 + bz^2)^a e^{-cz^2} z^{-2+\frac{2}{\nu}}$  where  $a = 4.046$ ,  $b = 0.01613 \text{ GeV}^2$ , and  $c = 0.227 \text{ GeV}^2$ . To respect the linear Regge trajectories, the gauge kinetic function for the HQ model is chosen in the form [55]  $f_0(z) = e^{-sz^2 - \mathcal{A}(z)}$ . In (2.5),  $q_1$  and  $q_3$  are constant “charges”.<sup>3</sup> See [59] for a discussion of the modifications of the LQ model.

The explicit form of the EOM's with ansatz (2.4)–(2.5) is given in Appendix

(A1)–(A7). Investigation of their selfconsistency shows that there is one dependent equation in the system and that all other equations are independent. Thus, the system (A1)–(A7) is self-consistent and the dilaton field equation (A1) serves as a constraint.

## B. Jet quenching

### 1. Isotropic case

The JQ parameter in the holographic models described by the isotropic metric

$$ds_s^2 = \frac{L^2 e^{2A_s(z)}}{z^2} (-g(z)dt^2 + \frac{dz^2}{g(z)} + dx_1^2 + dx_2^2 + dx_3^2), \quad (2.6)$$

is given by [8] (see also [26,39–41]),

$$\hat{q} = \frac{L^2}{\pi \alpha' a}, \quad (2.7)$$

where  $a$  is defined by the integral

$$a = \int_0^{z_h} dz \frac{z^2 e^{-2A_s(z)}}{\sqrt{g(z)(1-g(z))}}. \quad (2.8)$$

Note that, considering dimensionality, we have  $[a] = [L^3]$  and  $[\alpha'] = [L^2]$ . In a subsequent analysis, we study the dependence of  $\log a$  on various parameters, ignoring universal prefactors ( $2, \pi$ ) in the  $\hat{q}$  definition and setting  $L = 1$ . The  $\alpha'$ -dependence related to the 't Hooft coupling  $\lambda$  through  $\sqrt{\lambda} = L^2/\alpha'$  is incorporated into  $\mathcal{A}_s$  via (2.3).

<sup>2</sup>The authors of [38] considered the JQ parameter for the LQ model with slightly different parameters:  $a = 3.943$  and  $b = 0.0158 \text{ GeV}^2$ . The authors of [52] explored various LQ models defined by  $\mathcal{A} = -a \ln(bz^2 + 1) - a \ln(dz^4 + 1)$  for different parameters and flavor numbers. Specifically, for  $N_f = 2 + 1$ , they used  $a = 0.173$ ,  $b = 0.204 \text{ GeV}^2$ , and  $d = 0.013 \text{ GeV}^4$ .

<sup>3</sup>Roughly speaking, the value of the physical magnetic field scales as  $B_{\text{phys}} \sim q_3 e^{-L^2 c_B}$ ; see the discussion in [48,50,56–58].

In what follows, we refer to  $a$  as the IJQ parameter (denoting the inverse jet quenching), in direct contrast to the standard JQ parameter ( $\hat{q}$ ) for jet quenching, and  $\log a$  and  $-\log \hat{q}$  differ by an additive constant.

We considered the jet quenching parameter  $\hat{q}$  in holographic models that effectively describe heavy and light quarks. The holographic expression for jet quenching (2.7) includes some free parameters, such as anti-de Sitter (AdS) radius  $L$ , Newton constant  $G_5$ , and Regge slope  $\alpha'$ . In the numerical calculations, we set  $L = 1$ ,  $8\pi G_5 = 1$ , and  $\pi\alpha' = 1$ , but we keep them in our analytical calculations. These parameters can be changed to obtain some results for  $\hat{q}$  to be in correspondence with experimental results at RHIC and LHC [44,60]. The experimental results for the jet quenching parameter have been obtained for small values of chemical potential. The location of the first-order phase transition in the  $(\mu, T)$ -plane is independent of the choice of the boundary condition for the dilaton field and parameter  $\alpha'$ .

We can compare our results for the JQ parameter with the results from [40,61]:  $\mu = 0$ ,  $\hat{q} = 6 \text{ GeV}^2/\text{fm}$ , and  $T = 0.3 \text{ GeV}$ . For these parameters, we obtain for the LQ model  $\log a = 10$  and for HQ model  $\log a = 5$ . Using equation (2.7), we can estimate  $\alpha'$  for both LQ and HQ models,

$$\alpha'_{LQ} = \frac{5.07L^2}{3.14 \cdot 6 \cdot e^{10}} \approx 1.2 \times 10^{-5} L^2, \quad (2.9)$$

$$\alpha'_{HQ} = \frac{5.07L^2}{3.14 \cdot 6 \cdot e^5} \approx 1.8 \times 10^{-3} L^2. \quad (2.10)$$

### 2. Anisotropic case

We consider metrics with two types of anisotropy (2.2): the parameter  $\nu$  of the primary anisotropy and secondary anisotropy related to the magnetic field  $F_3$ . In this case, the JQ parameter depends on the orientation. Here, we consider a jet moving along the  $x_1$ -direction with a momentum broadening along  $x_2$  or  $x_3$ . The corresponding JQ parameters are defined as  $\hat{q}_2$  and  $\hat{q}_3$ . Standard holographic calculations for anisotropic cases give [18,40]

$$\hat{q}_i = \frac{L^2}{\pi \alpha' a_i}, \quad (2.11)$$

where

$$a_2 = \int_0^{z_h} dz \frac{e^{-2A_s(z)} (\frac{z}{L})^{2/\nu}}{\sqrt{g(z)(1-g(z))}}, \quad (2.12)$$

$$a_3 = \int_0^{z_h} dz \frac{e^{-2A_s(z) - c_B z^2} (\frac{z}{L})^{2/\nu}}{\sqrt{g(z)(1-g(z))}}. \quad (2.13)$$

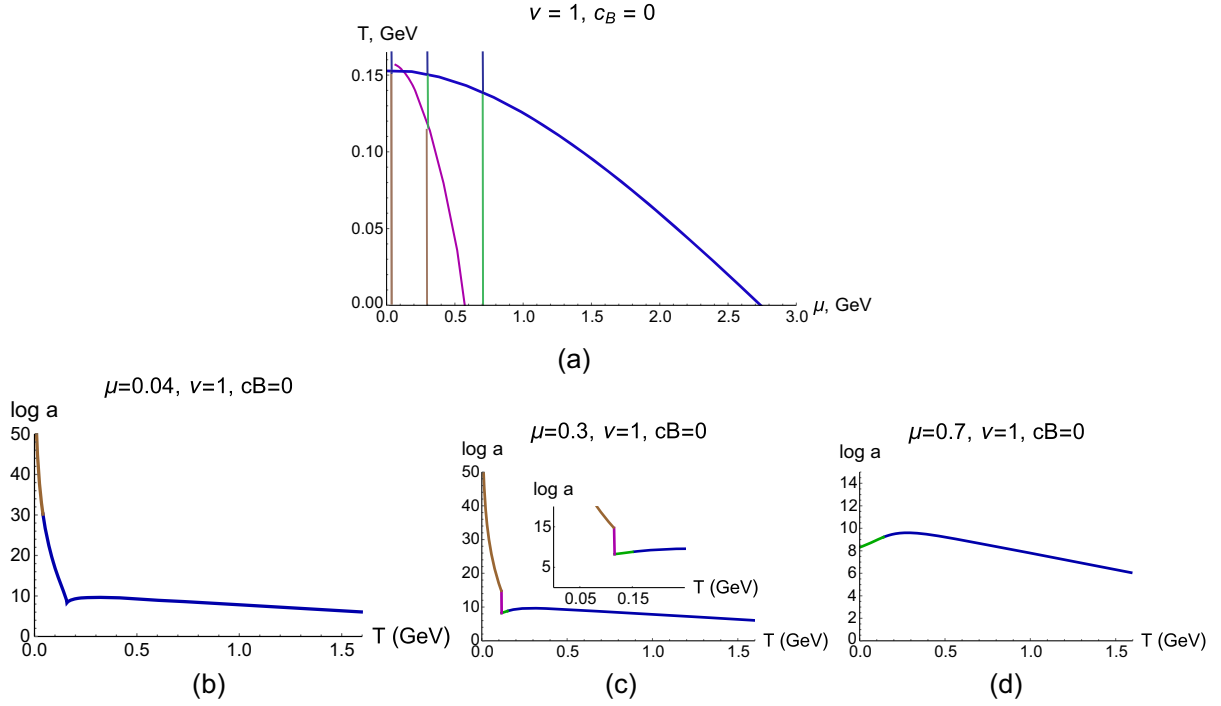


FIG. 1. (a) We calculate the JQ parameter for the LQ model, with  $\nu = 1$  and  $c_B = 0$ , along vertical lines (constant  $\mu$ ) in the  $(\mu, T)$ -plane at  $\mu = 0.04, 0.3$ , and  $0.7$  (GeV). Segments of these lines are colored blue (QGP), brown (hadronic), and green (quarkyonic) according to the phase traversed. The resulting plots of  $\log a$  versus the temperature are displayed in the bottom panels (b)–(d) using the same color scheme.

For  $\nu = 1$  and  $c_B = 0$ , (2.12) and (2.13) coincide with (2.8). As mentioned above, we call  $a_2$  and  $a_3$  as the IJQ parameters.

### III. JQ PARAMETER FOR LIGHT QUARKS: NUMERICAL RESULTS

Since holographic calculations involve an explicit dependence of physical quantities on the location of the horizon  $z_h$ , we depict the phase structure in the  $(\mu, z_h)$ -plane. Figure 41 shows this diagram for isotropic media composed of LQs (see [62,63]).

For anisotropic media in the presence of magnetic fields, the phase structure changes.

We will calculate the JQ parameter  $a$  along the lines of the constant chemical potential and present density plots of  $a$  in the physical parameter space  $(\mu, T)$ . These fixed- $\mu$  calculations provide detailed profiles of  $a$ , while the density plots illustrate the overall structure of  $a$  in the  $(\mu, T)$ -plane and help identify potential phase transitions.

#### A. Zero magnetic field

##### 1. Zero magnetic field, $\nu = 1$

In this subsection, we study the dependence of  $a$  on  $\mu$  and  $T$  for the isotropic case, i.e., for  $\nu = 1$ . The explicit formulas (2.12) and (2.13) give the values of the IJQ

parameter as a function of temperature  $T$  and chemical potential  $\mu$ . In the case of absence of the magnetic field, these two values  $a_2$  and  $a_3$  are identical.

Figure 1 depicts the  $\log a$  as a function of the temperature  $T$  along vertical lines [see Fig. 1(a)] in the  $(\mu, T)$ -plane<sup>4</sup> at fixed  $\mu = 0.04$  GeV,  $\mu = 0.3$  GeV, and  $\mu = 0.7$  GeV in panels [(b)–(d)], respectively. The blue and brown lines correspond to the QGP and hadronic phases, respectively. This synthesis reveals discontinuities in the JQ parameters at first-order phase transitions. Although no abrupt changes appear along the second-order transition line, detailed analysis uncovers a behavioral shift near this boundary: shortly after transitioning to the QGP phase,  $\log a$  decreases with increasing temperature, reversing its previous upward trend up to temperature  $T \approx 0.25$ – $0.3$  GeV. Note that Fig. 1 can be obtained as a comprehensive picture by combining Figs. 42–44 (see Appendix B), where higher values of  $T$  are admissible.

The dependence of  $\log a T^3$  on  $T$  for the isotropic LQ model for different values of  $\mu$  is presented in Fig. 2. We see that for large  $T$ ,  $\log a T^3$  does not depend on  $\mu$ .

The Density plots of  $\log a$  for light quarks at  $c_B = 0$  and  $\nu = 1$  are presented in Fig. 3(a), while Fig. 3(b) includes phase transition lines. The magenta line in panel (b) marks

<sup>4</sup>Throughout this paper, the dimensionality of quantities is as follows:  $[T] = [\mu] = [z]^{-1} = [z_h]^{-1} = [c_B]^{\frac{1}{2}} = \text{GeV}$ .

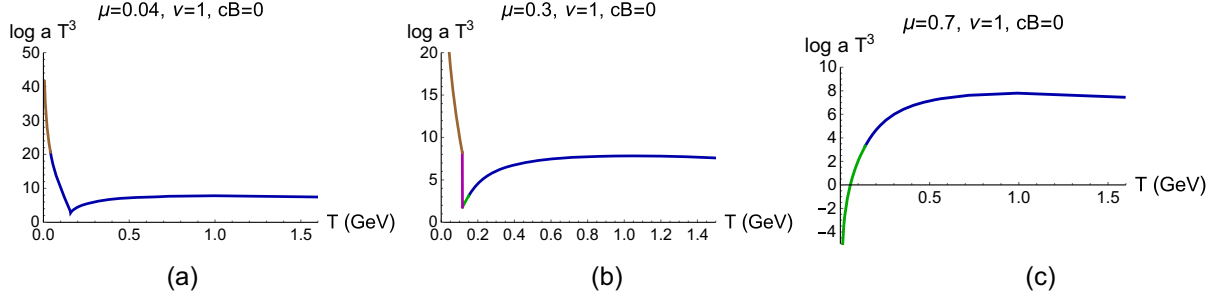


FIG. 2. Dependence of  $\log a T^3$  on  $T$  for the isotropic LQ model, with  $\nu = 1$  and  $c_B = 0$ , at fixed chemical potentials: (a)  $\mu = 0.04$  GeV, (b)  $\mu = 0.3$  GeV, and (c)  $\mu = 0.7$  GeV.

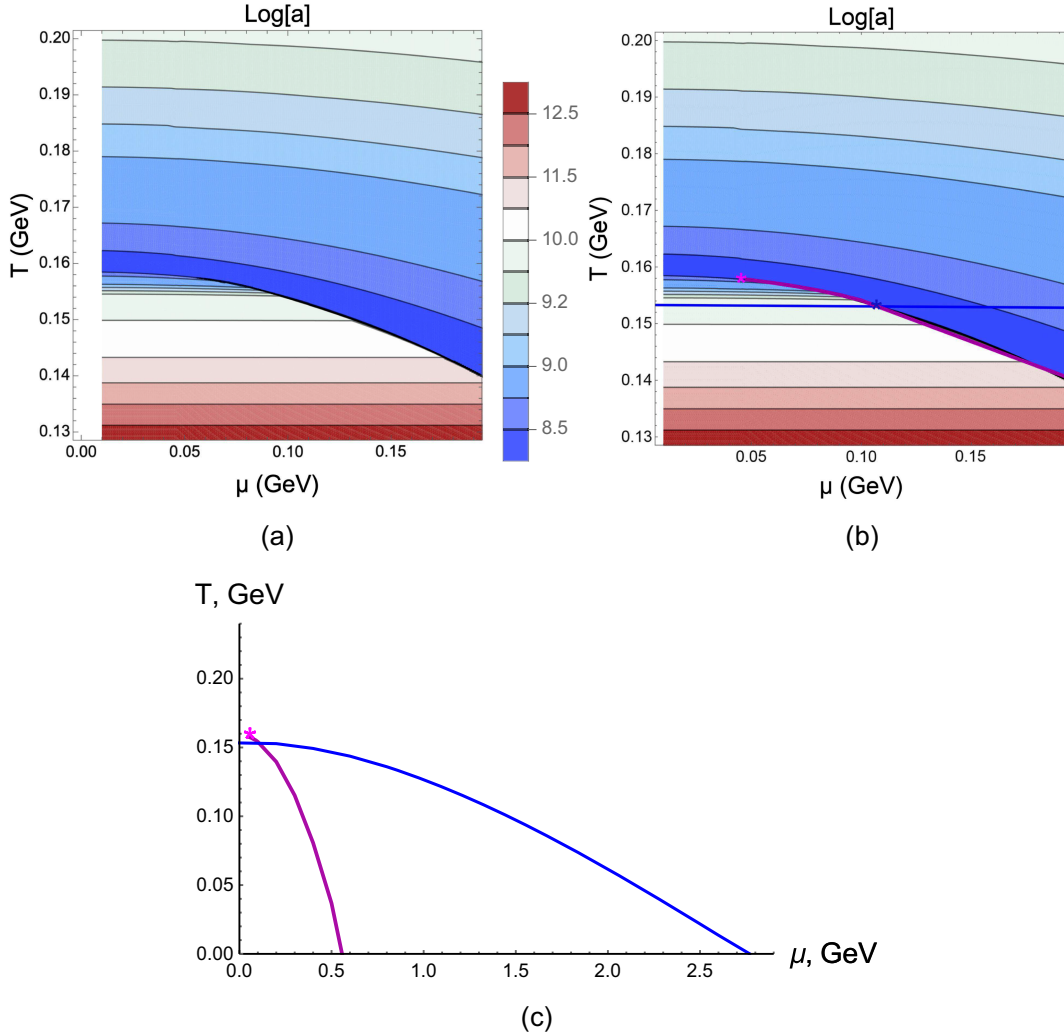


FIG. 3. (a) Density plots of  $\log a$  for the LQ model, with  $c_B = 0$  and  $\nu = 1$ . (b) The same with the phase transition lines: (i) the magenta line shows the first-order transition line that starts at CEP with coordinates  $(\mu_{\text{CEP}}, T_{\text{CEP}}) \simeq (0.046 \text{ GeV}, 0.158 \text{ GeV})$  shown by the magenta star, and (ii) the blue line shows the second-order phase transition line between the hadron and QGP phases (for  $0 < \mu < 0.095$  GeV) and between the quarkyonic and QGP phases ( $\mu > 0.095$  GeV). (c) The light quark phase diagram shows the hadronic phase below the magenta line, the quarkyonic phase between the magenta and blue lines, and the QGP phase above the blue line.

the first-order transition, starting at the critical endpoint (CEP) located at  $(\mu_{\text{CEP}}, T_{\text{CEP}}) \simeq (0.046, 0.158 \text{ GeV})$  and indicated by a magenta star. The blue line denotes the second-order transition: for  $0 < \mu < 0.095 \text{ GeV}$ , it separates the hadronic and QGP phases, while for  $\mu > 0.095 \text{ GeV}$ , it divides the quarkyonic and QGP phases. Figure 3(c) (adapted from [47]) shows the phase diagram, with the hadronic phase below the magenta line, the quarkyonic phase between the magenta and blue lines, and the QGP phase above the blue line.

Figures 3(a) and 3(b) reveal discontinuities in the JQ parameter along the first-order phase transition line, confirming the results shown in Fig. 1. In the hadronic phase, this parameter depends primarily on temperature  $T$  but shows a negligible dependence on the chemical potential  $\mu$ . In contrast, in both the QGP and quarkyonic phases, it exhibits a significant dependence on both  $T$  and  $\mu$ . A significant dependence of  $\log a$  on the chemical potential  $\mu$  in the quarkyonic phase is evident by examining the region between the blue and magenta lines in Fig. 3.

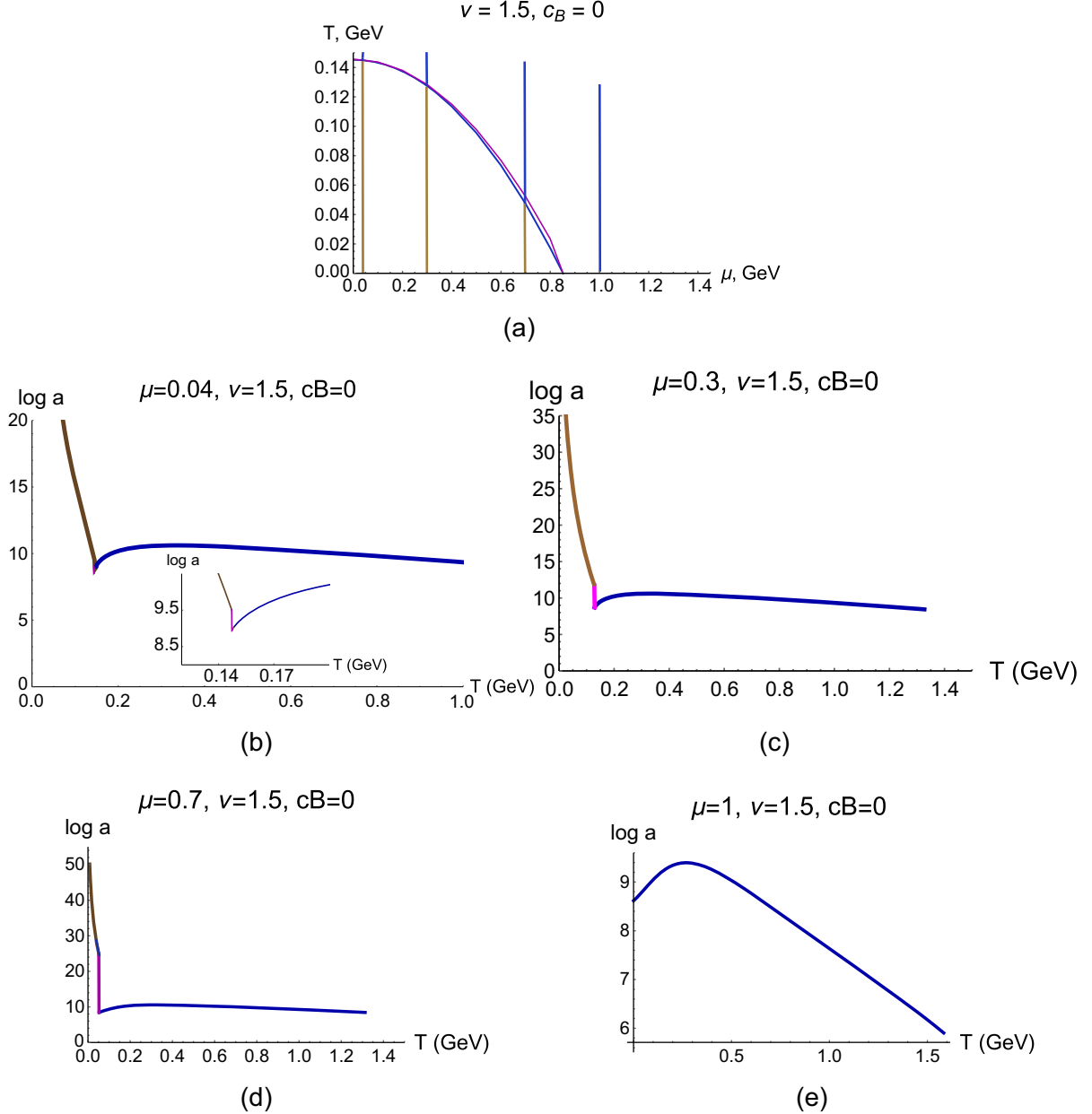


FIG. 4. (a) We calculate the JQ parameter for the LQ model, with  $\nu = 1.5$  and  $c_B = 0$ , along vertical lines (constant  $\mu$ ) in the  $(\mu, T)$ -plane at  $\mu = 0.04, 0.3, 0.7$  and  $\mu = 1$  (GeV), shown in this panel. Segments of these lines are colored blue (QGP), brown (hadronic), and green (quarkyonic) according to the phase traversed. The resulting plots of  $\log a$  versus the temperature are displayed in the bottom panels (b)–(e) using the same color scheme.

Here, in contrast to the region of the hadronic phase, where the lines of constant  $\log a$  lie predominantly horizontally, a significant slope of the lines of constant  $\log a$  is observed.

Notably, the JQ parameter shows no anomalous behavior along the confinement/deconfinement phase transition line. This is clearly demonstrated by comparing Figs. 3(a) and 3(c).

## 2. Zero magnetic field with spatial anisotropy: $\nu = 1.5, 3, 4.5$

The phase diagram in the  $(\mu, T)$ -plane for  $c_B = 0$  and  $\nu = 1.5$  is presented in Fig. 4(a). The magenta and blue lines describe the first-order and confinement/deconfinement phase transitions, respectively. We also draw here the vertical lines with fixed  $\mu = (0.04, 0.3, 0.7, 1)$  (GeV). The segments of these lines are colored blue (QGP), brown (hadronic), and green (quarkyonic), respectively.

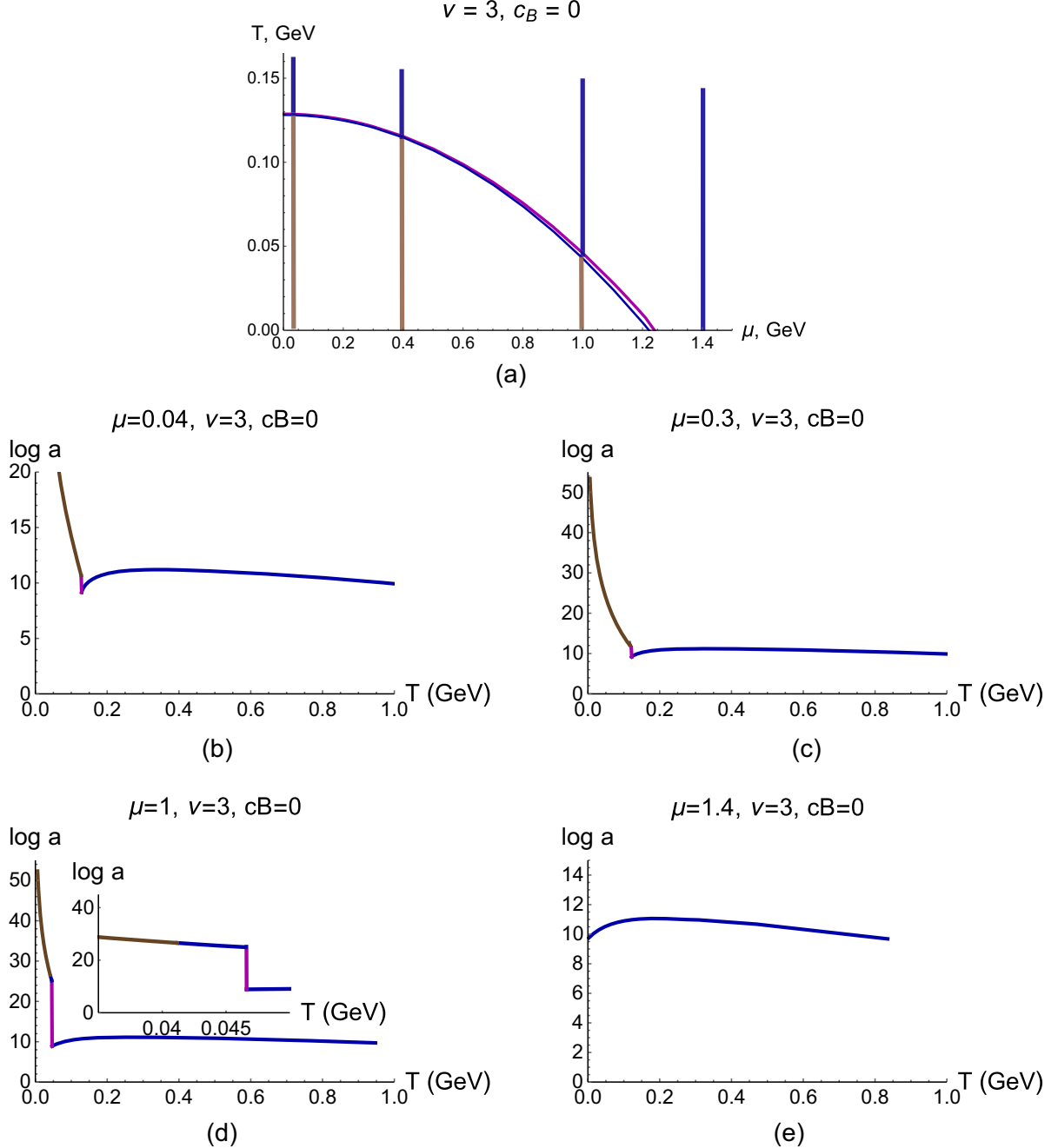


FIG. 5. (a) We calculate the JQ parameter for the LQ model, with  $\nu = 3$  and  $c_B = 0$ , along vertical lines at fixed  $\mu = 0.04, 0.3, 1$ , and  $\mu = 1.4$  (GeV) presented at the panel. Segments of these lines are colored blue, brown, and green corresponding to the QGP, hadronic, and quarkyonic phases they traverse, respectively. The resulting plots for log of the JQ parameter versus the temperature are presented in the two bottom panels (b)–(e) and are colored using the same scheme.

(hadronic), and green (quarkyonic) according to the phase traversed. The Plots of  $\log a$  versus the temperature are displayed in the bottom panels [(b)–(e)] using the same color scheme. Specifically,  $\log a$  versus  $T$  is shown for the chemical potential  $\mu = 0.04$  GeV in Fig. 4(b), for

$\mu = 0.3$  GeV in Fig. 4(c), for  $\mu = 0.7$  GeV in Fig. 4(d), and for  $\mu = 1$  GeV in Fig. 4(e).

Figure 4(b) shows that at  $\mu = 0.04$  GeV,  $\log a$  decreases monotonically in the hadronic phase, indicating enhanced JQ. Following a continuous transition to the QGP phase at

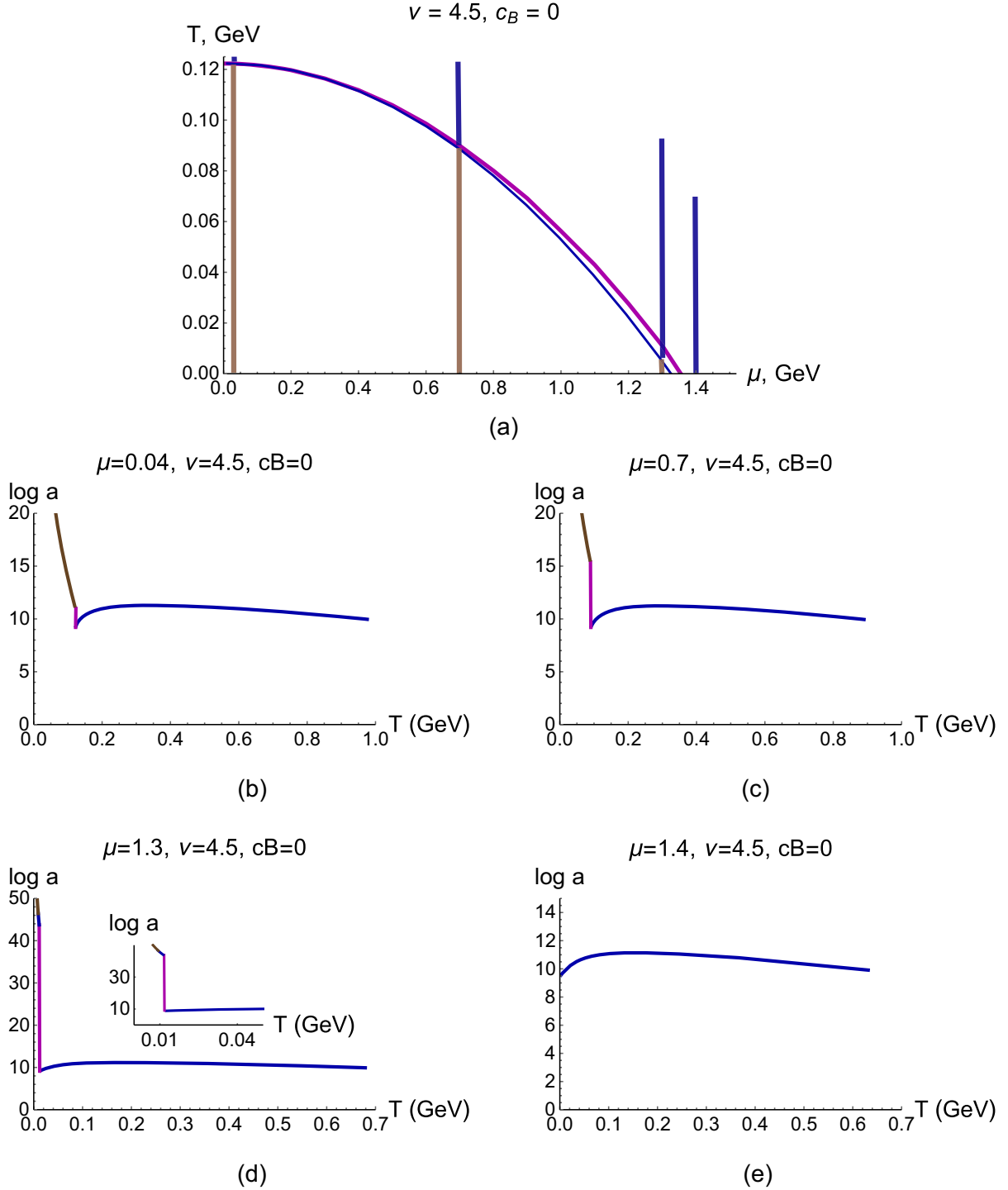


FIG. 6. (a) We calculate the JQ parameter for the LQ model, with  $\nu = 4.5$  and  $c_B = 0$ , along vertical lines at fixed  $\mu = 0.04, 0.7, 1.3$  and  $\mu = 1.4$  (GeV). Segments of these lines are colored blue, brown, and green corresponding to the QGP, hadronic, and quarkyonic phases they traverse, respectively. The resulting values for  $\log$  of the JQ parameter are presented in two bottom panels (b)–(e) and are colored using the same scheme.

$T \sim 0.153$  GeV and a subsequent first-order phase transition,  $\log a$  undergoes a jump [marked by the magenta line in the inset of Fig. 4(b)]. After this transition,  $\log a$  increases until  $T \approx 0.3$  GeV, then decreases with further increases in temperature. Figure 4(c) shows that  $\log a$  at  $\mu = 0.3$  GeV exhibits similar behavior to Fig. 4(b), differing only in the magnitude of the discontinuity at the first-order phase transition, which is larger in panel (c). In Fig. 4(d) ( $\mu = 0.7$  GeV),  $\log a$  decreases in the hadronic phase (indicating strengthening JQ) up to  $T \approx 0.025$  GeV, where a phase transition to the QGP occurs. Following this transition,  $\log a$  continues to decrease (further strengthening JQ) until  $T \sim 0.05$  GeV, where a first-order phase transition occurs. After the discontinuity at this transition,

$\log a$  increases (weakening JQ) until  $T \approx 0.25$  GeV and then slowly decreases (strengthening the JQ parameter again). Thus, at higher temperatures in the QGP phase, the JQ parameter strengthens with increasing  $T$ . In Fig. 4(f) ( $\mu = 1$  GeV), the JQ parameters weakens until  $T \sim 0.4$  GeV and then strengthens with further increases in temperature.

The phase diagrams in the  $(\mu, T)$ -plane for  $c_B = 0$  at  $\nu = 3$  and  $\nu = 4.5$  are shown in Figs. 5(a) and 6(a), respectively. Both figures employ the same color scheme as Fig. 4. While qualitatively similar to Fig. 4, these diagrams exhibit quantitative differences: chemical potential values, discontinuity magnitudes in the JQ parameters, and slopes of  $\log a$  variations differ.

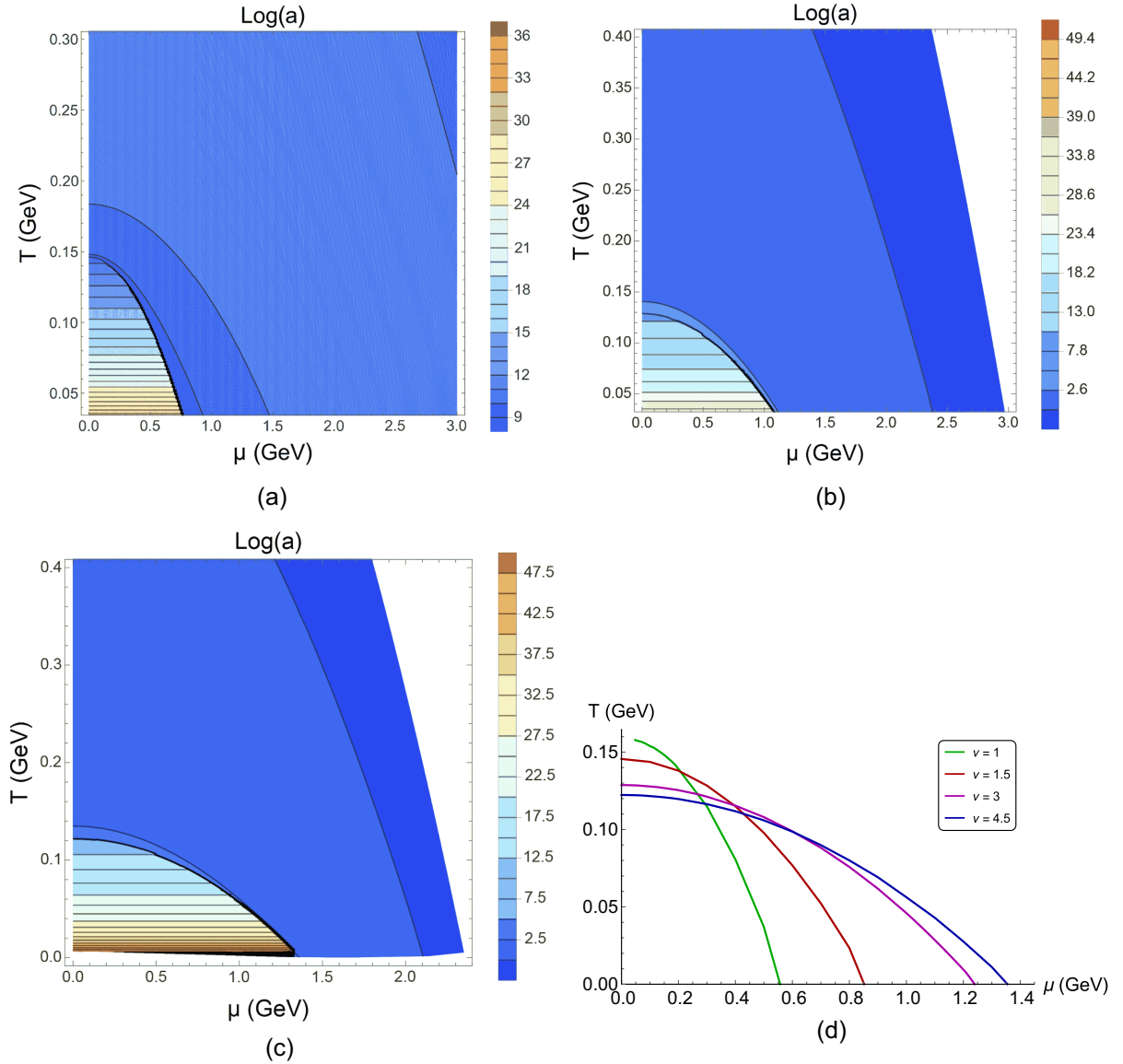


FIG. 7. Density plots for the LQ model at zero magnetic field ( $c_B = 0$ ) with spatial anisotropy parameters: (a)  $\nu = 1.5$ , (b)  $\nu = 3$ , and (c)  $\nu = 4.5$ . (d) First-order phase transition diagrams at  $c_B = 0$  for the isotropic case ( $\nu = 1$ ) and anisotropic cases ( $\nu = 1.5, 3, 4.5$ ), from [47].

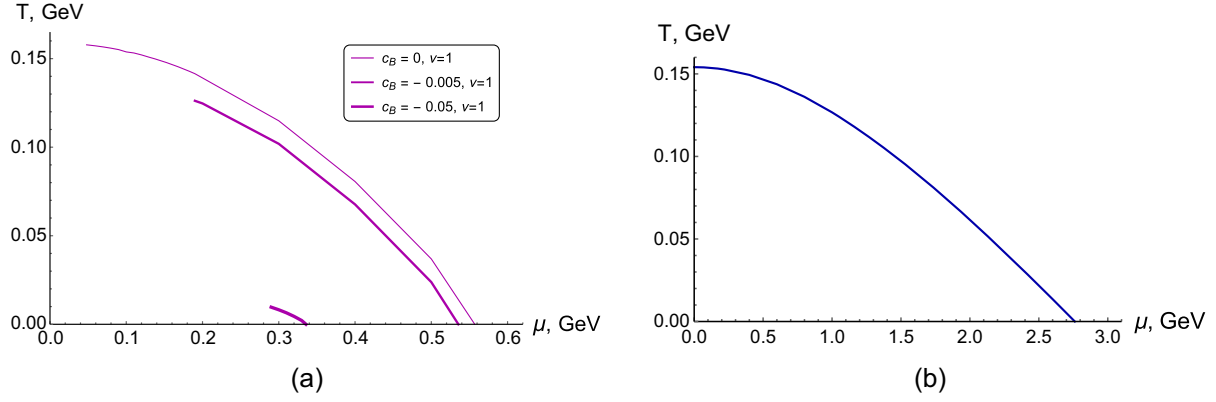


FIG. 8. (a) First-order phase transition lines for the LQ model with  $\nu = 1$  and different  $c_B$ . (b) The second-order phase transition line for the LQ model with  $\nu = 1$  at  $c_B = 0$ . The confinement/deconfinement phase transition disappears for  $c_B = -0.005 \text{ GeV}^2$  and  $c_B = -0.05 \text{ GeV}^2$ .

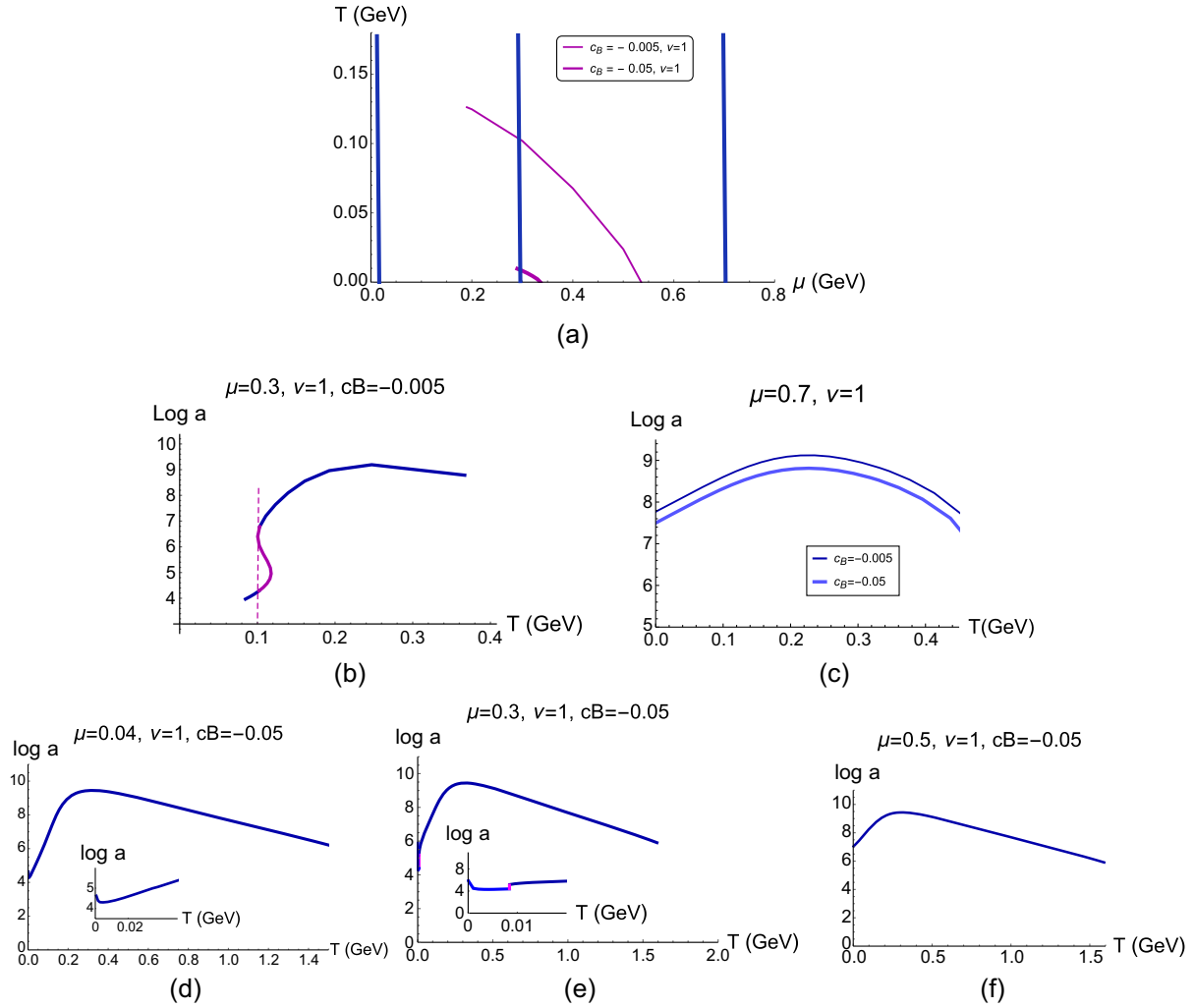


FIG. 9. (a) We calculate the JQ parameter for the LQ model with  $\nu = 1$  at  $c_B = -0.005 \text{ GeV}^2$  and  $c_B = -0.05 \text{ GeV}^2$  along vertical lines (constant  $\mu$ ) in the  $(\mu, T)$ -plane at  $\mu = 0.04, 0.3, 0.5$  and  $\mu = 0.7 \text{ GeV}$ , shown in the panel. Segments of these lines are colored blue (QGP), brown (hadronic), and green (quarkyonic) according to the phase traversed. The resulting plots of  $\log a$  versus the temperature are displayed in the bottom panels (b)–(f) using the same color scheme.

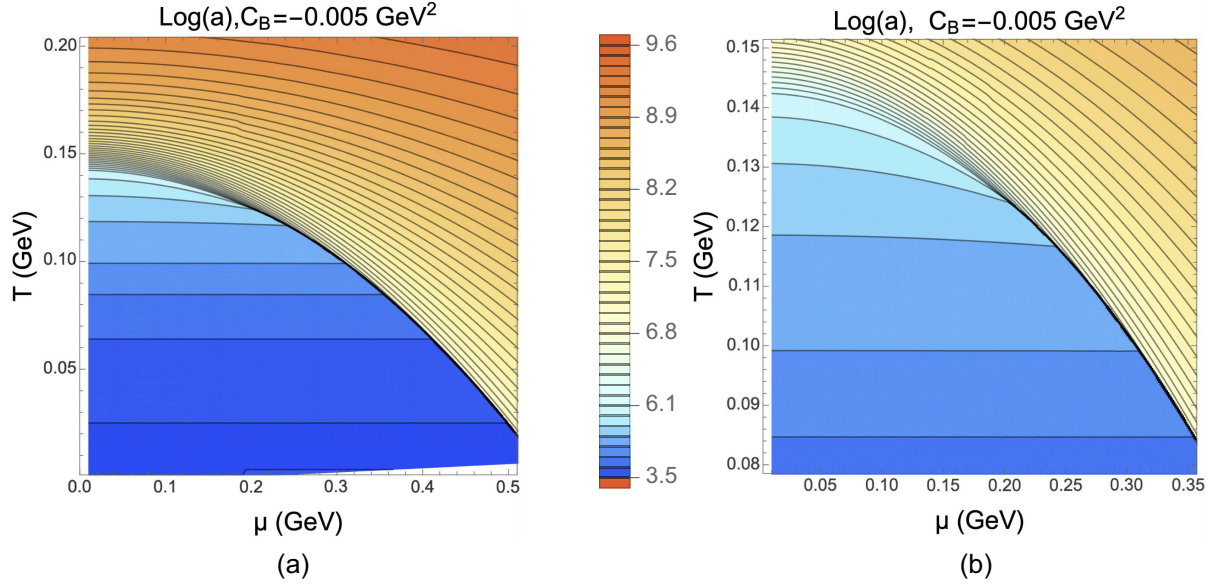


FIG. 10. (a) Density plots of  $\log a$  at  $\nu = 1$  for the LQ model in the presence of the magnetic field  $c_B = -0.005 \text{ GeV}^2$ . (b) The enlarged view of panel (a) near the CEP at ( $\mu \approx 0.19 \text{ GeV}$ ,  $T \approx 0.135 \text{ GeV}$ ).

The density plot for the LQ model at zero magnetic field  $c_B = 0$  with spatial anisotropies are depicted in Figs. 7(a)–7(c) corresponding to  $\nu = 1.5$ ,  $\nu = 3$ , and  $\nu = 4.5$ , respectively. In Fig. 7(d), the first-order phase transition diagrams at  $c_B = 0$  are depicted for the isotropic case  $\nu = 1$  and the anisotropic cases  $\nu = 1.5, 3, 4.5$  [47]. In the hadronic phase for different spatial anisotropy, the JQ parameter depends mainly on the temperature  $T$  and has no significant dependence on the chemical potential  $\mu$ . However, for the QGP phases the JQ parameter depends on both  $T$  and  $\mu$ .

Considering the graphs with density plots at zero magnetic field ( $c_B = 0$ ) in Fig. 1 for  $\nu = 1$  and Fig. 7 for  $\nu = 1.5, 3, 4.5$ , we observe that the JQ parameter behavior exhibits discontinuities precisely along the first-order phase transition lines in the  $(\mu, T)$ -planes. As the anisotropy parameter  $\nu$  increases, the phase transition location shifts, causing the transition lines to migrate rightward with a slight downward displacement.

## B. Nonzero magnetic field

Considering a magnetic field oriented along the  $x_3$ -axis, in this section, we mainly study jets propagating along the  $x_1$ -direction with momentum broadening perpendicular to it along the  $x_2$ -direction, i.e.,  $\hat{q}_2$ . In almost all plots in this section, we drop the index 2 for the JQ parameter as well as for the IJQ parameter  $a$ .

### 1. Nonzero magnetic field, $\nu = 1$

In this section, we examine magnetic field effects on the JQ parameters beginning with the  $\nu = 1$  case. Phase diagrams for different magnetic field strengths ( $c_B$  values)

are shown in Fig. 8. Panel (a) displays the first-order transition lines for the light quark holographic model, while panel (b) demonstrates the magnetic field modification of the confinement/deconfinement transition. Increasing  $|c_B|$  (for  $c_B < 0$ ) suppresses the confinement phase [47]. This aligns with [62], where increasing magnetic fields reduce the running coupling. Furthermore, [63] demonstrates that under physical boundary conditions, the running coupling decreases during hadronic-to-QGP phase transitions. Collectively, these findings indicate magnetic fields restrict systems exclusively to the QGP phase. This suppression of confinement is explicitly shown in Fig. 8 for  $c_B = -0.005 \text{ GeV}^2$  and  $c_B = -0.05 \text{ GeV}^2$ , where the confinement/deconfinement phase transition is absent.

Figure 9 shows<sup>5</sup> discontinuities in the JQ parameter at the first-order phase transitions for  $c_B = -0.005 \text{ GeV}^2$  and  $c_B = -0.05 \text{ GeV}^2$  at  $\mu = 0.3 \text{ GeV}$ . These discontinuities occur within the QGP phase. After each transition,  $\log a$  initially increases with the temperature (weakening JQ) until  $T \approx 0.25\text{--}0.3 \text{ GeV}$  (depending on  $c_B$  and  $\mu$ ) and then decreases (enhancing JQ) at higher temperatures. Figure 9 demonstrates that the JQ parameter behavior identifies first-order phase transitions even under external magnetic fields, consistent with the  $c_B = 0$  results in the previous subsection.

<sup>5</sup>Comparing the plots in panels (b) and (e), it is worth noting that panel (b) details the behavior within an unphysical region (which lacks essential physical significance), while panel (e) focuses on a jump occurring in the physical region. In what follows, one of these versions will be presented without further comment. This distinction arises from the different *Mathematica* codes used to generate the corresponding figures.

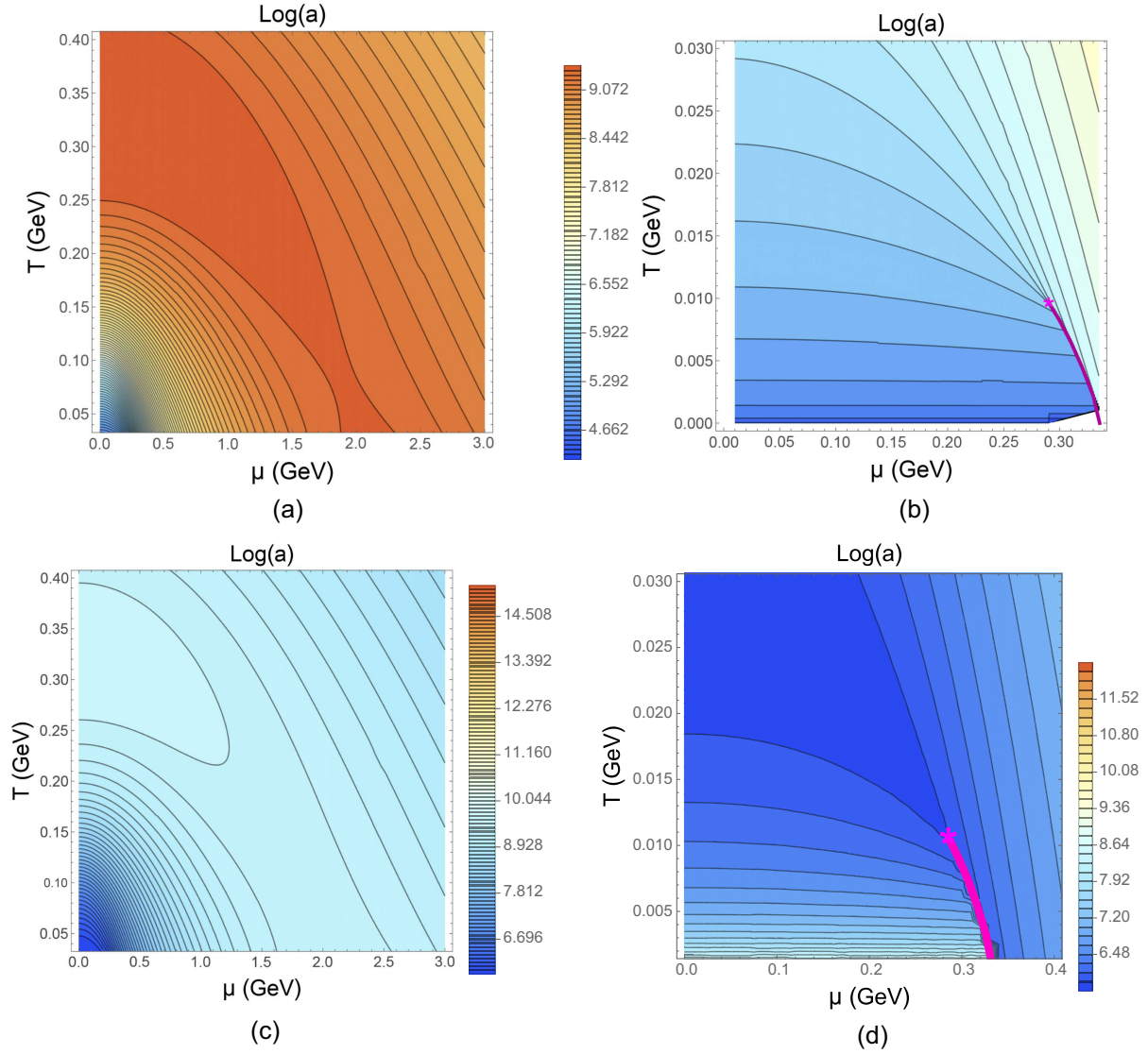


FIG. 11. (a) Density plots for  $\log a_2$  at  $\nu = 1$  for the LQ model in the presence of the magnetic field  $c_B = -0.05 \text{ GeV}^2$ . (b) The enlarged view of panel (a) with the phase transition line. (c) Density plots for  $\log a_3$  with the same  $\nu$  and  $c_B$ . (d) The enlarged view of panel (c) with the phase transition line. The magenta line shows the first-order transition line that starts at CEP with coordinates  $(\mu, T) \approx (0.29 \text{ GeV}, 0.01 \text{ GeV})$  shown by the magenta star.

Figures 10 and 11 show  $\log a$  density plots at  $\nu = 1$  for  $c_B = -0.005 \text{ GeV}^2$  and  $c_B = -0.05 \text{ GeV}^2$ , respectively. Figure 10(b) shows the enlarged view of panel (a) near the CEP at  $(\mu, T) \approx (0.19 \text{ GeV}, 0.135 \text{ GeV})$ . Figures 11(a) and 11(c) depict the density plots for  $\log a_2$  and  $\log a_3$  experiencing the hill-like structure located above the phase transition line.

Figures 11(b) and 11(d) are enlarged views of Figs. 11(a) and 11(b) with phase transition markers: magenta curves for the first-order transitions and a magenta star for the CEP at  $(\mu, T) \approx (0.29, 0.01) \text{ (GeV)}$ .

We investigated the directional dependence of the JQ parameter by comparing  $\hat{q}_3$  (momentum broadening

parallel to the magnetic field) and  $\hat{q}_2$  (momentum broadening perpendicular to the magnetic field). We observe a subtle but weak orientation dependence:

- (i) Figures 12 and 13 show that for  $\nu = 1$ , the JQ parameter exhibits anisotropy in the presence of a magnetic field:  $\log a_2 \neq \log a_3$  for  $c_B = -0.05 \text{ GeV}^2$ .
- (ii) For all values of the temperature, below and above the first-order phase transition we have  $\log a_2 < \log a_3$ .
- (iii) The behavior of the JQ parameter in the high-temperature regime is independent of orientations, i.e., increasing  $T$  leads to enhancement in the JQ parameter.

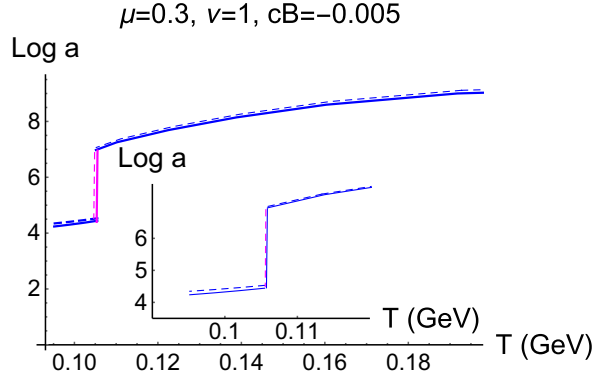


FIG. 12.  $\log a_2$  (solid lines) and  $\log a_3$  (dashed lines) for the LQ model with  $\nu = 1$  and  $c_B = -0.005 \text{ GeV}^2$ .

## 2. Nonzero magnetic field: $\nu = 1.5$

This section demonstrates that for  $\nu = 1.5$ , the behavior of the JQ parameter identifies first-order phase transitions [Fig. 14(a)] while remaining largely insensitive to confinement/deconfinement transitions [Fig. 14(b)]. We examine  $\mu = 0.3$  at  $c_B = -0.005 \text{ GeV}^2$ ,  $\mu = 0.04$ ,  $0.3$ , and  $\mu = 0.5 \text{ GeV}$  at  $c_B = -0.05 \text{ GeV}^2$ . Among these cases,

only the sets with  $c_B = -0.005 \text{ GeV}^2$ ,  $\mu = 0.3 \text{ GeV}$  and  $c_B = -0.05 \text{ GeV}^2$ , and  $\mu = 0.5 \text{ GeV}$ , cross the first-order transition line; however, only the case  $c_B = -0.005 \text{ GeV}^2$ ,  $\mu = 0.3 \text{ GeV}$ , crosses the confinement/deconfinement transition line [Figs. 15(a) and 15(b)].

In Fig. 15, we calculate the JQ parameter at  $\nu = 1.5$  for (a),  $c_B = -0.005 \text{ GeV}^2$  for (b), and  $c_B = -0.05 \text{ GeV}^2$  along the vertical lines (constant  $\mu$ ) in the  $(\mu, T)$ -plane at  $\mu = 0.04$ ,  $0.3 \text{ GeV}$  and  $\mu = 0.5 \text{ GeV}$ . The segments of these lines are colored blue (QGP) and brown (hadronic) according to the phase traversed. Magenta shows the jumps (in the forbidden area). The resulting plots of  $\log a$  versus the temperature are displayed in the bottom panels (c)–(f) using the same color scheme. Figure 15(c) shows that in the hadronic phase for  $c_B = -0.005 \text{ GeV}^2$ , the JQ parameter decreases until the first-order phase transition at  $T \sim 0.12 \text{ GeV}$ . Also, in the QGP phase, the JQ parameter decreases to  $T \sim 0.3 \text{ GeV}$ . Figure 15(d) shows that in the QGP phase for  $c_B = -0.05 \text{ GeV}^2$ , the JQ parameter decreases up to the minimum at  $T \sim 0.3 \text{ GeV}$ , and for  $T \gtrsim 0.3 \text{ GeV}$  the JQ parameter increases. The same behavior is seen at  $\mu = 0.3 \text{ GeV}$  in Fig. 15(e). In addition, the JQ parameter at Fig. 15(f) decreases until the jump at

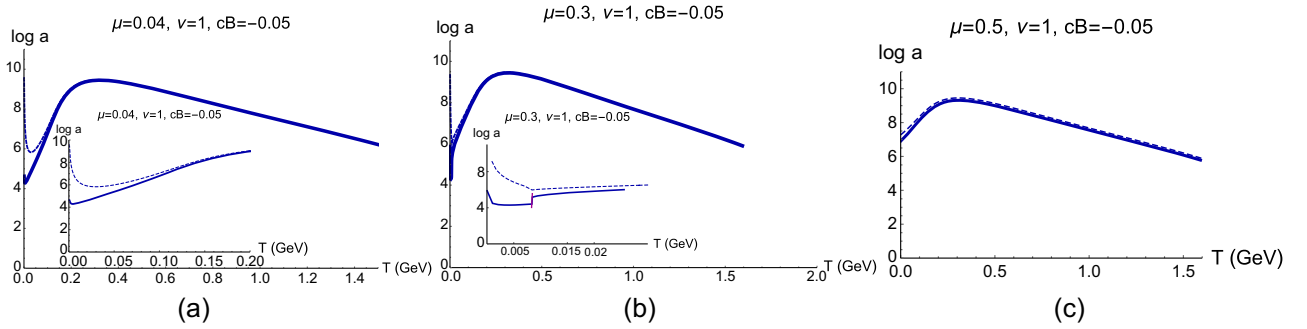


FIG. 13.  $\log a_2$  (solid lines) and  $\log a_3$  (dashed lines) for the LQ model with  $\nu = 1$  and  $c_B = -0.05 \text{ GeV}^2$  versus the temperature at fixed chemical potential: (a)  $\mu = 0.04 \text{ GeV}$ , (b)  $\mu = 0.3 \text{ GeV}$ , and (c)  $\mu = 0.5 \text{ GeV}$ . In panels (a) and (b) the enlarged plots are located inside the basic plots.

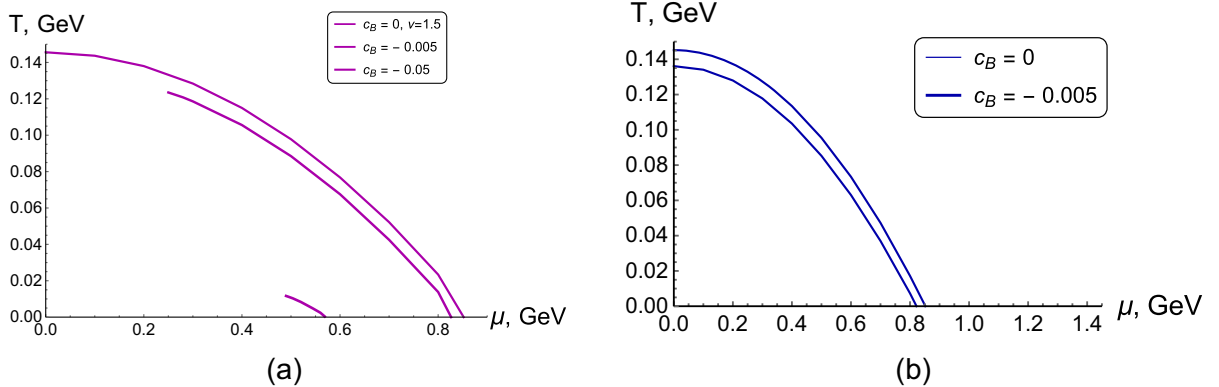


FIG. 14. (a) First-order phase transition lines with  $c_B = 0, -0.005, -0.05 \text{ GeV}^2$  and (b) confinement/deconfinement transition lines with  $c_B = 0, -0.005 \text{ GeV}^2$  for the LQ model with  $\nu = 1.5$ .

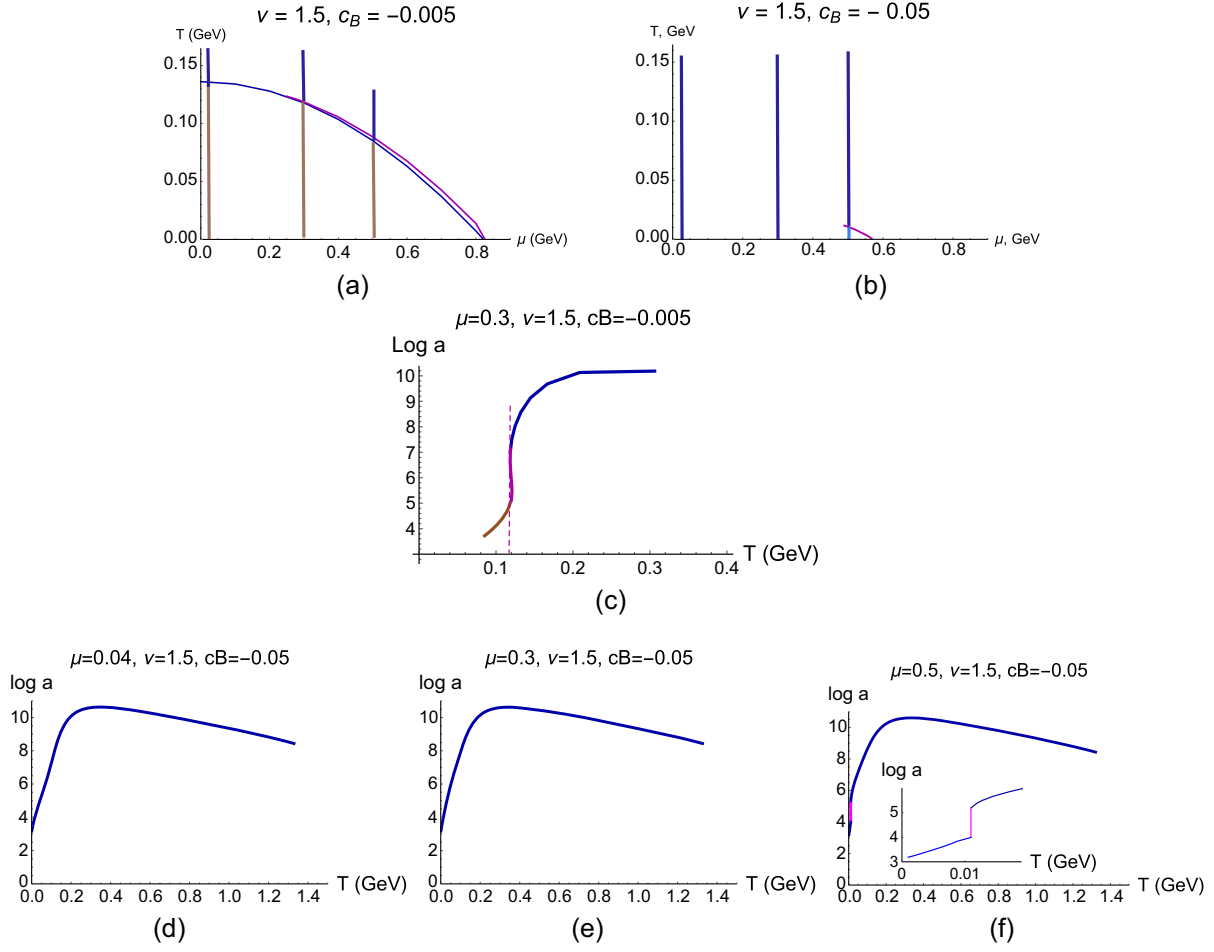


FIG. 15. We calculate the JQ parameter for the LQ model with  $\nu = 1.5$  along the vertical lines (constant  $\mu$ ) in the  $(\mu, T)$ -plane: (a) with  $c_B = -0.005$  GeV<sup>2</sup> at  $\mu = 0.04, 0.3, 0.5$  GeV and (b) with  $c_B = -0.05$  GeV<sup>2</sup> at  $\mu = 0.04, 0.3, 0.5$  GeV. The segments of these lines are colored blue, brown, and green corresponding to the QGP, hadronic, and quarkyonic phases they traverse. (c) The resulting values for  $\log a$  are presented on the bottom panels (for  $c_B = -0.005$  GeV<sup>2</sup>), and panels (d)–(f) (for  $c_B = -0.05$  GeV<sup>2</sup>) are colored using the same scheme.

$T \sim 0.01$  GeV and keeps decreasing behavior until reaching the minimum at  $T \sim 0.3$  GeV and then increases for  $T \gtrsim 0.3$  GeV in the QGP phase.

Figure 16 displays the density plots of  $\log a$  at  $\nu = 1.5$  for  $c_B = -0.05$  GeV<sup>2</sup>. Panel (b) enlarges into panel (a) near the critical endpoint at  $(\mu_{\text{CEP}}, T_{\text{CEP}}) \approx (0.49, 0.012)$  (GeV), with the magenta line marking the first-order transition. Panel (c) provides an enlarged view of panel (a) using increased contour density (100 contours), revealing a hill-like structure above the phase transition line. It is important to note that for these values of the magnetic field and anisotropy parameter, no hadronic phase exists; consequently, the second-order confinement/deconfinement phase transition is absent, and the system remains entirely in the QGP phase.

A detailed examination of the density plot in Fig. 16, and two-dimensional plots in Fig. 15 [panels (d)–(f)] reveals for  $c_B = -0.05$  GeV<sup>2</sup>

- (1) A discontinuity in  $\log a$  within  $0.52 \text{ GeV} < \mu < 0.59 \text{ GeV}$ .
- (2) For  $0.52 \text{ GeV} < \mu < 0.59 \text{ GeV}$ :
  - (a)  $\log a$  increases before the discontinuity.
  - (b) continues increasing in  $\log a$  immediately after the discontinuity.
  - (c)  $\log a$  decreases above  $T \gtrsim 0.4$  GeV due to a hill feature. For any fixed  $\mu$ , increasing  $T$  beyond this threshold reduces  $\log a$ , indicating  $\mu$ -independent behavior.
- (3) For  $\mu < 0.52$  GeV:
  - (a)  $\log a$  increases until  $T \approx 0.4$  GeV.
  - (b)  $\log a$  decreases at higher temperatures.
- (4) No discontinuity occurs at  $\mu < 0.52$  GeV, though  $\log a$  shifts from increasing to decreasing near  $T \approx 0.4$  GeV.
- (5) At high temperatures ( $T \gg 0.4$  GeV),  $\log a$  universally decreases with increasing  $T$ .

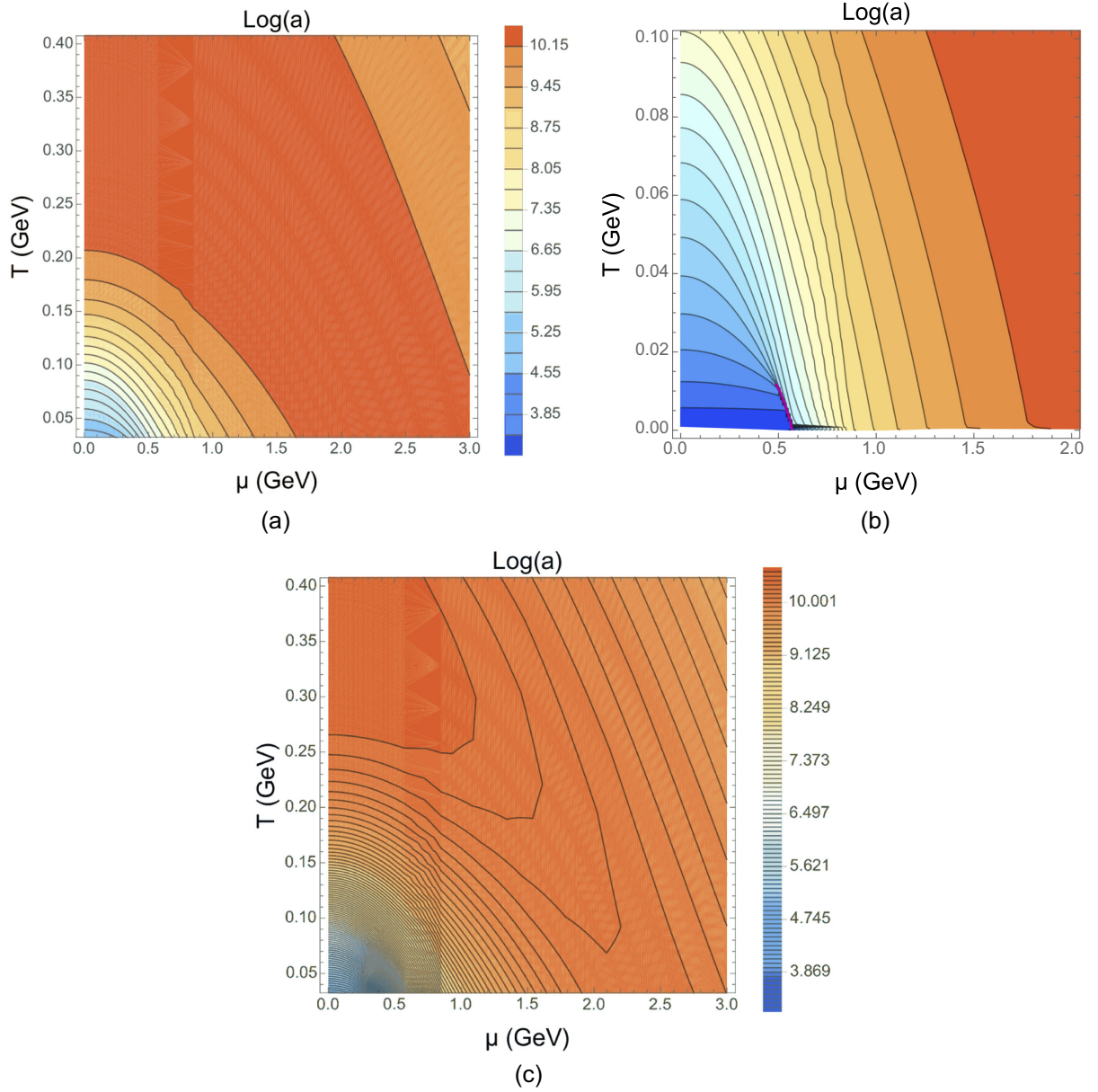


FIG. 16. (a) Density plots for  $\log a$  at  $\nu = 1.5$  for the LQ model in the presence of the magnetic field,  $c_B = -0.05 \text{ GeV}^2$ . Note that for these values of the magnetic field and anisotropy parameter, no hadronic phase exists; consequently, the second-order confinement/deconfinement phase transition is absent, and the system remains entirely in the QGP phase. (b) Enlarged view of panel (a) with the magenta line showing the first-order transition line. The CEP is located approximately at  $(\mu_{\text{CEP}}, T_{\text{CEP}}) \approx (0.49 \text{ GeV}, 0.012 \text{ GeV})$ . (c) Elargied view of panel (a) with increased contour density (100 contours), revealing the hill-like structure located above the phase transition line.

### 3. Nonzero magnetic field: $\nu = 3$

In this section, we demonstrate that for  $\nu = 3$  and for varying  $\mu$ , the behavior of the JQ parameter detects the first-order phase transitions shown in Fig. 17(a) while remaining practically insensitive to the confinement/deconfinement transition in Fig. 17(b).

We perform calculations for two sets of parameters:

- (i)  $c_B = -0.005 \text{ GeV}^2$ ,  $\mu = 0.3 \text{ GeV}$ .
- (ii)  $c_B = -0.05 \text{ GeV}^2$ ,  $\mu = 0.04, 0.3, 0.8 \text{ GeV}$ .

All cases include the confinement/deconfinement transition, but only the following exhibit first-order phase transitions:

- (i) Case (i) [Figs. 18(a) and 18(c)].
- (ii) The subcase of (ii) with  $\mu = 0.8 \text{ GeV}$  [Figs. 18(b) and 18(f)].

Results for  $c_B = -0.005 \text{ GeV}^2$  at  $\mu = 0.004, 0.7 \text{ (GeV)}$  are omitted, as they closely resemble the behavior shown in Fig. 18(c).

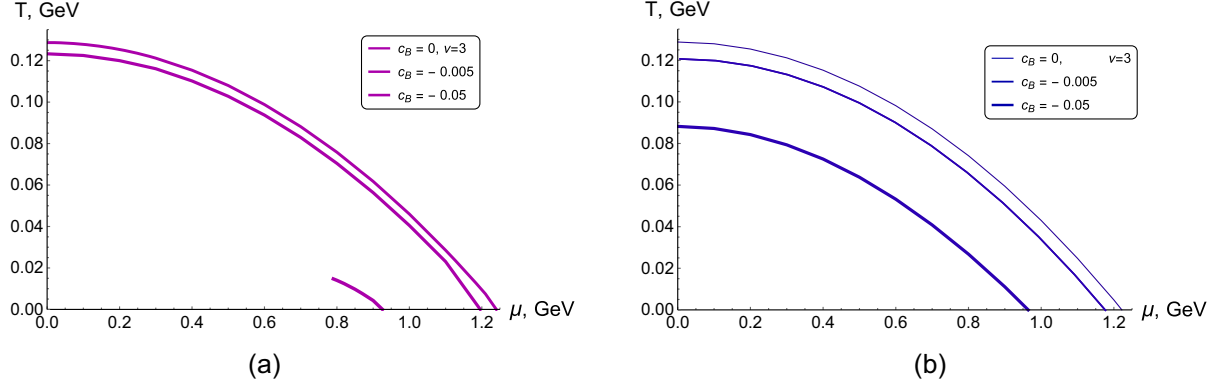


FIG. 17. (a) First-order phase transition lines and (b) confinement/deconfinement transition lines for the LQ model with  $\nu = 3$  and  $c_B = 0, -0.005, -0.05 \text{ GeV}^2$ .

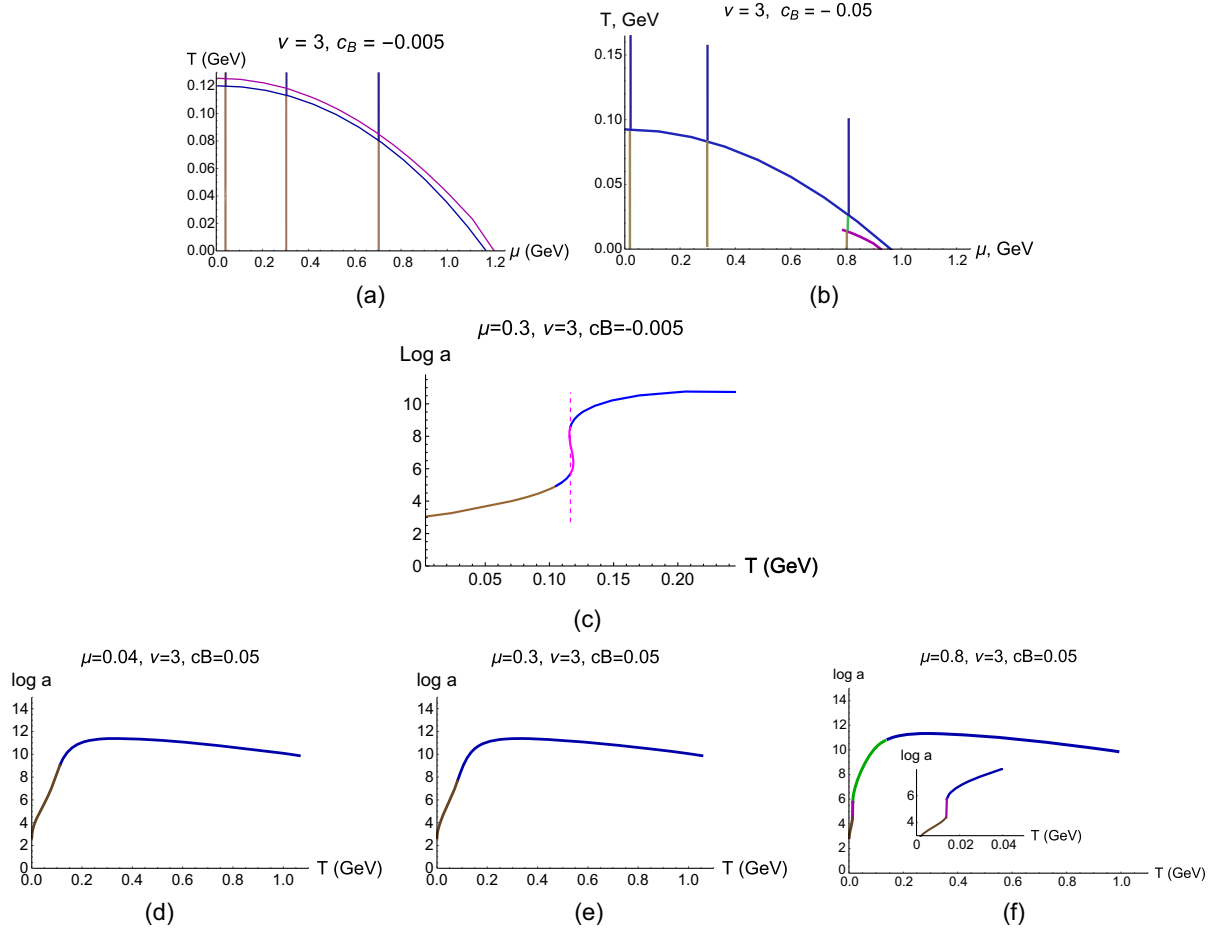


FIG. 18. We calculate the JQ parameter for the LQ model with  $\nu = 3$  along vertical lines (constant  $\mu$ ) in the  $(\mu, T)$ -plane: (a) with  $c_B = -0.005 \text{ GeV}^2$  at  $\mu = 0.04, 0.3, 0.7 \text{ GeV}$  and (b) with  $c_B = -0.05 \text{ GeV}^2$  at  $\mu = 0.04, 0.3, 0.8 \text{ GeV}$ . The segments of these lines are colored blue, brown, and green corresponding to the QGP, hadronic, and quarkyonic phases they traverse, respectively. (c) The resulting values for  $\log a$  are presented on the bottom panels (for  $c_B = -0.005 \text{ GeV}^2$ ), and panels (d)–(f) (for  $c_B = -0.05 \text{ GeV}^2$ ) are colored using the same scheme.

Our calculations show

- (i) A jump in  $\log a$  occurs at  $\mu = 0.3 \text{ GeV}$  with  $c_B = -0.005 \text{ GeV}^2$ , as illustrated in Fig. 18(c). This picture is similar to the plot presented in Fig. 15(c).

We observe that  $\log a$  increases rapidly before the jump and just after the jump and slowly decreases afterward, making the JQ parameter increase at high temperature regime.

$$\nu = 4.5$$

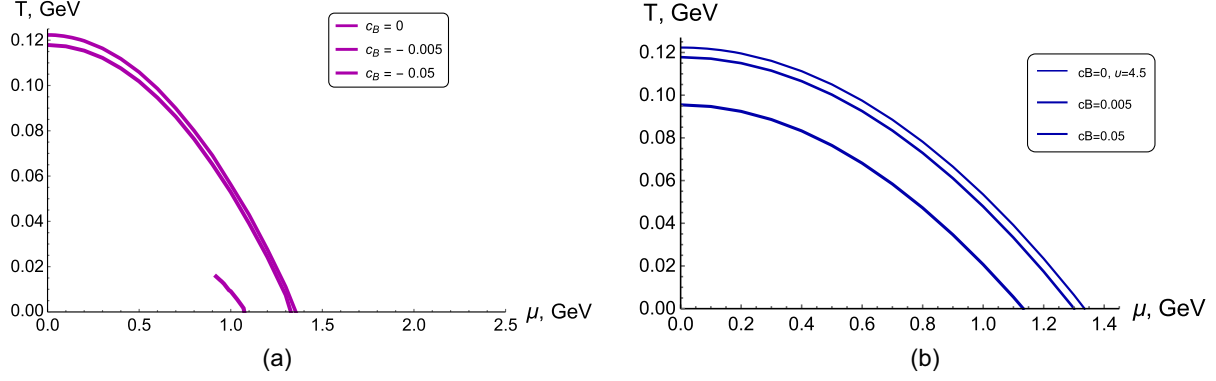


FIG. 19. (a) First-order phase transition lines and (b) confinement/deconfinement transition lines for the LQ model with  $\nu = 4.5$  and  $c_B = 0, -0.005 \text{ GeV}^2, -0.05 \text{ GeV}^2$ .

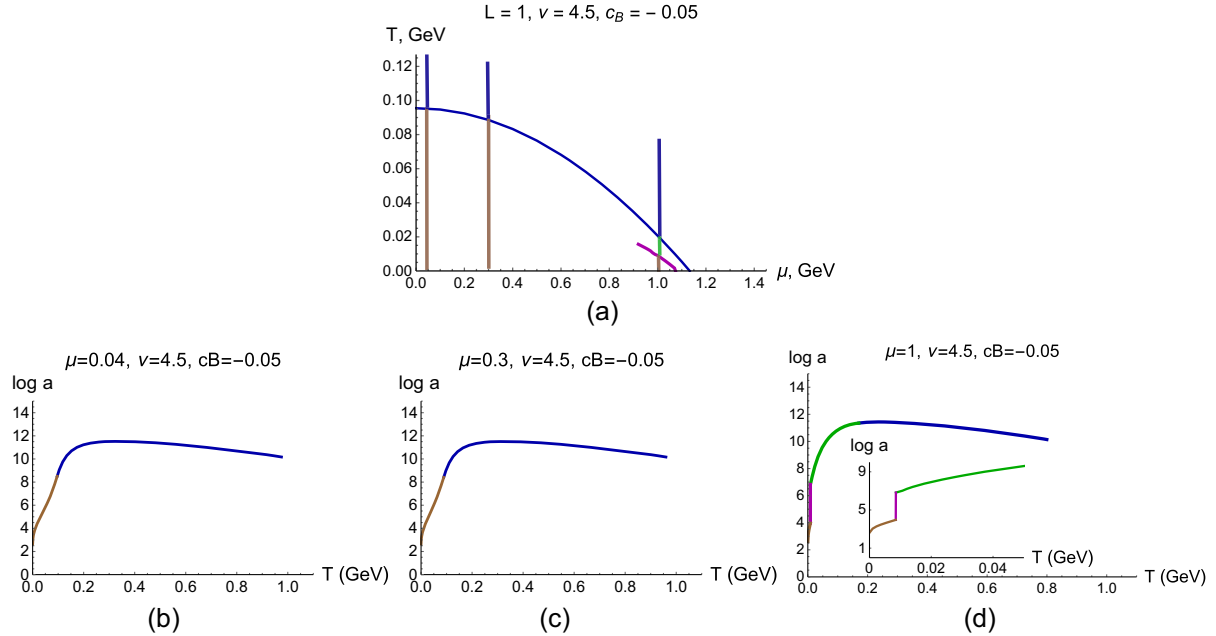


FIG. 20. Considering  $\nu = 4.5$  and  $c_B = -0.05 \text{ GeV}^2$ , we calculate the JQ parameter for the LQ model along vertical lines (constant  $\mu$ ) in the  $(\mu, T)$ -plane shown in panel (a). The calculations are performed at fixed chemical potentials: (b)  $\mu = 0.04 \text{ GeV}$ , (c)  $\mu = 0.3 \text{ GeV}$ , (d)  $\mu = 1 \text{ GeV}$ . The segments of these lines are colored blue, brown, and green corresponding to the QGP, hadronic, and quarkyonic phases they traverse, respectively (a). The resulting values for  $\log$  of the JQ parameter are presented on the bottom panels (b)–(d) and are colored using the same scheme.

- (ii) Although no discontinuity is observed in Figs. 18(d) and 18(e), a distinct change in slope occurs near the confinement/deconfinement transition. The color transition from brown (hadronic) to blue (QGP) occurs near the hills in these plots.

However, a similar change in slopes occurs even in cases without a confinement/deconfinement phase transition, as shown, for example, in panels (d) and (e) of Fig. 15. Naively, this suggests that for fixed parameters (while varying only the spatial anisotropy  $\nu$ ), the JQ parameter exhibits identical

temperature dependence. We expect universal asymptotic behavior in the JQ parameter at the high temperature regime.

- (iii) In all cases,  $\log a$  decreases, indicating enhancement in the JQ parameter with increasing temperature in the high-temperature regime.

#### 4. Nonzero magnetic field: $\nu = 4.5$

In this section, we study  $\nu = 4.5$  for various  $\mu$ . Figure 19 presents  $(\mu, T)$ -phase diagrams at  $\nu = 4.5$  and  $c_B = 0$ ,

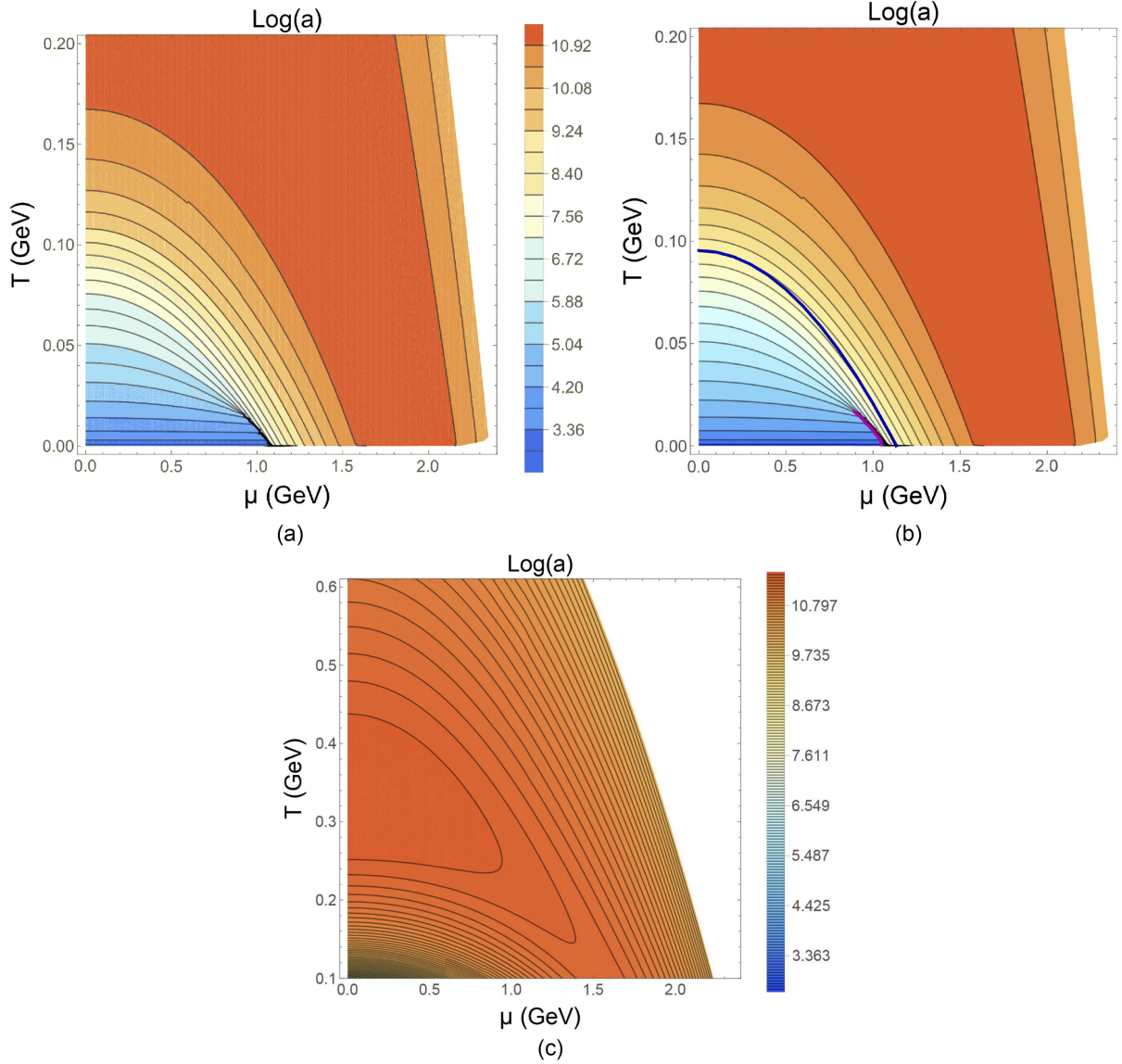


FIG. 21. (a) Density plots of  $\log a$  for the LQ model, with  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 4.5$ , including 20 contours. (b) The same with phase transition lines: (i) the magenta line presents the first-order transition line, and (ii) the blue one presents the confinement/deconfinement transition line. (c) Enlarged view of panel (a) with increased contour density (80 contours), revealing the hill-like structure located above the phase transition lines.

$-0.005, -0.05 \text{ (GeV}^2\text{)}$ , with first-order phase transitions in (a) and confinement/deconfinement transitions in (b). The behavior of the JQ parameter closely resembles that observed for  $\nu = 3$ , differing primarily in the chemical potential value where the first-order phase transition occurs. Our calculations show

- (1) No discontinuity appears in Figs. 20(b) and 20(c), but a distinct change in slope occurs near the confinement/deconfinement transition. This change is clearly visible in the density plot of Fig. 21(c).
- (2) For  $c_B = -0.05 \text{ GeV}^2$  at  $\mu = 1 \text{ GeV}$  [Fig. 20(d)],  $\log a$  exhibits:
  - (a) rapid increase before the discontinuity,
  - (b) a jump at  $T = 0.00757 \text{ GeV}$ ,

TABLE I. Schematic layout of two-dimensional plots of dependence of  $\log a$  of the JQ parameter for light quarks on  $T$  for fixed values of  $\mu$  at different  $c_B$  and  $\nu$ .

| $ c_B $ | $\nu$  |         |         |         |
|---------|--------|---------|---------|---------|
|         | 1      | 1.5     | 3       | 4.5     |
| 0       | Fig. 1 | Fig. 4  | Fig. 5  | Fig. 6  |
| 0.005   | Fig. 9 | Fig. 15 | Fig. 18 | ...     |
| 0.05    | Fig. 9 | Fig. 15 | Fig. 18 | Fig. 20 |

TABLE II. Density plots of  $\log a$  of the JQ parameter in the  $(\mu, T)$ -plane for the LQ model at different  $c_B$  and  $\nu$ .

| $ c_B $ | $\nu$   |           |           |           |
|---------|---------|-----------|-----------|-----------|
|         | 1       | 1.5       | 3         | 4.5       |
| 0       | Fig. 3  | Fig. 7(a) | Fig. 7(b) | Fig. 7(c) |
| 0.005   | Fig. 10 | ...       | ...       | ...       |
| 0.05    | Fig. 11 | Fig. 16   | ...       | Fig. 21   |

- (c) continued increase in the quarkyonic phase,  
 (d) slow decrease in the QGP phase.  
 (3) In all cases,  $\log a$  decreases, indicating enhancement in the JQ parameter with increasing temperature in the high-temperature regime.

### C. Summary of jet quenching results for the LQ model

We have calculated the JQ parameter  $\hat{q}$  for quarks propagating along lightlike trajectories in a QCD medium with light quarks at finite temperature ( $T$ ) and chemical potential ( $\mu$ ). Our results reveal a nontrivial dependence on both  $T$  and  $\mu$ , particularly near the phase transition temperature  $T_c$ .

From previous studies [55], we know that in the isotropic case, both first-order and second-order phase transitions occur. The corresponding phase structure is shown in Figs. 8(a) and 8(b). When varying parameters  $c_B$  and  $\nu$ , the following behavior emerges [47]:

- (1) As the magnetic field increases (controlled by  $c_B$ )  
 (a) the second-order phase transition disappears [Fig. 8(a)],

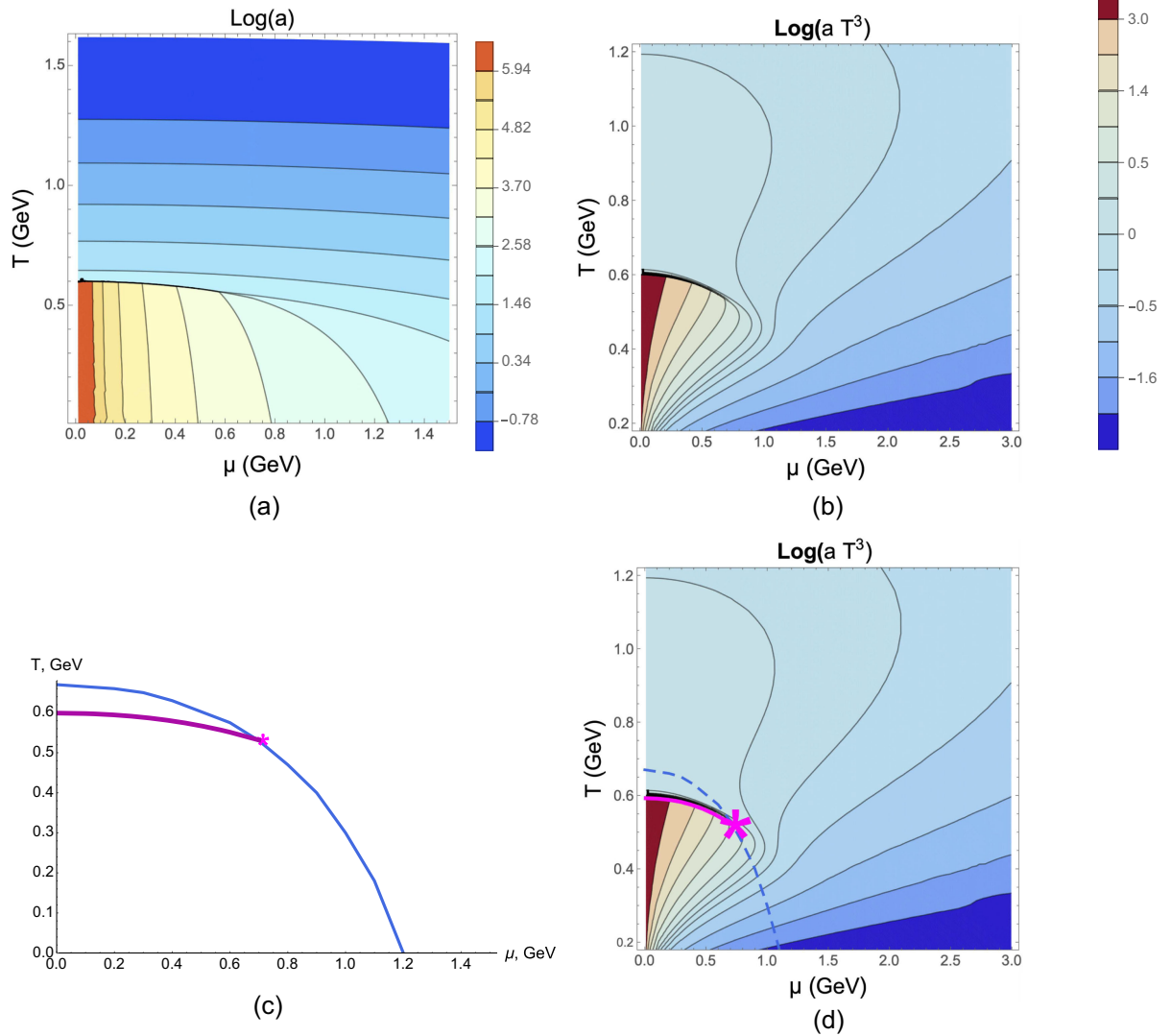


FIG. 22. (a) Density plot with contours for  $\log a$  of the HQ model,  $c_B = 0$  and  $\nu = 1$ , with boundary condition (A10). (b) Density plots with contours for  $\log(aT^3)$  for the same parameters as for the panel (a). (c) The phase diagram for the HQ model,  $c_B = 0$  and  $\nu = 1$ . (d) The same as panel (b) with lines indicating phase transition lines: the magenta line denotes first-order, the dashed blue line denotes second-order, and the magenta star presents the CEP.

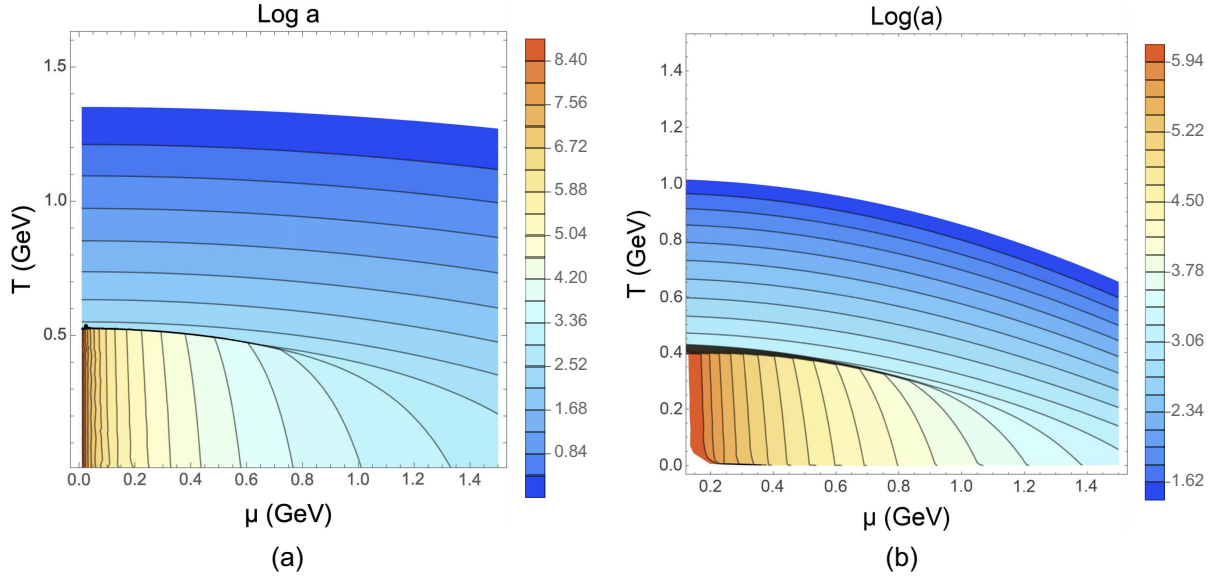


FIG. 23. Density plots for the HQ model in the  $(\mu, T)$ -plane with contours for  $\log a$  at  $c_B = 0$  with (a)  $\nu = 1.5$  and (b)  $\nu = 4.5$ .

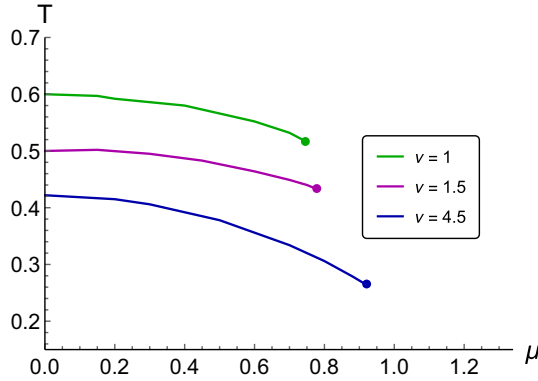


FIG. 24. First-order phase transition lines for the HQ model at  $c_B = 0$  considering spatial anisotropies  $\nu = 1, 1.5$  and  $\nu = 4.5$ .

(b) the first-order phase transition shifts toward lower temperatures and chemical potentials (inverse magnetic catalysis), becoming negligible at extreme values [Fig. 8(b)].

(2) As  $\nu$  increases

(a) the confinement/deconfinement transition reappears for large negative  $c_B$  (high magnetic fields).

(b) the first-order transition and critical endpoint shift rightward in the  $(\mu, T)$ -plane (Fig. 19).

Examining the JQ parameter for the LQ model across different  $c_B$  and  $\nu$  values, we observed

- (i) A discontinuity at the first-order phase transition line.
- (ii) A change in the slope of the  $\log a$  curve near second-order phase transition lines.

The corresponding plots are shown in the figures referenced in Tables I and II.

#### IV. JQ PARAMETER FOR HEAVY QUARKS: NUMERICAL RESULTS

In this section, we study the JQ parameter within a specialized holographic model for heavy quarks exhibiting magnetic catalysis. Experimentally, heavy-quark QCD displays quantum catalysis, where an external magnetic field increases the critical temperature of the first-order phase transition. This model was constructed in [48]. Our results for the JQ parameter for HQ models are different from other holographic HQ models just due to the difference of the models themselves.

The phase diagram of the HQ model in the  $(\mu, T)$ -plane describes two different types of phase transitions, i.e., the first-order phase transition and confinement/deconfinement phase transition [see Fig. 22(c)]. These phase transitions can also be presented in the  $(\mu, z_h)$ -plane. As noted in Sec. III, the phase structure in the  $(\mu, z_h)$ -plane provides a convenient basis for examining how physical quantities depend on thermodynamic parameters  $T$  and  $\mu$ .  $(\mu, z_h)$ -phase diagrams for our model are presented for two cases:

(i)  $\nu = 1$  with zero magnetic field, Fig. 47.

(ii)  $\nu \neq 1$  with nonzero magnetic field, Figs. 48 and 50.

The corresponding phase diagrams in the  $(\mu, T)$ -planes are presented in Figs. 22(c), 24, 48, and 50 (see [48,62,64]).

Our goal in this section is to show that a detailed examination of density plots of the JQ parameter on the  $(\mu, T)$ -plane reveals that the JQ parameter responds only at the first-order phase transition line, exhibiting a distinct change there. In contrast, the change of the JQ parameter across the second-order phase transition line is smooth.

##### A. Jet quenching for zero magnetic field

The plot in Fig. 22(a) shows the density plot for  $\log a$  of the isotropic HQ model (for  $\nu = 1$  and  $c_B = 0$ ) with the

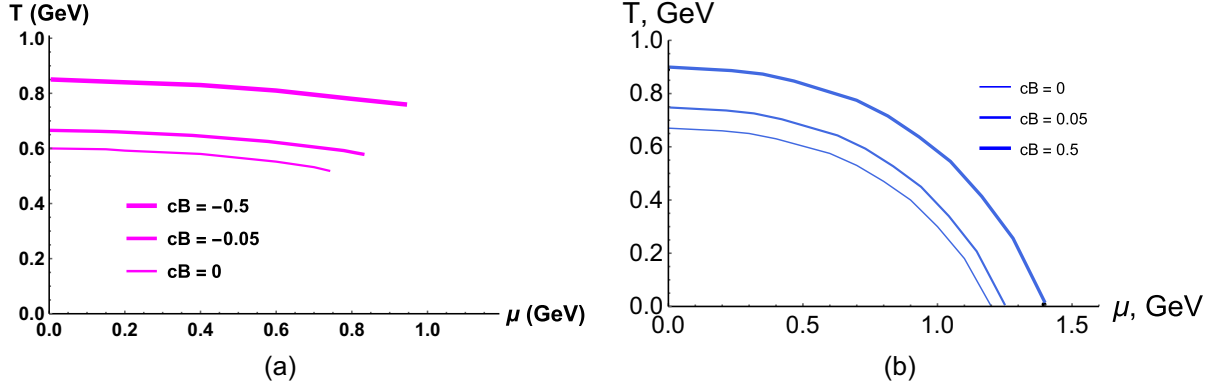


FIG. 25. (a) First-order phase transition lines for the HQ model with  $\nu = 1$  and  $c_B = 0, -0.05 \text{ GeV}^2$  and  $-0.5 \text{ GeV}^2$  and (b) the second-order phase transition lines for the HQ model with the same  $\nu$  and  $c_B$ .

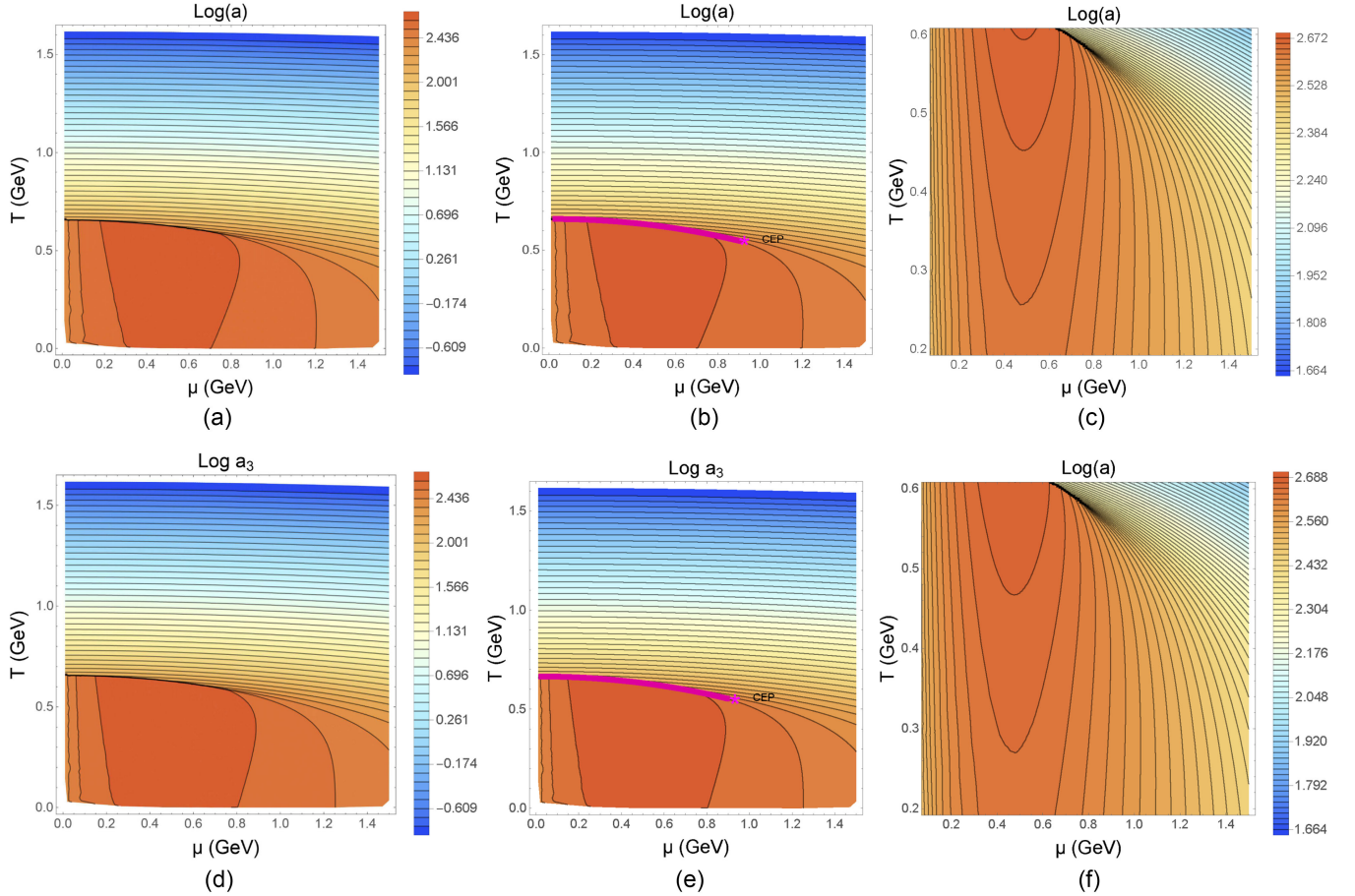


FIG. 26. Contour plots of (a)  $\log a_2$  and (d)  $\log a_3$  for the HQ model in the  $(\mu, T)$ -plane with nonzero magnetic field  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 1$ , both displaying 20 contours. Panels (b) and (e) replicate panels (a) and (d), respectively, with first-order phase transition lines in magenta and with magenta stars marking the critical endpoints (CEPs). Panels (c) and (f) show enlarged regions of panels (a) and (d) with higher contour density (200 contours), revealing hill-like structures beneath the phase transition lines. The first-order phase transition begins at  $(\mu, T) = (0, 0.65 \text{ GeV})$ . Increased contour density in (c) and (f) shows no significant differences between  $\log a_2$  (top row) and  $\log a_3$  (bottom row).

boundary condition (A10). Here are also some contours. Each contour corresponds to a fixed value of  $\log a$ . Figure 22(b) displays the density plot for  $\log(aT^3)$  with the same parameters as Fig. 22(a). Figure 22(d) is the same

as Fig. 22(b) but includes additional curves and stars marking the phase transition lines: magenta for the first-order, dashed blue for the second-order, and the magenta star presents the CEP. The arrangement of constant

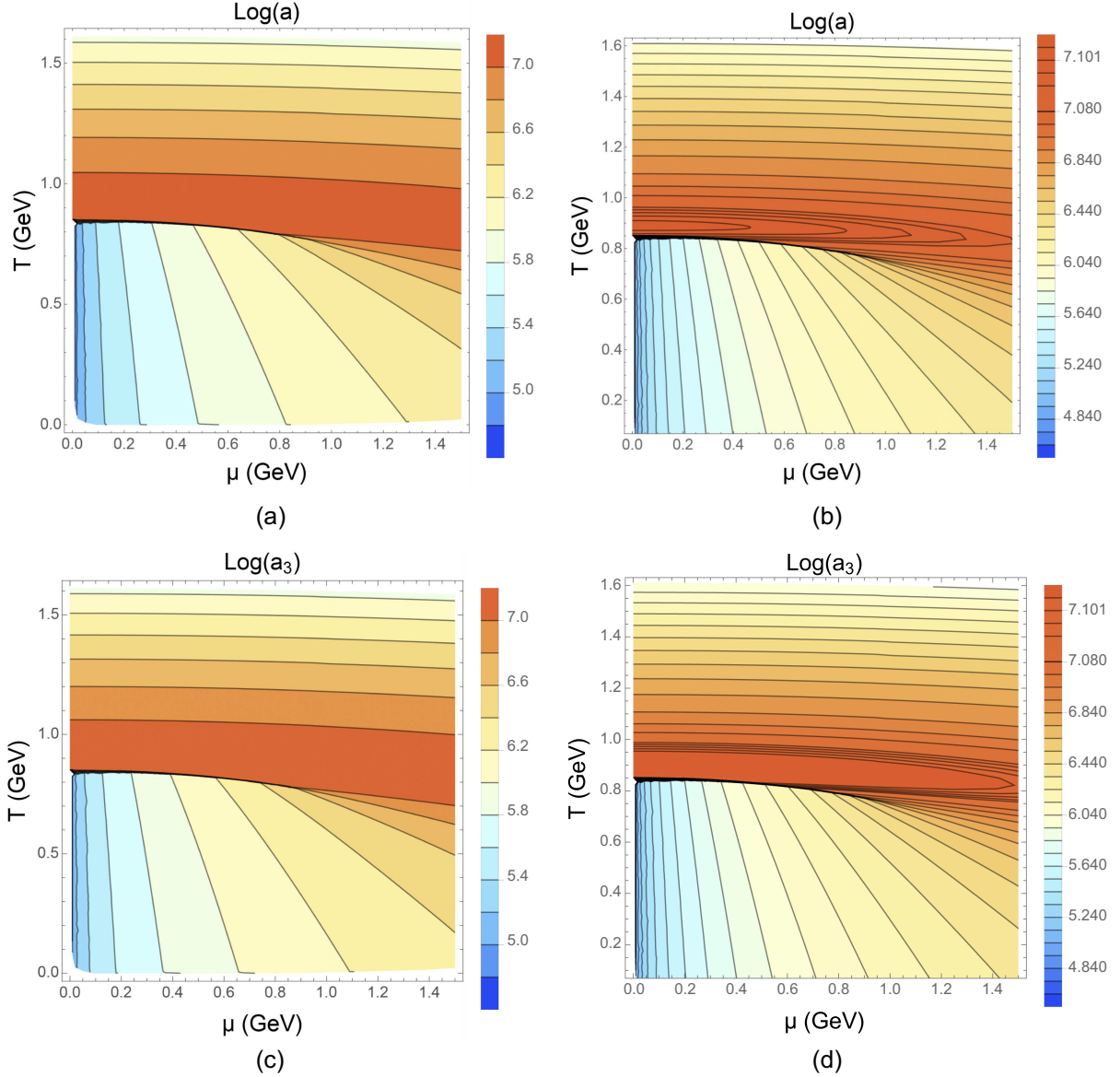


FIG. 27. Contour plots for the HQ model in the  $(\mu, T)$ -plane with  $c_B = -0.5$  GeV and  $\nu = 1$ : (a)  $\log a_2$  and (c)  $\log a_3$  at standard contour density and (b) and (d) show higher contour density versions of (a) and (c), respectively. Differences between  $\log a_2$  (top row) and  $\log a_3$  (bottom row) are only apparent at higher contour density, as seen in panels (b) and (d).

$\log(aT^3)$  contours allows us to determine the location of the first-order phase transition in the  $(\mu, T)$ -plane shown in Fig. 22(c).

The plots in Fig. 23 show density plots with contours for the HQ model with  $c_B = 0$  for  $\nu = 1.5$  and  $\nu = 4.5$ . The orientation of the contours differs between the regions below and above  $T \approx 0.5$  GeV [Fig. 23(a)] and  $T \approx 0.4$  GeV [Fig. 23(b)]. Furthermore, the density appears smoother for  $\mu > 0.81$  GeV in Fig. 23(a) and  $\mu > 0.95$  GeV in Fig. 23(b). Figure 23 shows that in the HQ model in the hadronic phase, the JQ parameter mostly depends on  $\mu$  and slightly on  $T$ , which is completely in opposite to the LQ model. These pictures reveal the

first-order phase transition in the  $(\mu, T)$ -plane, shown in Fig. 24, and the corresponding locations of the CEP.

### B. Nonzero magnetic field

In this subsection, with a magnetic field oriented along the  $x_3$ -axis, we focus on jets propagating along the  $x_1$ -direction and their momentum broadening primarily parallel to the magnetic field (along  $x_3$ ), as denoted  $\hat{q}_3$ . In all plots, we omit the subscript 3 for both the jet quenching parameter  $\hat{q}$  [Eq. (2.11)] and the parameter  $a$  [Eq. (2.13)]. When comparing  $\hat{q}$  values across different directions, we restore the corresponding indices (see Figs. 29, 32, and 36).

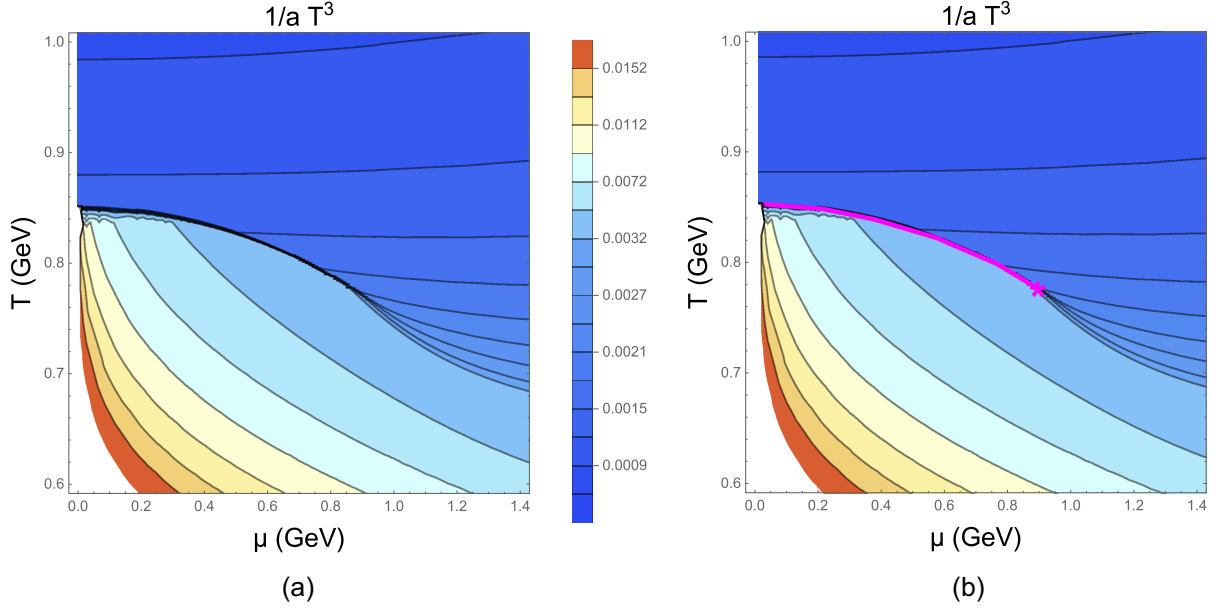


FIG. 28. (a) Density plot with contours for  $\log a_2 T^3$  for the HQ model with nonzero magnetic field  $c_B = -0.5 \text{ GeV}^2$  and  $\nu = 1$ . (b) The same as in panel (a), with the first-order transition indicated by a magenta line. Here, the magenta line starts at  $\mu = 0$ , and  $T = 0.85 \text{ GeV}$ .

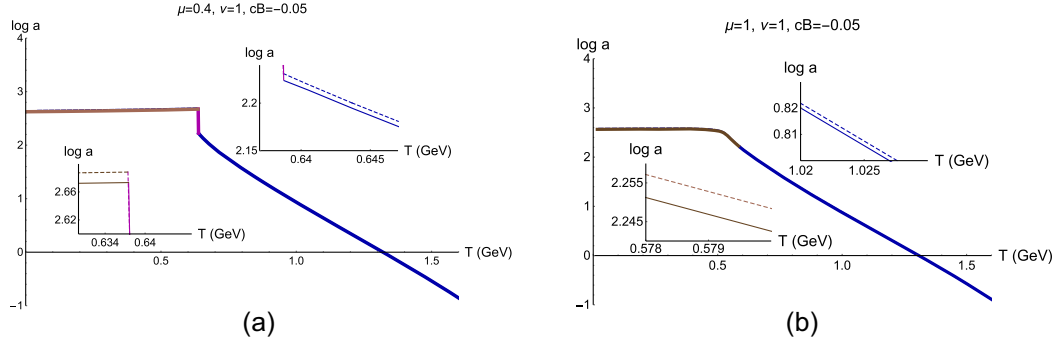


FIG. 29.  $\log a_2$  (solid lines) and  $\log a_3$  (dashed lines) for the HQ model, with  $\nu = 1$  and  $c_B = -0.05 \text{ GeV}^2$ , versus the temperature at fixed chemical potentials: (a)  $\mu = 0.4 \text{ GeV}$  and (b)  $\mu = 1 \text{ GeV}$ . The solid and dashed blue lines correspond to the QGP, and the solid brown and dashed gray lines correspond to the hadronic phases. The magenta lines (solid and dashed) correspond to the unstable regions. We see jumps at panel (a) and changes of the slopes near the second-order phase transitions at panel (b).

### 1. Nonzero magnetic field: $\nu = 1$

The plots in Fig. 25 show the phase transitions in the HQ model for  $\nu = 1$  and different magnetic fields.

The plots in Figs. 26–28 demonstrate that the behavior of the JQ parameter enables identification of the first-order phase transition position even in the presence of an external magnetic field. This conclusion is further supported by the two-dimensional graphs shown in Fig. 29.

Comparing the density plots of the JQ parameter in Fig. 22 (zero magnetic field) and Fig. 26 (nonzero magnetic field), significant differences emerge in the behavior of constant-value contours. Specifically, as noted in Sec. IV A, the contours in Fig. 22 are nearly vertical, while those in Fig. 26 are predominantly horizontal. This contrast concerns hadronic phases.

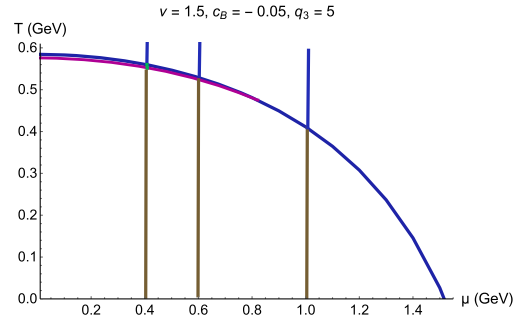


FIG. 30. Phase diagram for the HQ model in the  $(\mu, T)$ -plane for  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 1.5$ . Magenta and blue lines indicate the first-order and the second-order phase transitions, respectively. The vertical lines at fixed  $\mu = 0.4, 0.6, 1 \text{ (GeV)}$  show the paths we calculated for the JQ parameter.

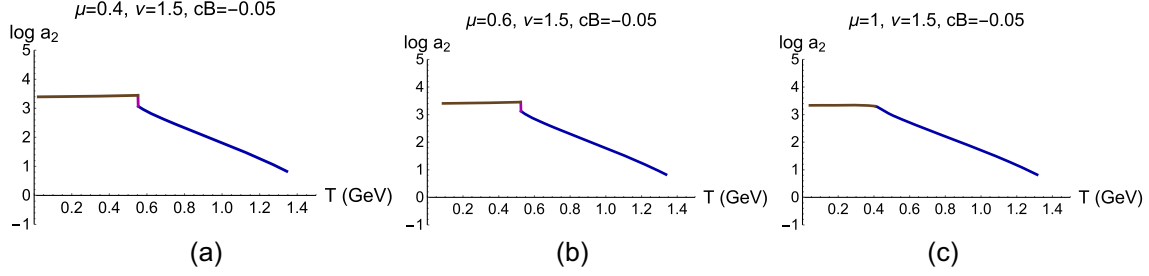


FIG. 31.  $\log a_2$  for the HQ model versus the temperature for  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 1.5$  at fixed chemical potentials: (a)  $\mu = 0.4 \text{ GeV}$ , (b)  $\mu = 0.6 \text{ GeV}$ , and (c)  $\mu = 1 \text{ GeV}$ . The blue and brown lines correspond to the QGP and hadronic phases, respectively. The magenta lines correspond to the unstable regions. We see jumps in panels (a) and (b) and a smooth change of the slopes near the second-order phase transition in panel (c).

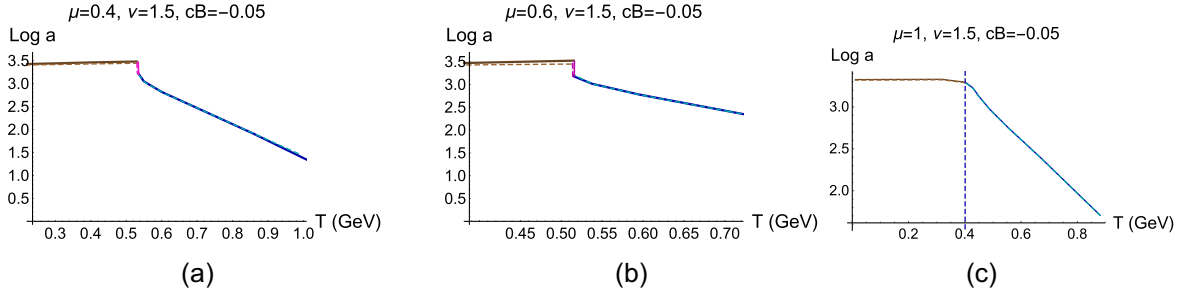


FIG. 32.  $\log a_2$  (solid lines) and  $\log a_3$  (dashed lines) are presented for the HQ model as a function of temperature for  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 1.5$  at fixed chemical potentials (a)  $\mu = 0.4 \text{ GeV}$ , (b)  $\mu = 0.6 \text{ GeV}$ , and (c)  $\mu = 1 \text{ GeV}$ . The blue and cyan correspond to the QGP and brown and gray lines correspond to the hadronic phases. The magenta lines (solid and dashed) correspond to unstable regions. We see jumps in panel (a) and a smooth change of the slopes near the second-order phase transition in panel (b).

- (i) The plots in Fig. 29 show that for  $\nu = 1$ , the JQ parameter exhibits anisotropy in the presence of a magnetic field:  $\log a_2 \neq \log a_3$  for  $c_B = -0.05 \text{ GeV}^2$ .
- (ii) Both  $\log a_2$  and  $\log a_3$  are nearly temperature-independent in the hadronic phase but decrease with temperature in the quark-gluon phase. Each exhibit jumps at first-order phase transitions with approximately equal magnitudes.
- (iii) For large chemical potentials, the slope of constant- $\mu$  curves changes near second-order phase transitions. These slopes are almost orientation-independent.
- (iv) Comparison of Figs. 27 and 28 shows that replacing  $\log(aT)$  with  $\log(aT^3)$  eliminates the hill-like characteristic.

## 2. Nonzero magnetic field: $\nu = 1.5$

The phase diagram in the  $(\mu, T)$ -plane, including the first-order and second-order phase transitions for the HQ model denoted by magenta and blue lines, in  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 1.5$  is depicted in Fig. 30.

- (i) The plots in Fig. 31 show that for  $\nu = 1.5$ ,  $\log a_2$  is nearly temperature-independent in the hadronic phase but decreases with the temperature in the quark-gluon phase. It exhibits jumps at the first-order phase transitions and the values of these jumps decrease with increasing the chemical potential until

it reaches a critical value. Indeed, the jump value in Fig. 31(a) is larger than the jump in Fig. 31(b), and the jump is absent in Fig. 31(c).

- (ii) Plots in Fig. 32 show that for  $\nu = 1.5$ , the JQ parameter exhibits anisotropy in the presence of a magnetic field:  $\log a_2 \neq \log a_3$  for  $c_B = -0.05 \text{ GeV}^2$ , but the anisotropy for  $c_B \neq 0$  is very small,

$$|\log a_2 - \log a_3|_{c_B=-0.05} \ll |\log a_2 - \log a_3|_{c_B=0}. \quad (4.1)$$

- (iii) Both  $\log a_2$  and  $\log a_3$  are nearly temperature-independent in the hadronic phase but decrease with the temperature in the quark-gluon phase. Each exhibit jumps at the first-order phase transitions with approximately equal magnitudes.
- (iv) For large chemical potentials the slope of constant- $\mu$  curves changes near the second-order phase transitions. These slopes are almost orientation-independent.

The plots in Fig. 33 show  $\log a$  for  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 1.5$  as a function of the chemical potential  $\mu$  at fixed temperatures below and above the first-order phase transition line:

- (i) *Below the first-order phase transition:* Figure 33(a) shows  $\log a$  versus  $\mu$  at fixed temperatures:  $T = 0.2$ ,

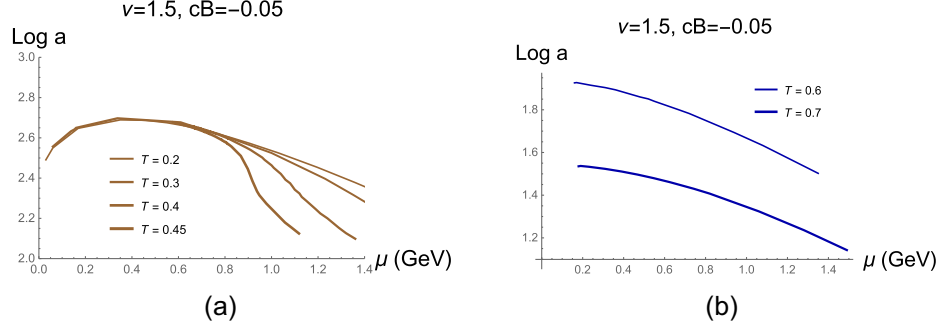


FIG. 33.  $\log a$  for the HQ model as a function of the chemical potential  $\mu$  for  $c_B = -0.05$   $\text{GeV}^2$  and  $\nu = 1.5$  at fixed temperatures: (a)  $T = 0.2, 0.3, 0.4$  (GeV) and  $T = 0.45$  GeV at below the first-order phase transition line, and (b) above the phase transition line at  $T = 0.6$  GeV and  $T = 0.7$  GeV. We see hills near  $\mu_{\text{hill}} \approx 0.5$  GeV.

0.3, 0.4 (GeV) and  $T = 0.45$  GeV. With increasing chemical potential,  $\log a$  first increases, reaches a maximum near  $\mu_{\text{hill}} \approx 0.5$  GeV, and then decreases. Below  $\mu_{\text{hill}}$ , the growth is nearly temperature-independent. Above  $\mu_{\text{hill}}$ , the decrease shows a clear

temperature dependence, occurring faster at lower temperatures.

(ii) *Above the first-order phase transition:* Figure 33(b) displays  $\log a$  versus  $\mu$  at fixed temperatures:  $T = 0.6$  GeV and  $T = 0.75$  GeV. Here,  $\log a$  decreases

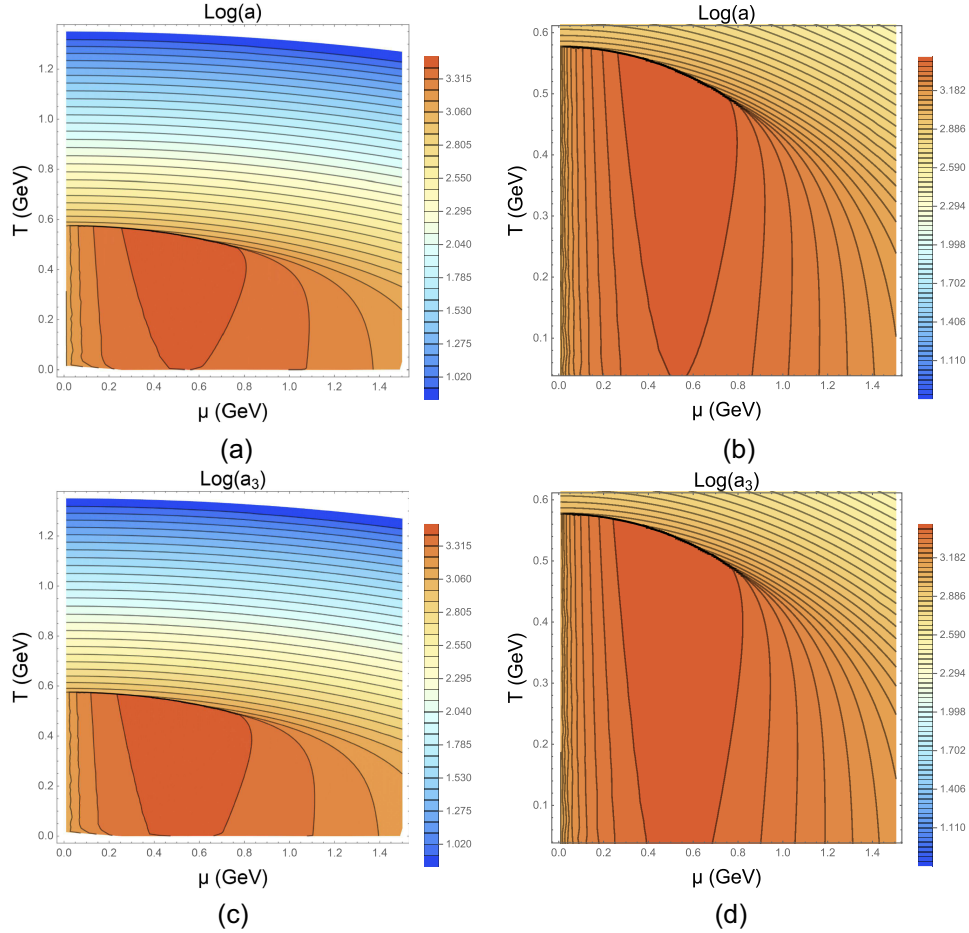


FIG. 34. Contour plots of (a)  $\log a_2$  and (c)  $\log a_3$  for the HQ model in the  $(\mu, T)$ -plane with nonzero magnetic field  $c_B = -0.05$   $\text{GeV}^2$  and  $\nu = 1.5$ . Panels (b) and (d) are enlarged views of panels (a) and (c) under the first-order phase transition line, respectively. The difference between the top and bottom panels is only visible in their enlarged versions (b) and (d).

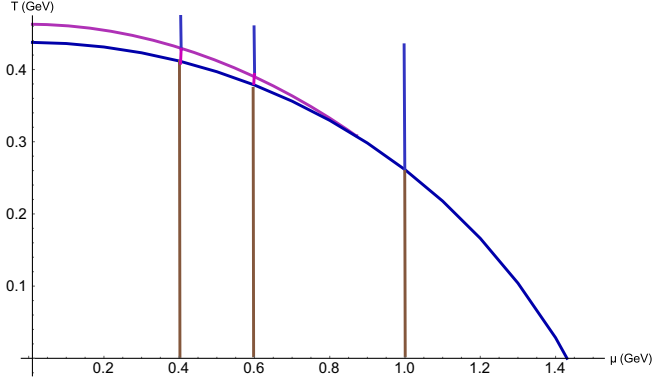


FIG. 35. First-order and second-order phase transitions for the HQ model denoted by magenta and blue lines in the  $(\mu, T)$ -plane for  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 4.5$ . The vertical lines with brown and blue colors indicate the hadronic and the QGP phases, respectively.

monotonically, with reduced magnitude at higher temperatures.

Figure 34 shows contour plots for the HQ model in the  $(\mu, T)$ -plane with  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 1.5$ : (a)  $\log a_2$ , (c)  $\log a_3$ , and (b) and (d) show enlarged views of regions (a) and (c), respectively, below the first-order phase transition line. Differences between  $\log a_2$  (top row) and  $\log a_3$  (bottom row) are only apparent in these enlarged regions [panels (b) and (d)].

### 3. Nonzero magnetic field: $\nu = 4.5$

The phase diagram in the  $(\mu, T)$ -plane, including the first-order and second-order phase transitions for the HQ model denoted by magenta and blue lines, at  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 4.5$  is depicted in Fig. 35.

- (i) The plots in Fig. 36 show that for  $\nu = 4.5$ , the JQ parameter exhibits anisotropy in the presence of a magnetic field:  $\log a_2 \neq \log a_3$  for  $c_B = -0.05 \text{ GeV}^2$ , but the anisotropy for  $c_B \neq 0$  is very small [see (4.1)].

- (ii) Both  $\log a_2$  and  $\log a_3$  are nearly temperature-independent in the hadronic phase but decrease with the temperature in the quark-gluon phase. Each exhibit jumps at the first-order phase transitions with approximately equal magnitudes.
- (iii) For large chemical potentials, the slope of constant- $\mu$  curves changes near second-order phase transitions. These slopes are almost orientation-independent.

The plots in Fig. 37 show  $\log a$  for  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 4.5$  as a function of the chemical potential  $\mu$  at fixed temperatures below and above the first-order phase transition line:

- (i) Below the first-order phase transition: Figure 37(a) shows  $\log a$  versus  $\mu$  at fixed temperatures:  $T = 0.2, 0.3$  and  $T = 0.4$  (GeV). For  $T = 0.2, 0.3$  (GeV), with increasing chemical potential,  $\log a$  first increases, reaches a maximum near  $\mu_{\text{hill}} \approx 4$  GeV, and then decreases. Below  $\mu_{\text{hill}}$ , the growth is nearly temperature-independent. Above  $\mu_{\text{hill}}$ , the decrease shows a clear temperature dependence, occurring faster at lower temperatures. For  $T = 0.4$  GeV, no clearly defined hill is observed; however, near  $\mu \approx 0.6$  GeV, a sharp drop occurs, followed by a gradual decrease.
- (ii) Above the first-order phase transition: Figure 37(b) displays  $\log a$  versus  $\mu$  at fixed temperatures:  $T = 0.5, 0.6$  and  $T = 0.7$  (GeV). Here,  $\log a$  decreases monotonically, with a reduced magnitude at higher temperatures.

The results presented in Fig. 38 summarize our discussions in Secs. IV B 2 and IV B 3, showing the density distribution of  $\log a_2$  in the  $(\mu, T)$ -plane.

- (i) These plots clearly reveal a first-order phase transition. Notably, the transition temperature for  $\nu = 4.5$  is lower than for  $\nu = 1.5$ . For comparison, Fig. 39 displays phase transition lines in the  $(\mu, T)$ -plane obtained from free energy analyses in our previous papers. Specifically, the plots in Fig. 39

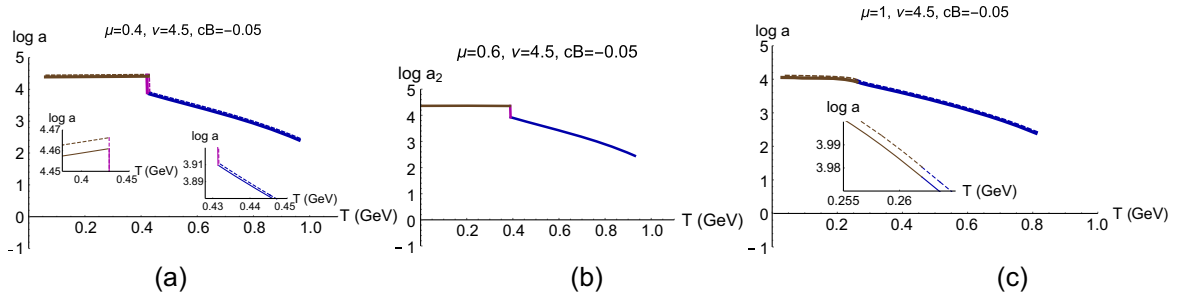


FIG. 36. Dependence of  $\log a_2$  (solid lines) and  $\log a_3$  (dashed lines) on  $T$  for the HQ model, with  $\nu = 4.5$  and  $c_B = -0.05 \text{ GeV}^2$ , at fixed chemical potentials (a)  $\mu = 0.4$  GeV, (b)  $\mu = 0.6$  GeV (only  $a_2$  and  $a_3$  is not included), and (c)  $\mu = 1$  GeV. The blue and cyan correspond to the QGP, and the brown and gray lines correspond to the hadronic phases. The magenta lines (solid and dashed) correspond to the unstable regions. We see jumps in panel (a) and a smooth change of the slopes near the second-order phase transition at panel (b).

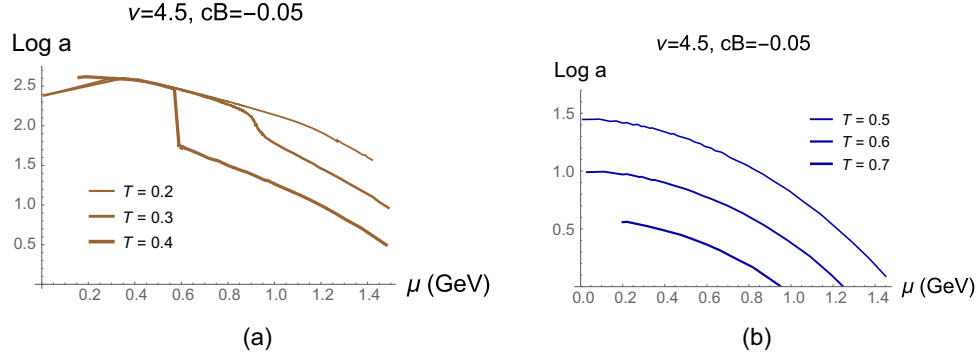


FIG. 37. (a) Dependence of  $\log a$  on  $\mu$  for the HQ model, with  $\nu = 4.5$  and  $c_B = -0.05$  GeV<sup>2</sup>, at fixed temperatures (a)  $T = 0.2, 0.3$  and  $T = 0.5$  (GeV) below the first-order phase transition line (b), and above the phase transition line for  $T = 0.5, 0.6$  and  $T = 0.7$  (GeV). We see hills near  $\mu \approx 0.4$  GeV.

correspond to those in Figs. 21(a) and 21(b) of [48] for  $c_B = 0, 0.05$  and  $0.5$  (GeV<sup>2</sup>).

- (ii) Figure 34 clearly demonstrates a prominent hill for  $\nu = 1.5$ . In contrast, Fig. 38 shows a less distinct hill but clearly exhibits a decrease in  $\log a$  with

increasing chemical potential for  $\mu \gtrsim 0.4$  GeV in Fig. 34(b) and  $\mu \gtrsim 0.42$  GeV in Fig. 38(b).

It is instructive to examine the dependence of discontinuity magnitudes on anisotropy. For this purpose, Fig. 40 plots  $\log a$  versus temperature  $T$  at fixed chemical potential

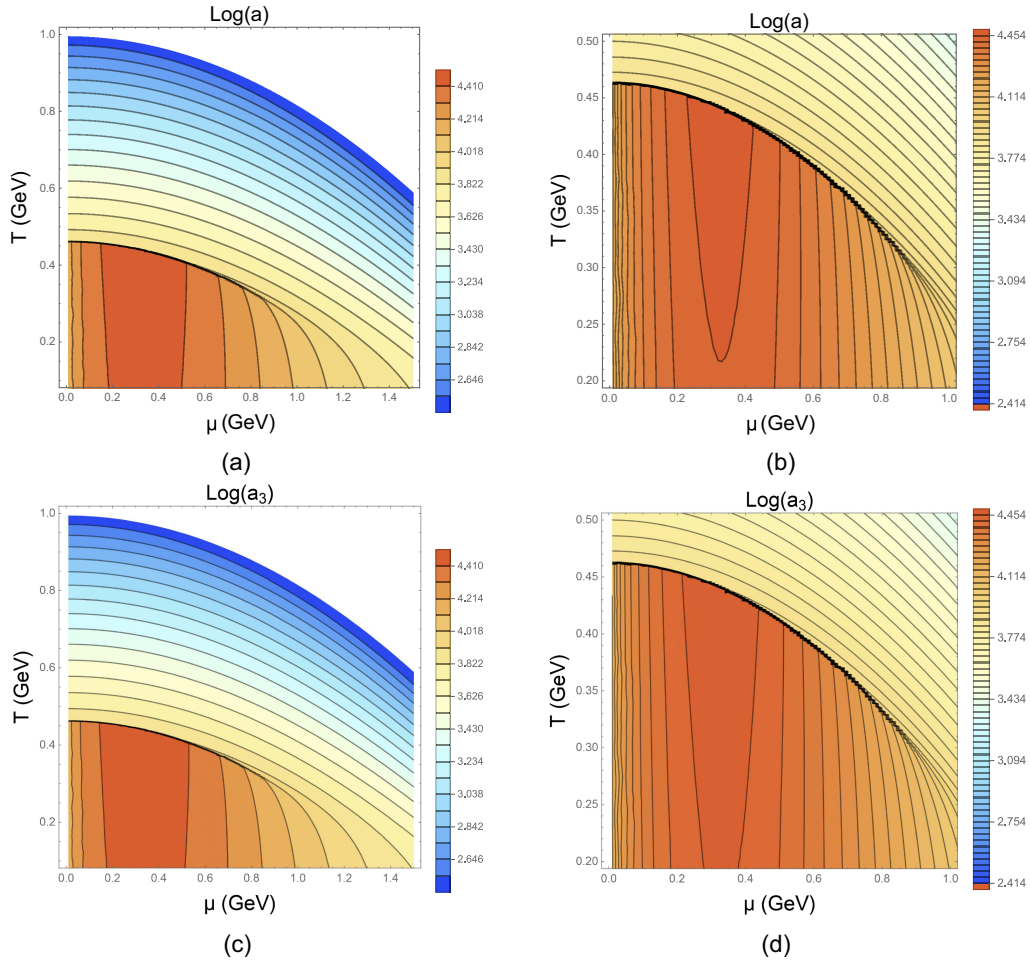


FIG. 38. Contour plots of (a)  $\log a_2$  and (c)  $\log a_3$  for the HQ model in the  $(\mu, T)$ -plane with nonzero magnetic field  $c_B = -0.05$  GeV<sup>2</sup> and  $\nu = 4.5$ . Panels (b) and (d) are enlarged views of panel panels (a) and (c) under the first-order phase transition line, respectively. The difference between the top and bottom panels is only visible in enlarged versions (b) and (d).

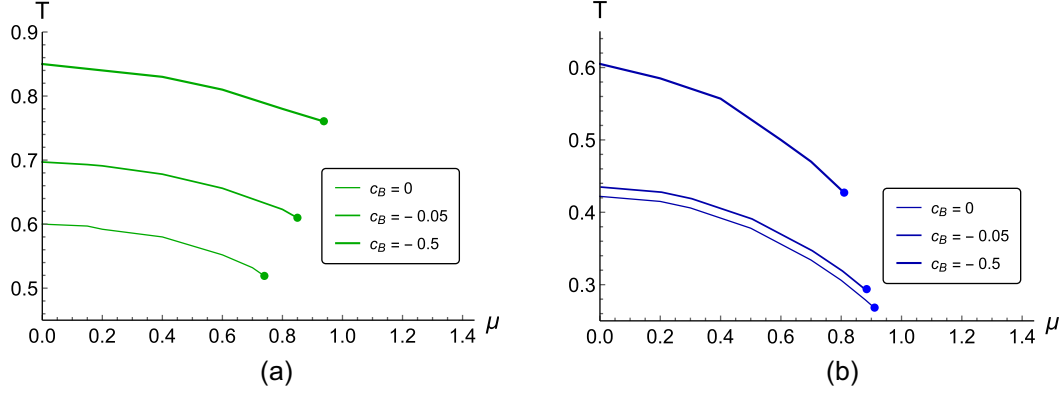


FIG. 39. First-order phase transition lines for the HQ model in the  $(\mu, T)$ -plane, with  $c_B = 0, -0.05$  and  $c_B = -0.5$  ( $\text{GeV}^2$ ) at fixed values of anisotropies (a)  $\nu = 1$  and (b)  $\nu = 4.5$ .

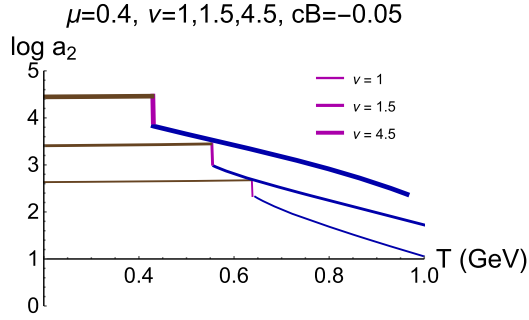


FIG. 40. Dependence of  $\log a_2$  on  $T$  for the HQ model, with  $\mu = 0.4$  and  $c_B = -0.05 \text{ GeV}^2$ , at fixed values of anisotropies  $\nu = 1, 1.5$ , and  $\nu = 4.5$ . The magnitude of the jumps in magenta color increases with the anisotropy parameter  $\nu$ .

$\mu = 0.4 \text{ GeV}$  for anisotropy parameters  $\nu = 1, 1.5$  and  $\nu = 4.5$ . Line thickness increases with  $\nu$  to visually distinguish the curves. Magenta segments indicate unstable regions and should be ignored. The observed

TABLE III. Schematic layout of two-dimensional plots of dependence of  $\log a$  of the JQ parameter for heavy quarks on  $T$  for fixed values of  $\mu$  at different  $c_B$  and  $\nu$ .

| $ c_B $ | $\nu$   |         |     |         |
|---------|---------|---------|-----|---------|
|         | 1       | 1.5     | 3   | 4.5     |
| 0       | Fig. 53 | ...     | ... | ...     |
| 0.05    | Fig. 29 | Fig. 31 | ... | Fig. 36 |

TABLE IV. Schematic layout of density plots for  $\log a$  showing the JQ parameter for heavy quarks at different  $c_B$  and  $\nu$ .

| $ c_B $ | $\nu$   |            |     |            |
|---------|---------|------------|-----|------------|
|         | 1       | 1.5        | 3   | 4.5        |
| 0       | Fig. 22 | Fig. 23(a) | ... | Fig. 23(b) |
| 0.05    | Fig. 26 | Fig. 34    | ... | Fig. 38    |
| 0.5     | Fig. 27 | ...        | ... | ...        |

discontinuities increase slightly with increasing anisotropy parameter  $\nu$ .

### C. Summary of jet quenching results for the HQ model

In this section, we have calculated the JQ parameter for quarks moving along lightlike trajectories in a QCD-like medium with heavy quarks at finite temperature  $T$ , chemical potential  $\mu$ , magnetic field, and the anisotropy parameter  $\nu$ . Our results are presented through multiple plots, where:

- (i) Table III summarizes the schematic layouts of two-dimensional plots showing the JQ parameter ( $\log a$ , in fact) of heavy quarks versus temperature  $T$  at fixed  $\mu$  values.
  - (ii) Table IV presents schematic layouts of density plots for the JQ parameter ( $\log a$ , in fact).
- These results reveal nontrivial dependencies of the JQ parameter on both the temperature and the chemical potential. Specifically, we observe

- (i) A discontinuous jump at the first-order phase transition line.
- (ii) A change in the slope of  $\log a$  versus temperature  $T$  (at fixed  $\mu$ ) at the second-order phase transition line.

The JQ parameter increases with  $T$  and  $\mu$  in the QGP phase. Conversely, the hadron phase displays complex behavior, featuring a hill-like structure in the  $(\mu, T)$ -plane below the first-order phase transition line at the nonzero magnetic field. We further analyzed the effects of magnetic field strength and anisotropy parameter  $\nu$  on these behaviors.

## V. CONCLUSION

In this paper, we investigate the behavior of the JQ parameter in an anisotropic holographic model under a strong magnetic field for light and heavy quark cases [47,48]. Both models are constructed using the Einstein-dilaton-three-Maxwell action and incorporate a five-dimensional metric with a warp factor, extending earlier

isotropic and partially anisotropic frameworks, specifically the HQ of [48,49,53–55,57,63–65] and the LQ model in [47,51,63,66].

The LQ and HQ models have both similarities and significant differences. First, both have first-order and second-order phase transitions. But locations of phase transitions in  $(\mu, T)$ -planes are quite different [see plots in Fig. 3(c) for light quarks and 22(c) for heavy quarks]. The locations of the phase transition depend on the anisotropy and the value of the magnetic field.

- (i) For light quarks, increasing the anisotropy parameter  $\nu$  reduces the temperature of the CEP and shifts it toward lower values of temperature and shifts the chemical potential down to zero [see Figs. 7(d), 8(a), 15(a), 18(a), and 19(a)].
- (ii) For heavy quarks, increasing the anisotropy parameter  $\nu$  reduces the temperature of the CEP and shifts it toward lower values of temperature and shifts the chemical potential down to zero, as in Fig. 39.
- (iii) The differences between the LQ and HQ models are exacerbated when magnetic fields are included. They respond differently to the magnetic field: the HQ model exhibits magnetic catalysis, whereas the LQ model demonstrates inverse magnetic catalysis [see Figs. 8(a), 18(a), and 19 for light quarks and Fig. 39 for heavy quarks].

Studies of the JQ parameter allow us to accurately, at least theoretically, localize the first-order phase transition. For this purpose, we studied the density plots of  $\log a$  and its dependence on the parameter  $\nu$  and the magnetic field parameter  $c_B$ .

- (1) Table II shows the map for the density plots of  $\log a$  of the JQ parameter for the LQ model. We see that
  - (a) the location of the first-order phase transition, which we can read from the plot for the isotropic case in Fig. 3(a), is in accordance with Fig. 3(c).
  - (b) When we increase  $\nu$ , we get a map for the first-order phase transition that is in accordance with expected features mentioned in (i) above.
- (2) The Table IV shows the map for the density plots of  $\log a$  of the JQ parameter for the HQ model. We see that
  - (a) the location of the first-order phase transition, we can read from the plot for isotropic case in Fig. 22(a), is in accordance with Fig. 22(c).
  - (b) When we increase  $\nu$  we get a map for the first-order phase transition that is in accordance with expected features mentioned in (ii) above.
- (3) The distinct positions of the first-order phase transition lines in the two models mentioned above in (iii) lead to contrasting behaviors in the discontinuity of  $\log a$  across the first-order phase transition.

- (a) For the LQ model, the jump in  $\log a$  increases with increasing  $\mu$ . We see this clearly for the totally isotropic case [Fig. 3(a)], zero magnetic field for different  $\nu$  [Fig. 7(a)–7(c)], and for the nonzero magnetic field in the cases where the first-order phase transition takes place, for example, in the cases presented in Figs. 10 and 11.
- (b) In the HQ model, the jump in  $\log a$  decreases with increasing  $\mu$  for the isotropic case [Fig. 22(a)] and for various  $\nu$  (Fig. 23). However, under nonzero magnetic fields, the behavior becomes more complex. As shown in Fig. 26 ( $c_B = -0.05 \text{ GeV}^2$ ,  $\nu = 1$ ), the jump initially increases until  $\mu = 0.45 \text{ GeV}$  and then decreases to zero at the CEP. Similar behavior occurs in Fig. 27(b) ( $c_B = -0.5 \text{ GeV}^2$ ,  $\nu = 1$ ), where the jump increases until  $\mu = 0.75 \text{ GeV}$  before decreasing to zero at the CEP. This pattern also holds for  $c_B = -0.05 \text{ GeV}^2$ ,  $\nu = 1.5$  (Fig. 34) and  $c_B = -0.05 \text{ GeV}^2$ ,  $\nu = 4.5$  (Fig. 38).
- (4) Moreover, in large  $T$ , the isotropic LQ model exhibits the behavior  $aT^3 \sim \text{const}$ , as shown in Fig. 2. This behavior is also exhibited by the LQ model in [40]. It is consistent with the predictions of the conformal invariant model [8] and shows good agreement with the experimental data from RHIC and LHC. The magnetic field breaks the conformal symmetry, as shown in Fig. 46.
- (5) Our isotropic HQ model at  $\mu = 0$  does not exhibit the behavior predicted by conformal invariance, where  $aT^3 \sim \text{const}$  at large  $T$  (see Fig. 54). This finding contrasts with the behavior of the JQ parameter for the HQ model in [39], which approaches conformality as  $T \rightarrow \infty$ , while ours does not. The deviation is expected because our HQ model lies far from conformality. Interestingly, the asymptotic behavior of  $\log(aT^3)$  in large  $T$  remains independent of the chemical potential  $\mu$ , as shown in Fig. 54(d).
- (6) However, there are also some similarities: for both light and heavy quarks,  $\log a$  decreases linearly at large temperatures, as shown in Figs. 9 and 15.
- (7) Considering a magnetic field oriented along the  $x_3$ -axis, we have studied jets propagating along the  $x_1$ -direction with momentum broadening both parallel to the magnetic field ( $\hat{q}_3$ ) and perpendicular to it along the  $x_2$ -direction ( $\hat{q}_2$ ). In Sec. III, our analysis focused specifically on the transverse broadening component  $\hat{q}_2$  for the LQ model, while in Sec. IV, we examined the parallel broadening component  $\hat{q}_3$  for the HQ model.
- (8) Nevertheless, in Secs. IIIB 1 and IVB 1 we compared the parameters  $\hat{q}$ , i.e.,  $\hat{q}_2$  and  $\hat{q}_3$  for the LQ and HQ models in a magnetic field, respectively.

- (a) In Sec. III B 1, we compared the  $\hat{q}$  parameters  $\hat{q}_2$  and  $\hat{q}_3$  for the LQ model in a magnetic field,  $c_B = -0.05 \text{ GeV}^2$ . We observe a subtle but weak orientation dependence:
- (i) Figures 12 and 13 demonstrate that, the JQ parameter exhibits anisotropy in the presence of a magnetic field:  $\log a_2 \neq \log a_3$  for  $\nu = 1$ .
  - (ii) Below the first-order phase transition temperature and above the transition,  $\log a_3 > \log a_2$  for  $\nu = 1$ .
- (b) In Sec. IV B 1, we compared the  $\hat{q}$  parameters  $\hat{q}_2$  and  $\hat{q}_3$  for the HQ model in a magnetic field,  $c_B = -0.05 \text{ GeV}^2$ :
- (i) For  $\nu = 1$ , at  $\mu = 0.4 \text{ GeV}$  and  $\mu = 1 \text{ GeV}$ , we observed  $\log a_3 > \log a_2$  (see Fig. 29).
  - (ii) For  $\nu = 1.5$ , at  $\mu = 1 \text{ GeV}$ , we observe  $\log a_2 > \log a_3$ . In the hadronic phase at  $\mu = 0.4$  and  $\mu = 0.6 \text{ GeV}$ ,  $\log a_2 > \log a_3$  similarly holds. However, in the QGP region, the relationship between  $\log a_2$  and  $\log a_3$  remains inconclusive (see Fig. 32).
  - (iii) For  $\nu = 4.5$ , we observe  $\log a_3 > \log a_2$  for  $\mu = 0.4 \text{ GeV}$  and  $\mu = 1 \text{ GeV}$ . However, at  $\mu = 0.6$ ,  $\log a_2$  and  $\log a_3$  nearly coincide (see Fig. 36).

Our study has analyzed the behavior of the JQ parameter under the idealized assumption that jet formation occurs within a medium consisting solely of light quarks (the LQ model) or solely of heavy quarks (the HQ model). In a physically realistic setting, however, the medium comprises both light and heavy quark species, with light quarks predominant at low energies and heavy quarks contributing significantly at high energies.

A natural extension of this work is therefore the investigation of jets within a hybrid model that incorporates both quark types, effectively interpolating between the LQ and HQ model limits. To construct the hybrid model it is needed to chose the proper warp-factor and

gauge kinetic function. The hybrid model should reproduce linear Regge trajectories that were observed experimentally and first-order phase transition lines for heavy and light quarks. This constitutes an important objective for future research.

## ACKNOWLEDGMENTS

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## DATA AVAILABILITY

Data are not publicly available. The data are available from the authors upon reasonable request.

## APPENDIX A: EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Together with the variations over the scalar field and first vector field (Maxwell field that serves a nonzero chemical potential and for which we have chosen the electric ansatz), second vector field (Maxwell field that serves a spatial anisotropy and for which we have chosen the magnetic ansatz), and third vector field (Maxwell field that serves a nonzero magnetic field and for which we have chosen the magnetic ansatz) we get the following EOMs:

$$\phi'' + \phi' \left( \frac{g'}{g} + \frac{3\mathfrak{b}'}{2\mathfrak{b}} - \frac{\nu + 2}{\nu z} + c_B z \right) + \left( \frac{z}{L} \right)^2 \frac{(A'_t)^2}{2\mathfrak{b}g} \frac{\partial f_0}{\partial \phi} - \left( \frac{L}{z} \right)^{2-\frac{4}{\nu}} \frac{e^{-c_B z^2} q_1^2}{2\mathfrak{b}g} \frac{\partial f_1}{\partial \phi} - \left( \frac{z}{L} \right)^{\frac{2}{\nu}} \frac{q_3^2}{2\mathfrak{b}g} \frac{\partial f_3}{\partial \phi} - \left( \frac{L}{z} \right)^2 \frac{\mathfrak{b}}{g} \frac{\partial V}{\partial \phi} = 0, \quad (\text{A1})$$

$$A''_t + A'_t \left( \frac{\mathfrak{b}'}{2\mathfrak{b}} + \frac{f'_0}{f_0} + \frac{\nu - 2}{\nu z} + c_B z \right) = 0, \quad (\text{A2})$$

$$(\text{I}) \quad g'' + g' \left( \frac{3\mathfrak{b}'}{2\mathfrak{b}} - \frac{\nu + 2}{\nu z} + c_B z \right) - \left( \frac{z}{L} \right)^2 \frac{f_0 (A'_t)^2}{\mathfrak{b}} - \left( \frac{z}{L} \right)^{\frac{2}{\nu}} \frac{q_3^2 f_3}{\mathfrak{b}} = 0, \quad (\text{A3})$$

$$(\text{II}) \quad \mathfrak{b}'' - \frac{3(\mathfrak{b}')^2}{2\mathfrak{b}} + \frac{2\mathfrak{b}'}{z} - \frac{4\mathfrak{b}}{3\nu z^2} \left( 1 - \frac{1}{\nu} + \left( 1 - \frac{3\nu}{2} \right) c_B z^2 - \frac{\nu c_B^2 z^4}{2} \right) + \frac{\mathfrak{b}(\phi')^2}{3} = 0, \quad (\text{A4})$$

$$(\text{III}) \quad 2g' \left( 1 - \frac{1}{\nu} \right) + 3g \left( 1 - \frac{1}{\nu} \right) \left( \frac{\mathfrak{b}'}{\mathfrak{b}} - \frac{4(\nu + 1)}{3\nu z} + \frac{2c_B z}{3} \right) + \left( \frac{L}{z} \right)^{1-\frac{4}{\nu}} \frac{L e^{-c_B z^2} q_1^2 f_1}{\mathfrak{b}} = 0, \quad (\text{A5})$$

$$\begin{aligned}
 \text{(IV)} \quad & 2g' \left( 1 - \frac{1}{\nu} + c_B z^2 \right) + 3g \left[ \left( 1 - \frac{1}{\nu} + c_B z^2 \right) \left( \frac{\mathfrak{b}'}{\mathfrak{b}} - \frac{4}{3\nu z} + \frac{2c_B z}{3} \right) - \frac{4(\nu - 1)}{3\nu z} \right] \\
 & + \left( \frac{L}{z} \right)^{1-\frac{4}{\nu}} \frac{L e^{-c_B z^2} q_1^2 f_1}{\mathfrak{b}} - \left( \frac{z}{L} \right)^{1+\frac{2}{\nu}} \frac{L q_3^2 f_3}{\mathfrak{b}} = 0,
 \end{aligned} \tag{A6}$$

$$\begin{aligned}
 \text{(V)} \quad & \frac{\mathfrak{b}''}{\mathfrak{b}} + \frac{(\mathfrak{b}')^2}{2\mathfrak{b}^2} + \frac{3\mathfrak{b}'}{\mathfrak{b}} \left( \frac{g'}{2g} - \frac{\nu + 1}{\nu z} + \frac{2c_B z}{3} \right) - \frac{g'}{3zg} \left( 5 + \frac{4}{\nu} - 3c_B z^2 \right) \\
 & + \frac{8}{3z^2} \left( 1 + \frac{3}{2\nu} + \frac{1}{2\nu^2} \right) - \frac{4c_B}{3} \left( 1 + \frac{3}{2\nu} - \frac{c_B z^2}{2} \right) + \frac{g''}{3g} + \frac{2}{3} \left( \frac{L}{z} \right)^2 \frac{\mathfrak{b}V}{g} = 0.
 \end{aligned} \tag{A7}$$

To solve the EOMs, we use the usual boundary conditions for the time component of the first gauge field  $A_t$  and blackening function  $g$ ,

$$A_t(0) = \mu, \quad A_t(z_h) = 0, \quad g(0) = 1, \quad g(z_h) = 0. \tag{A8}$$

The physical boundary conditions for the dilaton field are [63]

(i) for the LQ model

$$z_0 = 10e^{-z_h/4} + 0.1, \tag{A9}$$

(ii) for the HQ model

$$z_0 = e^{-z_h/4} + 0.1, \tag{A10}$$

where  $\phi(z_0) = 0$ .

## APPENDIX B: SUPPLEMENTARY FIGURES FOR LIGHT QUARKS

The phase structure of the LQ model in Fig. 41 shows different domains of phases, i.e., QGP, quarkyonic, and hadronic corresponding to the blue, green, and brown regions, respectively, [63]. The solid blue lines in Fig. 41 correspond to the confinement/deconfinement phase transition line obtained via the Wilson loop calculations [51]. The solid magenta lines in Fig. 41 correspond to the first-order phase transition line obtained via free energy calculations [51], and the thicker dark red line corresponds to the second horizon, where  $T = 0$ , while other solid darker red curves represent constant-temperature lines  $T = T(z_h, \mu)$ . Jumps for various physical quantities appear along two thin magenta curves that bounded the unstable region. The first-order phase transition manifests as a discontinuity in  $z_h$ , which takes place for  $\mu$  varying from the value corresponding to the magenta star to the zero-temperature endpoint. The unstable region between the magenta curves is highlighted in white in Fig. 41. Two magenta lines are mapping to a single transition line in the  $(\mu, T)$ -plane, denoted by the magenta curve in Fig. 1.

Figure 42(a) depicts the  $\log a$  as a function of the size of the horizon  $z_h$  and Fig. 42(b) as a function of the temperature  $T$  at  $\mu = 0.04$  GeV. The blue and brown lines correspond to the QGP and hadronic phases, respectively.  $\log a$  decreases up to  $T = 0.158$  GeV even after a continuous phase transition at  $T = 0.154$  GeV from the hadronic to the QGP phases, indicating an enhancement in the JQ parameter up to  $T = 0.158$  GeV and a decrease in the JQ parameter for  $0.158 \text{ GeV} < T < 0.3$  GeV. For higher values of temperature, i.e.,  $T \gtrsim 0.3$  GeV, the JQ parameter increases [see Fig. 1(b)].

Figure 43(a) shows the  $\log a$  as a function of the size of the horizon  $z_h$  and Fig. 43(b) as a function of the temperature  $T$  at  $\mu = 0.3$  GeV for  $c_B = 0$  and  $\nu = 1$ . The blue, green, and brown lines correspond to the QGP, quarkyonic, and hadronic phases, respectively. The magenta lines indicate the jump between the quarkyonic and hadronic

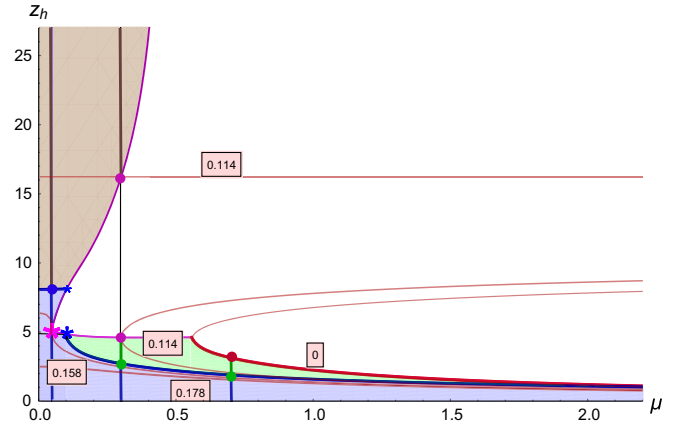


FIG. 41. Two dimensional plot in the  $(\mu, z_h)$ -plane for the LQ model in the isotropic case,  $c_B = 0$  and  $\nu = 1$  with different phases, i.e., QGP, quarkyonic, and hadronic corresponding to blue, green and brown regions, respectively. Solid orange lines show the temperature indicated in the squares. The intersection of the confinement/deconfinement and first-order phase transition lines is denoted by the blue stars. The magenta star indicates the CEP. Vertical lines indicate the lines along which we calculate the JQ parameter presented in the next figures, i.e., Figs. 42–44.

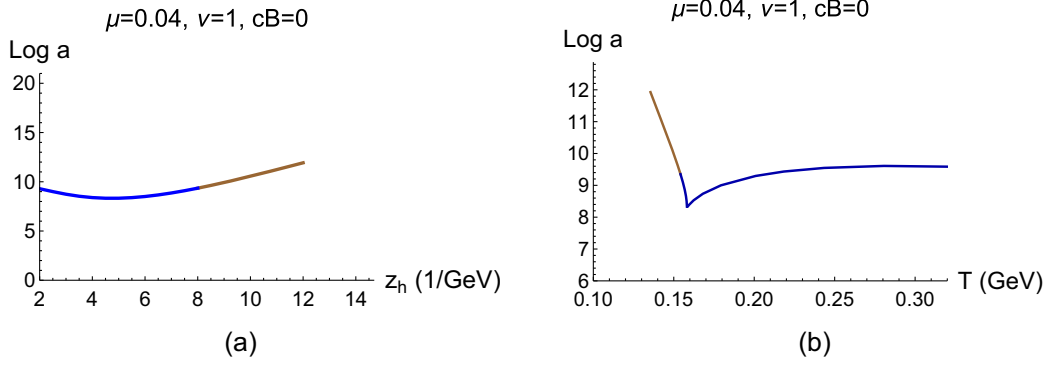


FIG. 42.  $\log a$  as a function of the size of the horizon  $z_h$  (a) and the temperature  $T$  (b) for the LQ model at  $\mu = 0.04$  GeV in the crossover region for  $c_B = 0$  and  $\nu = 1$ . The blue and brown lines correspond to the QGP and hadronic phases, respectively.

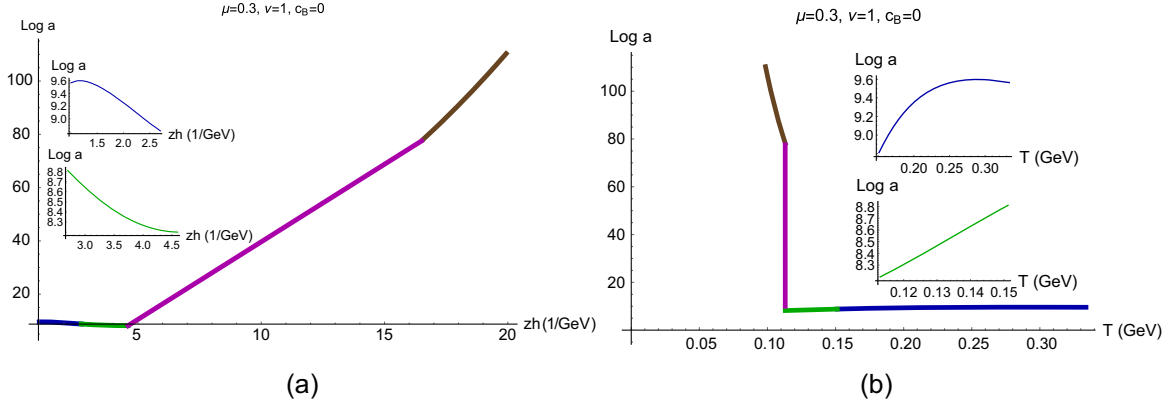


FIG. 43. (a)  $\log a$  as a function of the size of the horizon  $z_h$ , and (b) the temperature  $T$  for the LQ model, with  $c_B = 0$  and  $\nu = 1$ , at  $\mu = 0.3$  GeV. The blue, green, and brown lines correspond to the QGP, quarkyonic, and hadronic phases, respectively. The magenta lines indicate the jump between the quarkyonic and hadronic phases crossing the first-order phase transition line.

phases crossing the first-order phase transition line. In the hadronic phase, the increasing temperature increases the JQ parameter, while after the first-order phase transition at  $T = 0.11$  GeV the JQ parameter decreases at the quarkyonic and QGP phases by increasing the temperature to  $T = 0.28$  GeV. For higher values of  $T$  in the QGP phase, that is,  $T \gtrsim 0.28$  GeV, the JQ parameter increases.

Figures 42 and 43 show that for the zero magnetic field and spatially isotropic system, in the hadronic phase the JQ parameter increases by increasing  $T$ .

In Fig. 44(a),  $\log a$  as a function of the size of the horizon  $z_h$ , and in Fig. 44(b) as a function of the temperature  $T$  at  $\mu = 0.7$  GeV for  $c_B = 0$  and  $\nu = 1$  are depicted. The blue and green lines correspond to the QGP and quarkyonic

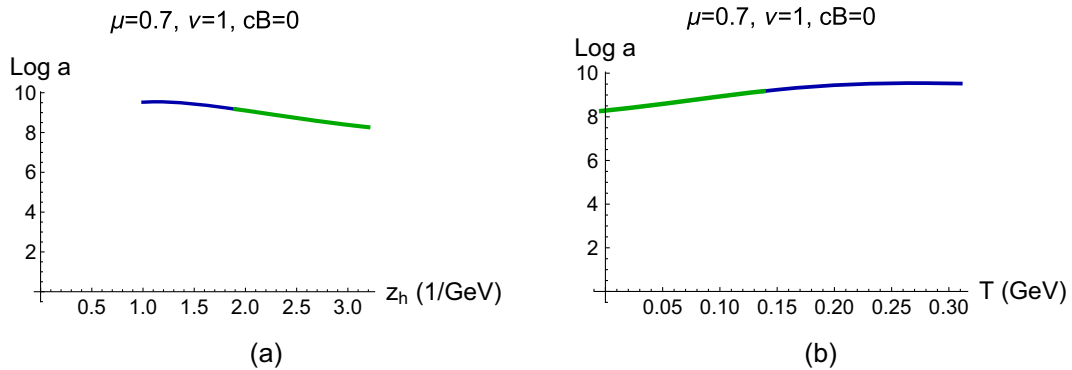


FIG. 44. (a)  $\log a$  as a function of the size of the horizon  $z_h$  and (b) the temperature  $T$  for the LQ model, with  $c_B = 0$  and  $\nu = 1$ , at  $\mu = 0.7$  GeV. The blue and green lines correspond to the QGP and quarkyonic phases, respectively.

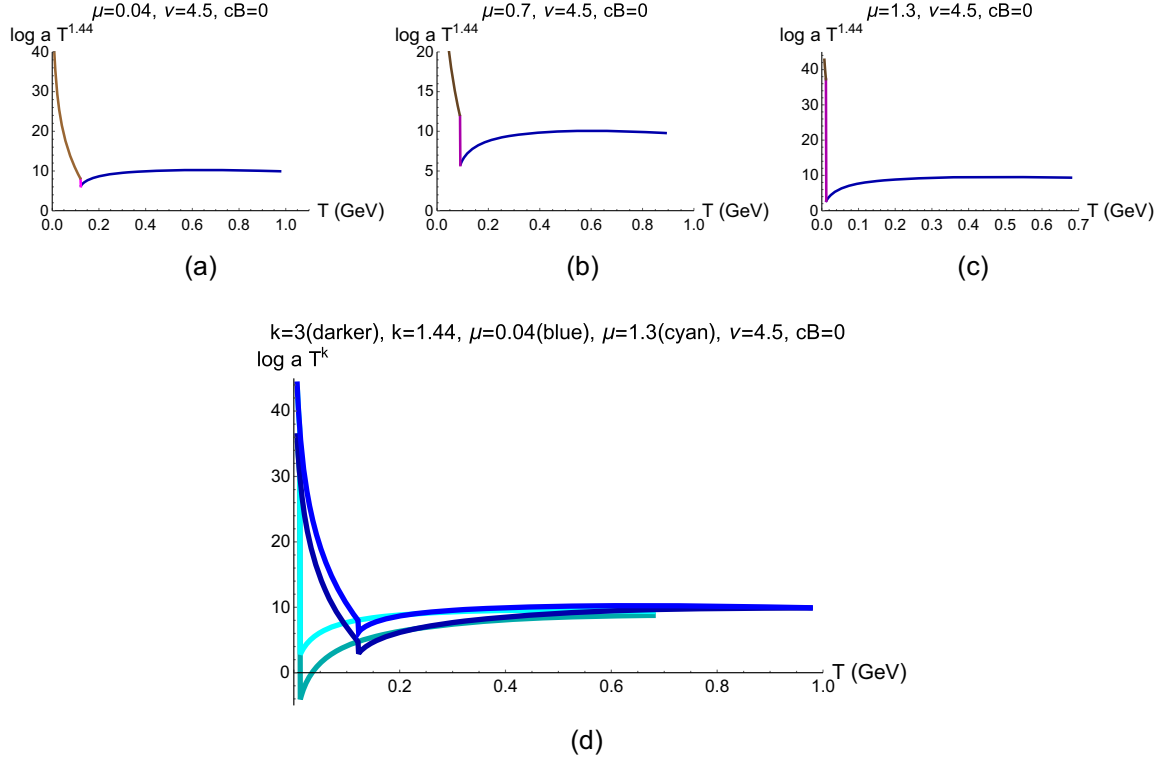


FIG. 45.  $\log a T^{1.44}$  as function of  $T$  for the LQ model, with  $\nu = 4.5$  and  $c_B = 0$ , at fixed chemical potentials: (a)  $\mu = 0.04$ , (b)  $\mu = 0.7$ , and (c)  $\mu = 1.3$  (GeV). Segments of lines are colored blue, brown, and green corresponding to the QGP, hadronic, and quarkyonic phases they traverse, respectively. (d) Comparison of  $\log a T^{1.44}$  and  $\log a T^3$  for the same  $\nu = 4.5$  and  $c_B = 0$  and  $\mu = 0.04$  GeV and  $\mu = 1.3$  GeV.

phases, respectively.  $\log a$  increases, with increasing temperature  $T$ , which indicates that the JQ parameter decreases even after the phase transition between quarkyonic and QGP phases. Figures 43(b) and 44(b) show that for the zero magnetic field and spatially isotropic system, in the quarkyonic phase, the JQ parameter decreases monotonically by increasing  $T$ .

In Fig. 45  $\log a T^{1.44}$  as function of  $T$  for the LQ model, with  $\nu = 4.5$  and  $c_B = 0$ , at fixed chemical potentials

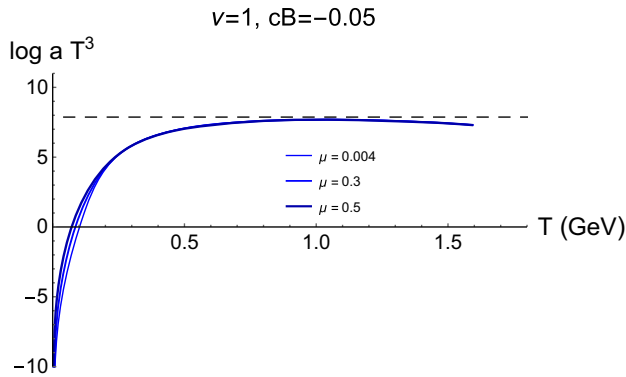


FIG. 46.  $\log a_2 T^3$  as a function of  $T$  for the LQ model, with  $\nu = 1$  and  $c_B = -0.05$  GeV<sup>2</sup>, at fixed chemical potentials:  $\mu = 0.04$ ,  $\mu = 0.3$ , and  $\mu = 0.5$  (GeV).

$\mu = 0.04$  GeV,  $\mu = 0.7$  GeV, and  $\mu = 1.3$  GeV are depicted in panels (a)–(c), respectively. Figure 45(c) shows a comparison between  $\log a T^{1.44}$  and  $\log a T^3$  for the same  $\nu = 4.5$  and  $c_B = 0$  and  $\mu = 0.04$  GeV and  $\mu = 1.3$  GeV. There is the same behavior at high-temperature regime.

Figure 46 shows that for the LQ model, with  $\nu = 1$  and  $c_B = -0.05$  GeV<sup>2</sup>, the JQ parameter does not admit the conformal asymptotics for large  $T$ .

### APPENDIX C: SUPPLEMENTARY FIGURES FOR HEAVY QUARKS

The phase diagram of the HQ model in the  $(\mu, z_h)$ -plane describes three different states: the blue QGP region, the brown hadronic region, and the quarkyonic green region; there is also the forbidden white region (the region of instability), Fig. 47. Let us calculate the JQ parameter for  $\nu = 1$  and  $c_B = 0$  for different fixed  $\mu$ , namely for  $\mu = 0.6, 0.8, 1$  (GeV) and for varying values of  $z_h$ . We indicate these values of the chemical potential parameter in Fig. 47 by thick lines. These lines depend on the value of  $\mu$  that crosses two, three, or four different colored regions.

The phase diagram in the  $(\mu, z_h)$ -plane in  $c_B = -0.05$  GeV<sup>2</sup> and  $\nu = 1.5$  that describes the HQ model is shown in Fig. 48, and its enlarged view can be found in Fig. 49. Different phases of Fig. 48 for three values of

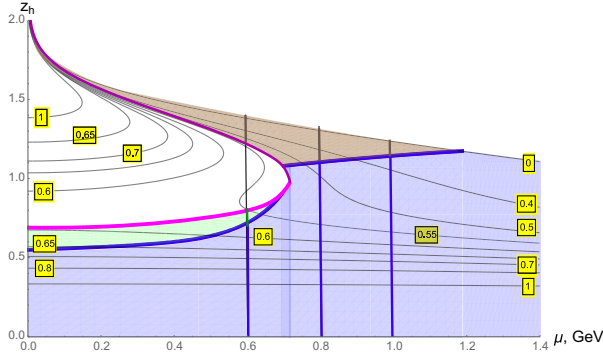


FIG. 47. Phase diagram in the  $(\mu, z_h)$ -plane at  $c_B = 0$  and  $\nu = 1$  describing the HQ model including phase transition lines (blue for the confinement/deconfinement transition and magenta for the first-order transition line) and the thick vertical lines along which we calculate the JQ parameter (see Fig. 53). The colors of the regions show blue (QGP), green (quarkyonic), and brown (hadronic) phases.

TABLE V. Points on  $(\mu, z_h, T)$  corresponding to change of phases. QGP, quark gluon phase; QP, quarkyonic phase; FA, forbidden area; HP, hadron phase.

| $\mu$ | QGP/QP         | QP/FA          | FA/HP          |
|-------|----------------|----------------|----------------|
| 0.4   | (0.589, 0.551) | (0.593, 0.552) | (1.245, 0.561) |
| 0.6   | (0.645, 0.531) | (0.646, 0.532) | (1.123, 0.525) |
| 1     |                | (0.881, 0.407) |                |

chemical potentials  $\mu = 0.4$  GeV,  $\mu = 0.6$  GeV, and  $\mu = 1$  GeV are presented in Table V.

The phase diagram in the  $(\mu, z_h)$ -plane in  $c_B = -0.05$  GeV<sup>2</sup> and  $\nu = 4.5$  that describes the HQ model is shown in Fig. 50, and its enlarged can be found in Fig. 51.

For our model, Fig. 52 shows  $\log aT^3$  varying slowly but non-constantly in the QGP phase contrasting with the  $N = 4$  SYM behavior reported in [8], where it remains constant. In the hadronic phase,  $\log aT^3$  increases universally with

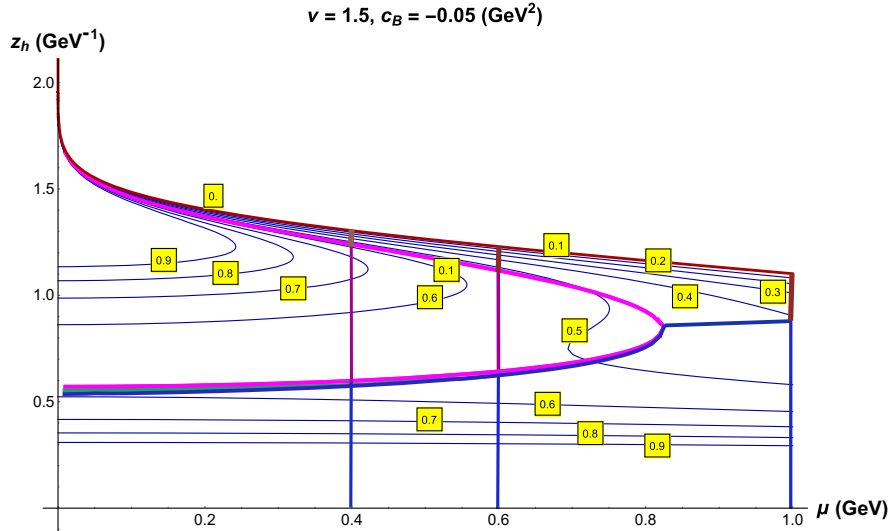


FIG. 48. Phase diagram for the HQ model in the  $(\mu, z_h)$ -plane for  $c_B = -0.05$  GeV<sup>2</sup> and  $\nu = 1.5$ . The magenta and blue lines indicate first-order and second-order phase transitions, respectively. Additional curves, i.e., blue, purple, and brown, mark paths for the JQ parameter calculations (see Fig. 31). The green parts on lines with  $\mu = 0.4$  GeV and  $\mu = 0.6$  GeV are very small.

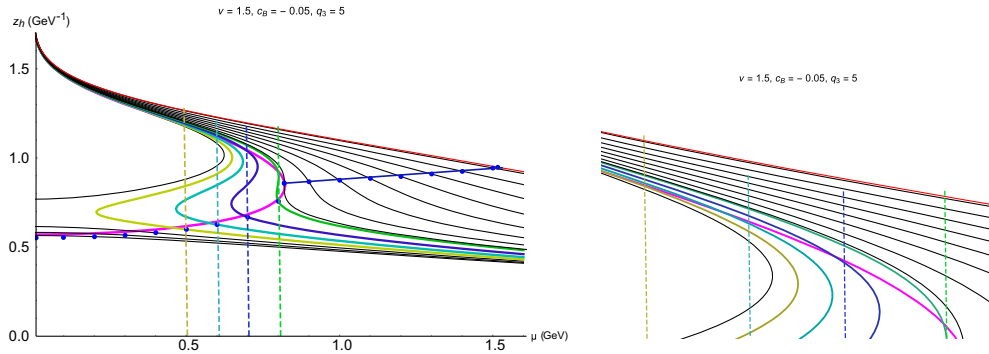


FIG. 49. Phase diagram for the HQ model in the  $(\mu, z_h)$ -plane for  $c_B = -0.05$  GeV<sup>2</sup> and  $\nu = 1.5$ . Magenta and blue lines indicate the first-order and the second-order phase transitions, respectively. Additional curves (colored in blue, purple, brown) indicate the paths along which the JQ parameters are calculated (see Fig. 31).

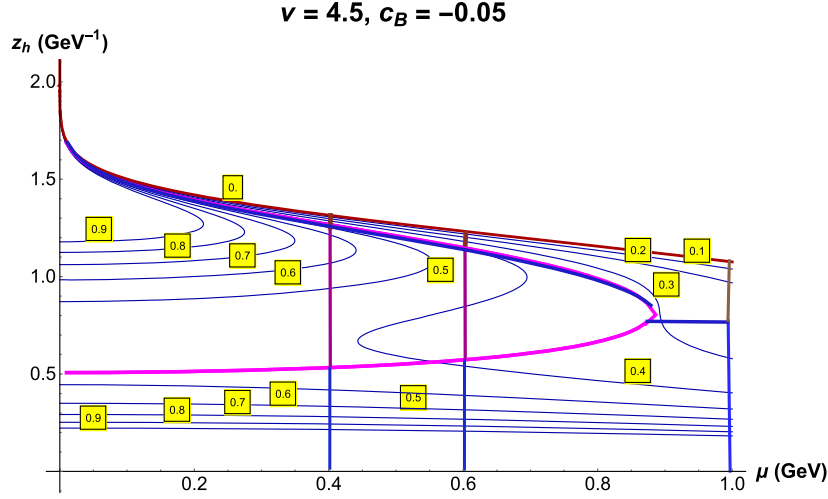


FIG. 50. Phase diagram in the  $(\mu, z_h)$ -plane at  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 4.5$  describing the HQ model. Magenta and blue lines indicate first-order and second-order phase transitions, respectively. Vertical lines colored in blue, purple, and brown indicate the paths along which the JQ parameters are calculated (see Fig. 47).

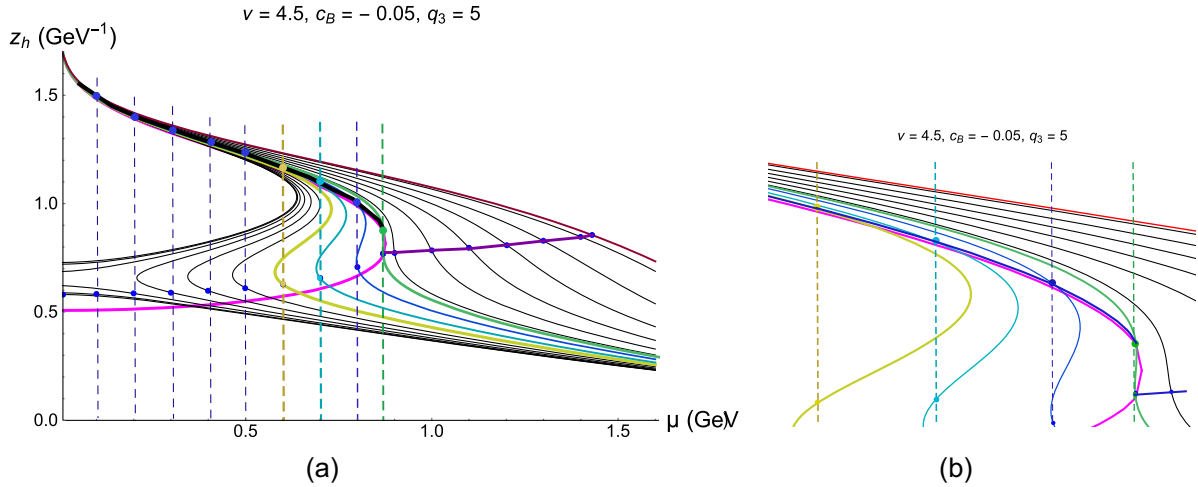


FIG. 51. (a) Confinement/deconfinement phase transition for the HQ model in the  $(\mu, z_h)$ -plane, with  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 4.5$  determined by stable points on the phase diagram. Unstable points (small dots) lie within the forbidden region. For each unstable point at given  $(\mu, z_h)$ -coordinates, a corresponding stable point exists at the same  $(\mu, z_h)$ , represented as a larger dots of the same color. (b) A enlarged view near the CEP.

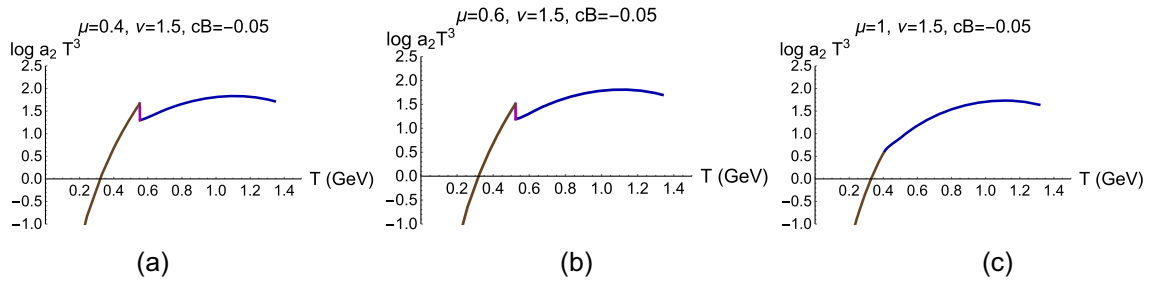


FIG. 52.  $\log(a_2 T^3)$  versus the temperature for the HQ model, with  $c_B = -0.05 \text{ GeV}^2$  and  $\nu = 1.5$ , at fixed chemical potentials: (a)  $\mu = 0.4 \text{ GeV}$ , (b)  $\mu = 0.6 \text{ GeV}$ , and (c)  $\mu = 1 \text{ GeV}$ . The blue and brown lines correspond to the QGP and hadronic phases, respectively. The magenta lines correspond to the unstable regions. We see jumps in panels (a) and (b), and a smooth change of the slopes near the second-order phase transition at panel (c).

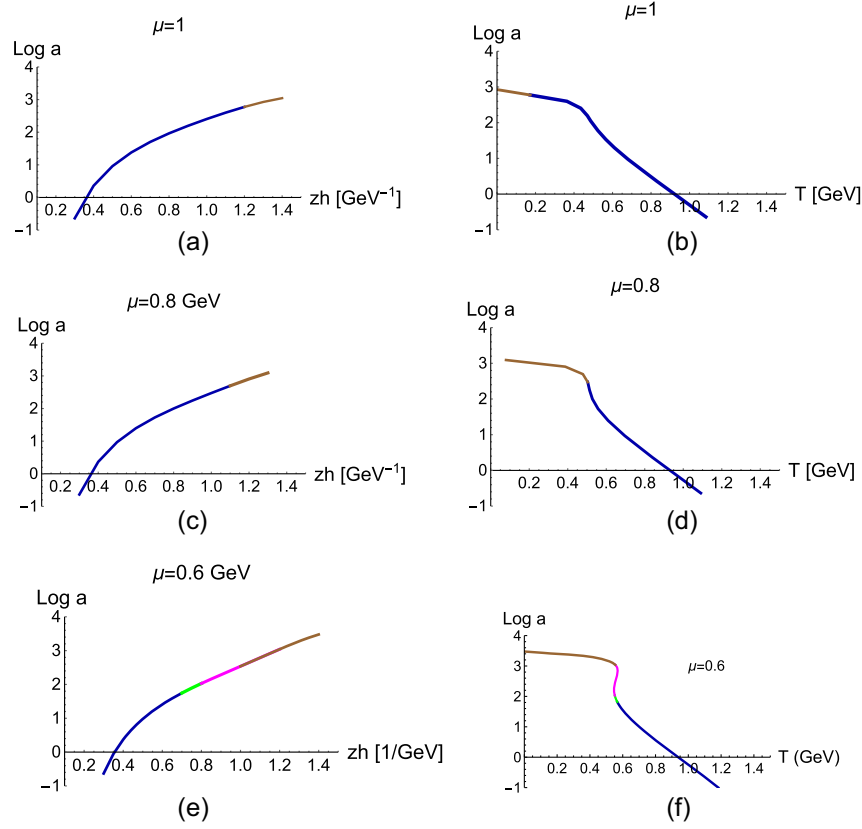


FIG. 53.  $\log a$  as a function of horizon size  $z_h$  [panels (a),(c),(e)] and temperature  $T$  [panels (b),(d),(f)] for the HQ model, with  $\nu = 1$  and  $c_B = 0$ , at chemical potentials  $\mu = 1, 0.8, 0.6$  GeV. The blue, brown, and green lines correspond to the QGP, hadronic and quarkyonic phases, respectively. The magenta lines correspond to the unstable regions. Panel (f) shows that the JQ parameter exhibits a jump at the first-order phase transition.

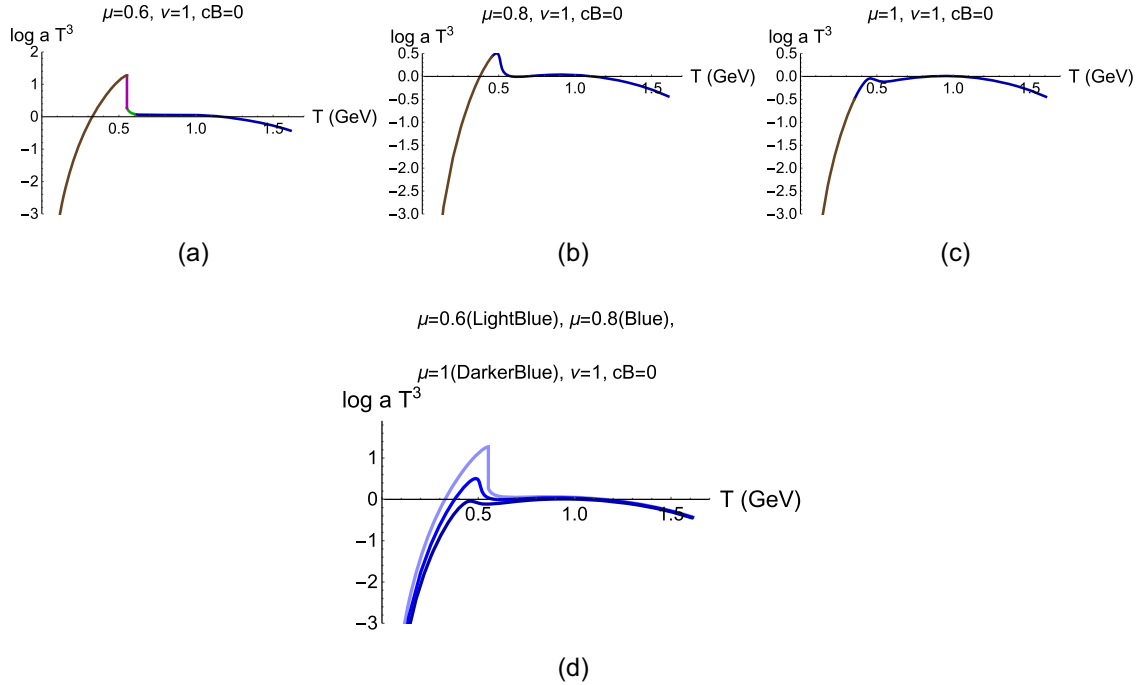


FIG. 54. The dependence of  $\log a T^3$  on  $T$  for the HQ model, with  $\nu = 1$  and  $c_B = 0$ , at fixed chemical potentials: (a)  $\mu = 0.6$  GeV, (b)  $\mu = 0.8$  GeV, and (c)  $\mu = 1$  GeV. The segments of these lines are colored blue (QGP), brown (hadronic), and green (quarkyonic) according to the phase traversed. (D) Comparison of  $\log a T^3$  for the same  $\nu = 1$  and  $c_B = 0$  at different  $\mu$ .

chemical potential. It is obvious that the jumps in Fig. 52 remain at the same temperatures as in Fig. 31.

We compute the logarithm of the integral (2.8) for fixed values of  $\mu$  and  $z_h$  and plot the results in Fig. 53 (left panels). The curves are color-coded according to the phase regions in Fig. 47: (i) brown segments correspond to  $z_h$  trajectories through the brown phase regions; (ii) blue segments represent trajectories through the blue regions; (iii) magenta segments indicate forbidden white areas; and (iv) green segments correspond to green phase regions.

The right panels of Fig. 53 show the temperature dependence of  $\log a$ . As evident in the left panels, only

one curve crosses the forbidden region specifically in Fig. 53(e). The corresponding right panel [Fig. 53(f)] exhibits a jump at  $T = T_c(\mu)$ , reflecting the removal of the magenta curve segment in this graph.

Figure 54 shows the  $T$ -dependence of  $\log aT^3$  in the HQ model for fixed parameters  $\nu = 1$  and  $c_B = 0$ , at chemical potentials  $\mu = 0.6$  GeV [panel (a)],  $\mu = 0.8$  GeV [panel (b)], and  $\mu = 1$  GeV [panel (c)]. Figure 54(d) compares  $\log aT^3$  for fixed parameters  $\nu = 1$  and  $c_B = 0$  at different values of  $\mu$ . The behavior of  $\log aT^3$  in the high-temperature regime is independent of  $\mu$ , is not constant, and decreases as  $T$  increases.

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