

Fragmentation functions for axial-vector heavy tetraquarks: A TQ4Q1.1 update

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We present and discuss the release of novel sets of collinear, variable-flavor-number-scheme fragmentation functions for axial-vector, fully heavy $T_{4c}(1^{+-})$ and $T_{4b}(1^{+-})$ tetraquarks. Working within the single-parton approximation at leading power and employing nonrelativistic QCD factorization adapted for tetraquark Fock states, we model the initial-scale fragmentation input using a recent calculation for the constituent heavy-quark channel. Standard Dokshitzer-Gribov-Lipatov-Altarelli-Parisi evolution is then applied, ensuring consistent implementation of evolution thresholds. To support phenomenology, we investigate NLL/NLO⁺ rates for tetraquark-jet systems at the High-Luminosity LHC and Future Circular Collider from (sym)JETHAD. This study further integrates exotic matter explorations with precision QCD calculations.

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Introduction. Can high-energy precision quantum chromodynamics (QCD) provide a coherent framework for describing the production of exotic matter at colliders? Despite significant progress, the fundamental mechanisms driving exotic matter formation remain an open question. However, recent advancements in (all-order) perturbative techniques and QCD factorization could offer new and unexpected perspectives. Mesons and baryons, made of minimal valence-quark configurations, are *ordinary* hadrons. QCD color neutrality, however, allows bound states with different valence content. When their quantum numbers defy conventional models, they are called *exotics* [1–5]. Their structure is still debated. Two main scenarios exist: lowest Fock states with active gluons (*hybrids* [6–8] and *glueballs* [9–12]) and compact *multi*quark states [13–16]. The first exotic hadron, $X(3872)$, was discovered by Belle in 2003 [17] and later confirmed by multiple experiments. Classified as a hidden-flavor state made of heavy-quark pairs [18–20], its discovery marked the beginning of the second quarkonium revolution, following the first triggered by the J/ψ in 1974. Although $X(3872)$ has nonexotic quantum numbers, its isospin-violating decays suggest a structure beyond pure quarkonium [21]. Competing interpretations include a meson molecule

[22–27], a double-diquark configuration [28–32], or a hadroquarkonium composed of a quarkonium core and an orbiting light meson [33–37]. For years, $X(3872)$ was the only exotic hadron observed in prompt proton collisions, until the discovery of the doubly charmed T_{cc}^+ [38,39]. The $X(6900)$ resonance [40] is considered a candidate for the $[J^{PC} = 0^{++}]$ ground or the $[2^{++}]$ radial state of the fully charmed tetraquark T_{4c} [41,42]. Since the heavy-quark mass m_Q exceeds the perturbative threshold, a fully heavy tetraquark, T_{4Q} , can be described as a non-relativistic $|Q\bar{Q}Q\bar{Q}\rangle$ state, similar to $|Q\bar{Q}\rangle$ quarkonia. Thus, methods developed for quarkonia extend naturally to heavy tetraquarks. As charmonia are QCD “hydrogen atoms” [43], T_{4c} states may resemble QCD “heavier nuclei” or “molecules” [44]. Exotic hadron production remains poorly understood, despite progress on their spectra and decays since $X(3872)$. Only model-dependent methods, like color evaporation [45], multiparticle interactions [46,47], and high-energy dynamics [48], have been explored. Large- p_T $X(3872)$ yields at the LHC [49–51] suggest that structural scenarios may act as initial-scale proxies for QCD-based mechanisms like single-parton fragmentation. A new set of collinear fragmentation functions (FFs), TQ4Q1.1, was recently introduced [52] to describe S -wave color-singlet $T_{4Q}(0^{++})$ and $T_{4Q}(2^{++})$ production at moderate-to-large p_T within the variable-flavor number scheme (VFNS) [53,54]. These FFs are based on initial-scale nonrelativistic QCD (NRQCD) inputs for the gluon [55] and charm [56] channels [57,58]. Leveraging key aspects of the newly developed heavy-flavor nonrelativistic evolution (HF-NRevo) scheme

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[59,60], a proper Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution of these inputs is performed, ensuring a consistent treatment of all parton thresholds.

In this Letter, we present new FFs for axial-vector heavy tetraquarks, $T_{4Q}(1^{+-})$, extending them to both charmed and bottomed states. Tetraquarks with $[J = 1^{+-}]$ are of special interest due to their structural richness: they may appear as axial-vector states or as mixed configurations involving quarkonia and tetraquarks. In the light sector, the active role of chiral symmetry breaking enhances $[1^{++} \leftrightarrow 1^{+-}]$ mixing [61–65], making pure tetraquark states hard to isolate [66,67]. In contrast, 1^{+-} fully heavy tetraquarks experience strongly suppressed mixing. Heavy-quark spin symmetry (HQSS) [68,69] implies weak spin-dependent interactions between charm and bottom quarks, reducing $[1^{++} \leftrightarrow 1^{+-}]$ overlap [70]. Lattice QCD and potential models confirm a rigid spin structure, with 1^{+-} nearly a pure eigenstate [71]. Unlike heavy-light tetraquarks, which mix moderately [72], they offer a clean probe of HQSS and exotic production mechanisms. Electron-ion collider [73] predictions for $T_{4c}(1^{+-})$ photoproduction, based on single-parton [56] or two-parton [74] nonrelativistic QCD (NRQCD) fragmentation, indicate promising yields. In hadroproduction, $T_{4c}(1^{+-})$ rates at the LHC are lower than for $T_{4c}(0^{++})$ and $T_{4c}(2^{++})$, but still significant [75]. A preliminary TQ4Q1.1 update was used to describe indirect production via Higgs and electroweak decays [76]. A full VFNS set of DGLAP-evolving FFs for 0^{++} , 1^{+-} , and 2^{++} heavy tetraquarks provides a robust tool to investigate their high-energy production mechanisms. In this light, the timely release of the TQ4Q1.1 update [77] provides a valuable resource to the community, driving progress in this rapidly evolving field. Axial-vector fully heavy tetraquarks exhibit unique dynamical and phenomenological features. Unlike scalar and tensor states, often affected by mixing with quarkonia or molecular configurations [1,67,78], they offer a cleaner probe of multi-quark dynamics [70,71,79]. Their leading-order production is suppressed by the Landau-Yang theorem and C -parity selection rules [80], making them ideal to isolate strong-interaction effects. Theoretical studies [75,81,82] further highlight their distinctive production and decay patterns. Mixing is reduced in the fully heavy sector due to HQSS [62,79]. This motivates a dedicated exploration of the 1^{+-} channel, especially at the High-Luminosity LHC (HL-LHC) and Future Circular Collider (FCC), where rare exotic states may become accessible. Accurate theoretical predictions are essential to support these searches.

TQ4Q1.1 functions for $T_{4Q}(1^{+-})$ tetraquarks. Heavy-flavored hadron fragmentation differs substantially from the light sector due to the large heavy-quark masses, which lie above the perturbative-QCD threshold. While light-hadron FFs are purely nonperturbative, heavy-hadron FFs include perturbative parts [54,83–87]. For D , B , and $\Lambda_{c,b}$

hadrons, fragmentation proceeds in two steps: perturbative parton-to- Q splitting [53], followed by nonperturbative hadronization. To build VFNS FFs, DGLAP evolution must be applied to the nonperturbative inputs, yielding μ_F -dependent distributions at fixed accuracy. The fragmentation ansatz for heavy-light hadrons becomes a formal property in quarkonium studies via NRQCD, which factorizes short- and long-distance effects into short-distance coefficients (SDCs) and long-distance matrix elements (LDMEs). At high p_T , single-parton fragmentation dominates, despite starting at higher order, due to a $[p_T/m_Q]^2$ enhancement [88–92], surpassing two-parton mechanisms [93–95]. NLO NRQCD input enabled the first VFNS FFs for vector quarkonia [96] and $B_c(^1S_0, ^3S_1)$ mesons [97,98]. NRQCD factorization provides a robust framework to model T_{4Q} fragmentation [79,99]. Initial-scale leading-order inputs for parton-to- T_{4Q} fragmentation were computed in [55,56,100]. Using these as model proxies, the first VFNS FFs for $T_{4Q}(0^{++})$ and $T_{4Q}(2^{++})$ were released in TQ4Q1.1, extending TQ4Q1.0 [101], which was based on the Suzuki model [102,103]. For a fully heavy tetraquark, $T_{4Q}(J^{PC})$, with total angular momentum, parity, and charge J^{PC} , the LO NRQCD initial-scale input for the $[i \rightarrow T_{4Q}]$ collinear FF reads

$$D_i^{T_{4Q}(J^{PC})}(z, \mu_{F,0}) = \frac{1}{m_Q^2} \sum_{[\tau]} \mathcal{S}_i^{(J^{PC})}(z, [\tau]) \times \langle \mathcal{O}^{T_{4Q}(J^{PC})}([\tau]) \rangle, \quad (1)$$

with i a generic parton. Here, $\mathcal{S}_i^{(J^{PC})}(z, [\tau])$ represents the SDC describing the perturbative component of the $[i \rightarrow (Q\bar{Q}Q\bar{Q})]$ fragmentation, while $\langle \mathcal{O}^{T_{4Q}(J^{PC})}([\tau]) \rangle$ denotes the color-composite LDMEs [55] characterizing the purely nonperturbative hadronization of $T_{4Q}(J^{PC})$. The composite quantum number $[\tau]$ spans the configurations: $4[3, 3]$, $[6, 6]$, $[3, 6]$, and $6, 3]$. In the diquark-antidiquark basis, the color-singlet tetraquark appears as $[\bar{3} \otimes 3]$ or $[6 \otimes \bar{6}]$. Fermi-Dirac statistics allow spin 0, 1, or 2 for $[\bar{3} \otimes 3]$, but only spin 0 for $[6 \otimes \bar{6}]$. Thus, only the $[3, 3]$ channel contributes to the fragmentation of the axial-vector $T_{4Q}(1^{+-})$. The gluon FF is suppressed at leading order (LO). Because of the Landau-Yang theorem, an on-shell, color-singlet gluon cannot fragment into a 1^{+-} tetraquark, as it cannot decay into two identical massless vector bosons. In contrast, transitions into scalar $[0^{++}]$ or tensor $[2^{++}]$ states are allowed. The nonconstituent (light or heavy) quark FF is also suppressed at LO, not by Landau-Yang, but by C -parity conservation [100]. Hence, the only allowed LO channel is $[Q \rightarrow T_{4Q}(1^{+-})]$ (Fig. 1). Full symmetry between Q and $\bar{Q}$ channels is assumed. The $[Q \rightarrow T_{4Q}(1^{+-})]$ SDC was first computed in [56] and rederived via (sym)JETHAD [104–106]. Its full form is given in Eq. (S1) of the Supplemental Material [107].

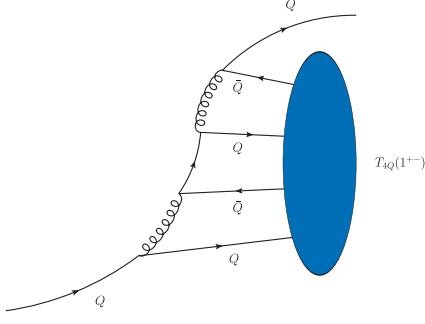


FIG. 1. LO diagram for S -wave color-singlet $[Q \rightarrow T_{4Q}(1^{+-})]$ fragmentation at $\mu_{F,0}$. The SDC is shown on the left. The blue oval denotes the LDME. The $[\bar{Q} \rightarrow T_{4Q}(1^{+-})]$ channel is omitted.

LDMEs are based on Cornell-inspired potential models [55,56]. To estimate uncertainty, we average results from [108,109], following validation in [52,101], and use their spread as variation:

$$\begin{aligned} \mathcal{O}^{T_{4c}(1^{+-})}([3,3]) &= (0.0878 \pm 0.0098) \text{ GeV}^9, \\ \mathcal{O}^{T_{4b}(1^{+-})}([3,3]) &\simeq (35.1 \pm 3.9) \text{ GeV}^9. \end{aligned} \quad (2)$$

More details on LDMEs are given in Sec. 1 of [107]. The final step in building our TQ4Q1.1 collinear FFs for $T_{4Q}(1^{+-})$ states [77] is implementing a threshold-consistent DGLAP evolution. Kinematics sets $\mu_{F,0} = 5m_Q$ as the minimal invariant mass for the $[Q \rightarrow (Q\bar{Q}Q\bar{Q}) + Q]$ splitting in Fig. 1, which we adopt as the threshold for Q fragmentation. In the HF-NRevo scheme [59,60], DGLAP evolution involves two steps: an expanded and semianalytic decoupled evolution (EDevo), which handles channel-dependent thresholds, followed by a numerically implemented all-order evolution (AOevo). However, since only the $[Q \rightarrow T_{4Q}(1^{+-})]$ channel is modeled at $\mu_{F,0}$, we skip EDevo and directly apply AOevo. Starting from the $[Q \rightarrow T_{4Q}(1^{+-})]$ input defined at the *evolution-ready* scale $\mu_{F,0}$, we construct the TQ4Q1.1 sets for axial-vector heavy tetraquarks via NLO DGLAP evolution using APFEL++ [110], and release them in LHAPDF format [111]. Plots in Fig. 2 show the z dependence of the Q quark (upper) and gluon (lower) FFs to $T_{4c}(1^{+-})$ (left) and $T_{4b}(1^{+-})$ (right). We consider representative factorization scales: $\mu_{F,0} \equiv 5m_Q$, 40, 80, and 160 GeV. The moderate- z peak rises with energy, as expected from DGLAP evolution. At $\mu_{F,0}$, only the Q -quark FF is nonzero, since the gluon FF vanishes at this scale and

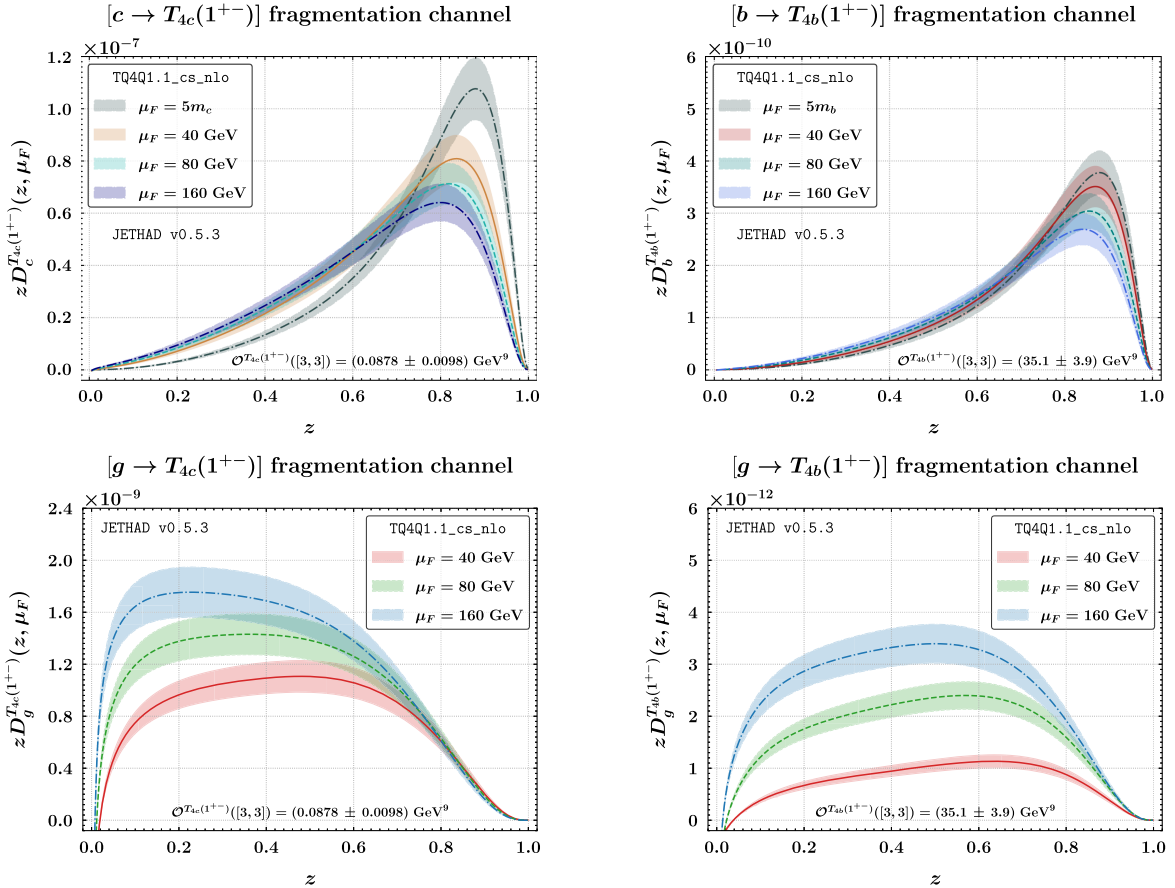


FIG. 2. z dependence of TQ4Q1.1 FFs for $T_{4c}(1^{+-})$ (left) and $T_{4b}(1^{+-})$ (right) at different μ_F values. Upper (lower) plots show the heavy-quark (gluon) channel. Shaded bands indicate LDME uncertainty.

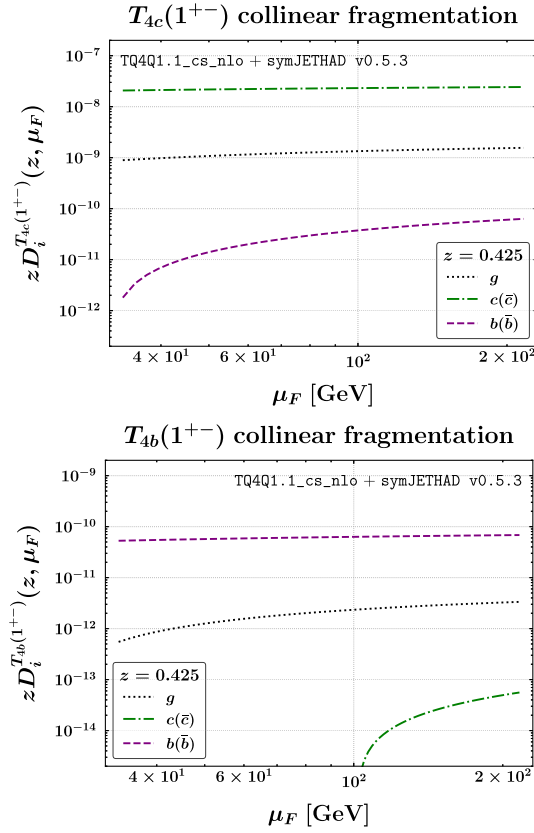


FIG. 3. Energy dependence of TQ4Q1.1 FFs for the $T_{4c}(1^{+-})$ (upper) and $T_{4b}(1^{+-})$ (lower) states at $z = \langle z \rangle \simeq 0.425$.

emerges only above $\mu_{F,0}$. We use $m_c = 1.5$ GeV and $m_b = 4.9$ GeV. Q -quark FFs peak near $z \simeq 0.85$, consistent with heavy-light trends [112,113]. Gluon FFs show a broad distribution at moderate z , with a sharp rise at low z and slow falloff at large z . Both channels exhibit shapes different from scalar and tensor cases [107], highlighting the dynamics of axial states. FFs for 1^{+-} are suppressed compared to 0^{++} and 2^{++} , particularly for gluons. This pattern follows from NRQCD expectations based on HQSS, selection rules, and production dynamics. HQSS enhances symmetric diquark-antidiquark combinations, favoring 0^{++} and 2^{++} over the antisymmetric 1^{+-} [70]. Collinear NRQCD selection rules also disfavor 1^{+-} , often requiring orbital excitation, unlike the favored S -wave or spin-aligned channels [114–116]. Production mechanisms are also less efficient, as 1^{+-} states couple more weakly to LO gluon channels [66,75,117]. A softer fragmentation spectrum further reduces high- p_T production [115,118]. This suppression hierarchy is thus well motivated within NRQCD. Figure 3 shows gluon, charm, and bottom FFs at $z \simeq 0.425$, a typical hadroproduction value [96–98,104,119,120]. Charm and bottom FFs dominate, while the gluon FF grows with μ_F . This behavior stabilizes high-energy observables against NLO corrections and missing higher-order uncertainties (MHOUs). One might question the absence of initial-scale inputs for light

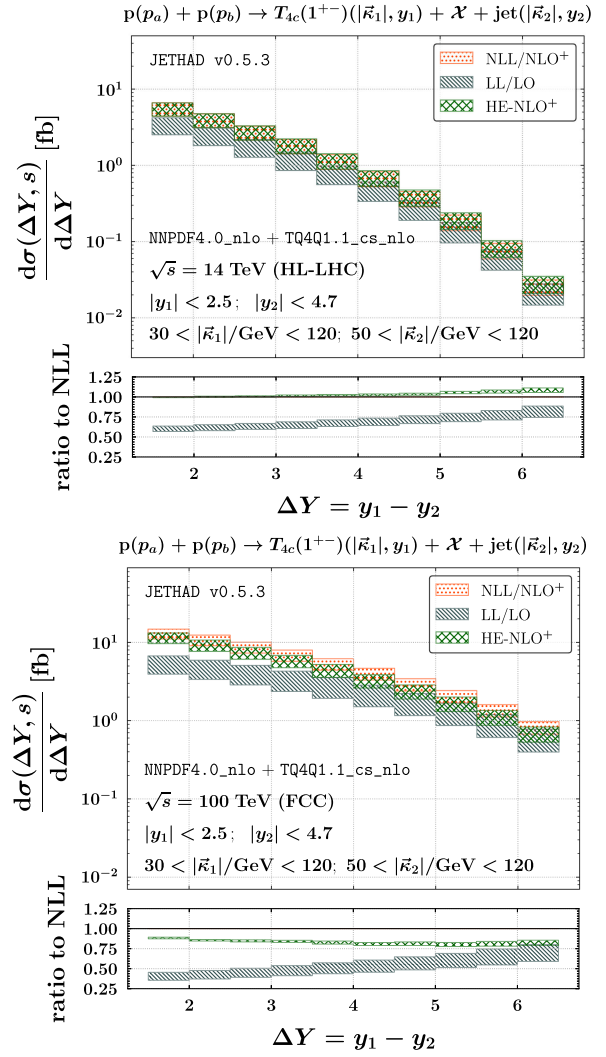


FIG. 4. $T_{4c}(1^{+-})$ plus jet ΔY rates at $\sqrt{s} = 14$ TeV (LHC, left) and 100 TeV (FCC, right). Shaded bands in main panels show MHOUs, LDMEs, and phase-space combined uncertainties. Ancillary panels display LL/LO or HE-NLO+ to NLL/NLO+ ratios, with bands for MHOUs only.

or nonconstituent heavy partons. However, since these SDCs vanish at LO and no NLO results exist, our NLO-evolved approach is the most complete framework available for $T_{4Q}(1^{+-})$. Moreover, the gluon FF vanishes as $z \rightarrow 1$, as required for leading-power channels. Thus, our method overcomes the well-known issue of NRQCD gluon FFs, which often fail to satisfy this condition [55,88,92].

Hadron-collider phenomenology. We present predictions for rapidity-interval distributions in the semi-inclusive hadroproduction of a $T_{4c}(1^{+-})$ plus a light jet, working at next-to-leading-logarithmic/next-to-leading-order (NLL/NLO+) accuracy in the hybrid factorization framework. This approach resums high-energy logarithms [121,122] within the collinear NLO scheme [119,120,123–133]. Numerical results were obtained with the JETHAD interface and (sym)

JETHAD symbolic engine [104–106] at 14 TeV HL-LHC and 100 TeV (nominal) FCC. Figure 4 shows the differential cross section in $\Delta Y = y_1 - y_2$, computed using CMS-consistent rapidity cuts [134] and asymmetric transverse-momentum bins: $30 < |\vec{k}_1| < 120$ GeV for the tetraquark and $50 < |\vec{k}_2| < 120$ GeV for the jet. These kinematic selections enhance the discriminating power between the resummed NLL signal and the high-energy NLO⁺ limit [107], particularly in forward configurations. Shaded bands represent the combined effect of MHOUs, LDME variations, and numerical uncertainties. Event rates, although reduced by axial-vector FF suppression [52], remain promising. The NLL/NLO⁺ and LL/LO bands partially overlap at large ΔY , indicating good stability. This agrees with previous analyses on exotic states [52,101,103] and reflects the known μ_F -driven growth of gluon FFs [96–98,119,120]. While earlier studies [97,106] questioned the discriminating power of ΔY observables, here we find that the NLL/NLO⁺ to LL/LO ratio rises from 50% to near unity, confirming the axial-vector channel as a sensitive probe of high-energy dynamics.

Conclusions. We presented and released new sets of VFNS FFs for axial-vector, fully heavy $T_{4c}(1^{+-})$ and $T_{4b}(1^{+-})$ tetraquarks. Starting from NRQCD-modeled perturbative inputs for the constituent heavy-quark channel, and evolving them through threshold-matched DGLAP equations, the TQ4Q1.1 sets offer a realistic tool for collider phenomenology. We employed them to generate high-energy predictions for tetraquark-jet observables at HL-LHC and FCC, working at NLL/NLO⁺ accuracy within the (sym) JETHAD framework. Advancing hadron structure studies

requires deeper insight into exotic-matter dynamics, supported by next-generation collider data. To this end, we aim to refine heavy-tetraquark fragmentation by improving MHOUs quantification [135–137] and by including color-octet contributions once available. Recent evidence for (valence [138]) intrinsic charm [139,140] may also motivate future studies of intrinsic bottom. Understanding exotic and ordinary bottom physics remains essential, as it was for charm [141]. The release of TQ4Q1.1 [77], now extended to axial-vector states, supports future investigations of heavy-tetraquark production via VFNS collinear fragmentation. The newly available VFNS, DGLAP-evolving FFs for axial-vector fully heavy tetraquarks provide a novel foundation for high-energy exotic-matter studies and open a new direction in probing the core effects of strong interactions. The 1^{+-} channel, with its distinctive dynamics and suppressed production, offers a clean window into exotic spectroscopy. Our work timely contributes to the exploration of exotic QCD at present and future colliders.

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Data availability. The data that support the findings of this article are openly available [77].

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