

Anomalous spontaneous induction of magnetic and electric fields in dense quark matter

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In this paper, we will demonstrate that a dense quark-matter system in the dual chiral density wave (DCDW) phase behaves as a ferromagnet in the sense that its magnetic-field dependent magnetization remains different from zero even at $B \rightarrow 0$. The corresponding permanent magnetization is a function of the baryonic chemical potential μ , decreasing up to zero as μ increases in the range of intermediate densities ($312 \text{ MeV} \leq \mu \leq 342 \text{ MeV}$) and then increasing from zero in the higher density interval $490 \text{ MeV} \leq \mu \leq 550 \text{ MeV}$. We will show that this system's ability to generate permanent magnetization, together with the existence of the axial anomaly, open up the possibility of spontaneously generating a magnetic field coupled to a collinear electric field. The generated magnetic field can reach values up to 10^{16} G , depending on μ , and the electric field will be 3 orders smaller. The fact that the DCDW phase is able to induce a magnetic field can be seen as its spontaneous tendency to remove the so called Landau-Peierls instability that is present in this single-modulated phase in the absence of a magnetic field. The spontaneous induction of a strong magnetic field at intermediate to high densities can be of interest for the astrophysics of compact stellar objects exhibiting strong magnetic fields as magnetars.

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I. INTRODUCTION

Quark matter at finite density has an interesting peculiarity. Once the Fermi sphere at zero temperature is created, the condensates associated with both the chiral pairs [1,2] and the color-superconducting Cooper pairs [3] are energetically favorable when they are inhomogeneous. It has been found, on the other hand, that while the inhomogeneity of the Cooper pairs becomes favored when it has a higher-dimensional modulation, the inhomogeneous chiral condensate is favored when it is single modulated. However, single-modulated condensates suffer from the so-called Landau-Peierls (LP) instability [4]. This instability appears at any nonzero temperature, no matter how small. Due to this instability, the average of the order parameter vanishes, and the long-range order is washed out preserving only a quasilong-range order [5], in close analogy with smectic liquid crystals [6].

Nevertheless, when a magnetic field is present, the inhomogeneous chiral condensate phase known as the dual chiral density wave (DCDW) phase becomes energetically favored with respect to other known chiral phases [7,8], and

moreover it allows the formation of new structures in the generalized Ginzburg-Landau (GL) expansion of the thermodynamic potential [9] that remove the LP instability. That is, when considering the phonon fluctuations about the inhomogeneous condensate, these new field-induced structural terms give rise to a linear transverse mode in the spectrum of the fluctuations which prevents the existence of the LP instability [9,10]. This is in sharp contrast with what happens at $B = 0$ in the DCDW phase, where the fluctuation spectrum is soft in the transverse direction and therefore, the system exhibits the LP instability. The new magnetized phase [11] satisfies an axion electrodynamics with anomalous electromagnetism and is called the magnetic dual chiral density wave (MDCDW) phase [12].

We must emphasize that not all single modulated inhomogeneous phases in the presence of a magnetic field will be free from LP instability. In Ref. [10] it was shown that another necessary condition in removing the LP instability is the existence of a ground state that breaks time-reversal symmetry. This is the case for the ground state of the MDCDW phase, but not for other well-known single-modulated cases in the literature such as the real-kink crystal (RKC) [13] and quarkyonic matter [14] that can also be realized at finite density.

The phenomenology we are describing takes place in the range of moderate to high densities, that is, approximately between $1.8n_s$ and $2.3n_s$, or between $6.7n_s$ and $10n_s$ (n_s is the saturation density), as can be obtained from the values

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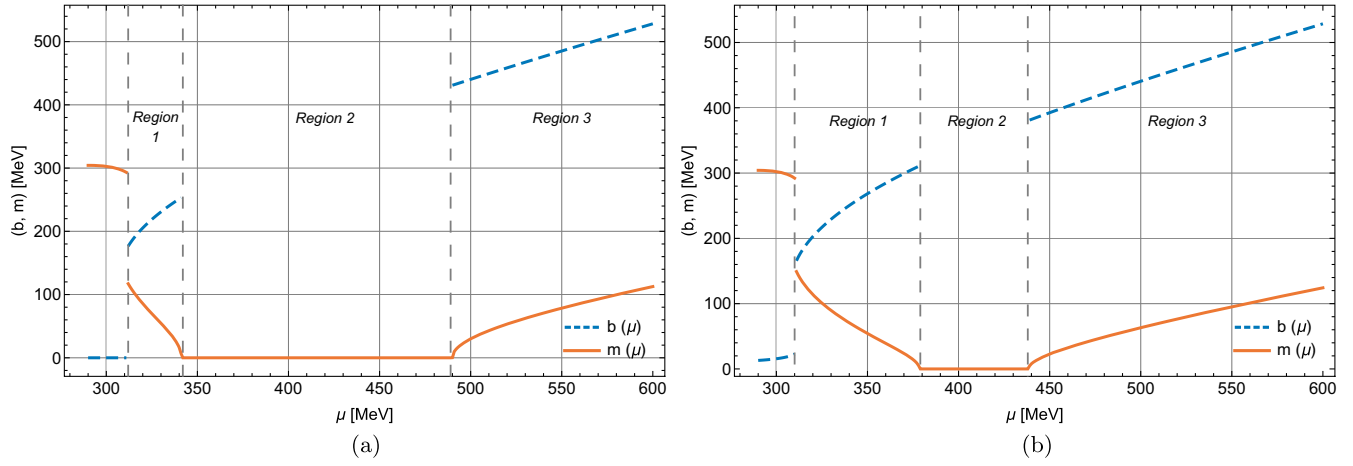


FIG. 1. Minimum solutions at $T = 0$ of the condensate amplitude m and modulation b as function of chemical potential, at $B = 0$ in (a) and at $B = 1.5 \times 10^{18}$ G in (b).

of the baryonic chemical potential where the condensate amplitude m and modulation b are not vanishing as can be seen in Fig. 1. These are densities that could well be found inside neutron stars (NS). Thus, the results we will report in this paper can be of interest to understand the physics of these compact objects taking into account, in addition, that strong magnetic fields populate the vast majority of the astrophysical compact objects.

In this regard, the subclass of NS known as magnetars [15] exhibits magnetic fields that can reach values of the order of 10^{15} G on the star surface [16]. Even magnetic fields of the order of 10^{17} G and higher are expected to be sustained in stars interiors, as suggested by various recent observations (see, e.g., [17] and references therein), as well as from theoretical estimations. Estimates based on the energy equipartition theorem, for instance, give upper estimates of the order of 10^{18} G for nuclear matter [18] and 10^{20} G for quark matter [19]. Nevertheless, fields of the order and higher than 10^{19} G create vanishing pressures that will make the star unstable [20]. Other investigations of star stability taking into account the matter pressure and the star's gravitational pressure in magnetohydrostatic equilibrium point to fields of the order of 10^{17} G [21].

One mechanism widely considered to generate the star magnetic field is the so called dynamo mechanism [22]. This mechanism is responsible for the conversion of kinetic energy of an electrically conducting fluid into magnetic energy. Nevertheless, when this mechanism is considered taking the star spin periods from the McGill Online Magnetar Catalog [23], the magnetic-field strengths found are of the order of those observed in millisecond pulsars [24], which are far from the 10^{17} G inner field expected for magnetars [25]. This result suggests that the origin of this field could be associated with a microscopic mechanism that should be able to boost the inner field to larger values.

In this paper we will discuss how if the dense stellar medium is in the DCDW phase, it can behave as a

ferromagnet that can generate, depending on the star inner density, magnetic fields much larger than those obtained by the dynamo mechanism in magnetars.

The paper is organized as follows. In Sec. II, we express in a compact way the GL expansion formula for the thermodynamic potential introduced in Ref. [26]. Using this formulation, we study the magnetic field dependence of the minimum solutions of the condensate amplitude and modulation for different density values. In Sec. III, we investigate the anomalous ferromagnetic properties of the MDCDW phase. We show that the magnetization as a function of the magnetic field tends as $B \rightarrow 0$ to a permanent magnetization that depends on the density. From Secs. II and III we see that both the minimum solutions for the condensate amplitude and modulation, as well as the magnetization have analytical behaviors in the $B \rightarrow 0$ limit. In Sec. IV, we show that in the DCDW phase of dense quark matter, the breaking of time-reversal and rotational symmetries by the ground state allows the induction of a magnetic field that can be as high as 10^{14} – 10^{16} G depending on the medium density. Taking into account the electromagnetic anomaly that is present in this phase, we show in Sec. V that together with the magnetic field a collinear electric field that is three orders smaller than B can be induced. In Sec. VI, we present the concluding remarks and discuss how the findings we are reporting can affect the physics of NS.

II. AMPLITUDE AND MODULATION OF THE INHOMOGENEOUS CONDENSATE OF THE MDCDW PHASE VERSUS MAGNETIC FIELD IN THE GINZBURG-LANDAU EXPANSION

It has been shown that at finite baryon density the two condensates,

$$\langle \bar{\psi} \psi \rangle = \Delta \cos(q_\mu x^\mu), \quad \langle \bar{\psi} i \tau_3 \gamma_5 \psi \rangle = \Delta \sin(q_\mu x^\mu), \quad (1)$$

get expectation values different from zero forming a dual chiral density wave condensate [7]. In the presence of a magnetic field, the modulation vector is energetically favored when oriented along the field direction [11,27]. The phase with a ground state given by the inhomogeneous condensate (1) in the presence of a magnetic field is called the MDCDW phase. The condensate with γ^5 matrix spontaneously breaks time-reversal symmetry.

Under an external uniform magnetic field the system original global SO(3) symmetry decreases to SO(2) contributing to critical topological effects [12,27]. First of all, the magnetic field produces an asymmetry with respect to the zero energy level in the spectrum of the lowest Landau level (LLL) [11],

$$E_0 = \epsilon \sqrt{m^2 + p_3^2} + b, \quad l = 0, \quad \epsilon = \pm, \quad (2)$$

while the highest Landau levels (HLL) are symmetric,

$$E_l = \epsilon \sqrt{\left(\epsilon \sqrt{m^2 + p_3^2} + b\right)^2 + 2|e_f B|l}, \quad l = 1, 2, 3, \dots, \quad (3)$$

Here, $m = -2G\Delta$ and $b = q_3/2$ are the condensate amplitude and modulation, respectively.

The HLL spectrum ($l \geq 1$) has four branches, with $\epsilon = \pm$ indicating spin projections and $\epsilon = \pm$ the energy sign. In contrast, the LLL ($l = 0$) has only two branches because only one spin projection contributes to the LLL modes. This asymmetry, consequently, gives rise to a topological term in the thermodynamic potential that significantly enhances the window of inhomogeneity [11,27]. Furthermore, in the presence of an electric field with a nonzero component in the direction of the background magnetic field, new topological effects emerge due to the lack of invariance of the path-integral fermion measure under the local chiral transformation. This lack of invariance gives rise to an ill-defined Jacobian that requires a proper regularization [12]. Using Fujikawa's method [28] it was extracted in [12] the regularized contribution to the effective action, which turned out to be the chiral anomaly in the electromagnetic sector $(\kappa/8)\theta F_{\mu\nu}^* F^{\mu\nu}$, with $\kappa/8 = \alpha/4\pi$. This interaction couples the electromagnetic strength tensor and its dual to $\theta = 2bz$, with z the spatial coordinate in the direction of the modulation. This chiral-anomalous term leads to anomalous electric transport properties (for more details see the review [2]).

Considering that the condensate parameters m and b are smaller than the baryonic chemical potential μ , it was carried out in [26] the GL expansion of the thermodynamic potential in the weak-field approximation. To order $2N$ with $N = 1, 2, 3, \dots$, it can be expressed as

$$\Omega^{(2N)} = \sum_{i=1,2,3,\dots}^N m^{2i} \left[\sum_{j=0,1,2,\dots}^{N-i} \alpha_{2(i+j),2j} b^{2j} + \sum_{j=0,1,2,\dots}^{N-(i+1)} \beta_{2(i+j)+1,2j+1} b^{2j+1} \right], \quad (4)$$

where the expansion coefficients α and β depends at $T = 0$ on the chemical potential and magnetic field. In [26] it was found a relationship that gives the option of finding all the coefficients as a numerical factor times the coefficient with the smallest subindices' values,

$$c_{n,n_q} = \frac{(n-2)! 2^{1-\frac{(n-n_b)}{2}}}{\left(\frac{n-n_b}{2}\right)! n_b! (n-n_b-2)!} c_{n,n-2}, \quad (5)$$

here, c is a generic coefficient representing α or β without distinction and in our notation $n = 2(i+j)$ when $c = \alpha$ and $n = 2(i+j) + 1$ when $c = \beta$, and n_b denotes the exponent of b in each case. The smallest values in the subindices in (4) are for $j = 0$ and $i = 1$, which gives for the α coefficient, $\alpha_{2,0}$; and for the β coefficient, $\beta_{3,1}$. The generic coefficients for the last member of each family (i.e., for the family formed by the coefficients with the same first index and with the higher second index) in the right-hand side of (4), are given by

$$\alpha_{2(j+1),2j} \approx \frac{\delta_{0,2j}}{4G} + \sum_{r=0,2,4,\dots} |eB|^r \frac{B_r}{r!} \cdot \frac{1+2^r}{2\pi^2 3^{r-1}} \cdot \frac{1}{(2j-1)!} I_{2(j+r-1)}(\mu), \quad (6)$$

$$\beta_{2(j+1)+1,2j+1} = -\frac{3|eB|}{(2\pi)^2} \frac{1}{(2j+1)\mu^{(2j+1)}}, \quad (7)$$

where the β coefficients exclusively depends on odd powers of μ and the α coefficients are described by the whole range of j values. In (6) the following notation was introduced:

$$I_{-2}(\mu) = -\frac{1}{4}\Lambda^2 + \frac{1}{2}\mu^2, \quad (8)$$

$$I_0(\mu) = -\frac{\gamma}{2} - \ln\left(\frac{2\mu}{\Lambda}\right), \quad (9)$$

$$I_{p>0}(\mu) = -\frac{1}{p} \left(\frac{i\sqrt{2}}{\Lambda}\right)^p + \frac{1}{p!} (p-1)! \mu^{-p}. \quad (10)$$

Here, we are considering the values for the theory cutoff Λ and coupling constant G as in [11], given, respectively, by $\Lambda \approx 636.790$ MeV and $G\Lambda^2 = 6$. Also, we have that B_r with $r = 0, 2, 4, \dots$ are the Bernoulli numbers and γ is the Euler-Mascheroni constant ($\gamma \approx 0.577216$).

We would like to draw attention to the fact that the terms with beta coefficients in Eq. (4) depend only on linear powers in the field. This means that their contributions come only from the LLL. The terms with alpha coefficients have field powers greater than one, so they come from the HLL. This means that any anomalous effects that may occur will be associated with terms from the beta part of (4), which are associated with the asymmetric energy spectrum (2).

The minimum equations from where to obtain the condensate amplitude and modulation are, respectively, given by

$$\frac{\partial \Omega^{(2N)}}{\partial m} = \sum_{i=1}^N 2im^{2i-1} \left[\sum_{j=0,1,2,\dots}^{N-i} \alpha_{2(i+j),2j} b^{2j} + \sum_{j=0,1,2,\dots}^{N-(i+1)} \beta_{2(i+j)+1,2j+1} b^{2j+1} \right] = 0, \quad (11)$$

$$\frac{\partial \Omega^{(2N)}}{\partial b} = \sum_{i=1}^N m^{2i} \left[\sum_{j=0,1,2,\dots}^{N-i} 2j\alpha_{(i+j),2j} b^{2j-1} + \sum_{j=0,1,2,\dots}^{N-(i+1)} (2j+1)\beta_{2(i+j)+1,2j+1} b^{2j} \right] = 0. \quad (12)$$

As a function of the chemical potential the minimum solutions for m and b are given at $B = 0$ in panel (a) of Fig. 1 and at $B = 1.5 \times 10^{18}$ G in panel (b). From Fig. 1(a), it can be seen that in the interval $312 \text{ MeV} \leq \mu \leq 342 \text{ MeV}$ the condensate amplitude m decreases with μ , while for $\mu \geq 490 \text{ MeV}$ it increases. At the same time, b increases in all intervals where $m \neq 0$, stressing that the increase of the chemical potential favors the inhomogeneity.

Here, the following comment is in order. As discussed in Ref. [26], in the old picture (see Ref. [1]), it was considered that once μ is large enough to form a Fermi sphere (i.e., when $\mu > E$) the system undergoes a first-order phase transition to the chirally symmetric phase, indicating that the energy cost to pair antiquarks from the Dirac sea with quarks at the Fermi surface becomes energetically unfavored. However, this picture overlooks the fact that once a Fermi surface is formed, a new pairing mechanism becomes feasible. In this case, particle-hole pairs can be excited with negligible energy cost near the Fermi surface, since quarks and quark-holes co-moving there with the same spatial direction exchange only small momenta. The nonvanishing total momentum of these pairs is the genesis of the condensate's spatial inhomogeneity, manifested through the emergence of a nonzero condensate's modulation b . Due to this new pairing mechanism, the DCDW inhomogeneous condensate becomes energetically preferable over the chirally restored phase at a large range of

chemical potentials where it was previously considered that chiral symmetry was no longer broken.

The above scenario is captured by the behavior of the condensate parameters shown in Fig. 1(a). Notice that the emergence of the modulation b indicates the activation of the particle-hole pairing mechanism. Moreover, since this type of pairing is strengthened as the Fermi surface becomes larger, one would expect that with increasing μ , the condensate magnitude m would also increase. Such a behavior occurs in Region 3 ($\mu > 490 \text{ MeV}$), but not in Region 1 ($312 \text{ MeV} \leq \mu \leq 342 \text{ MeV}$). A reasonable explanation is to assume that in Region 1, the particle-antiparticle pairing is still active and driving the net decreasing of m , while in Region 3, is the opposite, or more likely, the particle-antiparticle pairing is completely inactive due to its large energy cost at those higher densities. The intermediate Region 2 ($342 \text{ MeV} \leq \mu \leq 490 \text{ MeV}$) has no condensate at all, a sign that the contribution of the two pairing mechanisms compensate each other.

On the other hand, the role of a strong magnetic field is to shrink Region 2 as can be seen comparing Fig. 1(a) with Fig. 1(b). Looking at Fig. 1(b), we see that the effect of the magnetic field is to shift the boundary between Regions 1 and 2 to the right, meaning that the phase where the particle-antiparticle contribution is more relevant can now survive at larger chemical potentials. Also, from Fig. 2(a) we see that while the condensate amplitude m increases with the magnetic field the modulation b decreases. These results are compatible with the well-known magnetic-field effect on particle-antiparticle pairs, characteristic of the magnetic catalysis of chiral symmetry breaking (MCCSB) phenomenology [29]. In the MCCSB at $\mu = 0$ the magnetic field tends to strengthen the homogeneous condensate by producing a stronger alignment of the magnetic moments of the particle and antiparticles forming the homogeneous chiral condensate. Thus, a magnetic field tends to favor opposite momenta for the particles and antiparticles forming the pair, respectively weakening the inhomogeneity. In conclusion, the magnetic field acting on the particle-antiparticle component of the inhomogeneous phase try to regain the ground state of the magnetic catalysis scenario. In Region 3, with a larger presence of particle-hole pairs, we observe [see Fig. 2(b)] a milder effect but favorable of the magnetic field on m and b . The shift to the left of the boundary between Regions 2 and 3 at a strong magnetic field, as can be seen comparing Fig. 1(a) and (b), is due to the fact that with the magnetic field the particle-hole density of state increases with the magnetic field and once the particle-antiparticle pairs' participation is weakened as it is raised the Fermi energy, the presence of the particle-hole pairs is showed earlier.

A similar effect for pairs taking place close to the Fermi sphere was found in color superconductivity under a rotated magnetic field where the magnetic moments of the quarks

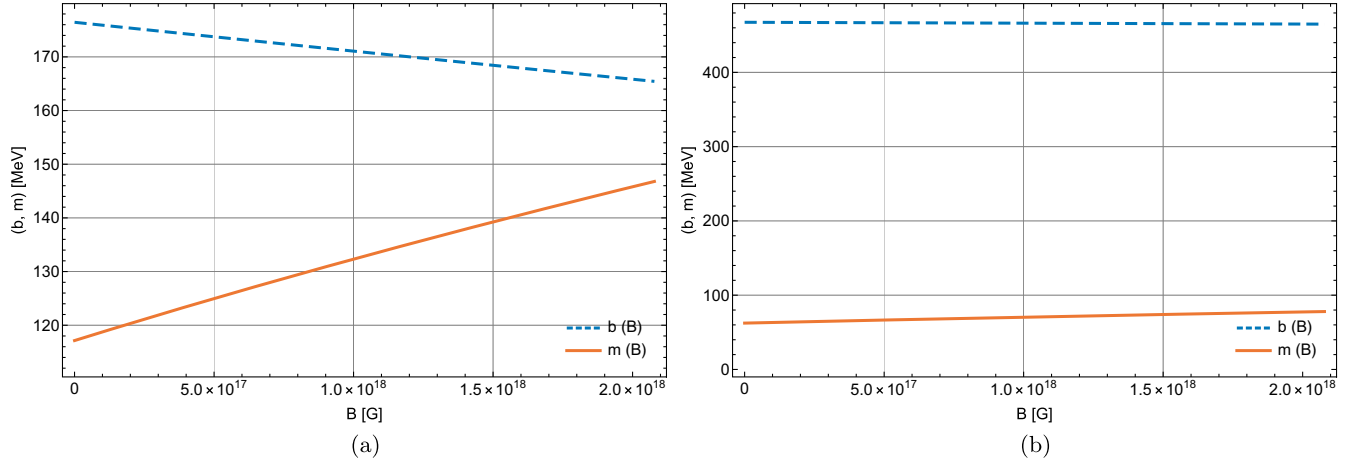


FIG. 2. Minimum solutions at $T = 0$ of the condensate amplitude m and modulation b as function of the magnetic field, at $\mu = 312$ MeV in (a) and at $\mu = 530$ MeV in (b).

forming the Cooper pairs were also aligned and hence the magnetic field strengthens the couple [30]. In a conventional superconductor, on the contrary, because the particles forming the pair have opposite spins and equal charges, their magnetic moments are pointing in opposite directions and a magnetic field tends to break the pairs by rotating one of the magnetic moments.

The minimum solutions for m and b we are reporting in Figs. 1–2 will be used in the calculations to be performed in the following sections. Here, we want to underline that in the calculations done in Figs. 1–2, as well as in all other numerical calculations done in this paper, the GL expansion (4) is taken up to $N = 10$, which, as demonstrated in [26], is the order required for the GL expansion to closely approximate the exact thermodynamic potential at $m, b < \mu$ given in Eqs. (70)–(75) of Ref. [12]b. Calculations done at any order higher than 10 becomes practically undistinguishable from those at $N = 10$ and therefore from the exact numerical calculation.

III. PERMANENT MAGNETIZATION: FERROMAGNETISM IN DENSE QUARK MATTER

The magnetization as a function of the magnetic field is given by the formula obtained from (4),

$$M^{(2N)}(B) = -\frac{\partial \Omega^{(2N)}}{\partial B} = M_0^{(2N)} + \sum_{n=1}^{\infty} M_n^{(2N)} B^{2n-1}. \quad (13)$$

Indeed, the series in the magnetic field is an expansion in powers of B/μ^2 and B/Λ^2 , which in the limit we are considering, $B < \mu^2 < \Lambda^2$, which is the suitable one for NS astrophysics, turns appropriate for a weak-field approximation truncation. To see how this happens, let us consider the expansion of the thermodynamic potential (4) up to the sixth order [9] as an example. In this case we have

$$\Omega^{(6)} = m^2[\alpha_{2,0} + \alpha_{4,2}b^2 + \alpha_{6,4}b^4 + \beta_{3,1}b + \beta_{5,3}b^3] + m^4[\alpha_{4,0} + \alpha_{6,2}b^2 + \beta_{5,1}b] + m^6\alpha_{6,0}. \quad (14)$$

Then, the first term in the magnetization (13) at order 6 is given by

$$M_0^{(6)} = m^2 \left[\left(\frac{b}{\mu} \right) \bar{\beta}_{3,1} + \left(\frac{m^2}{\mu^2} \right) \left(\frac{b}{\mu} \right) \bar{\beta}_{5,1} + \left(\frac{b^3}{\mu^3} \right) \bar{\beta}_{5,3} \right], \quad (15)$$

where $\bar{\beta}_{i,j}$ are numerical coefficients.

For the rest of the terms in (13), we start by taking into account that, as it is known, the first subindices in the α and β coefficients are equal to the sum of the powers of the factors m plus those of b ; and the second subindices are equal to the powers of b . From the terms with β coefficients, it is obtained only linear in B contributions multiplying a series in powers of $(b/\mu)^{2j+1}$ as seen from (7) and (14). On the other hand, the α coefficients, $\alpha_{2,0}$, $\alpha_{4,2}$ and $\alpha_{6,4}$, which are the corresponding last members of each family, can be found using formula (6). The rest of the α coefficients in (14) are given by a numerical factor multiplied by the corresponding last member of the family as given in (5). Hence, to see how the magnetic field enters in each last member of each family, we have to analyze how it enters in the coefficients $\alpha_{2,0}$, $\alpha_{4,2}$ and $\alpha_{6,4}$ of (14). Now, from (6) and (10), we can see that the B dependence determining their contributions to the magnetization is given by

$$\begin{aligned} \alpha_{2,0} &= B \left[C_1 \frac{B}{\mu^2} + C_2 \frac{B^3}{\mu^6} + C_3 \frac{B^5}{\mu^{10}} + \dots + (\mu \rightarrow \Lambda) \right], \\ \alpha_{4,2}b^2 &= B \frac{b^2}{\mu^2} \left[C'_1 \frac{B}{\mu^2} + C'_2 \frac{B^3}{\mu^6} + C'_3 \frac{B^5}{\mu^{10}} + \dots + (\mu \rightarrow \Lambda) \right], \\ \alpha_{6,4}b^4 &= B \frac{b^4}{\mu^4} \left[C''_1 \frac{B}{\mu^2} + C''_2 \frac{B^3}{\mu^6} + C''_3 \frac{B^5}{\mu^{10}} + \dots + (\mu \rightarrow \Lambda) \right], \end{aligned} \quad (16)$$

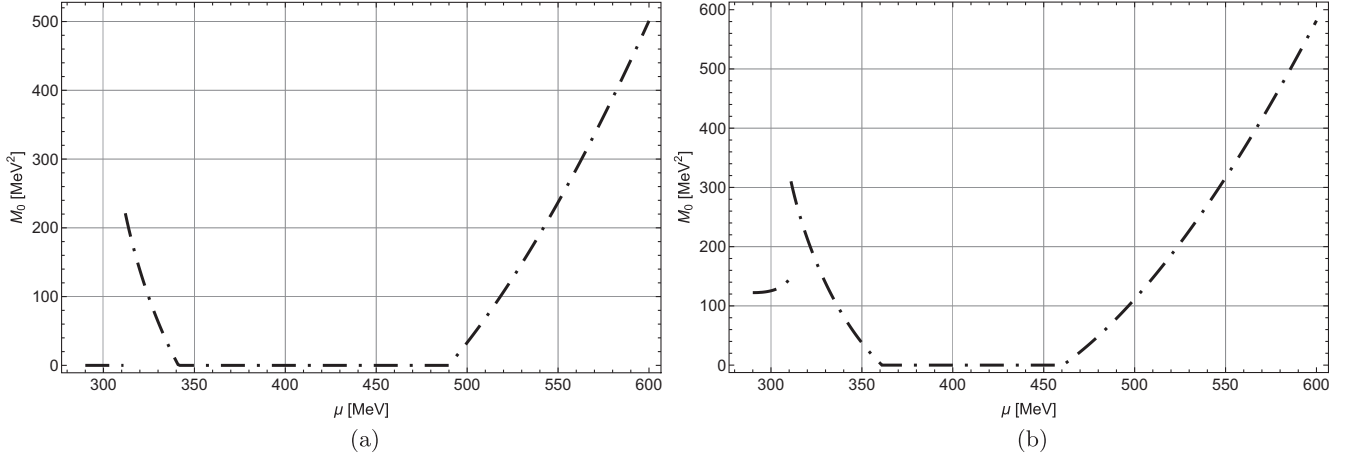


FIG. 3. The permanent magnetization, $M_0(B=0)$, as a function of the chemical potential μ is given in panel (a), and $M_0(B)$ at $B = 1.5 \times 10^{18}$ G in (b). Both graphs are obtained from the GL expansion (4) at $N = 10$ and with condensate parameters m and b found from the minimum equations (11) and (12), respectively.

where C_i , C'_i and C''_i are numerical coefficients. From (15) and (16), we see that M^6 is indeed expressed as an expansion in the small parameters $\frac{b}{\mu}$, $\frac{b}{\Lambda}$, $\frac{B}{\mu^2}$ and $\frac{B}{\Lambda^2}$ under the approximation $\Lambda^2 > \mu^2 > b^2 > B$. This fact facilitates the truncation of the series in power of the magnetic field in (13).

We should notice moreover that the expansion coefficients $M_i^{(20)}$, $i = 0, 2, \dots$ in (13) implicitly depend on B through the condensate parameters m and b . But as seen from Figs. 1(a) and 2, even at $B = 0$ those parameters are different from zero. Therefore, if $M_0^{(20)} \neq 0$ when m and b are evaluated at their minimum solutions for $B = 0$, we can say that there exists a permanent magnetization in the system. That is, the system retains a magnetization that prevails even when $B \rightarrow 0$. From now on, to simplify the notation we will skip the supraindex (20) in $M_i^{(20)}$.

The magnetization M_0 is given as a function of the baryonic chemical potential at $B = 0$ in Fig. 3(a) and at $B \neq 0$ in Fig. 3(b). In doing those plots the parameters m and b are taking as the solutions of Eqs. (11) and (12) at $B = 0$ and $B \neq 0$, respectively. From Figs. 1(a) and 3(a), we see that once the Fermi sphere is formed and the condensate modulation arises, the system acquires a permanent magnetization characterizing a ferromagnetic behavior. Therefore, at a critical value of the baryonic chemical potential $\mu_0 = 312$ MeV there is a discontinuous jump in M_0 from zero to a different from zero value. Being M_0 a first derivative of the thermodynamic potential with respect to the magnetic field, the jump is signaling a first-order phase transition which is related to the emergence of the inhomogeneous ground state (i.e., with $b \neq 0$).

The result reported in Fig. 3(a) up to $\mu = 342$ MeV coincides, apart from some numerical factors, with the one found in Ref. [31] that was obtained in the small m limit from the exact thermodynamic potential. The

numerical discrepancy between the two results is due to the fact that we are including the color number $N_c = 3$ and considering the values for the NJL model parameters Λ and G , used in Ref. [11], which differ from those used in [31].

To avoid the possible suspicion of a nonanalytic behavior of $M(B)$ at $B \rightarrow 0$, we are showing in Fig. 4 the graphs for $M(B)$ vs B at $\mu = 312$ MeV in panel (a) and at $\mu = 530$ MeV in panel (b), which correspond to chemical potentials in Regions 1 and 3 of Fig. 1, respectively. In both plots we can corroborate that the magnetization smoothly tends to $M_0(B=0)$ when $B \rightarrow 0$, so showing that this limit is analytic. Thus, this result is indicative that there exists a permanent magnetization in the system when the magnetic field tends to zero.

Here, the following discussion is in order. The existence of the found permanent magnetization $M_0(B=0)$ is related to the anomalous effect produced by the quarks in the LLL. Nevertheless, it could seem that there is a contradiction by the fact that $M_0 \neq 0$ when evaluated at $B = 0$. But, we should notice in this regard that the contribution to M_0 from the thermodynamic potential (4) comes from the β -coefficient terms, which, as we already pointed out, are directly related to the LLL contribution to $\Omega^{(2N)}$. Thus, what is determining the permanent magnetization is the structure of the LLL contribution to the thermodynamic potential $\Omega^{(2N)} \approx M_0 B$, which still remains in the $B \rightarrow 0$ limit of M_0 , since m and b are different from zero in the $B \rightarrow 0$ limit as seen from Fig. 2. From a physical point of view, we have that in the presence of a magnetic field the quarks (antiquarks) occupying the LLL have only one spin orientation (with the antiquarks having the opposite orientation). Thus, the magnetic field orients the corresponding magnetic moment so to minimize the system energy (the magnetic moment of the antiquarks will have the same orientation since they have opposite spin and opposite

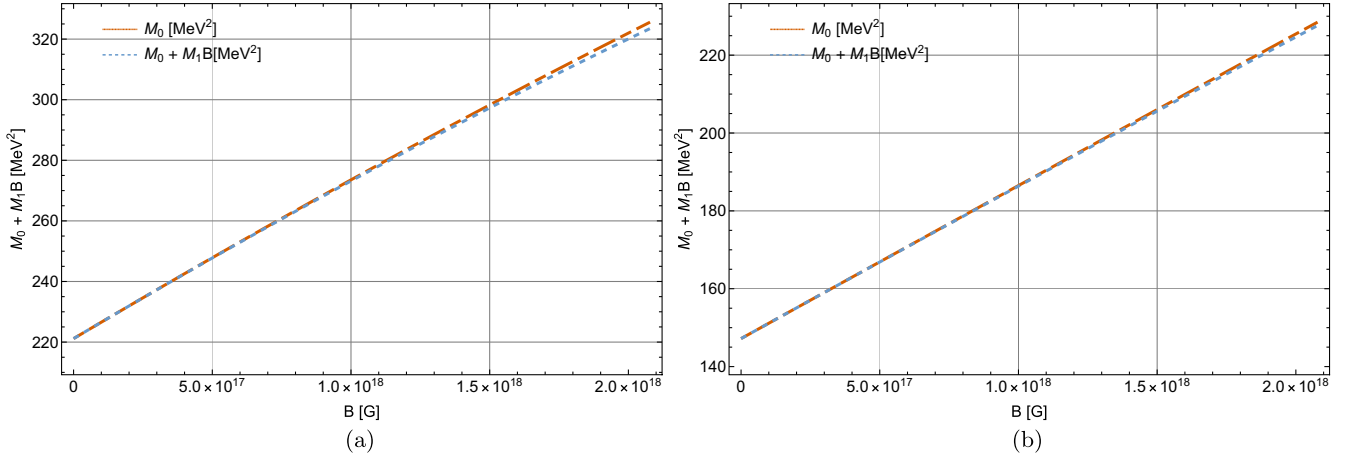


FIG. 4. Magnetization $M(B)$ as a function of magnetic field at $\mu = 312$ MeV in (a) and at $\mu = 530$ MeV in (b). In each graph we plotted M_0 and $M_0 + M_1B$ versus B .

charge). Disconnecting the magnetic field, as suggested in Fig. 4 by taking the $B \rightarrow 0$ limit, the magnetic moment of the pair that minimized the energy is preserved so keeping the system magnetization $M_0(B=0)$, similarly to what happens with ferromagnetic materials that acquire a permanent magnetization once the magnetic field is turned off. Hence, we can see the important role that the LLL is playing in this phenomenology. We must also emphasize that not every system under a magnetic field can exhibit this effect due to the contribution of LLL. We should underline, that in the MDCDW case the joint effect of the existence of an inhomogeneous condensate that breaks time-reversal symmetry, thus allowing a term M_0B in the free energy, and the asymmetry of the quark spectrum in the LLL, which determines the analytical structure of M_0 , played a crucial role in the realization of this anomalous magnetization effect.

Finally, to facilitate a more solid explanation of the paradoxical physical picture under discussion, we must point out that in this dense quark phase there will always be present a magnetic field spontaneously induced once

m and b are different from zero, as we will see in the next section. Thus, M_0 will always depends on the magnetic field through m and b in this phase.

To determine where to cut the expansion in powers of B in (13), we plot in Fig. 5(a) and (b) the magnetization contributions linear and cubic in B versus the magnetic field, respectively. Comparing the values of those terms with the ones of Fig. 4, we see that the contribution of terms beyond M_0 is negligibly small. Since the terms M_i with $i \geq 1$ get only contributions from the HLL, while M_0 only depends on the LLL, we conclude that the magnetization of this system is mainly driven by the dynamics of the LLL. In Fig. 4 we showed that adding the term M_1B to M_0 does not make a significant change in the magnetization versus magnetic field curves.

IV. INDUCED MAGNETIC FIELD IN THE DCDW PHASE OF DENSE QUARK MATTER

The expansion of the thermodynamic potential up to second order in powers of the magnetic field is given by

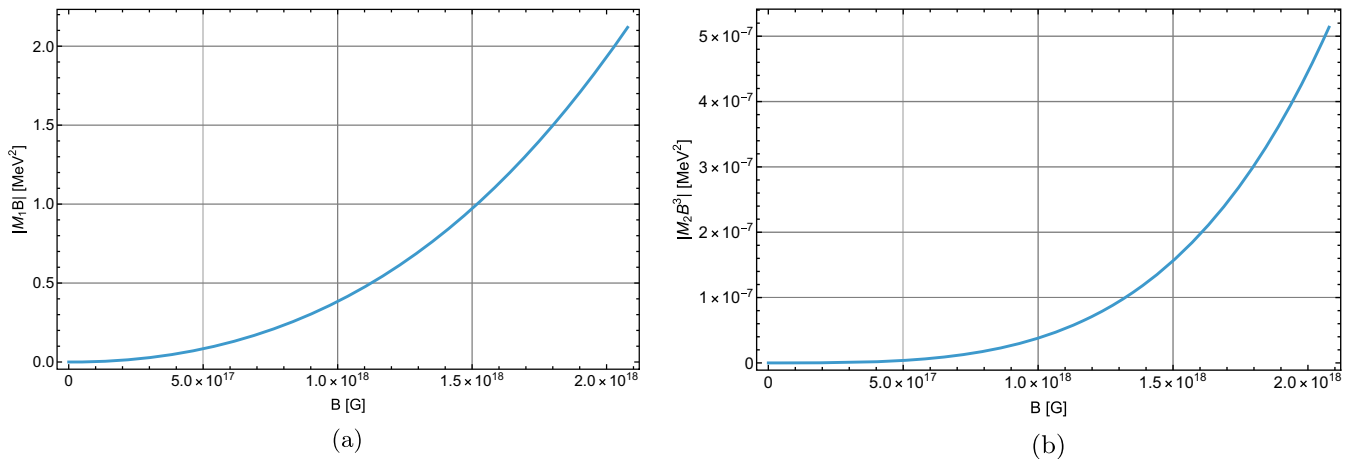


FIG. 5. Contributions of M_1B in (a) and M_2B^3 in (b) versus magnetic field for $\mu = 312$ MeV.

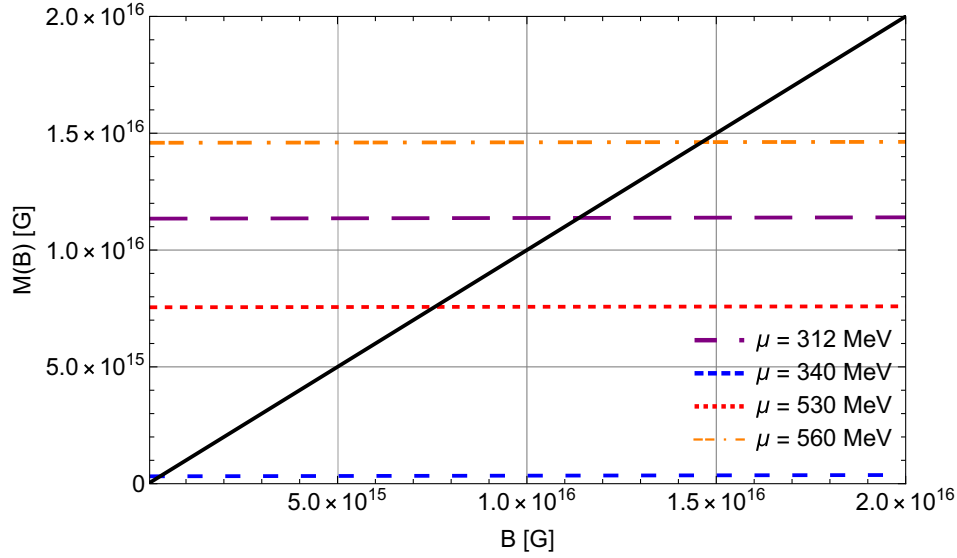


FIG. 6. Graphical method to get the solutions of the induced magnetic field for different chemical potentials. The found solutions are, respectively, ($\mu = 312$ MeV, $B = 1.19 \times 10^{16}$ G), ($\mu = 340$ MeV, $B = 3.19 \times 10^{14}$ G), ($\mu = 530$ MeV, $B = 7.5 \times 10^{15}$ G), ($\mu = 560$ MeV, $B = 1.45 \times 10^{16}$ G).

$$\Omega(B, E) \simeq \frac{1}{2}(B^2) - M_0 B, \quad (17)$$

where the first term in the right-hand side is the Maxwell term, and the second term is the medium term obtained from (4) to order $N = 10$. Our goal now is to investigate the possibility to have an induced magnetic field in the DCDW phase where the inhomogeneous condensate (1) has been spontaneously generated. From a symmetry point of view, since the DCDW inhomogeneous condensate spontaneously breaks time-reversal and rotation $O(3)$ symmetries, an induced magnetic field does not break any symmetry that has not already been broken by the condensate forming the ground state, so lacking of a mechanism that protect the system against the spontaneous induction of a magnetic field.

Since, M_0 implicitly depends on B , to solve the minimum equation for B

$$\frac{\partial \Omega}{\partial B} = B - M_0 = 0, \quad (18)$$

we use the graphical method shown in Fig. 6, where the dashed lines correspond to $M_0(B)$ versus B for different chemicals potentials and the continue line representing the $B = B$ graph, which is drawn at 45° . The interception point for each $M_0(B)$ gives the solution of (20) for the corresponding chemical potential. We selected two chemical potentials corresponding to Region 1 and two other corresponding to Region 3 and the corresponding values of the induced magnetic field are for $\mu = 312$ MeV, $B = 1.19 \times 10^{16}$ G, for $\mu = 340$ MeV, $B = 3.19 \times 10^{14}$ G,

for $\mu = 530$ MeV, $B = 7.5 \times 10^{15}$ G, and for $\mu = 560$ MeV, $B = 1.45 \times 10^{16}$ G.

From this set of values we observe that when increasing the chemical potential in Region 1, the magnitude of the induced magnetic field decreases, at the same time that the amplitude of the condensate decreases; while in Region 3 when the chemical potential increases, the condensate magnitude increases as well as the value of the corresponding induced magnetic field. Hence, we see that there is a direct proportionality between the magnitude of the induced magnetic field and the magnitude of the condensate in both regions. Since the magnitude of the condensate is related to the pair condensation energy we conclude that the energy needed to create a magnetic field is supplied by the lowering of the system energy due to the condensation process.

V. INDUCED ELECTRIC FIELD IN THE DCDW PHASE OF DENSE QUARK MATTER

As it was explained in Sec. II, in the presence of collinear electric and magnetic fields, a new term is raised in the regularized effective action of the MDCDW phase [12]. The new contribution in the regularized effective action turned out to be the chiral anomaly in the electromagnetic sector $(\kappa/8)\theta F_{\mu\nu}^* F^{\mu\nu}$, with $\kappa/8 = \alpha/4\pi$. Thus, the interaction between the electromagnetic strength tensor and its dual are coupled through $\theta = 2bz$, with z the spatial coordinate in the direction of the condensate modulation.

Thus, the expansion of the thermodynamic potential up to second order in powers of the fields is given by

$$\Omega(B, E) \simeq \frac{1}{2}(E^2 + B^2) - \frac{\kappa}{2}\theta \vec{B} \cdot \vec{E} - M_0 B. \quad (19)$$

To have a term linear in B in $\Omega(B, E)$, the system ground state should break the time reversal symmetry, which is the case in the MDCDW phase under study [32]. In addition, to have such a term, we need covariant scalar structures that allow for such linear term in the field. For the magnetization times magnetic field term, we can see that the thermodynamic potential in this system can have a structure proportional to $\epsilon_{\mu\nu\rho\lambda} u^\mu n^\nu F^{\rho\lambda}$, where u^μ is the medium four-velocity, which in the rest frame takes the form $u_\mu = (1, 0, 0, 0)$, and n^ν is a four vector associated with the modulation of the inhomogeneous condensate, which in the rest frame takes the form $n_\nu = (0, 0, 0, 1)$. The magnetic field along the z -direction is given by the tensor F_{21} . But, to have an electric polarization term entering in $\Omega(B, E)$ as $P_0 E$, parity symmetry should be broken, which is not the case in the system under consideration. Moreover, to sustain such term there is no tensor covariant structure that can represent a term linear in the electric field of the form $\epsilon_{\mu\nu\rho\lambda} u^\mu n^\nu F^{\rho\lambda}$. Thus, the only term linear in the electric field which preserves parity symmetry is of the form $(\kappa/8)\theta \epsilon_{\mu\nu\rho\lambda} F^{\mu\nu} F^{\rho\lambda}$, which is the anomalous second term in the right-hand side of (19).

Now, to find if the B and E can be spontaneously induced, we need to consider the minimum equations,

$$\frac{\partial \Omega}{\partial B} = 0, \quad \frac{\partial \Omega}{\partial E} = 0, \quad (20)$$

which give rise, respectively, to the couple equations

$$\frac{\partial \Omega}{\partial B} = B - M_0 - \frac{\kappa}{2}\theta E = 0, \quad (21)$$

$$\frac{\partial \Omega}{\partial E} = E - \frac{\kappa}{2}\theta B = 0, \quad (22)$$

where we are considering that both B and E are along the z direction and we are neglecting M_i , $i \geq 0$ since they are too much smaller than M_0 , as seen from Fig. 5 and discussed previously.

Substituting with (22) in (21) we obtain

$$\frac{\partial \Omega}{\partial B} = B - M_0 - \frac{\kappa^2}{4}\theta^2 B = \left(1 - \frac{\kappa^2}{4}\theta^2\right)B - M_0 = 0. \quad (23)$$

Taking into account that $\theta = 2bz$, we will consider a domain of the order of the wavelength of the inhomogeneous condensate $\lambda_b = 1/2b$. Then, the average value of B in that domain is

$$B = \frac{1}{\lambda_b} \int_0^{\lambda_b} dz \frac{M_0}{(1 - (b\kappa z)^2)} = \frac{2M_0}{\kappa} \int_0^{\frac{\kappa}{2}} \frac{dz'}{1 - z'^2} \approx M_0, \quad (24)$$

and from (22) we have

$$E = \frac{1}{\lambda_b} \int_0^{\lambda_b} dz \frac{M_0 \kappa b z}{(1 - (b\kappa z)^2)} = 2M_0 \int_0^{\frac{\kappa}{2}} \frac{dz'}{1 - z'^2} = \kappa B, \quad (25)$$

where we did the variable change $z' = b\kappa z$.

We have that Eq. (24) coincides with (21), which was already solved numerically in Sec. IV. On the other hand, from (25) we see that $E \ll B$, what is expected from the fact that E has a quantum origin of order α , which is produced through the anomaly by radiative effects that can take place once a magnetic field is generated in the system. Hence, we have that $|E| \approx 10^{-3}|B|$.

The values of B and E depend on μ through M_0 . For example, in the mid-density region (Region 1: from $\mu = 312$ MeV to $\mu = 342$ MeV) the induced magnetic field, for instance, can go from zero to 10^{16} G, with E damped in three orders.

The spontaneous induction of the permanent magnetization M_0 , together with a magnetic and an electric field have significant consequences. It implies that the transit between the two phases $DCDW \rightarrow MDCDW$ occurs spontaneously once the ground state is realized by the condensate (1) that breaks time-reversal and rotational symmetries as needed for the presence of the induced E and B fields, respectively. This transit, on the other hand, removes the LP instability, which could otherwise disable the realization of this single-modulated DCDW phase. All these anomalous behaviors can be traced back to the asymmetry with respect to the zero energy level in the spectrum of the LLL (2), which is created once a magnetic field is induced on that ground state. Thus, dense quark matter in this phase exhibits an intrinsic mechanism for generating electromagnetic fields, which can be of great interest for magnetar physics.

VI. CONCLUDING REMARKS

The effect of a magnetic field on the chiral condensate was exhaustively studied at $\mu = 0$ and neglecting the field effect on the pairing coupling in Ref. [29]. There, it was found that the magnetic field strengthen the chiral condensation in the weak coupling limit. That phenomenon was known as the Magnetic Catalysis of Chiral Symmetry Breaking. Later on, adding the effect of the magnetic field on the pairing coupling it was found that the magnetic field can weaken the chiral condensation [33]. This phenomenon is called the Inverse Magnetic Catalysis. When the baryonic chemical potential is added as a new factor, it was first found that at a critical value $\mu_c \gtrsim m_{dyn}$, where m_{dyn} is the dynamical mass due to the chiral condensate at $\mu = 0$, the constant chiral condensate disappears [34]. However, it was shown later that at $\mu \neq 0$ the phase with the inhomogeneous condensate (1) is energetically preferred over the homogeneous chiral phases (broken and unbroken) in a wide range of the parameter space at supercritical coupling and

even at subcritical coupling [5,7,8,11]. Also it was found that the condensate under sufficiently strong magnetic fields, is favored up to temperatures of tens of MeV, which is several orders of magnitude higher than typical temperatures of old NSs [10,26].

The inhomogeneous chiral condensate (1) is realized once $\mu \gtrsim m_{dyn}$ even at $B = 0$ forming what is known as the DCDW phase [7]. Nevertheless, single-modulated condensates, as the one of the DCDW phase (1), suffer from the LP instability [4] that takes place due to the thermal fluctuations at any nonzero temperature. Consequently, the LP instability undermines the long-range order of single-modulated inhomogeneous phases in $3 + 1$ dimensions. However, as it was shown in [9,10], the situation changes for the DCDW phase in the presence of a magnetic field. Indeed, the joint effects of the rotational symmetry breaking produced by the presence of the magnetic field together with the fact that the inhomogeneous condensate (1) breaks time-reversal symmetry open the possibility for free-energy terms which are odd in the condensate modulation, to exist. These odd-in- q terms ensure the absence of soft transverse modes in the spectrum of the fluctuations, hence the lack of LP instability [9,10].

With the results we are reporting in this paper we are showing that once the Fermi sphere is formed at moderate baryonic density and the ground state of quark matter takes the inhomogeneous form (1) the system spontaneously undergoes the phase transition DCDW $\rightarrow$ MDCDW with the induction of a magnetic and a much weaker electric fields.

Under these conditions, the system is spontaneously released from the thermal LP instability. Once the pair condenses, spontaneously selecting a modulation direction, the induced magnetic field will arise pointing in the same direction as the one of the condensate modulation, since the alignment of both is energetically preferred [11,27]. Moreover, because of the chiral anomaly term, an electric field will be induced, also collinear to the modulation direction, but with a magnitude three orders lower than that of the magnetic field. Thus, our result is proposing a new mechanism to generate magnetic and electric fields in NSs endowed with dense quark matter in their interiors, without the necessity of any magnetohydrodynamic mechanism.

Let us discuss a possible mechanism that can serve to create and propel inner magnetic fields in magnetars. For this analysis we will restrict ourselves to Region 1 and assume that the inner density of NS is in that range of densities, which is not unreasonable. The arguments can be also extended to Region 3.

From the values of the induced magnetic fields obtained from Fig. 6, we see that as the density approaches $\mu \simeq 340$ MeV the value of the induced inner magnetic field will be smaller. But if it turns out that the MDCDW phase could coexist with the 2SC color-superconducting phase, then we would have two complementary mechanisms one due to the inhomogeneous chiral condensate that can generate a magnetic field, does not matter how weak, and the other, produced by the color superconductor ground state in the presence of a magnetic field that will boost the magnetic field to the expected inner values for NSs. In this case, the inhomogeneous chiral condensate would serve to generate the magnetic field that, in the color-superconducting phase, would be amplified by the zero-mode mechanism [25].

The zero-mode mechanism refers to the generation of a tachyonic mode as a consequence of the interaction of a magnetic field with the anomalous magnetic moment of spin-1 charged fields. This instability occurs when the field becomes greater than the square of the Meissner mass of the spin-1 fields, and the solution is given by the restructuring of the ground state of the system, giving rise to vortex solutions that, as a result, produce a boosting of the magnetic field. This effect has been reported in different systems of spin-1 charged fields [35] and specifically in color superconductivity where certain gluons acquire a rotated charge [36]. The fact that at moderate densities the 2SC color superconducting phase shows the so-called chromomagnetic instability contributes substantially to the possibility of this scenario being realized [25]. In this regard it is still an open question if the MDCDW phase and the magnetized 2SC phase of color superconductivity can coexist at moderate densities.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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