

AdS perspective on the nonlinear tails in black hole ringdowns

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Black holes gradually settle into their static configuration by emitting gravitational waves, whose amplitudes diminish over time according to a power-law decay at fixed spatial locations. We show that the nonlinear tails in the presence of a quadratic source, which have been recently found to potentially dominate over the linear ones, can be simply derived from the $\text{AdS}_2 \times \text{S}^2$ spacetime perspective with their amplitudes being related to the Aretakis constants.

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I. INTRODUCTION

The rise of gravitational wave (GW) astronomy has inaugurated a transformative period for performing high-precision tests of strong-field gravity [1–4]. Ground-based interferometers—LIGO, VIRGO, and KAGRA—together with the upcoming space-based mission LISA and the third-generation GW detectors Einstein Telescope and Cosmic Explorer, are attaining the sensitivities necessary to probe the ringdown phase of black hole (BH) mergers in detail [5–9]. A key component of this investigation involves characterizing the behavior of BH perturbations at late times, where both linear and nonlinear dynamics impart distinctive features onto the emitted GWs.

Historically, linear perturbation theory has formed the backbone of our understanding of BH ringdowns. This framework predicts that, at late times, BHs exhibit a characteristic pattern of exponentially decaying oscillations, known as quasinormal modes (QNMs) (for a review, see Ref. [10]), followed by a late-time decay that adheres to an inverse power law in time. Price’s well-known result exemplifies this decay: for massless fields propagating on a nonrotating BH background, the signal at a fixed radius decays as $\sim t^{-2\ell-3}$ in the asymptotic regime, where ℓ denotes the multipole index [11–20].

To better capture the features of BH ringdowns, substantial work has gone into refining predictions for QNM spectra and late-time tails [21–33]. These linear results

have been strongly supported by numerical relativity and increasingly sensitive GW measurements.

More recently, attention has shifted toward the nonlinear regime of BH perturbations [34–65], uncovering new phenomena beyond linear predictions, such as second-order QNMs and their power-law tails. The emergence of nonlinear tails holds significant interest, as they stem naturally from the outgoing QNM structures inherent in ringdown processes [66,67], reflecting the underlying nonlinear dynamics of gravity itself. Advances in numerical techniques have recently uncovered distinct power-law decay patterns in the Weyl scalar associated with these nonlinear features, setting them apart from traditional Price tails [68]. Furthermore, full 3 + 1-dimensional simulations of black hole mergers in numerical relativity have detected power-law decays that diverge from the classical Price tail expectations [69,70]. These developments enrich our understanding of nonlinear phenomena in black hole perturbation theory and underscore the importance of advancing experimental strategies—such as refined gravitational wave modeling—to probe their observational imprints.

Numerical analyses have also shown that a second-order $\ell = 4$ mode, sourced by two interacting $\ell = 2$ modes, exhibits a late-time decay proportional to $\sim t^{-10}$ [68] pointing toward a generalized behavior of the form $\sim t^{-2\ell-2}$. This nonlinear law has been shown to arise from the propagation of the quadratic QNMs in an asymptotically flat spacetime [71,72]. In this paper, we take a further step ahead and show that such nonlinear tails may be understood from a symmetric perspective once it is realized that the spacetime in which the nonlinear tails are generated is effectively an $\text{AdS}_2 \times \text{S}^2$ spacetime. This allows one to simply derive the nonlinear tail power law from first principles and relate its amplitude to the so-called Aretakis constants in AdS_2 .

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II. THE SCHWARZSCHILD BH, THE $\text{AdS}_2 \times \text{S}^2$ SYMMETRY AT INFINITY, THE LADDER OPERATORS, AND THE GREEN FUNCTION

Our starting point is a Schwarzschild BH with mass M whose spacetime is described by the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \quad f(r) = 1 - r_s/r, \quad (1)$$

where $r_s = 2G_N M$ is the Schwarzschild radius. In this BH background, a massless scalar field Ψ evolves according to $\square\Psi(r, t) = 0$, which after expanding Ψ as

$$\Psi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\psi_{\ell}(r, t)}{r} Y_{\ell m}(\theta, \phi), \quad (2)$$

where $Y_{\ell m}$ are scalar spherical harmonics, we find that $\psi_{\ell}(r, t)$ satisfies

$$\left[\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - \left(1 - \frac{r_s}{r}\right) \left(\frac{\ell(\ell+1)}{r^2} + \frac{r_s}{r} \right) \right] \psi_{\ell} = 0. \quad (3)$$

As usual, r_* is the tortoise coordinate defined as $dr_* = dr/f$. At large r , we have that $r = r_*$, and Eq. (3) is written as

$$\left(\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - \frac{\ell(\ell+1)}{r_*^2} \right) \psi_{\ell} \approx 0, \quad (4)$$

whereas the corresponding Green function satisfies

$$\left(\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - \frac{\ell(\ell+1)}{r_*^2} \right) G(t, r_*; t', r'_*) = \frac{\delta(t-t')\delta(r_*-r'_*)}{\sqrt{-g}}. \quad (5)$$

Note that although we have considered a scalar field above, at spatial infinity Eqs. (4) and (5) also describe the equation and Green function of tensor perturbations (GW) at large r . In other words, scalar and tensor perturbations exhibit identical tails, simply because the latter are determined by scattering off the asymptotic potential. Let us now consider an exact $\text{AdS}_2 \times \text{S}^2$ spacetime with metric in Poincaré coordinates for the AdS_2

$$\begin{aligned} ds^2 &= g_{ij} dx^i dx^j + d\Omega_2^2 \\ &= \frac{-dt^2 + dr_*^2}{r_*^2} + d\Omega_2^2 \quad (i, j = 0, 1). \end{aligned} \quad (6)$$

A massless scalar field Ψ on $\text{AdS}_2 \times \text{S}^2$ obeys

$$\square\Psi = \square_{\text{AdS}_2}\Psi + \square_{\text{S}^2}\Psi = 0. \quad (7)$$

By using the metric (6) and expanding Ψ as $\Psi = \sum \psi_{\ell}(r_*, t) Y_{\ell m}(\theta, \phi)$ in spherical harmonics, we find that Eq. (7) turns out to be

$$\square_{\text{AdS}_2} \psi_{\ell} - \ell(\ell+1) \psi_{\ell} = 0. \quad (8)$$

Therefore, the modes ψ_{ℓ} are massive scalars on AdS_2 , with square mass $m^2 = \ell(\ell+1)$. The scaling dimension of a field of mass m in AdS_2 is

$$\Delta = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2}, \quad (9)$$

which for $m^2 = \ell(\ell+1)$ gives

$$\Delta = \ell + 1. \quad (10)$$

Since on AdS_2 we have

$$\square_{\text{AdS}_2} = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j) = r_*^2 \left(\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} \right), \quad (11)$$

Eq. (8) is written as

$$r_*^2 \left(\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} \right) \psi_{\ell} - \ell(\ell+1) \psi_{\ell} = 0, \quad (12)$$

which coincides with Eq. (4). In other words, the equation that dominates the dynamics at large r of a massless scalar in the Schwarzschild BH background is identical to the equation which governs a massive scalar with mass square $m^2 = \ell(\ell+1)$ on an AdS_2 spacetime.

Now, the Green function of Eq. (8) satisfies [73,74]

$$(\square_{\text{AdS}_2} - m^2) G_{\text{AdS}_2} = \frac{1}{\sqrt{-g}} \delta(t-t') \delta(r_*-r'_*). \quad (13)$$

The solution to the above equation has been found at various occasions, and it is identical to the Green function of the BH Green function for large r of Eq. (5). For the conformal weight (10), it is given by

$$\begin{aligned} G_{\text{AdS}_2}(\chi) &= \frac{1}{2^{\ell} \sqrt{\pi}} \frac{\Gamma(\ell+1)}{\Gamma\left(\ell + \frac{3}{2}\right)} \chi^{-\ell-1} \\ &\quad \times {}_2F_1\left(\frac{\ell}{2} + \frac{1}{2}, \frac{\ell}{2} + 1, \ell + \frac{3}{2}, \frac{1}{\chi^2}\right), \end{aligned} \quad (14)$$

where χ is the AdS_2 invariant distance defined in Appendix A. For $(t-t') \gg r_* > r'_*$ we find

$$\chi \approx -\frac{(t-t')^2}{2r_* r'_*}, \quad (15)$$

so that Eq. (14) gives

$$G_{\text{AdS}_2}(t, r_*, t', r'_*) \approx -(-1)^\ell \frac{2}{\sqrt{\pi}} \frac{\Gamma(\ell+1)}{\Gamma(\ell+\frac{3}{2})} \frac{(r_* r'_*)^{\ell+1}}{(t-t')^{2\ell+2}}. \quad (16)$$

Note that the Green function in Eq. (14) is invariant under the AdS_2 isometries, namely dilations, time translations, and conformal inversions. One may then wonder where such symmetries are in the standard determination of the tails from the Green function branch cut. The latter is given by (setting $t' = 0$) [75]

$$G^C(t, r; r') = 4iMrr' \int_0^{-i\infty} d\omega \omega^2 j_\ell(\omega r) j_\ell(\omega r'). \quad (17)$$

As we will show below, the above branch-cut contribution $G^C(t, r; r')$ is subleading with respect to G_{AdS_2} . Evaluating the above integral (17) leads to

$$G^C(t, r; r') = -\frac{2M}{\pi} \frac{t}{rr'} \frac{Q_\ell(\chi)}{\sqrt{\chi^2 - 1}}, \quad (18)$$

where $Q_\ell(\chi)$ is the Legendre polynomial. Equation (18) can be written as

$$G^C(t, r; r') = -\frac{M}{2^\ell \sqrt{\pi}} \frac{t}{rr'} \frac{\Gamma(\ell+1)}{\Gamma(\ell+\frac{3}{2})} \frac{\chi^{-\ell-1}}{\sqrt{\chi^2 - 1}} \times {}_2F_1\left(\frac{\ell}{2} + \frac{1}{2}, \frac{\ell}{2} + 1, \ell + \frac{3}{2}, \frac{1}{\chi^2}\right). \quad (19)$$

Comparing Eqs. (19) and (14) at $r \simeq r_*$, we see that

$$G^C(t, r; r') = \frac{Mt}{rr'} \frac{G(\chi)}{\sqrt{\chi^2 - 1}}. \quad (20)$$

In the limit $t \gg r$, or $\chi^2 \gg 1$, we find

$$G^C(t, r_*; r'_*) = \frac{2M}{t} G_{\text{AdS}_2}(\chi) \sim \frac{1}{t^{2\ell+3}} \ll G_{\text{AdS}_2}(\chi) \sim \frac{1}{t^{2\ell+2}}, \quad (21)$$

and we see that the contribution to the Green function from the branch cut is subleading with respect to the flat spacetime contribution when $t \gg r$.

Let us return now to the Green function in Eq. (14). The latter is actually the two-point function calculated in the $\text{SO}(2,1)$ -invariant vacuum in Poincaré coordinates $|0_{\text{Poincaré}}\rangle$ [76]. Then, by using the standard relation between Legendre and hypergeometric functions

$$Q_\ell(z) = 2^{-\ell-1} \frac{\Gamma(\ell+1)}{\Gamma(\ell+\frac{3}{2})} z^{-\ell-1} \times {}_2F_1\left(1 + \frac{\ell}{2} + \frac{1}{2} + \frac{\ell}{2}, \ell + \frac{3}{2}, \frac{1}{z^2}\right), \quad |z| > 1, \quad (22)$$

we can express Eq. (14) as

$$G_{\text{AdS}_2}(\chi) = G_{\text{AdS}_2}^\ell(\chi) = \frac{1}{2\pi} Q_\ell(\chi). \quad (23)$$

Using the recurrence relation of the Legendre polynomials, we can define raising and lowering ladder operators $L_\pm$ as

$$L_+ = (\chi^2 - 1) \frac{d}{d\chi} + (\ell + 1)\chi, \\ L_- = (1 - \chi^2) \frac{d}{d\chi} + \ell\chi, \quad (24)$$

which satisfy

$$L_+ G_{\text{AdS}_2}^\ell(\chi) = (\ell + 1) G_{\text{AdS}_2}^{\ell+1}(\chi), \\ L_- G_{\text{AdS}_2}^\ell(\chi) = \ell G_{\text{AdS}_2}^{\ell-1}(\chi). \quad (25)$$

In addition, if we further define the operator L_0 such that

$$L_0 G_{\text{AdS}_2}^\ell(\chi) = \left(\ell + \frac{1}{2}\right) G_{\text{AdS}_2}^\ell(\chi), \quad (26)$$

then it is easy to verify that $L_\pm$ and L_0 satisfy the $\text{SO}(2,1)$ algebra

$$[L_0, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = -2L_0. \quad (27)$$

Therefore, starting with the lowest $\ell = 0$ Green function $G_{\text{AdS}_2}^0(\chi)$, we can construct all $G_{\text{AdS}_2}^\ell(\chi)$'s by repeated applications of the raising operator L_+ as

$$G_{\text{AdS}_2}^\ell(\chi) = \frac{1}{\ell!} L_+^\ell G_{\text{AdS}_2}^0(\chi). \quad (28)$$

In addition, let us note that the $\text{SO}(2,1)$ spectrum generating symmetry mentioned above is related to the hidden AdS_2 structure underlying our problem.

To proceed, let us notice that $Q_\ell(z)$ is analytic in the whole complex z plane with branch points at $z = \pm 1, \infty$, whereas the standard branch cut is along the $(-\infty, 1]$. Actually, in the complex z plane $Q_\ell(z)$ is defined by Eq. (22). The hypergeometric function has a branch cut from $z = 1$ to $z = \infty$ along the real axis, and defining $z^{-\ell-1} = e^{-(\ell+1) \log z}$, the branch cut is taken to be $(-\infty, 1]$.

Then, the Feynman propagator is the limit of $G_{\text{AdS}_2}(\chi + i0)$ above the cut, whereas the retarded Green function is

$$G_{\text{AdS}_2}^{\text{ret}}(\chi) = i\theta(t-t')[G_{\text{AdS}_2}(\chi + i0) - G_{\text{AdS}_2}(\chi - i0)]. \quad (29)$$

Then by using the relation [77]

$$Q_\ell(z + i0) = Q_\ell(z - i0) - i\pi P_L(\chi), \quad (30)$$

we find that the retarded Green function is given by

$$G_{\text{AdS}_2}^{\text{ret}}(\chi) = \frac{1}{2}\theta(t-t')P_L(\chi). \quad (31)$$

III. THE NONLINEAR QUADRATIC SOURCE

Imagine now that there is a second-order source S_L , composed of the combination of two linear QNMs (e.g., an $L = 4$ mode made out of two linear $\ell = 2$ modes). As the second-order source originates from the product of two linear modes, each of frequency ω , it will necessarily take the form at spatial infinity [41]

$$S_L \approx \frac{A_L}{r_*^2} e^{-2i\omega u}. \quad (32)$$

We assume that such a source is initiated at a specific moment t_0 and radial coordinate r_0 and then propagates outward at the speed of light. Its influence is confined in retarded time, denoted by $u' \equiv (t' - r') \simeq u_0 \pm \delta$, where δ indicate the small support of the propagating source. In contrast, the advanced time variable $v' \equiv (t' + r')$ spans the range from $(t - r)$ to $(t + r)$; see Fig. 1. We also assume that the observer's position is such that the time coordinate is significantly larger than the radial coordinate, i.e., $t \gg r$. Under these circumstances [66,71,72], Eq. (4) will take the form

$$\left(\frac{\partial^2}{\partial r_*^2} - \frac{\partial^2}{\partial t^2} - \frac{L(L+1)}{r_*^2} \right) \psi_L \approx S_L, \quad (33)$$

since now the second-order perturbation ψ_L will be sourced by S_L . Therefore, from the $\text{AdS}_2 \times \text{S}^2$ perspective, the nonlinear quadratic L -mode satisfies an equation of the form

$$(\square_{\text{AdS}_2} - L(L+1))\psi_L = A_L e^{-2i\omega_\ell u}. \quad (34)$$

The nonlinear quadratic tail is therefore fully dictated by the behavior of the AdS_2 Green function

$$\psi_L = \int_{u_0-\delta}^{u_0+\delta} du' e^{-2i\omega_\ell u'} \int_{t-r}^{t+r} \frac{1}{2} P_L(\chi) \left(1 - \frac{2M}{r'} \right)^{-1} dv', \quad (35)$$

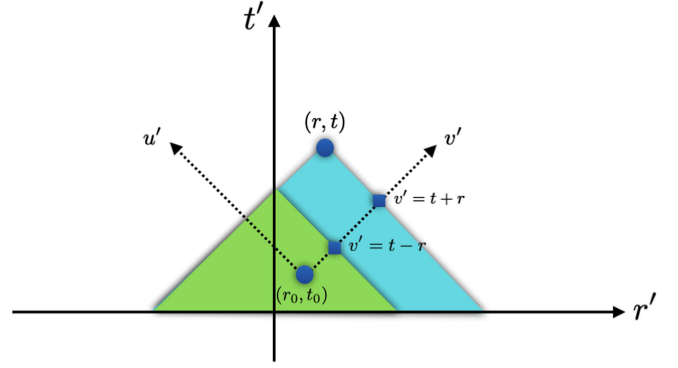


FIG. 1. The causality diagram concerning an outgoing signal propagating from the time t_0 , emitted at the point r_0 , at the speed of light along the null direction $u' = (t_0 - r_0) = u_0$.

where we have gone back to the variable r , and we write the second integral as

$$\begin{aligned} I_2(u') &= \int_{t-r}^{t+r} \frac{1}{2} P_L(\chi) \left(1 - \frac{2M}{r'} \right)^{-1} dv' \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \int_{t-r}^{t+r} P_L(\chi) \left(\frac{2M}{r'} \right)^n dv'. \end{aligned} \quad (36)$$

In terms of u', v', χ is written as

$$\chi = \frac{r^2 - (t - u')(t - v')}{t(u' - v')}, \quad (37)$$

and if we trade v' for χ in the integral of Eq. (36), we get

$$\begin{aligned} I_2 &= \sum_{n=0}^{\infty} r[r^2 - (t - u')^2]^{-1-n} (2M)^n \\ &\times \int_{-1}^1 d\chi P_L(\chi) (r\chi + t - u')^n. \end{aligned} \quad (38)$$

Then, using the relation

$$\int_{-1}^1 d\chi P_L(\chi) (a\chi + b)^n = 0, \quad \text{for } n < L, \quad (39)$$

and

$$\int_{-1}^1 P_L(\chi) (a\chi + b)^L d\chi = \frac{2^L L! (L-1)!}{(2L+1)(2L-1)!!} a^L, \quad (40)$$

we find the first nonzero contribution appears for $n = L$ for which

$$\begin{aligned} I_2 &= r[r^2 - (t - u')^2]^{-1-L} (2M)^L \int_{-1}^1 P_L(\chi) (r\chi + t - u')^L d\chi \\ &= \frac{2^L L! (L-1)!}{(2L+1)(2L-1)!!} \frac{(2M)^L r^{L+1}}{[r^2 - (t - u')^2]^{1+L}}. \end{aligned} \quad (41)$$

For $t \gg r$ we find

$$I_2 \approx (-)^{L+1} \frac{2^L L! (L-1)! (2M)^L}{(2L+1)(2L-1)!!} \frac{r^{L+1}}{t^{2L+2}}, \quad (42)$$

which reproduces the findings of Refs. [71,72] as the remaining integral in u' amounts to replacing u' with u_0 for $u_0 \ll \delta$. The generation mechanism for the nonlinear quadratic tail differs from that of the linear Price's tail. The latter is sourced by QNMs backscattered from the potential at infinity; this potential peaks at small, nonvanishing frequencies, resulting in an amplitude proportional to the BH mass. Conversely, the nonlinear tail originates from the quadratic QNM itself. Because this mode is constant along the edge $(t' - r'_*) \simeq u_0$, it does not show damped oscillations.

IV. RELATING THE NONLINEAR QUADRATIC RINGDOWN TAILS TO THE ARETAKIS CONSTANTS

Let us now show the relation of our result with the so-called Aretakis constants [78–80]. First, we define new coordinates

$$R = \frac{1}{r}, \quad v = t - \frac{1}{R}, \quad (43)$$

so that the AdS_2 part of the metric in Eq. (6) is written as

$$ds^2 = -R^2 dt^2 + \frac{dR^2}{R^2} = -R^2 dv^2 + 2dvdr. \quad (44)$$

Then, the Aretakis number $\mathcal{H}_L$ [78,79] for a field in AdS_2 of squared mass $L(L+1)$ (more details in Appendix B) reads

$$\mathcal{H}_L = (\partial_R^{L+1} \psi)|_{R=0} \quad (45)$$

and is constant

$$\partial_v \mathcal{H}_L = 0. \quad (46)$$

In particular, for the scalar field ψ_L , the Aretakis number is expressed as [81]

$$\mathcal{H}_L = (\partial_R^{L+1} \psi_L)|_{R=0} = (\partial_R^{L+1} I_1 I_2)|_{R=0}. \quad (47)$$

Since in the new coordinates (R, v) , the integral I_2 in Eq. (42) is written as

$$I_2 \approx - \frac{(-)^L 2^L L! (L-1)!}{(2L+1)(2L-1)!!} \frac{(2M)^L R^{L+1}}{(Rv-1)^{2L+2}}, \quad (48)$$

and we find that

$$(\partial_R^{L+1} I_2)|_{R=0} = (-)^{L+1} \frac{2^L (L!)^2 (L-1)! (2M)^L}{(2L+1)(2L-1)!!}, \quad (49)$$

so that

$$\mathcal{H}_L = -(-)^L \frac{2^L (L!)^2 (L-1)! (2M)^L}{(2L+1)(2L-1)!!} I_1. \quad (50)$$

Then, in the original (t, r) coordinates, we can write

$$I_2 \approx \frac{L!}{I_1} \mathcal{H}_L \frac{r^{L+1}}{t^{2L+2}}. \quad (51)$$

Therefore

$$\psi_L \approx L! \mathcal{H}_L \frac{r^{L+1}}{t^{2L+2}}, \quad (52)$$

and the nonlinear source induces a tail proportional to the Aretakis constant.

V. CONCLUSIONS

Nonlinear tails of the QNMs are recently getting attention due to the fact that their power law is smaller than the one dictated by Price's law at the linear level, and therefore they might dominate at sufficiently long times. In this paper we have shown that, at the quadratic level, the dynamics of the tails can be understood from an AdS perspective. The question is if this conclusion holds at any order in perturbation theory as higher-order contributions, even though with smaller amplitudes, it might, in principle, alter the power law. We intend to address this issue in the near future. Moreover, typically, $\text{AdS}_2 \times S_2$ arises as the near-horizon limit of the extremal Reissner-Nordström black hole (the Bertotti-Robinson solution). In our case, however, we are dealing with empty spacetime, and yet an $\text{AdS}_2 \times S_2$ structure still emerges. Importantly, this appears only at the level of perturbations and not in the underlying geometry, most likely indicating the presence of an associated symmetry. To what extent this symmetry influences the physics remains an open question under investigation.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: THE $\text{AdS}_2 \times \text{S}^2$ INVARIANT DISTANCE

The AdS_2 spacetime (of unit radius) is the hyperboloid

$$X_0^2 - X_1^2 + X_2^2 = 1, \quad (\text{A1})$$

embedded into three-dimensional spacetime with metric

$$ds^2 = \eta_{MN} dX^M dX^N = -dX_0^2 + dX_1^2 - dX_2^2, \quad (\text{A2})$$

so that $\eta_{MN} = (-1, 1, -1)$, $(M, N = 0, 1, 2)$. The parametrization of the AdS_2 hyperboloid (A1) by the Poincaré coordinates (t, r)

$$\begin{aligned} X_0 &= \frac{1}{2r}(r^2 - t^2 + 1), \\ X_1 &= \frac{1}{2r}(r^2 - t^2 - 1), \\ X_2 &= \frac{t}{r}, \end{aligned} \quad (\text{A3})$$

leads to the induced metric on the AdS_2

$$ds^2 = \frac{-dt^2 + dr^2}{r^2}. \quad (\text{A4})$$

The invariant distance on the AdS_2 hyperboloid (A1) between the points (X_0, X_1, X_2) and (X'_0, X'_1, X'_2) is

$$\sigma(X, X') = \frac{1}{2} \eta_{MN} (X - X')^M (X - X')^N, \quad (\text{A5})$$

which by using the parametrization (A3) is written as

$$\sigma = \frac{-(t - t')^2 + (r - r')^2}{2rr'}. \quad (\text{A6})$$

One quantity that appears frequently is

$$\chi = 1 + \sigma = \frac{-(t - t')^2 + r^2 + r'^2}{2rr'}. \quad (\text{A7})$$

Both σ and χ are invariant under the $\text{SL}(2)$ isometry group of AdS_2 corresponding to the invariance of the metric (A4) under dilations $x^i \rightarrow \lambda x^i$, time shift $t \rightarrow t + \text{const}$, and conformal inversions $x^i \rightarrow x^i/x_j x^j$.

APPENDIX B: ARETAKIS NUMBER

Consider Eq. (4) which we write here as

$$\left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2} - \frac{\ell(\ell+1)}{r^2} \right) \psi_\ell = 0. \quad (\text{B1})$$

One can recognize the above equation as the Klein-Gordon in AdS_2 with mass square $\ell(\ell+1)$. Such a mass is the

result of a massless scale on $\text{AdS}_2 \times \text{S}_2$. Indeed, the $\text{AdS}_2 \times \text{S}_2$ spacetime has the metric

$$\begin{aligned} ds^2 &= g_{ij} dx^i dx^j + d\Omega_2^2 \\ &= \frac{-dt^2 + dr^2}{r^2} + d\Omega_2^2 \quad (i, j = 0, 1), \end{aligned} \quad (\text{B2})$$

for which the AdS_2 part has inverse $g^{ij} = \text{diag}(-r^2, r^2)$ and $\sqrt{-g} = 1/r^2$. In AdS_2 we have

$$\begin{aligned} \square_{\text{AdS}_2} &= \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j) \\ &= r^2 \left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2} \right). \end{aligned} \quad (\text{B3})$$

Equation (B1) is written as

$$r^2 \left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2} \right) \psi_\ell - \ell(\ell+1) \psi_\ell = 0. \quad (\text{B4})$$

In Poincaré coordinates, the AdS_2 metric can be written as

$$ds^2 = -R^2 dT^2 + \frac{dR^2}{R^2}, \quad (\text{B5})$$

for which $g^{ij} = \text{diag}(-1/R^2, R^2)$ and $\sqrt{-g} = 1$. In these coordinates we have

$$\begin{aligned} \square_{\text{AdS}_2} &= \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j) = \partial_R (R^2 \partial_R) - \frac{1}{R^2} \partial_T^2 \\ &= \frac{1}{R^2} [R^2 \partial_R (R^2 \partial_R) - \partial_T^2]. \end{aligned} \quad (\text{B6})$$

Setting $R = 1/r$ and $T = t$, one has

$$dR = -\frac{dr}{r^2} \quad \text{and} \quad R^2 \partial_R = -\partial_r, \quad (\text{B7})$$

and

$$\frac{1}{R^2} [R^2 \partial_R (R^2 \partial_R) - \partial_T^2] = r^2 (\partial_r^2 - \partial_t^2). \quad (\text{B8})$$

We define

$$u = T + \frac{1}{R} = t + r \quad \text{and} \quad v = T - \frac{1}{R} = t - r, \quad (\text{B9})$$

for which $dT = dv - dR/R^2$, and the metric becomes

$$\begin{aligned} ds^2 &= -R^2 dT^2 + \frac{dR^2}{R^2} = -R^2 \left(dv^2 + \frac{dR^2}{R^4} - 2dv \frac{dR}{R^2} \right) \\ &\quad + \frac{dR^2}{R^2} = -R^2 dv^2 + 2dv dR. \end{aligned} \quad (\text{B10})$$

The equation for a massive scalar field of mass square $m^2 = \ell(\ell + 1)$ becomes

$$2\partial_v \partial_R \psi_\ell + \partial_R (R^2 \partial_R \psi_\ell) - \ell(\ell + 1) \psi_\ell = 0. \quad (\text{B11})$$

Operating both sides with ∂_R^ℓ , one encounters

$$\partial_R^{\ell+1} (R^2 \partial_R \psi_\ell) = \sum_{k=0}^{\ell+1} \binom{\ell+1}{k} \partial_R^{\ell+1-k} (R^2) \partial_R^{k+1} \psi_\ell. \quad (\text{B12})$$

For $R \simeq 0$, the sum does not vanish only for $(\ell + 1 - k) = 2$, and one has

$$\begin{aligned} \partial_R^{\ell+1} (R^2 \partial_R \psi_\ell) &\simeq 2 \binom{\ell+1}{\ell-1} \partial_R^\ell \psi_\ell \\ &= \ell(\ell + 1) \partial_R^\ell \psi_\ell, \end{aligned} \quad (\text{B13})$$

from which we deduce the Aretakis constant [80]

$$\partial_v \partial_R^{\ell+1} \psi_\ell|_{R=0} = 0 \Rightarrow \mathcal{H}_\ell \equiv \partial_R^{\ell+1} \psi_\ell|_{R=0} = \text{const.} \quad (\text{B14})$$

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