

Role of $a_0(980)$ in the decays $D^0 \rightarrow K^+ K^- \eta$ and $\pi^+ \pi^- \eta$

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In the present work, we study the reactions $D^0 \rightarrow K^+ K^- \eta$ and $D^0 \rightarrow \pi^+ \pi^- \eta$, and find that the $a_0(980)$ state plays a dominant role. At the quark level, the external and internal W -emission mechanisms are taken into account, which can hadronize into the final states, and then the $a_0(980)$ state is generated from the final state interaction. In addition, the contributions of other intermediate resonances, such as $\rho(770)$ and $\phi(1020)$, are also considered. We make a combined fit of the invariant mass spectra measured by the Belle and BESIII Collaborations, where the results are in good agreement with the experiments, and the signal of $a_0(980)$ shows great significance. Furthermore, the antisymmetry data for the production of $a_0(980)^+$ and $a_0(980)^-$ are described well in the combined fit.

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I. INTRODUCTION

Even though the states $a_0(980)$ and $f_0(980)$ were found a long time ago [1–3], the nature and structure of them remain topics of active research and debate in particle physics; see more discussions in the review of the Particle Data Group (PDG) [4] for the meson resonances below 1 GeV. As already known, they exhibit unique properties that challenge conventional understanding. The $a_0(980)$ state is an isovector ($I = 1$), while $f_0(980)$ is an isoscalar ($I = 0$), of which their masses are approximately 980 MeV and just below the $K\bar{K}$ threshold, leading to various interpretations concerning their properties. In the quark model, these resonances might be viewed as conventional scalar $q\bar{q}$ states [5–7]. However, the low masses of $a_0(980)$ and $f_0(980)$ raise questions about their compatibility with this picture, particularly since the expected mass of the lightest scalar $q\bar{q}$ state, the σ or $f_0(500)$, is significantly different. On the other hand, some theoretical points of view suggested these particles could be tetraquarks composed of four quarks ($qq\bar{q}\bar{q}$) [8–10], which could explain some of their unusual properties and decay patterns. An

alternative and increasingly popular interpretation was that $a_0(980)$ and $f_0(980)$ could be considered as molecular states, composed primarily of $K\bar{K}$ pairs [11–15]; see more discussions in the review [16]. This interpretation is supported by the proximity of their masses to the $K\bar{K}$ threshold and a strong coupling to the $K\bar{K}$ channel. Furthermore, some models proposed that $a_0(980)$ and $f_0(980)$ could possess more complex structures involving contributions from other resonances, or even nonperturbative effects that lead to a more nuanced picture than a simple quark pair or a molecular model [17,18]. Thus, many experimental efforts, including the analysis of their decay modes, scattering experiments, and resonant production mechanisms in various reactions, aim to clarify the nature of $a_0(980)$ and $f_0(980)$. Moreover, the measurements of branching ratios in different decay modes, studying how these states couple to different final states and probing their production in high-energy collisions can help distinguish between different theoretical frameworks. Their productions in the ϕ meson radiative decays were searched for in the decays $\phi \rightarrow \gamma\pi^+\pi^-$ [19], $\phi \rightarrow \gamma\pi^0\pi^0$ [20], and $\phi \rightarrow \gamma\pi^0\eta$ [21], with the corresponding branching ratios measured. Analogous productions in the J/ψ meson radiative decays were done in the decays $J/\psi \rightarrow \gamma\pi^0\pi^0$ [22], $J/\psi \rightarrow \gamma\pi^0\eta$ [23], $J/\psi \rightarrow \gamma(\pi\pi, K\bar{K}, \eta\eta, \omega\phi)$ [24], and so on, where the resonant signal was found. In the hadronic decay modes, the production of $a_0(980)$ and $f_0(980)$ can be easy in some certain decay processes. Three-body decays of charmed mesons, such as those involving D mesons, represent a good opportunity to

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explore low-lying scalar resonances, since the presence of three mesons in the final state provides a fertile ground for studying the properties of various resonances, including the $a_0(980)$ and $f_0(980)$ states. The LHCb Collaboration employed a Dalitz plot method to analyze information about the resonance contributions, such as $a_0(980)$, $f_0(980)$, $\phi(1020)$, and $f_0(1370)$ in the decay $D^+ \rightarrow K^- K^+ K^+$ [25], which was a doubly Cabibbo-suppressed process. Reference [26] reported the results of the first measurements of the absolute branching fractions for 14 hadronic $D^{0(+)}$ decays with η in the exclusive final states, for example, $D^0 \rightarrow K^+ K^- \eta$, $K_S^0 K_S^0 \eta$, and so on. Aiming at hinting the CP violation effect, the singly Cabibbo-suppressed decays $D^0 \rightarrow K^+ K^- \eta$, $\pi^+ \pi^- \eta$, $\phi \eta$ were investigated in Ref. [27], where the branching fractions and CP asymmetries were measured, and the invariant mass distributions were also given with the resonance signals, such as the $a_0(980)$, appearing. The CP asymmetries were also tested in the decays $D^0 \rightarrow \pi^+ \pi^- \eta$ and $D^+ \rightarrow \pi^+ \pi^0 \eta$ with the absolute branching fractions measured [28], where no CP violation was found and which were measured for first time by the CLEO Collaboration [29]. Recently, the BESIII Collaboration reported the updated results of the first amplitude analyses for the decays $D^{0(+)} \rightarrow \pi^+ \pi^{-(0)} \eta$ in Ref. [30], where they found that the production of $a_0(980)^+$ was much larger than the one of $a_0(980)^{-(0)}$ in the intermediate processes of these two decays, indicating final state isospin symmetry breaking, and the contribution of the $\rho^{0(+)}$ resonances was important. Furthermore, $a_0(980)$ and $f_0(980)$ can also be reproduced in the certain final state interactions of three-body hadronic decays of the D_s , $B_{(s)}$ mesons, and of course, some other D meson decay processes, too; see more results in PDG [4].

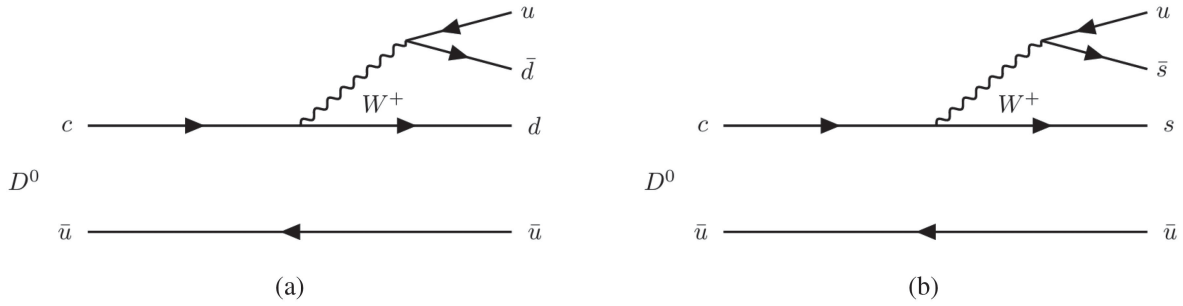
On the theoretical side, there are also many works with different models or approaches devoting great effort to shed light on the dynamics of strong interactions and the resonance nature. Using a covariant quark model, the properties of the $f_0(980)$ resonance were investigated in Ref. [31] in the decays $D_{(s)} \rightarrow f_0(980) \pi$, $f_0(980) K$. Within the perturbative QCD approach, Ref. [32] systematically analyzed the charmless decays $B_{(s)} \rightarrow V \pi \pi$ with V representing vector mesons, where the branching fractions were evaluated by considering the $f_0(980)$ resonance contribution. In Ref. [33], the decay $D^+ \rightarrow K^- K^+ K^+$ was investigated using the isobar model alongside the coupled-channel K -matrix approach, allowing for a detailed examination of the $a_0(980)$ and $f_0(980)$ resonance contributions. This double Cabibbo-suppressed decay, $D^+ \rightarrow K^- K^+ K^+$, was also examined by exploring the N/D method in Ref. [34], where the data of the invariant mass distributions were well fitted. Applying a coupled-channel framework that accounted for the final state interactions, Ref. [35] analyzed the decay $D^+ \rightarrow K^- \pi^+ \pi^+$, which was also studied in Refs. [36,37] with the dispersion theory based

on the Khuri-Treiman formalism. Furthermore, based on the final state interaction from the chiral unitary approach (ChUA) [13,38–40], the results of Ref. [41] indicated that the resonance contributions from $f_0(500)$, $f_0(980)$, and $a_0(980)$ in the $\pi^+ \pi^-$ and $\pi^0 \eta$ invariant mass distributions of the decays $D^0 \rightarrow \bar{K}^0 \pi^+ \pi^-$ and $D^0 \rightarrow \bar{K}^0 \pi^0 \eta$, respectively, were important, where an obtained ratio of the branching fractions was consistent with the experimental measurement within the uncertainties. With similar formalism, the singly Cabibbo-suppressed processes $D^0 \rightarrow \pi^0 \pi^0 \pi^0$, $\pi^0 \pi^0 \eta$, $\pi^0 \eta \eta$ were studied in Ref. [42], where the resonances $f_0(500)$, $f_0(980)$, and $a_0(980)$ were dynamically reproduced in the final state interactions based on the ChUA, and the data of the $\pi^0 \eta$ invariant mass spectrum of the decay $D^0 \rightarrow \pi^0 \eta \eta$, measured by the BESIII Collaboration [43], were described well.

We know that the isobar model can describe the resonant contributions well, but its resonance parameters do not provide much information on the resonant properties. It looks like the K -matrix formalism is a bit better since its kernels can contain the bare poles of the resonances, and it imposes the two-body unitarity constraint compared to the isobar model, whereas there are some approximations that deal with the particle propagators in the loops of the K -matrix amplitudes [33], where the resonant interpretation is still missing. The Khuri-Treiman formalism primarily employs dispersive relationships based on the principles of analyticity, unitarity, and crossing symmetry to parametrize the scattering amplitudes of three-body decays at low-energy dynamics [36], where the scattering phase shifts should be taken as input, and the asymptotic behavior of the related phase shifts in the high-energy region is unknown and needs to make an assumption for it in the calculation of the inhomogeneities. For the ChUA, one can dynamically reproduce the resonances from the two-body interaction scattering amplitudes, where the scattering information and the decay properties of the resonant poles can be obtained. Thus, in the present work, aiming at understanding the production properties of $a_0(980)$ and $f_0(980)$, we analyze the decays $D^0 \rightarrow K^+ K^- \eta$ and $D^0 \rightarrow \pi^+ \pi^- \eta$ based on the experimental measurements of Refs. [27,30] applying the final state interaction framework within the ChUA. Our manuscript is organized as follows: In the next section, our formalism of the final state interaction is presented. Then, the results for the decays $D^0 \rightarrow K^+ K^- \eta$ and $D^0 \rightarrow \pi^+ \pi^- \eta$ are discussed in detail. Finally, we provide a short summary.

II. FORMALISM

In the present work, we investigate the weak decay process of the $D^0 \rightarrow K^+ K^- \eta$ and $\pi^+ \pi^- \eta$ reactions based on the ChUA for the final state interaction. First, the resonance $a_0(980)$ is dynamically generated by the S -wave pseudo-scalar-pseudoscalar meson interaction. Then, the states ϕ


 FIG. 1. External W -emission mechanism for the processes (a) $c \rightarrow W^+ d$ and (b) $c \rightarrow W^+ s$.

and ρ are contributed from the P wave, which are calculated with the Breit-Wigner amplitudes as done in Refs. [44–46]. Finally, the double differential widths of the $D^0 \rightarrow K^+ K^- \eta$ and $\pi^+ \pi^- \eta$ processes are evaluated and fitted to the experimental data. It should be mentioned that the goal of Refs. [27,28] in the experiments was to hint the CP violation, which is not the motivation of the present work. We just focus on how the molecular states produce in the final state interactions. Note that the final state interaction at the hadron level was introduced in the formalism to investigate the CP violation in Refs. [47,48]. In our formalism, we just assume the short-distance part of the weak interaction is unknown parameters, which are determined by the fitting of experimental data; the details will be provided later. Thus, the essential information of the weak phases for the CP violation is lacking in our case, which in fact is done by the QCD factorization approach; see more detail in Refs. [49–51].

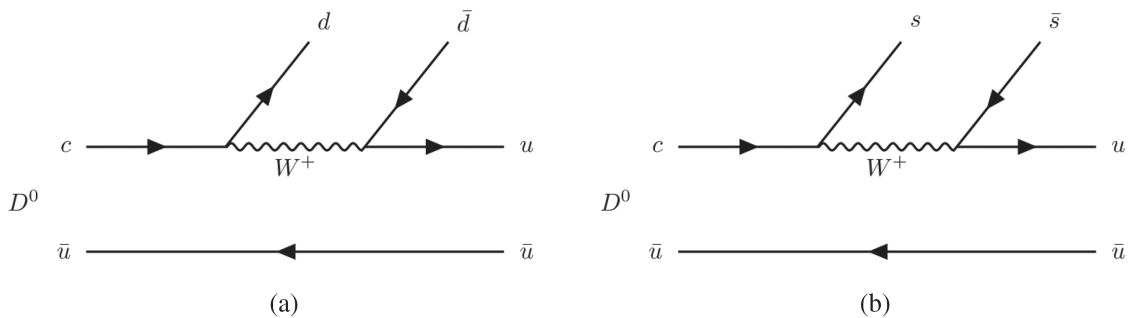
There are several cases of weak decay topography contributions [52–55], and only the dominant mechanisms for the external and internal W emission are considered, which are depicted in Figs. 1 and 2. Note that these diagrams can all contribute to the two decays that we are concerned about. For the W -external emission at the quark level, as shown in Fig. 1, the quark c of the initial meson D^0 can decay into a W^+ boson and $d(s)$ quark, while the $\bar{u}$ quark of the D^0 meson remains as a spectator. Then, there are two cases for the hadronization procedures. The first one is that, the quark pair $u\bar{d}(u\bar{s})$ created by the W boson can form the $\pi^+(K^+)$ meson directly, and the other

pairs $d\bar{u}(s\bar{u})$ generate two final states through hadronization with quark pairs $q\bar{q} = u\bar{u} + d\bar{d} + s\bar{s}$ produced from vacuum. The second one is that, the quark pairs $d\bar{u}(s\bar{u})$ become $\pi^-(K^-)$, and the other quark pair $u\bar{d}(u\bar{s})$ undergoes hadronization. Then, the corresponding decay procedures for these hadronizations can be written as

$$\begin{aligned} H^{(1a)} &= V_P V_{cd} V_{ud} \{ (u\bar{d} \rightarrow \pi^+) [d\bar{u} \rightarrow d\bar{u}(u\bar{u} + d\bar{d} + s\bar{s})] \\ &\quad + (d\bar{u} \rightarrow \pi^-) [u\bar{d} \rightarrow u\bar{d}(u\bar{u} + d\bar{d} + s\bar{s})] \} \\ &= V_P V_{cd} V_{ud} \{ (u\bar{d} \rightarrow \pi^+) [M_{21} \rightarrow (M \cdot M)_{21}] \\ &\quad + (d\bar{u} \rightarrow \pi^-) [(M \cdot M)_{12}] \}, \end{aligned} \quad (1)$$

$$\begin{aligned} H^{(1b)} &= V'_P V_{cs} V_{us} (u\bar{s} \rightarrow K^+) \{ s\bar{u} \rightarrow s\bar{u}(u\bar{u} + d\bar{d} + s\bar{s}) \} \\ &\quad + (s\bar{u} \rightarrow K^-) [u\bar{s} \rightarrow u\bar{s}(u\bar{u} + d\bar{d} + s\bar{s})] \} \\ &= V'_P V_{cs} V_{us} \{ (u\bar{s} \rightarrow K^+) [M_{31} \rightarrow (M \cdot M)_{31}] \\ &\quad + (s\bar{u} \rightarrow K^-) [M_{13} \rightarrow (M \cdot M)_{13}] \}. \end{aligned} \quad (2)$$

The W -internal emission mechanism is shown in Fig. 2. Similarly, the quark c of the initial meson D^0 can decay into a W^+ boson and $d(s)$ quark, while the $\bar{u}$ quark of the D^0 meson remains as a spectator. But, the two cases of hadronization processes occur between two quark pairs of $u\bar{u}$ and $d\bar{d}(s\bar{s})$. Analogously, these hadronization processes can be written as


 FIG. 2. Internal W -emission mechanism for the processes (a) $c \rightarrow W^+ d$ and (b) $c \rightarrow W^+ s$.

$$\begin{aligned}
H^{(2a)} &= \beta V_P V_{cd} V_{ud} \left[\left(d\bar{d} \rightarrow -\frac{1}{\sqrt{2}}\pi^0 \right) [u\bar{u} \rightarrow u\bar{u}(u\bar{u} + d\bar{d} + s\bar{s})] + \left(d\bar{d} \rightarrow \frac{1}{\sqrt{6}}\eta \right) [u\bar{u} \rightarrow u\bar{u}(u\bar{u} + d\bar{d} + s\bar{s})] \right. \\
&\quad \left. + \left(u\bar{u} \rightarrow \frac{1}{\sqrt{2}}\pi^0 \right) [d\bar{d} \rightarrow d\bar{d}(u\bar{u} + d\bar{d} + s\bar{s})] + \left(u\bar{u} \rightarrow \frac{1}{\sqrt{6}}\eta \right) [d\bar{d} \rightarrow d\bar{d}(u\bar{u} + d\bar{d} + s\bar{s})] \right] \\
&= \beta V_P V_{cd} V_{ud} \left[\left(d\bar{d} \rightarrow -\frac{1}{\sqrt{2}}\pi^0 \right) [M_{11} \rightarrow (M \cdot M)_{11}] + \left(d\bar{d} \rightarrow \frac{1}{\sqrt{6}}\eta \right) [M_{11} \rightarrow (M \cdot M)_{11}] \right. \\
&\quad \left. + \left(u\bar{u} \rightarrow \frac{1}{\sqrt{2}}\pi^0 \right) [M_{22} \rightarrow (M \cdot M)_{22}] + \left(u\bar{u} \rightarrow \frac{1}{\sqrt{6}}\eta \right) [M_{22} \rightarrow (M \cdot M)_{22}] \right], \tag{3}
\end{aligned}$$

$$\begin{aligned}
H^{(2b)} &= \beta V'_P V_{cs} V_{us} \left[\left(s\bar{s} \rightarrow -\frac{2}{\sqrt{6}}\eta \right) [u\bar{u} \rightarrow u\bar{u}(u\bar{u} + d\bar{d} + s\bar{s})] + \left(u\bar{u} \rightarrow \frac{1}{\sqrt{2}}\pi^0 \right) [s\bar{s} \rightarrow s\bar{s}(u\bar{u} + d\bar{d} + s\bar{s})] \right. \\
&\quad \left. + \left(u\bar{u} \rightarrow \frac{1}{\sqrt{6}}\eta \right) [s\bar{s} \rightarrow s\bar{s}(u\bar{u} + d\bar{d} + s\bar{s})] \right] \\
&= \beta V'_P V_{cs} V_{us} \left[\left(s\bar{s} \rightarrow -\frac{2}{\sqrt{6}}\eta \right) [M_{11} \rightarrow (M \cdot M)_{11}] + \left(u\bar{u} \rightarrow \frac{1}{\sqrt{2}}\pi^0 \right) [M_{33} \rightarrow (M \cdot M)_{33}] \right. \\
&\quad \left. + \left(u\bar{u} \rightarrow \frac{1}{\sqrt{6}}\eta \right) [M_{33} \rightarrow (M \cdot M)_{33}] \right], \tag{4}
\end{aligned}$$

where V_P and V'_P are the vertex factors of the weak interaction strength, which are assumed to be constants [45,56–58]. β is the relative weight coefficient between the internal and external W -emission mechanisms [59,60]. The factors $-1/\sqrt{2}$, $1/\sqrt{2}$, $1/\sqrt{6}$, and $-2/\sqrt{6}$ come from the flavor components of π^0 and η , which are given as $|\pi^0\rangle = \frac{1}{\sqrt{2}}|u\bar{u} - d\bar{d}\rangle$ and $|\eta\rangle = \frac{1}{\sqrt{6}}|u\bar{u} + d\bar{d} - 2s\bar{s}\rangle$. $V_{q_1 q_2}$ is the element of the Cabibbo-Kabayashi-Maskawa (CKM) matrix for the transition of the quark $q_1 \rightarrow q_2$, and the matrix M is the $SU(3)$ matrix of the pseudoscalar mesons given by

$$M = \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}, \tag{5}$$

where we take $\eta \equiv \eta_8$ as done in Ref. [44]. Then, the hadronization processes at the quark level in Eqs. (1)–(4) can be rewritten at the hadron level as

$$\begin{aligned}
(M \cdot M)_{11} &= \frac{1}{2}\pi^0\pi^0 + \frac{1}{6}\eta\eta + \frac{1}{\sqrt{3}}\pi^0\eta + \pi^+\pi^- + K^+K^-, \\
(M \cdot M)_{12} &= \frac{2}{\sqrt{6}}\pi^+\eta + K^+\bar{K}^0, \\
(M \cdot M)_{13} &= \frac{1}{\sqrt{2}}\pi^0K^+ - \frac{1}{\sqrt{6}}K^+\eta + \pi^+K^0, \\
(M \cdot M)_{21} &= \frac{2}{\sqrt{6}}\pi^-\eta + K^0K^-, \\
(M \cdot M)_{22} &= \pi^+\pi^- + \frac{1}{2}\pi^0\pi^0 + \frac{1}{6}\eta\eta - \frac{1}{\sqrt{3}}\pi^0\eta + K^0\bar{K}^0, \\
(M \cdot M)_{31} &= \frac{1}{\sqrt{2}}K^-\pi^0 - \frac{1}{\sqrt{6}}\eta K^- + \bar{K}^0\pi^-, \\
(M \cdot M)_{33} &= K^+K^- + K^0\bar{K}^0 + \frac{2}{3}\eta\eta. \tag{6}
\end{aligned}$$

Therefore, one can get all the possible final states with $\pi^+(\pi^-)$, $K^+(K^-)$, π^0 , and η directly produced after the hadronization,

$$H^{(1a)} = V_P V_{cd} V_{ud} \left(\frac{4}{\sqrt{6}} \pi^+ \pi^- \eta + \pi^+ K^0 K^- + \pi^- K^+ \bar{K}^0 \right), \quad (7)$$

$$H^{(1b)} = V'_P V_{cs} V_{us} \left(\frac{2}{\sqrt{2}} K^+ K^- \pi^0 - \frac{2}{\sqrt{6}} \eta K^+ K^- + K^+ \bar{K}^0 \pi^- + \pi^+ K^0 K^- \right), \quad (8)$$

$$H^{(2a)} = \beta V_P V_{cd} V_{ud} \left(\frac{1}{3\sqrt{6}} \eta \eta \eta - \frac{1}{\sqrt{2}} \pi^0 K^+ K^- + \frac{2}{\sqrt{6}} \pi^+ \pi^- \eta + \frac{1}{\sqrt{6}} K^+ K^- \eta \right. \\ \left. - \frac{1}{\sqrt{6}} \pi^0 \pi^0 \eta + \frac{1}{\sqrt{2}} \pi^0 K^0 \bar{K}^0 + \frac{1}{\sqrt{6}} K^0 \bar{K}^0 \eta \right), \quad (9)$$

$$H^{(2b)} = \beta V'_P V_{cs} V_{us} \left(-\frac{1}{\sqrt{6}} \pi^0 \pi^0 \eta + \frac{1}{3\sqrt{6}} \eta \eta \eta - \frac{2}{\sqrt{6}} \pi^+ \pi^- \eta - \frac{1}{\sqrt{6}} K^+ K^- \eta \right. \\ \left. + \frac{1}{\sqrt{2}} K^+ K^- \pi^0 + \frac{1}{\sqrt{2}} \pi^0 K^0 \bar{K}^0 + \frac{1}{\sqrt{6}} K^0 \bar{K}^0 \eta \right). \quad (10)$$

Because the elements of the CKM matrix have the relationship $V_{cd} V_{ud} = -V_{us} V_{cs}$ [4], we can obtain the total contributions in the S wave of the weak decay processes $D^0 \rightarrow \pi^+ \pi^- \eta$ and $K^+ K^- \eta$, having

$$H = H^{(1a)} + H^{(1b)} + H^{(2a)} + H^{(2b)} \\ = \frac{2}{\sqrt{6}} (2V_P + \beta V_P + \beta V'_P) V_{cd} V_{ud} \pi^+ \pi^- \eta + (V_P - V'_P) V_{cd} V_{ud} \pi^+ K^0 K^- + (V_P - V'_P) V_{cd} V_{ud} \pi^- K^+ \bar{K}^0 \\ - \frac{1}{\sqrt{2}} (2V'_P + \beta V_P + \beta V'_P) V_{cd} V_{ud} K^+ K^- \pi^0 + \frac{1}{\sqrt{6}} (2V'_P + \beta V_P + \beta V'_P) V_{cd} V_{ud} K^+ K^- \eta \\ + \frac{1}{3\sqrt{6}} \beta (V_P - V'_P) V_{cd} V_{ud} \eta \eta \eta + \frac{1}{\sqrt{6}} \beta (V'_P - V_P) V_{cd} V_{ud} \pi^0 \pi^0 \eta + \frac{1}{\sqrt{2}} \beta (V_P - V'_P) V_{cd} V_{ud} \pi^0 K^0 \bar{K}^0 \\ + \frac{1}{\sqrt{6}} \beta (V_P - V'_P) V_{cd} V_{ud} K^0 \bar{K}^0 \eta \\ = \frac{2}{\sqrt{6}} (2C_1 + \beta C_1 + \beta C_2) \pi^+ \pi^- \eta + (C_1 - C_2) \pi^+ K^0 K^- + (C_1 - C_2) \pi^- K^+ \bar{K}^0 \\ - \frac{1}{\sqrt{2}} (2C_2 + \beta C_1 + \beta C_2) K^+ K^- \pi^0 + \frac{1}{\sqrt{6}} (2C_2 + \beta C_1 + \beta C_2) K^+ K^- \eta \\ + \frac{1}{3\sqrt{6}} \beta (C_1 - C_2) \eta \eta \eta + \frac{1}{\sqrt{6}} \beta (C_2 - C_1) \pi^0 \pi^0 \eta + \frac{1}{\sqrt{2}} \beta (C_1 - C_2) \pi^0 K^0 \bar{K}^0 + \frac{1}{\sqrt{6}} \beta (C_1 - C_2) K^0 \bar{K}^0 \eta, \quad (11)$$

where the factors C_1 and C_2 are defined as $C_1 = V_P V_{cd} V_{ud}$ and $C_2 = V'_P V_{cd} V_{ud}$, which are also absorbed the normalization factor of the data events and determined by fitting the experimental data later.

Note that the terms $K^+ K^- \eta$ and $\pi^+ \pi^- \eta$ can not only be obtained from the final states on the tree-level

hadronization processes, but also produced through the rescattering procedures, as depicted in Figs. 3 and 4. Thus, after considering the rescattering processes, the full amplitude for the decays $D^0 \rightarrow K^+ K^- \eta / \pi^+ \pi^- \eta$ in the S wave can be obtained as follows:

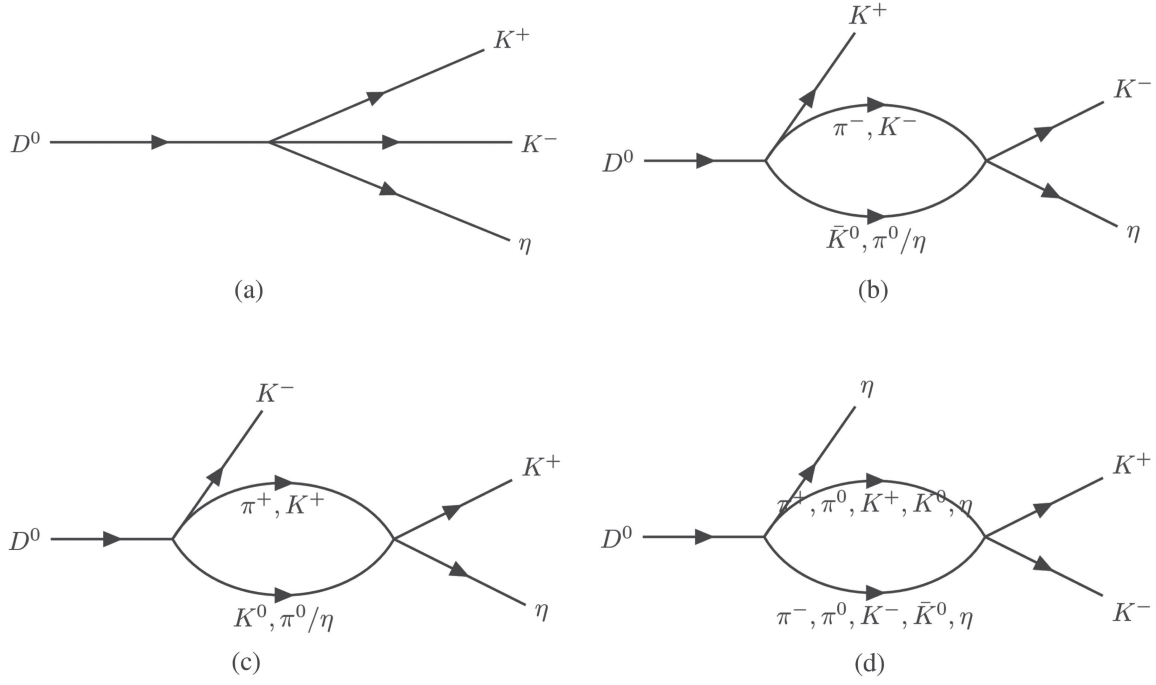


FIG. 3. Diagrammatic representations of rescattering for the decay $D^0 \rightarrow K^+ K^- \eta$. (a) Tree-level production. (b) Rescattering of $\pi^- \bar{K}^0$, $K^- \pi^0$, and $K^- \eta$. (c) Rescattering of $\pi^+ K^0$, $K^+ \pi^0$, and $K^+ \eta$. (d) Rescattering of $\pi^+ \pi^-$, $\pi^0 \pi^0$, $K^+ K^-$, $K^0 \bar{K}^0$, and $\eta \eta$.

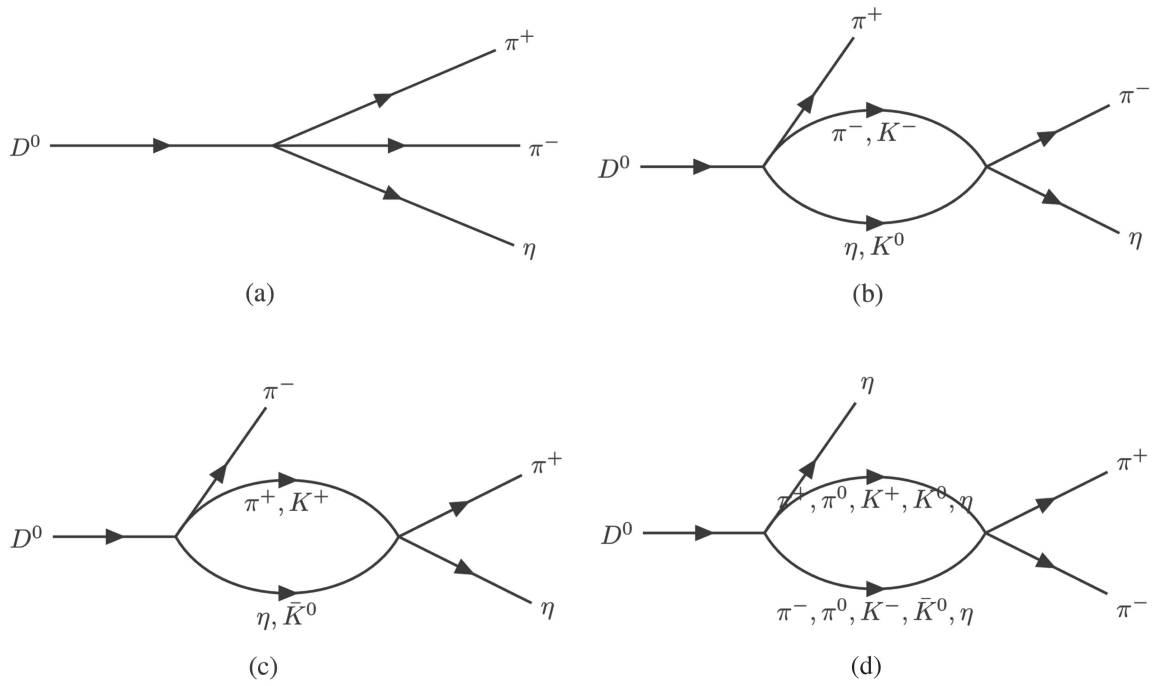


FIG. 4. Diagrammatic representations of rescattering for the decay $D^0 \rightarrow \pi^+ \pi^- \eta$. (a) Tree-level production. (b) Rescattering of $\pi^- \bar{K}^0$ and $K^- \pi^0$. (c) Rescattering of $\pi^+ \eta$ and $K^+ \bar{K}^0$. (d) Rescattering of $\pi^+ \pi^-$, $\pi^0 \pi^0$, $K^+ K^-$, $K^0 \bar{K}^0$, and $\eta \eta$.

$$\begin{aligned}
t_{D^0 \rightarrow K^+ K^- \eta}(s_{12}, s_{23}) = & \frac{2}{\sqrt{6}}(2C_1 + \beta C_1 + \beta C_2)G_{\pi^+ \pi^-}(s_{12})T_{\pi^+ \pi^- \rightarrow K^+ K^-}(s_{12}) + \frac{1}{\sqrt{6}}(2C_2 + \beta C_1 + \beta C_2) \\
& + \frac{1}{\sqrt{6}}(2C_2 + \beta C_1 + \beta C_2)G_{K^+ K^-}(s_{12})T_{K^+ K^- \rightarrow K^+ K^-}(s_{12}) + \frac{1}{3\sqrt{6}}\beta(C_1 - C_2)G_{\eta\eta}(s_{12})T_{\eta\eta \rightarrow K^+ K^-}(s_{12}) \\
& - \frac{1}{\sqrt{6}}\beta(C_1 - C_2)G_{\pi^0 \pi^0}(s_{12})T_{\pi^0 \pi^0 \rightarrow K^+ K^-}(s_{12}) + \frac{1}{\sqrt{6}}\beta(C_1 - C_2)G_{K^0 \bar{K}^0}(s_{12})T_{K^0 \bar{K}^0 \rightarrow K^+ K^-}(s_{12}) \\
& + (C_1 - C_2)G_{\pi^- \bar{K}^0}(s_{23})T_{\pi^- \bar{K}^0 \rightarrow K^- \eta}(s_{23}) - \frac{1}{\sqrt{2}}(2C_2 + \beta C_1 + \beta C_2)G_{K^- \pi^0}(s_{23})T_{K^- \pi^0 \rightarrow K^- \eta}(s_{23}) \\
& + \frac{1}{\sqrt{6}}(2C_2 + \beta C_1 + \beta C_2)G_{K^- \eta}(s_{23})T_{K^- \eta \rightarrow K^- \eta}(s_{23}) + (C_1 - C_2)G_{\pi^+ K^0}(s_{13})T_{\pi^+ K^0 \rightarrow K^+ \eta}(s_{13}) \\
& - \frac{1}{\sqrt{2}}(2C_2 + \beta C_1 + \beta C_2)G_{K^+ \pi^0}(s_{13})T_{K^+ \pi^0 \rightarrow K^+ \eta}(s_{13}) + \frac{1}{\sqrt{6}}(2C_2 + \beta C_1 + \beta C_2)G_{K^+ \eta}(s_{13})T_{K^+ \eta \rightarrow K^+ \eta}(s_{13}),
\end{aligned} \tag{12}$$

$$\begin{aligned}
t_{D^0 \rightarrow \pi^+ \pi^- \eta}(s_{12}, s_{23}) = & \frac{2}{\sqrt{6}}(2C_1 + \beta C_1 + \beta C_2) + \frac{2}{\sqrt{6}}(2C_1 + \beta C_1 + \beta C_2)G_{\pi^+ \pi^-}(s_{12})T_{\pi^+ \pi^- \rightarrow \pi^+ \pi^-}(s_{12}) \\
& + \frac{1}{\sqrt{6}}(2C_2 + \beta C_1 + \beta C_2)G_{K^+ K^-}(s_{12})T_{K^+ K^- \rightarrow \pi^+ \pi^-}(s_{12}) + \frac{1}{3\sqrt{6}}\beta(C_1 - C_2)G_{\eta\eta}(s_{12})T_{\eta\eta \rightarrow \pi^+ \pi^-}(s_{12}) \\
& - \frac{1}{\sqrt{6}}\beta(C_1 - C_2)G_{\pi^0 \pi^0}(s_{12})T_{\pi^0 \pi^0 \rightarrow \pi^+ \pi^-}(s_{12}) + \frac{1}{\sqrt{6}}\beta(C_1 - C_2)G_{K^0 \bar{K}^0}(s_{12})T_{K^0 \bar{K}^0 \rightarrow \pi^+ \pi^-}(s_{12}) \\
& + \frac{2}{\sqrt{6}}(2C_1 + \beta C_1 + \beta C_2)G_{\pi^- \eta}(s_{23})T_{\pi^- \eta \rightarrow \pi^- \eta}(s_{23}) + (C_1 - C_2)G_{K^0 K^-}(s_{23})T_{K^0 K^- \rightarrow \pi^- \eta}(s_{23}) \\
& + \frac{2}{\sqrt{6}}(2C_1 + \beta C_1 + \beta C_2)G_{\pi^+ \eta}(s_{13})T_{\pi^+ \eta \rightarrow \pi^+ \eta}(s_{13}) + (C_1 - C_2)G_{K^+ \bar{K}^0}(s_{13})T_{K^+ \bar{K}^0 \rightarrow \pi^+ \eta}(s_{13}),
\end{aligned} \tag{13}$$

where s_{ij} is the energy of the two-body system in the center-of-mass frame, and the lower indices $i, j = 1, 2, 3$ represent the final states $K^+/\pi^+(1)$, $K^-/\pi^-(2)$, and $\eta(3)$, respectively. In addition, the $G_{pp'}$ is a loop function of the two meson propagator, which is logarithmically divergent. There are two methods to solve this problem: the three-momentum cutoff method [13,38,61,62] and the dimensional regularization method [40,63–65]. Of course, the choice of these two regularization schemes does not affect our final conclusions. In this work, we adopt the three-momentum cutoff method, and the function $G_{pp'}$ can be reexpressed as in Refs. [66,67], written as

$$G(s) = \frac{1}{16\pi^2 s} \left\{ \sigma \left(\arctan \frac{s + \Delta}{\sigma \lambda_1} + \arctan \frac{s - \Delta}{\sigma \lambda_2} \right) - \left[(s + \Delta) \ln \frac{(1 + \lambda_1)q_{\max}}{m_1} + (s - \Delta) \ln \frac{(1 + \lambda_2)q_{\max}}{m_2} \right] \right\}, \tag{14}$$

where $\sigma = [-(s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2)]^{1/2}$, $\Delta = m_1^2 - m_2^2$, and $\lambda_i = \sqrt{1 + m_i^2/q_{\max}^2}$ ($i = 1, 2$). $q_{\max}$ is the cutoff momentum, and we take $q_{\max} = 600$ MeV as used in Refs. [68–70]. Furthermore, the $T_{pp \rightarrow pp}$ is the two-body scattering amplitude, which can be obtained by solving the coupled-channel Bethe-Salpeter equation of the ChUA [38,63] given by

$$T = [1 - VG]^{-1}V, \tag{15}$$

where V is the potential matrix constructed by the scattering potentials for every coupled channel. For the $I = 0$ sector, there are five coupled channels: $\pi^+ \pi^-(1)$, $\pi^0 \pi^0(2)$, $K^+ K^-(3)$, $K^0 \bar{K}^0(4)$, $\eta\eta(5)$, and the elements of the 5×5 symmetric V matrix can be written as [45,56,64,69]

$$\begin{aligned}
V_{11} &= -\frac{1}{2f^2}s, & V_{12} &= -\frac{1}{\sqrt{2}f^2}(s - m_\pi^2), & V_{13} &= -\frac{1}{4f^2}s, \\
V_{14} &= -\frac{1}{4f^2}s, & V_{15} &= -\frac{1}{3\sqrt{2}f^2}m_\pi^2, & V_{22} &= -\frac{1}{2f^2}m_\pi^2, \\
V_{23} &= -\frac{1}{4\sqrt{2}f^2}s, & V_{24} &= -\frac{1}{4\sqrt{2}f^2}s, & V_{25} &= -\frac{1}{6f^2}m_\pi^2, \\
V_{33} &= -\frac{1}{2f^2}s, & V_{34} &= -\frac{1}{4f^2}s, & V_{35} &= -\frac{1}{12\sqrt{2}f^2}(9s - 6m_\eta^2 - 2m_\pi^2), \\
V_{44} &= -\frac{1}{2f^2}s, & V_{45} &= -\frac{1}{12\sqrt{2}f^2}(9s - 6m_\eta^2 - 2m_\pi^2), & V_{55} &= -\frac{1}{18f^2}(16m_K^2 - 7m_\pi^2).
\end{aligned} \tag{16}$$

For the $I = 1/2$ sector, there are three channels: $K^-\pi^+(1)$, $\bar{K}^0\pi^0(2)$, $\bar{K}^0\eta(3)$, and then the elements of the matrix V are given by [44,45,67]

$$\begin{aligned}
V_{11} &= \frac{-1}{6f^2} \left(\frac{3}{2}s - \frac{3}{2s}(m_\pi^2 - m_K^2)^2 \right), \\
V_{12} &= \frac{1}{2\sqrt{2}f^2} \left(\frac{3}{2}s - m_\pi^2 - m_K^2 - \frac{(m_\pi^2 - m_K^2)^2}{2s} \right), \\
V_{13} &= \frac{1}{2\sqrt{6}f^2} \left(\frac{3}{2}s - \frac{7}{6}m_\pi^2 - \frac{1}{2}m_\eta^2 - \frac{1}{3}m_K^2 + \frac{3}{2s}(m_\pi^2 - m_K^2)(m_\eta^2 - m_K^2) \right), \\
V_{22} &= \frac{-1}{4f^2} \left(-\frac{s}{2} + m_\pi^2 + m_K^2 - \frac{(m_\pi^2 - m_K^2)^2}{2s} \right), \\
V_{23} &= -\frac{1}{4\sqrt{3}f^2} \left(\frac{3}{2}s - \frac{7}{6}m_\pi^2 - \frac{1}{2}m_\eta^2 - \frac{1}{3}m_K^2 + \frac{3}{2s}(m_\pi^2 - m_K^2)(m_\eta^2 - m_K^2) \right), \\
V_{33} &= -\frac{1}{4f^2} \left(-\frac{3}{2}s - \frac{2}{3}m_\pi^2 + m_\eta^2 + 3m_K^2 - \frac{3}{2s}(m_\eta^2 - m_K^2)^2 \right).
\end{aligned} \tag{17}$$

In addition, for the $I = 1$ sector, three channels are coupled, $K^+K^-(3)$, $K^0\bar{K}^0$, $\pi^0\eta(3)$, and the elements of V can be obtained as follows [13]:

$$\begin{aligned}
V_{11} &= -\frac{1}{2f^2}s, & V_{12} &= -\frac{1}{4f^2}s, & V_{13} &= -\frac{\sqrt{3}}{12f^2} \left(3s - \frac{8}{3}m_K^2 - \frac{1}{3}m_\pi^2 - m_\eta^2 \right), \\
V_{22} &= -\frac{1}{2f^2}s, & V_{23} &= \frac{\sqrt{3}}{12f^2} \left(3s - \frac{8}{3}m_K^2 - \frac{1}{3}m_\pi^2 - m_\eta^2 \right), & V_{33} &= -\frac{1}{3f^2}m_\pi^2,
\end{aligned} \tag{18}$$

where f is the pion decay constant which is taken as 0.093 GeV [13,44], and m_p is the mass of the corresponding pseudoscalar meson. Meanwhile, with the isospin multiplets $K = (K^+, K^0)$, $\bar{K} = (\bar{K}^0, -K^-)$, and $\pi = (-\pi^+, \pi^0, \pi^-)$, we can obtain the other potentials with the isospin relationship, which are used in Eqs. (12) and (13), having

$$\begin{aligned}
t_{\pi^-\eta \rightarrow \pi^-\eta} &= t_{\pi^+\eta \rightarrow \pi^+\eta} = t_{\pi^0\eta \rightarrow \pi^0\eta}, \\
t_{K^0K^- \rightarrow \pi^-\eta} &= t_{K^+\bar{K}^0 \rightarrow \pi^+\eta} = \sqrt{2}t_{K^+K^- \rightarrow \pi^0\eta} = -\sqrt{2}t_{K^0\bar{K}^0 \rightarrow \pi^0\eta}, \\
t_{\bar{K}^0\pi^- \rightarrow K^-\eta} &= t_{K^0\pi^+ \rightarrow K^+\eta} = t_{K^-\pi^+ \rightarrow \bar{K}^0\eta} = -\sqrt{2}t_{\bar{K}^0\pi^0 \rightarrow \bar{K}^0\eta}, \\
t_{K^-\pi^0 \rightarrow K^-\eta} &= t_{K^+\pi^0 \rightarrow K^+\eta} = \frac{1}{\sqrt{2}}t_{K^-\pi^+ \rightarrow \bar{K}^0\eta} = -t_{\bar{K}^0\pi^0 \rightarrow \bar{K}^0\eta}, \\
t_{K^-\eta \rightarrow K^-\eta} &= t_{K^+\eta \rightarrow K^+\eta} = t_{\bar{K}^0\eta \rightarrow \bar{K}^0\eta}.
\end{aligned} \tag{19}$$

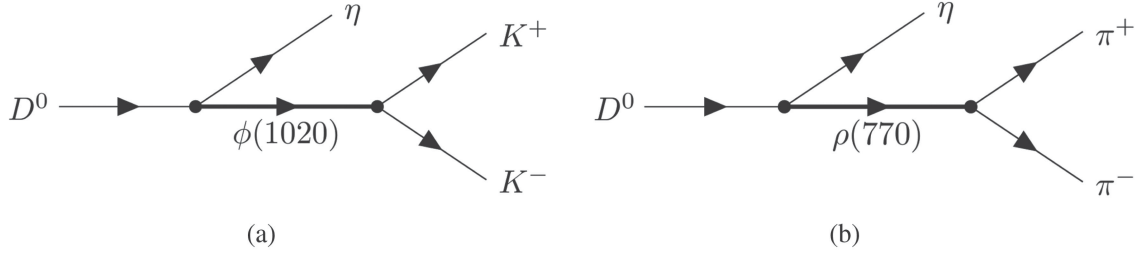


FIG. 5. Mechanisms for the $D^0 \rightarrow K^+ K^- \eta$ and $\pi^+ \pi^- \eta$ processes via the intermediate vector mesons. (a) Mechanism via $\phi(1020)$. (b) Mechanism via $\rho(770)$.

Furthermore, we also consider the contribution of the P -wave resonances. For the $D^0 \rightarrow K^+ K^- \eta$ processes, the case of W -internal emission as shown in Fig. 2, not only can obtain the $K^+ K^-$ final states in the S wave through hadronization, but also produce the $\phi(1020)$ meson in the P wave, which decays into $K^+ K^-$. Similarly, for the $D^0 \rightarrow \pi^+ \pi^- \eta$ processes, the final states also can be obtained by the ρ intermediate state, followed by $\rho \rightarrow \pi^+ \pi^-$. These production mechanisms are depicted in Fig. 5, and the full relativistic amplitudes can be written as [34,44,46]

$$M_\phi(s_{12}, s_{23}) = \frac{D_\phi e^{i\alpha_\phi}}{s_{12} - m_\phi^2 + im_\phi \Gamma_\phi} (s_{23} - s_{13}), \quad (20)$$

$$M_\rho(s_{12}, s_{23}) = \frac{D_\rho e^{i\alpha_\rho}}{s_{12} - m_\rho^2 + im_\rho \Gamma_\rho} (s_{23} - s_{13}), \quad (21)$$

where $D_{\phi/\rho}$ and $\alpha_{\phi/\rho}$ are the normalization constants and phases, respectively, of which the values can be calculated by fitting the experimental data. $\Gamma_{\phi/\rho}$ and $m_{\phi/\rho}$ are the total widths and masses of the corresponding intermediate states, of which the values are taken from PDG [4].

Finally, the double differential width distributions for these two three-body decays can be calculated by

$$\frac{d^2\Gamma}{ds_{12}ds_{23}} = \frac{1}{(2\pi)^3} \frac{1}{32m_{D^0}^3} |t_{D^0 \rightarrow \pi^+ \pi^- \eta}(s_{12}, s_{23}) + M_\rho(s_{12}, s_{23})|^2, \quad (22)$$

$$\frac{d^2\Gamma}{ds_{12}ds_{23}} = \frac{1}{(2\pi)^3} \frac{1}{32m_{D^0}^3} |t_{D^0 \rightarrow K^+ K^- \eta}(s_{12}, s_{23}) + M_\phi(s_{12}, s_{23})|^2, \quad (23)$$

where we have taken into account the interference effects between the S -wave final state interaction amplitudes and the ones of the P wave with the vector meson contributions. Thus, one can separate the contributions of the S and P waves just by removing the other one in Eqs. (22) and (23); the results will be provided later. Furthermore, the single invariant mass distributions $d\Gamma/ds_{12}$ and $d\Gamma/ds_{23}$ can be evaluated by integrating the other invariant mass

invariable in Eqs. (22) and (23). On the other hand, the $d\Gamma/ds_{13}$ can be obtained through the constraint condition $s_{12} + s_{13} + s_{23} = m_{D^0}^2 + m_{K^+/\pi^+}^2 + m_{K^-/\pi^-}^2 + m_\eta^2$. The explicit form for the limits of integration can be taken from PDG and other literature [4,46,71]. Furthermore, as discussed in Refs. [44,45,71], the effective energy range in the ChUA has limitations. In order to make reliable predictions up to the higher-energy region, we need to expand the scattering amplitudes above the energy cut $\sqrt{s_{\text{cut}}} = 1.1$ GeV smoothly [71],

$$G(s)T(s) = G(s_{\text{cut}})T(s_{\text{cut}})e^{-\alpha(\sqrt{s} - \sqrt{s_{\text{cut}}})}, \quad \text{for } \sqrt{s} > \sqrt{s_{\text{cut}}}, \quad (24)$$

where G is the loop function of the meson-meson propagator, which is introduced in Eq. (14). T is the two-body scattering amplitude of the corresponding coupled channel, which is calculated by the ChUA and presented in Eq. (15). α is the extrapolation parameter, which is determined by fitting the experimental data. Note that since we are just interested in the low-energy region, where the resonances show up, for the nonresonant high-energy region, we take the exponential extrapolation of Eq. (24). This extrapolation is Omnès-like, based on the dispersion relation, which is commonly used in the literature [72,73]. Normally, the energy cut $\sqrt{s_{\text{cut}}}$ is taken around 1.1–1.2 GeV, which has tiny effects on the final results.

III. RESULTS

In this section, we present our results obtained with the above formalism. Actually, we have six free parameters for each three-body process, that is, C_1 , C_2 , β , and α in the S wave, $D_{\phi/\rho}$ and $\alpha_{\phi/\rho}$ for the contributions of intermediate resonances from the P wave, of which the values can be determined by fitting the experimental data. As shown in Eqs. (22) and (23), we consider the coherence of the amplitudes between the S and P waves.

For the $D^0 \rightarrow K^+ K^- \eta$ process, we perform a combined fit to the experimental data of the Belle Collaboration [27] to simultaneously describe the invariant mass distributions of $K^+ K^-$, $K^+ \eta$, and $K^- \eta$. The fitted parameters are listed in Table I, and the obtained invariant mass spectra are shown

TABLE I. Values of the parameters from the combined fit for the data of decay $D^0 \rightarrow K^+ K^- \eta$ measured by the Belle Collaboration [27].

Parameters	C_1	C_2	β	α	$D_{\phi(1020)}$	$\alpha_{\phi(1020)}$	$\chi^2/\text{d.o.f.}$
Fit	-3448.24 ± 221.07	99.07 ± 91.70	0.77 ± 0.04	6.28 ± 0.05	123.75 ± 2.02	2.89 ± 0.12	1.48

in Fig. 6, where one can see that our fitting results are in good agreement with the experimental data, considering the contributions from the S and P waves. Additionally, with these parameters, we also can distinguish the contributions of the S wave and the intermediate resonance $\phi(1020)$ from the P wave in the invariant mass distributions, as presented in different lines in Fig. 6; see the dashed (blue) and dashed-dot (purple) lines, respectively. For the $K^+ K^-$ mass spectrum as shown in Fig. 6(a), we can see a clear peak around 1.0 GeV^2 , which corresponds to the contribution from $\phi(1020)$, and a bit enhancement close to the $K\bar{K}$ threshold, which is a signal of $a_0(980)$ and/or $f_0(980)$. For the $K^+ \eta$ and $K^- \eta$ invariant mass distributions as depicted in Figs. 6(b) and 6(c), their results are similar and show a

double hump structure around the energy regions of 1.4 and 1.8 GeV^2 . One can find that the hump structure comes from the interference effect between the S wave and the $\phi(1020)$ resonance of the P wave, where $\phi(1020)$ is dominant. Finally, it should be mentioned that the error bars of Fig. 6 are not visible even though we have taken into account the errors of the fitting parameters, as shown in Table I, and 10% uncertainties of the cutoff q_{max} [see Eq. (14)], which is not a free parameter in our fits. Note that in all the cases, including Figs. 7 and 8 later, the error bars evaluated with the errors of the fitting parameters and combined with the covariance matrix from the fits are also not visible. Thus, when one considers 10% uncertainties of the cutoff q_{max} , one can obtain obvious error bars in Figs. 7 and 8, except

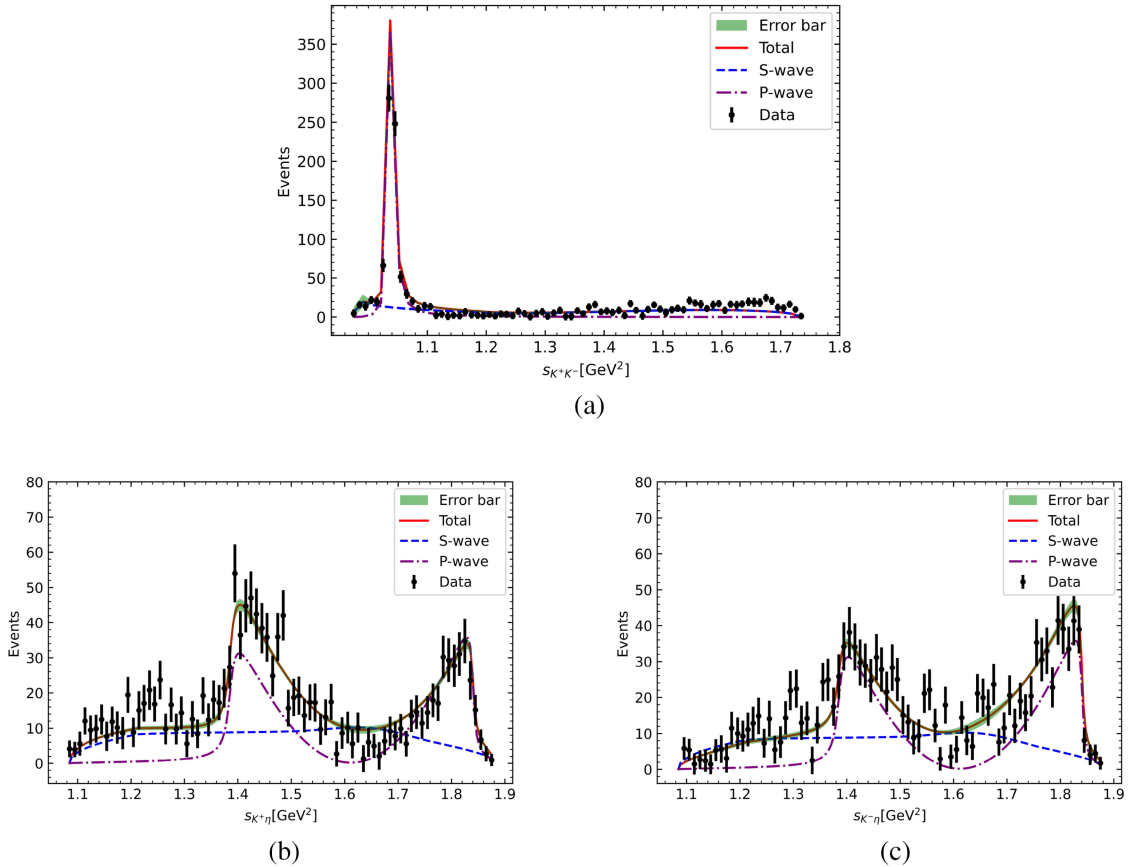


FIG. 6. Invariant mass distributions of the decay $D^0 \rightarrow K^+ K^- \eta$. (a) $K^+ K^-$ invariant mass distribution. (b) $K^+ \eta$ invariant mass distribution. (c) $K^- \eta$ invariant mass distribution. The solid (red) line is the total contributions of the S and P waves, the dashed (blue) line represents the S -wave contributions, and the dashed-dot (purple) line is the contributions of $\phi(1020)$ in the P wave. The dots (black) are the experimental data measured by the Belle Collaboration [27]. The (green) error bars are evaluated with both parameters' errors combined with the covariance matrix from the fits and 10% uncertainties of the cutoff q_{max} .

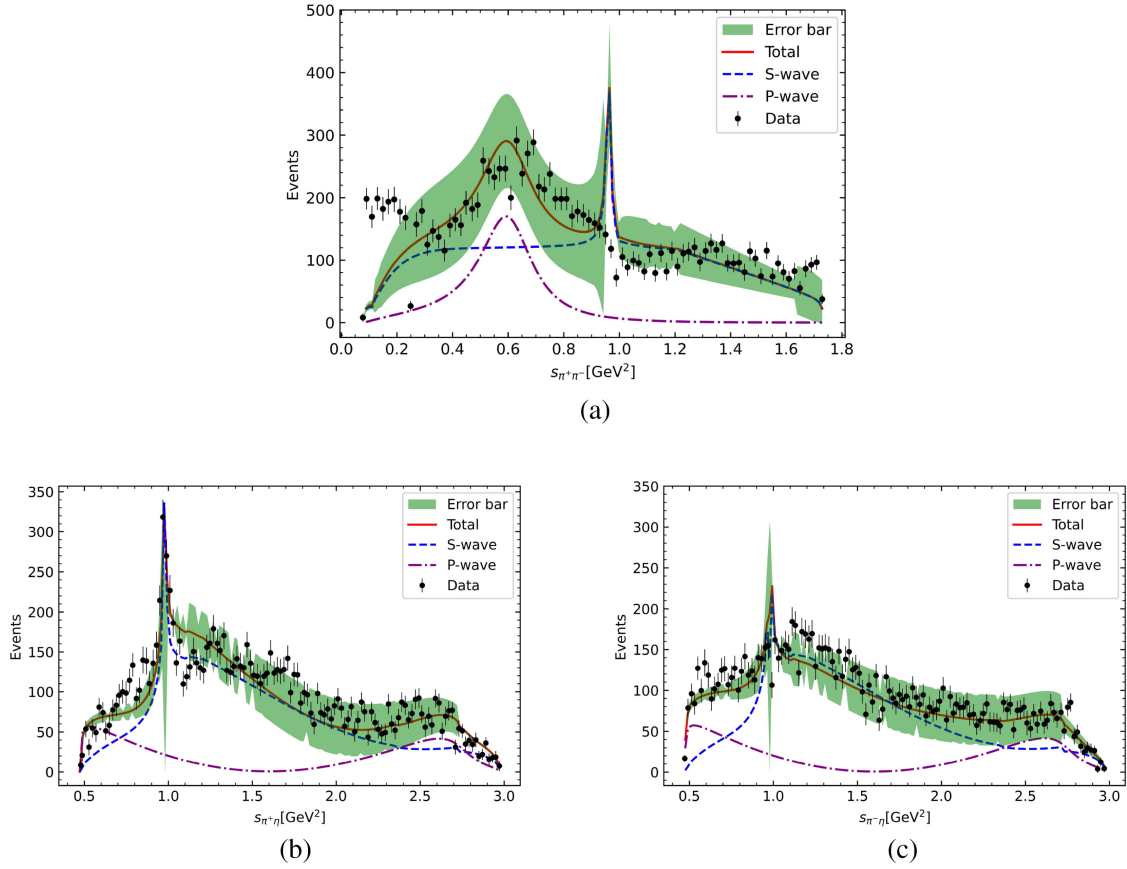


FIG. 7. Invariant mass distributions of the decay $D^0 \rightarrow \pi^+\pi^-\eta$. (a) $\pi^+\pi^-$ invariant mass distribution. (b) $\pi^+\eta$ invariant mass distribution. (c) $\pi^-\eta$ invariant mass distribution. The solid (red) line is the total contributions of the S and P waves, the dashed (blue) line represents the S -wave contributions, the dashed-dot (purple) line is the contributions of $\rho(770)$ in the P wave. The dots (black) are the experimental data measured by the Belle Collaboration [27]. The (green) error bars are evaluated with both parameters' errors combined with the covariance matrix from the fits and 10% uncertainties of the cutoff $q_{\max}$.

for the one in Fig. 6. This indicates once again that the results of Fig. 6 are dominant by $\phi(1020)$ in the P wave, since the changes of the cutoff $q_{\max}$ only affect the results of the S -wave contributions.

For the $D^0 \rightarrow \pi^+\pi^-\eta$ process, we first make a combined fit with the data of the $\pi^+\pi^-$, $\pi^+\eta$, and $\pi^-\eta$ invariant mass distributions measured by the Belle Collaboration [27]. The fitted parameters are given in Table II, and the corresponding invariant mass spectra with these parameters are shown in Fig. 7. From Fig. 7, one can find that, except for the data at the low-energy region above the threshold in the $\pi^+\pi^-$ distribution,¹ which leads to a bit larger value $\chi^2/\text{d.o.f.}$, the other fitted results agree with the experimental data within the error ranges. In Fig. 7(a), there is a distinct peak appearing in the $\pi^+\pi^-$ spectrum around 1.0 GeV² corresponding to the $f_0(980)$ resonance, which is the contribution of the $I = 0$ sector encoded in Eq. (13), while this

¹This part cannot be described well even when doing a single fit for the $\pi^+\pi^-$ invariant mass distributions.

signal is not visible in the experimental results [27]. Note that the contributions of the state $f_0(500)$ from the S wave can be found around the region of 0.2–0.4 GeV², and those of the ρ in P wave are dominant in the region of 0.6 GeV². In Fig. 7(b), the data of the $\pi^+\eta$ invariant mass distribution are well fitted, and the peak around 1.0 GeV² is dominant by the contribution of $a_0(980)$, which comes from the interaction of the $I = 1$ sector. Note that the contribution from $a_0(980)$ is relatively larger than the one from $\rho(770)$. In addition, the invariant mass distribution of $\pi^-\eta$ is exhibited in Fig. 7(c), which is similar to what we have for $\pi^+\eta$. As one can see from the features in Figs. 7(b) and 7(c), with one set of the combined fit parameters, the $\pi^+\eta$ and $\pi^-\eta$ mass distributions are both described well, even though they do not have isospin symmetry, as indicated by the BESIII Collaboration [30], where the production of $a_0(980)^+$ was larger than that of $a_0(980)^-$.

Since recently the BESIII Collaboration also investigated the $D^0 \rightarrow \pi^+\pi^-\eta$ decay and reported corresponding invariant mass distributions [30], we make a further combined fit with their measured data. Note that, even though the decay

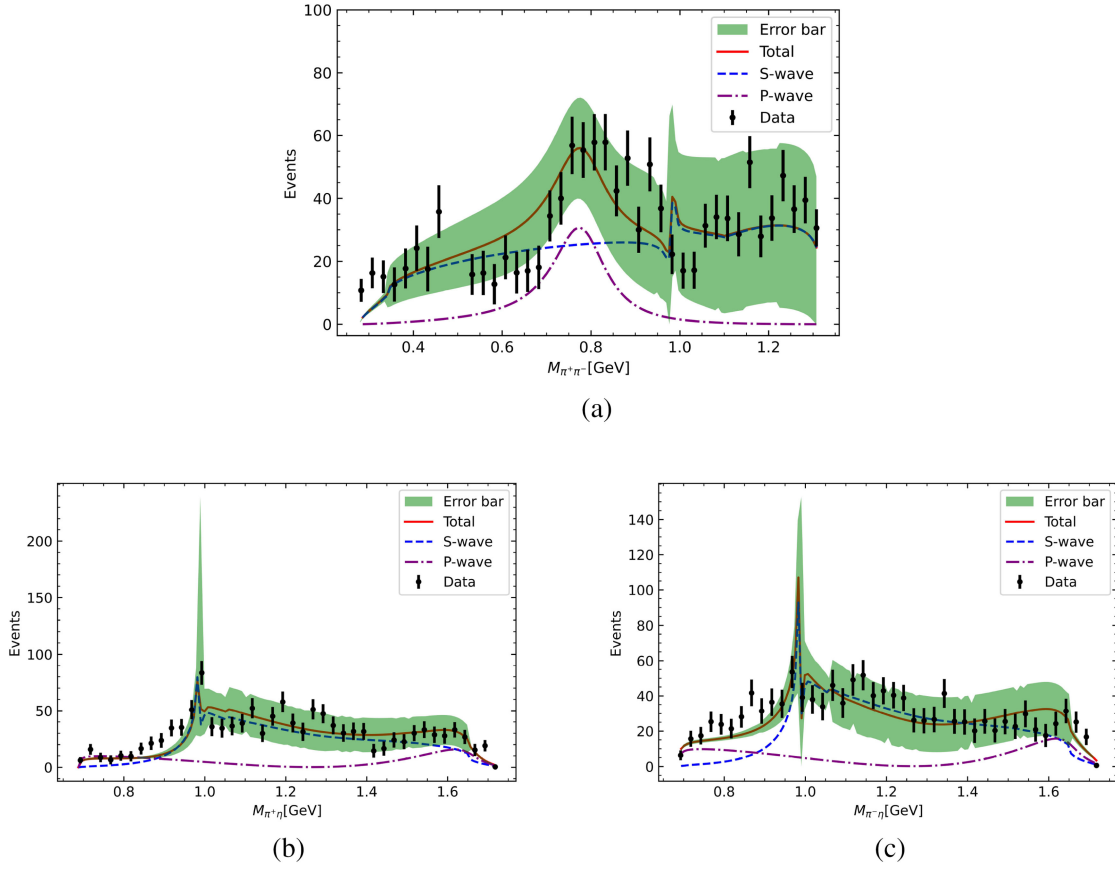


FIG. 8. Combined fit for the invariant mass distributions of the decay $D^0 \rightarrow \pi^+ \pi^- \eta$. (a) Invariant mass distribution of $\pi^+ \pi^-$. (b) Invariant mass distribution of $\pi^+ \eta$. (c) Invariant mass distribution of $\pi^- \eta$. The dots (black) are the experimental data measured by the BESIII Collaboration [30]; all others are the same as Fig 7.

TABLE II. Values of the parameters from the combined fit for the data of decay $D^0 \rightarrow \pi^+ \pi^- \eta$ measured by the Belle Collaboration [27].

Parameters	C_1	C_2	β	α	$D_{\rho(770)}$	$\alpha_{\rho(770)}$	$\chi^2/\text{d.o.f.}$
Fit	419.79 ± 23.61	2196.24 ± 57.25	1.00 ± 0.006	0.38 ± 0.10	-170.97 ± 2.90	2.20 ± 0.03	5.23

TABLE III. Values of the parameters from the combined fit for the data of decay $D^0 \rightarrow \pi^+ \pi^- \eta$ measured by the BESIII Collaboration [30].

Parameters	C_1	C_2	β	α	$D_{\rho(770)}$	$\alpha_{\rho(770)}$	$\chi^2/\text{d.o.f.}$
Fit	282.74 ± 15.25	-516.33 ± 26.32	1.00 ± 0.03	2.47 ± 0.44	58.41 ± 2.98	4.36 ± 0.13	2.52

$D^+ \rightarrow \pi^+ \pi^0 \eta$ was also measured in Ref. [30], we only focus on the two D^0 decay channels to simplify the formalism of our work,² since both of them have the same dynamical mechanisms at the quark level, as shown in

²From our primary results of the decay $D^+ \rightarrow \pi^+ \pi^0 \eta$, we get consistent conclusions.

Figs. 1 and 2. The fitted parameters are listed in Table III, and the reproduced mass spectra are presented in Fig. 8, where our fit results show great agreement with experiments within the error range. One can see that the fitted results for these three distributions are similar to the results in Fig. 7. In Fig. 8(a), the contributions of the state $f_0(500)$ in the S wave and ρ in the P wave are also found in the corresponding energy regions. But, now the signal of the

$f_0(980)$ resonance in the $\pi^+\pi^-$ spectrum around 1.0 GeV [see Fig. 8(a)] is not so strong compared to the one in Fig. 7(a), and thus, the value $\chi^2/\text{d.o.f.}$ in Table III becomes one-half smaller than the one in Table II. Indeed, the theoretical signal for the $f_0(980)$ resonance is not so visible compared with the one of the state $f_0(500)$, where these signals were claimed as the contributions of $\pi^+\pi^-$ S -wave scattering in the experiment [30]. Of course, the feature where the isospin antisymmetry data for the production of $a_0(980)^+$ and $a_0(980)^-$ is also fitted well with one set of the combined fit parameters, still remains, as shown in Figs. 8(b) and 8(c).

IV. SUMMARY

Based on the experimental results for the $D^0 \rightarrow K^+ K^- \eta$ and $\pi^+\pi^-\eta$ three-body weak decays measured by the Belle and BESIII Collaborations, we adopt the chiral unitary approach and final state interaction formalism to investigate these two processes. Taking into account the contribution of the S -wave interactions between pseudoscalar and pseudoscalar mesons, the $a_0(980)$ resonance is dynamically generated in the final state interactions. In addition, we also consider the contributions of intermediate resonances in the P wave, that is, the ϕ meson for $D^0 \rightarrow K^+ K^- \eta$ and the ρ meson for $D^0 \rightarrow \pi^+\pi^-\eta$. Considering the coherence effect between the amplitudes of the S and P waves, we do a combined fit of the experimental data to

determine the free parameters of the formalism. Most of results are in good agreement with the experimental data, indicating the contributions of $a_0(980)$ in the S wave and ϕ or ρ in the P wave are of great significance for these two decays. Note that the antisymmetry data of the $\pi^+\eta$ and $\pi^-\eta$ mass distributions are described well in our combined fit.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [27,30].

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