

Parton distribution and fragmentation functions with massive gluons

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The correct description of the hadron's structure requires understanding how quarks and gluons form the observable hadrons and how they are distributed within them. Two key nonperturbative quantities encapsulate this information: parton distribution functions (PDFs) and fragmentation functions (FFs). The former define a probabilistic light-front momentum distribution of partons within a hadron, whereas the latter describe the hadronization process of high-energy partons. Computing these functions analytically poses significant challenges, as it demands models that accurately incorporate the nonperturbative infrared dynamics of quantum chromodynamics (QCD). In this work, we compute the pion PDF and its elementary and full FFs using the Curci-Ferrari (CF) model. This model enables the exploration of nonperturbative QCD effects by introducing a gluon-mass scale within the Landau gauge QCD Lagrangian. The two-point quark and gluon correlation functions derived from the CF model agree well with lattice-QCD results and reproduce the pion decay constant in the chiral limit, consistent with chiral perturbation theory. The resulting pion PDF and FFs computed with the CF quark propagator and pion Bethe-Salpeter amplitude are in good qualitative and quantitative agreement with those obtained using the Qin-Chang model, a benchmark approach to nonperturbative QCD. These findings support the broader applicability of the Curci-Ferrari model in hadron phenomenology.

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I. INTRODUCTION

Parton distribution functions (PDFs) and fragmentation functions (FFs) play a crucial role in describing the internal structure of hadrons and their formation in high-energy processes. While the former characterize the momentum distributions of partons inside a hadron, the latter describe how a high-energy parton hadronizes into observable particles. Both objects arise in the factorization of hard scattering cross sections in quantum chromodynamics (QCD), allowing for the separation of perturbative short-distance and nonperturbative long-distance effects [1]. Thus, these distributions are inherently of nonperturbative

nature and the past years have seen considerable efforts to obtain them indirectly in Euclidean space with lattice-QCD simulations [2–5] and with continuum functional methods using the Dyson-Schwinger equations (DSEs) with different interaction models [6–11], including the Nambu–Jona-Lasinio (NJL) model [12–14].

PDFs are defined in the operator product expansion of the hadronic tensor $W_{\mu\nu}$ that enters the cut diagram of the forward Compton amplitude [15] applying the optical theorem. In particular, the cut enforces an on-shell condition for the quarks, thereby ensuring that only physical partons contribute to the process and allowing for a probabilistic interpretation of the partons. This diagrammatic approach has taken advantage of using quark propagators and Bethe-Salpeter wave functions in both the NJL model [12–14] and the functional DSE approach [6,8,11]. Moreover, the quark-to-pion FF and the PDF of a valence quark in a pion obey a Drell-Levy-Yan relation, as charge conjugation and crossing symmetry relate both functions in different kinematic regions [16–18]. A notable feature of Ref. [11] is that the DSE and NJL predictions for the elementary quark-to-pion FF are rather alike despite the pointlike nature of the pion's wave function in the latter model. More precisely, the

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FF obtained in the DSE framework is about 10%–12% enhanced for momentum fractions $z \lesssim 0.8$, when compared to an earlier NJL model calculation [13], but otherwise in good qualitative agreement. The pion PDFs and FFs were also computed with renormalization-group-invariant interaction kernels consistent with expectations based on QCD [9,10].

A common feature of the NJL model and the DSE applied to QCD is “dynamical chiral symmetry breaking,” that is, the quark gap equation possesses a nonperturbative solution even in the chiral limit. The strength of the DSE kernel comes from both the quark-gluon vertex and the fully dressed gluon propagator, which obeys its own gap equation. Massive solutions to the latter exist and have been the subject of intense debates [19–27]. In particular, the “decoupling” solution of the gluon DSE and the associated effective gluon mass have inspired calculations with a higher-derivative Podolsky Lagrangian [28] and in a particular version [29–35] of the Curci-Ferrari (CF) model [36]. In the latter model, the standard Faddeev-Popov QCD Lagrangian is augmented by a gluon-mass term. This term clearly violates the Becchi-Rouet-Stora-Tyutin (BRST) symmetry of the Faddeev-Popov Lagrangian, though it is still symmetric under a modified BRST symmetry responsible for its renormalizability. One may also think of this procedure as a minimal extension to take into account the effects of Gribov copies [34].

The CF model has been used to explore the infrared behavior of Green’s functions in QCD [29–35,37–41], and at two-loop order the dressed ghost and gluon propagators in Landau gauge are in rather good qualitative agreement with the propagators obtained in lattice QCD [20–25] and with the gluon DSE [26,27]. However, as dynamical chiral symmetry breaking in the chiral limit is truly a nonperturbative phenomenon, even next-to-leading order perturbation theory only reproduces predicted quark-mass functions with a large gluon mass and an unphysical current-quark mass [35]. An alternative approach combines an expansion in the Yang-Mills coupling and in the inverse number of colors N_c , without expanding in the quark-gluon coupling [32,42,43], also known as rainbow-improved loop expansion. At leading order in this expansion, a quark-antiquark bound state corresponds to the resummation of a ladder diagram of massive gluon exchanges. Fitting the CF model parameters to the lattice-QCD predictions of the gluon, ghost, and quark dressing functions, a mass function is predicted that yields a reasonable Euclidean constituent quark mass and the perturbative tail of the mass function is qualitatively in agreement with the lattice data.

Our goal in this work is to apply and evaluate the capability of the CF model to describe physical observables beyond two-point Green’s functions. To this end, we employ dressed quark propagators and Bethe-Salpeter amplitudes (BSAs), derived in the CF model in the chiral limit [43], to compute the PDF and FF of a pion with the

aforementioned cut diagram. Finally, we compare both distribution functions with those of a DSE calculation [11] to assess whether the CF model reproduces their qualitative and quantitative behavior as a function of the quark’s momentum fraction.

This paper is organized as follows. Section II introduces the definitions of the PDF and FF and discusses the quark-jet model. Section III briefly summarizes the DSE formalism with both the Qin-Chang (QC) and CF gluon models. Our numerical results are discussed in Sec. IV and conclusions are given in Sec. V.

II. DISTRIBUTIONS AND FRAGMENTATION FUNCTIONS

The quest of understanding the internal structure of hadrons has led to the definitions of nonperturbative functions that encode how quarks and gluons are distributed inside a hadron and how they hadronize into observable particles [44]. Among others, two fundamental one-dimensional quantities fulfill this task: PDFs and FFs. The former describes the light-front momentum distribution of partons within a hadron, while the latter describes their hadronization process. These functions are central to factorization theorems and the interpretation of high-energy scattering experiments.

A. Parton distribution function

The PDF of a hadron h is defined through the hadronic matrix element of a quark-field bilinear operator,

$$f_q^h(x) = \int \frac{d\omega^-}{4\pi} e^{ixp-\omega^-} \langle p(h) | \bar{\psi}(0) \gamma^+ \psi(\omega^-) | p(h) \rangle, \quad (1)$$

where $x = k^+/p^+$ is defined as the light-front momentum fraction of a quark with momentum k in a hadron with total momentum p . We compute this distribution function with the relevant cut diagram [12,13,15] and use the conventions of Ref. [45] for the light-cone variables in Euclidean space. The Feynman rules of the cut diagram in Euclidean metric yield [13]

$$f_q^\pi(x) = N_c C_q^\pi \int_k \delta(k^+ - xp^+) \text{Tr}_D \left[S(k) \gamma^+ S(k) \right. \\ \left. \times \bar{\Gamma}_\pi \left(-k + \frac{p}{2}, -p \right) S_{\text{os}}(l) \Gamma_\pi \left(k - \frac{p}{2}, p \right) \right], \quad (2)$$

where $\int_k = \int d^4k/(2\pi)^4$, $l = k - p$, $N_c = 3$, $S(k)$ is a dressed quark propagator, $S_{\text{os}}(l)$ is the on-shell quark propagator, $\Gamma_\pi(k, p)$ is the BSA of the pion, and $C_q^\pi = \frac{1}{2}(1 + \tau_q \tau_\pi)$ is a product of isospin factors using the basis $(\tau_u, \tau_d) = (1, -1)$ and $(\pi^+, \pi^0, \pi^-) = (1, 0, -1)$. In the on-shell propagator,

$$S_{\text{os}}(l) = 2\pi i \delta(l^2 + M_E^2) [-i\gamma \cdot \ell + M(\ell^2)],$$

we define a Euclidean constituent mass with $M^2(k^2) = k^2$ [46]. The relevant cut diagram for the PDF is illustrated in Fig. 1(a) and the PDF is normalized as

$$\int_0^1 \hat{f}_u^\pi(x) dx = 1. \quad (3)$$

B. Elementary fragmentation functions

A simple relation exists between the PDF for unphysical $x > 1$ and the elementary FF for physical $z \leq 1$, based on charge conjugation and crossing symmetry, namely, the Drell-Levy-Yan relation [16–18]

$$d_q^h(z) = (-1)^{2(s_q + s_h) + 1} \frac{z}{d_q} f_q^h\left(x = \frac{1}{z}\right), \quad (4)$$

where s_q and s_h are the spins of the quark and hadron, respectively, and d_q denotes the spin-color degeneracy of the quark. Applied to the pion FF, we have $d_q = 6$, $s_q \equiv s_u = 1/2$ and $s_h = s_\pi = 0$, so that Eq. (4) becomes

$$d_q^\pi(z) = \frac{z}{6} f_q^\pi\left(x = \frac{1}{z}\right), \quad (5)$$

which yields

$$d_q^\pi(z) = \frac{N_c C_q^\pi z}{6} \int_k \delta\left(k^+ - \frac{p^+}{z}\right) \text{Tr}_D \left[S(k) \gamma^+ S(k) \times \bar{\Gamma}_\pi\left(-k + \frac{p}{2}, -p\right) S_{\text{os}}(l) \Gamma_\pi\left(k - \frac{p}{2}, p\right) \right]. \quad (6)$$

We illustrate the relevant cut diagram of this FF in Fig. 1(b) and note that Eqs. (5) and (6) refer to a frame in which the produced hadron has no transverse momentum, $p_T = 0$, while the fragmenting quark does: $k_T \neq 0$. To interpret this result as a distribution of a pion in the quark, one must boost to a frame where $k_\perp = 0$ and $p_\perp \neq 0$. This is discussed in detail in Refs. [47,48] and we can simply substitute $k_T = -p_\perp/z$. The elementary FF is normalized according to

$$\sum_m \int_0^1 \hat{d}_q^m(z) dz = 1, \quad m = \pi^-, \pi^0, \pi^+, \quad (7)$$

which ensures that the isospin and momentum sum rules for the full FF in Sec. II C are satisfied [13,49].

C. Quark-jet model

With the elementary FF (6) in hand, one must also consider the possibility that a quark can fragment via a cascading process of hadron emissions. The quark-jet

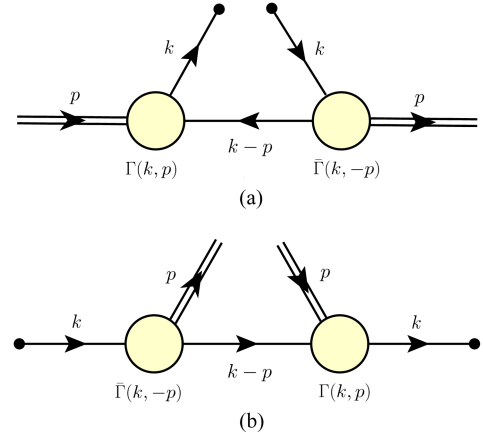


FIG. 1. Cut-diagram representation of the parton distribution function (a) and elementary fragmentation function (b). The shaded circle with outgoing/incoming double solid lines denote the pion in the fragmentation process, while solid lines are quark propagators and solid dots represent the γ^+ between the two quarks with momentum k .

model of Field and Feynman [50] assumes that the meson observed in a semi-inclusive process is merely one among many in a jet of hadrons. The probability of finding a pion with light-front momentum fraction z in the jet is described by the full FF $D_q^\pi(z)$, also known as jet function.

The full FF can be conveniently written in terms of the isospin decomposition [13],

$$D_q^\pi(z) \equiv \frac{1}{3} [D_0(z) + \tau_q \tau_\pi D_1(z)], \quad (8)$$

where the functions D_0 and D_1 are given by Volterra integral equations,

$$\frac{2}{3} D_0^\pi(z) = \hat{d}_q^\pi(z) + \int_z^1 \frac{dy}{y} \hat{d}_u^\pi\left(1 - \frac{z}{y}\right) D_0^\pi(y), \quad (9)$$

$$\frac{2}{3} D_1^\pi(z) = \hat{d}_q^\pi(z) - \frac{1}{3} \int_z^1 \frac{dy}{y} \hat{d}_u^\pi\left(1 - \frac{z}{y}\right) D_1^\pi(y). \quad (10)$$

In case of a u quark fragmenting into a π^+ , using the isospin conventions in Sec. II A, Eq. (8) becomes $D_u^{\pi^+} = (D_0 + D_1)/3$. In the limit $z \rightarrow 1$, the full fragmentation function reduces to the elementary one, $D_u^{\pi^+}(z) \xrightarrow{z \rightarrow 1} \hat{d}_u^{\pi^+}(z)$, as all of the quark's momentum is transferred to the initial pion and no momentum is left for a jet. Finally, the moments of the full FFs are defined as

$$\langle z \rangle_{D_q^\pi} = \int_0^1 z D_q^\pi(z) dz. \quad (11)$$

III. THEORETICAL FRAMEWORK

The quark gap equation in QCD is described by a DSE, a nonlinear integral equation that couples the two-point Green's function of quark fermion fields, namely, the fully dressed quark propagator, with an infinite tower of higher-order Green's functions [51],

$$S^{-1}(p) = Z_2(i\not{p} + m_{\text{bm}}) + Z_1 g^2 \int_k^\Lambda D_{\mu\nu}^{ab}(l) \frac{\lambda^a}{2} \gamma_\mu S(k) \Gamma_\nu^b(k, p), \quad (12)$$

where $D_{\mu\nu}^{ab}(l)$ is the dressed gluon propagator with $l = k - p$, $\Gamma_\nu^b(k, p)$ is the dressed quark-gluon vertex, λ^a are SU(3) color matrices, m_{bm} is the bare current-quark mass, and $Z_1(\mu, \Lambda)$ and $Z_2(\mu, \Lambda)$ are the vertex and wave function renormalization constants at the renormalization scale μ , respectively. The regularization scale $\Lambda \gg \mu$ can be taken to infinity. The covariant solution to this equation can generally be expressed as

$$S(p) = \frac{Z(p^2)}{i\not{p} + M(p^2)}, \quad (13)$$

where $Z(p^2)$ is the wave renormalization function and $M(p^2)$ is the quark mass function.

The bound-state interactions of a quark-antiquark pair in a pion are described by the Bethe-Salpeter equation (BSE) and in its homogeneous form is written as

$$[\Gamma_\pi(k, p)]_{tu} = \int_q [K(k, q, p)]_{tu}^{rs} [S(q_\eta) \Gamma_\pi(q, p) S(q_{\bar{\eta}})]_{rs}, \quad (14)$$

where $K(k, q, p)$ is the fully amputated quark-antiquark scattering kernel and $r, \dots, u$ denote color, Dirac, and flavor matrix indices. The quark and antiquark momenta are given by $q_\eta = q + \eta p$ and $q_{\bar{\eta}} = q - \eta p$, respectively, and the momentum-partitioning parameters satisfy $\eta + \bar{\eta} = 1$. We stress that in a Poincaré covariant approach the mass spectrum and other static observables cannot depend on them due to translational invariance.

The solution of the BSE (14) for a bound state with quantum numbers $J^{PC} = 0^{-+}$, relative momentum k , and total momentum p is constrained by Lorentz covariance, parity, and charge conjugation, and the most general BSA for a pseudoscalar meson reads [52]

$$\Gamma_\pi(k, p) = \gamma_5 [iE_\pi(k, p) + \not{p}F_\pi(k, p) + \not{k} \cdot p G_\pi(k, p) + \sigma_{\mu\nu} k^\mu p^\nu H_\pi(k, p)], \quad (15)$$

where E_π , F_π , G_π , and H_π are Lorentz-invariant scalar amplitudes. The conjugate BSA is defined as

$\bar{\Gamma}_\pi(k, p) = C \Gamma_\pi^T(-k, p) C^T$, where C is the charge-conjugation operator. Finally, we remind the reader that the pion's weak decay constant is defined as

$$f_\pi p_\mu = \frac{Z_2 N_c}{\sqrt{2}} \int_q \text{Tr}_D [\gamma_5 \gamma_\mu S(q_\eta) \Gamma_\pi(q, p) S(q_{\bar{\eta}})]. \quad (16)$$

Several models and truncation schemes can be employed to calculate the interaction kernel entering both the quark self-energy equation (12) and the BSE (14). Within the functional approach, the simplest and most widely used truncation is the rainbow-ladder (RL) approximation, where the product of the fully dressed gluon propagator and quark-gluon vertex is replaced by an effective interaction kernel that features infrared enhancement and asymptotic freedom. This scheme preserves the axial-vector Ward-Takahashi identity. A representative example is the Maris-Tandy model [53], which has provided numerous quantitative predictions for vacuum and hadronic properties, including the quark condensate, decay constants, and hadron spectra. However, its infrared behavior does not fully reflect the nonvanishing interaction strength observed in modern lattice and continuum studies. To address this, Qin *et al.* [54] introduced an improved parametrization that incorporates the infrared saturation of the quark-gluon coupling, aligning it more closely with modern results.

More sophisticated kernel constructions have been developed, including those based on the Ball-Chiu ansatz for the quark-gluon vertex [55], which automatically satisfies the Ward-Takahashi identity and ensures multiplicative renormalizability. The Curtis-Pennington ansatz [56] extends this framework by including a transverse component that preserves multiplicative renormalizability to all orders in leading and next-to-leading logarithmic accuracy within perturbation theory. In this continuum framework, Cornwall introduced the concept of a dynamically generated gluon mass and an infrared-finite effective charge [57]. Subsequent developments, which combined the pinch technique with the background-field method, yielded a consistent set of DSEs for gluons and ghosts. These studies demonstrated that the gluon propagator saturates in the infrared, the ghost propagator remains finite, and the resulting effective charge exhibits infrared saturation, all in agreement with lattice-QCD results [27, 58, 59]. Notably, this formalism has also been successful in describing hadronic observables, thus bridging the gap between first-principles QCD and phenomenological modeling [60].

In general, the use of such improved kernels increases the computational complexity of solving the DSEs and BSEs. Given the simplicity of the Qin-Chang¹ [54] model kernel and its robust description of a broad range of hadron

¹The name is a short reference to the authors Qin *et al.* of Ref. [54].

properties, we will use it as a benchmark to evaluate results obtained with the CF model. In the following, we provide a detailed exposition of these two models.

A. Qin-Chang model

Working in the leading rainbow-ladder truncation, one simplifies $\Gamma_\mu^a \rightarrow Z_2 \frac{\lambda^a}{2} \gamma_\mu$. This truncation is characterized by an Abelianized Ward identity which, as in QED, implies $Z_1 = Z_2$ [51]. With that, the kernel in the self-energy integral (12) can be written as

$$Z_1 g^2 D_{\mu\nu}(q) \Gamma_\nu^b(k, p) = Z_2^2 \mathcal{G}(q^2) D_{\mu\nu}^{\text{free}}(q) \frac{\lambda^b}{2} \gamma_\nu, \quad (17)$$

where the free-gluon propagator in Landau gauge is

$$D_{\mu\nu}^{\text{free}}(q) = \frac{1}{q^2} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right). \quad (18)$$

The dressing function $\mathcal{G}(q^2)$ in Eq. (17) is composed of two pieces,

$$\frac{\mathcal{G}(q^2)}{q^2} = \mathcal{G}_{\text{IR}}(q^2) + 4\pi\tilde{\alpha}_{\text{PT}}(q^2), \quad (19)$$

where a factor $1/q^2$ of the free-gluon propagator is included in the definition. The first term in Eq. (19) dominates the infrared momentum region, $q^2 < \Lambda_{\text{QCD}}^2$, and is suppressed at large momenta. The second term describes a monotonically decreasing continuation of the perturbative coupling that dominates at large momenta: $\mathcal{G}_{\text{IR}}(q^2) \ll \tilde{\alpha}_{\text{PT}}(q^2) \forall q^2 \gtrsim 2 \text{ GeV}^2$. In the QC model [54],

$$\mathcal{G}_{\text{IR}}(q^2) = \frac{8\pi^2}{\omega^4} D e^{-q^2/\omega^2} \quad (20)$$

$$4\pi\tilde{\alpha}_{\text{PT}}(q^2) = \frac{8\pi^2 \gamma_m \mathcal{E}(q^2)}{\ln \left[\tau + (1 + q^2/\Lambda_{\text{QCD}}^2)^2 \right]}, \quad (21)$$

where $\gamma_m = 12/(33 - 2N_f)$ is the mass anomalous dimension, N_f is the active flavor number, $\Lambda_{\text{QCD}} = 0.234 \text{ GeV}$, $\tau = e^2 - 1$, $\mathcal{E}(q^2) = [1 - \exp(-q^2/4m_t^2)]/q^2$, and $m_t = 0.5$, $\omega = 0.5$, and $\kappa = (\omega D)^{1/3} = 0.8 \text{ GeV}$.

The fully amputated scattering kernel compatible with rainbow truncation is given by the ladder truncation [61],

$$[K(k, q, p)]_{tu}^{rs} = -Z_2^2 \mathcal{G}(l^2) D_{\mu\nu}^{\text{free}}(l) \left(\frac{\lambda^a}{2} \gamma_\mu \right)_{ir} \left(\frac{\lambda^a}{2} \gamma_\nu \right)_{su}, \quad (22)$$

with the gluon momentum $l = k - q$. Inserting Eq. (22) in Eq. (14), we can rewrite the BSE as

$$\begin{aligned} \Gamma_\pi(k, p) = & -Z_2^2 C_F \int_q \mathcal{G}(l^2) D_{\mu\nu}^{\text{free}}(l) \gamma_\mu S(q_\eta) \\ & \times \Gamma_\pi(q, p) S(q_\eta) \gamma_\nu, \end{aligned} \quad (23)$$

where $C_F = (N_c^2 - 1)/2N_c$ is the Casimir operator.

B. Curci-Ferrari model

The CF model [36], originally proposed as an alternative to the Higgs mechanism, can be understood as a consistent massive vector-boson extension of QCD. It has been revived in the past decade to explore nonperturbative effects and the infrared regime of QCD. In the original CF approach, a mass term [36,62] is introduced at tree level in a gauge-fixed version of the Lagrangian, which guarantees its perturbative renormalizability and regulates the long-distance behavior.

We here consider the CF Lagrangian in Euclidean space and in Landau gauge with the presence of a Nakanishi-Lautrup field h_a that simplifies the writing of a modified BRST symmetry,

$$\begin{aligned} \mathcal{L} = & \sum_{i=1}^{N_f} \bar{\Psi}_i (\not{D} + \mathcal{M}) \Psi_i + \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a \\ & + i h^a \partial_\mu A_\mu^a + \partial_\mu \bar{c}^a (D_\mu c)^2 + \frac{1}{2} m^2 (A_\mu^a)^2. \end{aligned} \quad (24)$$

In this Lagrangian, the usual covariant derivative is defined as $D_\mu = \partial_\mu - ig A_\mu^a$, $F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$ is the field-strength tensor, c^a and $\bar{c}^a$ are ghost and antighost fields, ψ_i and $\bar{\psi}_i$ are quark and antiquark fields of flavor i , and $\mathcal{M}$ is the current-quark mass. The additional gluon-mass term, $\frac{1}{2} m^2 (A_\mu^a)^2$, is motivated by the emergence of a “dynamical” gluon mass observed in lattice and functional methods in the continuum [20–27].

A more sophisticated approach to the CF model combines an expansion in the inverse number of colors N_c with an expansion of the strong coupling in the gauge sector, while keeping a realistic running of the quark-gluon coupling and gluon mass at next-to-leading order. At leading order, the well-known rainbow truncation emerges from this expansion with constant quark-gluon coupling and gluon mass. The systematic implementation of the running of the renormalization group is known as “rainbow-improved” loop expansion [32,42,43]. In the DSE language, this corresponds partially to the truncation beyond “rainbow-ladder” that has received considerable attention over the past years [63–75].

The tree-level gluon propagator corresponding to the Lagrangian in (24) is given by

$$D_{\mu\nu}^{\text{CF}}(q) = \frac{1}{q^2 + m^2} \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right). \quad (25)$$

The computed quark self-energy in Ref. [43] leads to momentum-dependent wave function renormalization $Z(p^2)$ and mass function $M(p^2)$ which are used to compute the Bethe-Salpeter amplitudes.

The BSE for a pion in the CF model was solved in the chiral limit in Ref. [43] and reads for a given isospin,

$$\gamma_\pi(k, p) = -C_F g^2 \int_q D_{\mu\nu}^{\text{CF}}(l) \gamma_\mu S(q_\eta) \gamma_\pi(q, p) S(q_{\bar{\eta}}) \gamma_\nu, \quad (26)$$

where g is the renormalized strong coupling and again $C_F = (N_c^2 - 1)/2N_c \sim N_c/2$ for $N_c \gg 1$. Working in the chiral limit, the quark propagator (13) is expanded around $p^2 = 0$ up to linear order in p in Eq. (26), which reduces the BSE to a set of one-dimensional coupled integral equations that can be solved numerically. As before, the most general BSA can be decomposed in terms of covariant amplitudes,

$$\gamma_\pi(k, p) = -i\gamma_5 \left[\gamma_P(k^2) + i\not{p}\gamma_A(k^2) + i\sigma_{\mu\nu} p_\mu k_\nu \gamma_T(k^2) + 2i\not{k} \frac{k \cdot p}{k^2} (\gamma_A(k^2) - \gamma_B(k^2)) \right] + \mathcal{O}(p^2), \quad (27)$$

and $\gamma_P, \gamma_A, \gamma_B$, and γ_T are real scalar amplitudes that depend solely on the relative momentum k for $p^2 \rightarrow 0$, while the integral equation for $\gamma_P(k^2)$ decouples from the remaining, coupled ones. Numerical results for these scalar amplitudes are given in Figs. 5 and 6 of Ref. [43].

The BSA in Eq. (27) can be related to the one in Eq. (15), i.e., we write $E_\pi(k, P)$, $F_\pi(k, P)$, $G_\pi(k, P)$, and $H_\pi(k, P)$ in terms of $\gamma_P, \gamma_T, \gamma_A$, and γ_B ,

$$E_\pi^{\text{CF}}(k^2) \equiv -\frac{\sqrt{2}}{f_\pi} \gamma_P(k^2), \quad (28)$$

$$F_\pi^{\text{CF}}(k^2) \equiv \frac{\sqrt{2}}{f_\pi} \gamma_A(k^2), \quad (29)$$

$$G_\pi^{\text{CF}}(k^2) \equiv \frac{2\sqrt{2}}{f_\pi k^2} [\gamma_A(k^2) - \gamma_B(k^2)], \quad (30)$$

$$H_\pi^{\text{CF}}(k^2) \equiv -\frac{\sqrt{2}}{f_\pi} \gamma_T(k^2). \quad (31)$$

IV. RESULTS

In order to compute the PDF (2) and FF (6) with the quark propagators and the pion BSA from either approaches in Sec. III A or Sec. III B, we rewrite the integration measure in terms of light-cone variables [76],

$$\int d^4k = \frac{1}{2} \int dk^+ dk^- dk_T, \quad (32)$$

and integrate over k^+ and k^- using both Dirac δ functions in Eq. (2). The second δ function yields

$$\delta(l^2 + M_E^2) = \delta(k_T^2 - k^+ k^- - m_\pi^2 + k^+ p^- + k^- p^+ + M_E^2) = \frac{1}{p^+(1-x)} \delta(k^- - f(k_T^2, x)), \quad (33)$$

where $f(k_T^2, x)$ is defined as

$$f(k_T^2, x) = \frac{k_T^2 + M_E^2 - (1-x)m_\pi^2}{p^+(1-x)} \equiv k^-, \quad (34)$$

and the remaining k_T integration can be performed numerically. The same procedure is used to integrate over k^+ and k^- in the FF (6) where we apply a boost to the frame where the pion has transverse momentum, $k_T = -p_\perp/z$; see Refs. [11,13] for details.

Since the solutions of the quark propagators and the BSAs are obtained numerically in Euclidean space, a convenient algebraic parametrization of their scalar covariant amplitudes is required to perform the integrals over the Dirac δ functions. The quark dressing functions $Z(k)$ and $M(k)$ can be parametrized with a Padé approximant [77], such as

$$M(k^2) = \frac{a_0 + a_1 k^2 + a_2 k^4}{1 + b_1 k^2 + b_2 k^4 + b_3 k^6}, \quad (35)$$

$$Z(k^2) = \frac{c_0 + c_1 k^2 + c_2 k^4}{1 + d_1 k^2 + d_2 k^4}. \quad (36)$$

The coefficients of both Eqs. (35) and (36) are listed in Table I.

The BSA, on the other hand, can be parametrized with a Nakanishi integral representation [78,79] of the form

$$\mathcal{F}_i(k, p) = \sum_{j=1}^N \int_{-1}^1 \frac{U_j \Lambda_j^{2n_j} \rho_j(\alpha)}{(k^2 + \alpha k \cdot p + \Lambda_j^2)^{n_j}} d\alpha, \quad (37)$$

in which we define the spectral density $\rho_j(\alpha)$ as [80]

TABLE I. Parameters for the quark mass function (35) and wave renormalization function (36).

	a_0	a_1	a_2	b_1	b_2	b_3
CF	0.332	4.949	-0.284	16.58	55.74	-4.812
QC	0.500	-0.076	0.0126	1.324	0.636	0.328
	c_0	c_1	c_2	d_1	d_2	
CF	0.985	30.40	222.2	223.5	22.78	
QC	0.717	2.518	1.413	3.469	1.170	

TABLE II. Parameters of the BSA in the CF model.

$\mathcal{F}_i$	Λ_1	Λ_2	Λ_3	U_1	U_2	U_3	n_1	n_2	n_3
E_π	-67.08	-1.134	3.15	-0.199	-3.943	-1.374	4	5	6
F_π	-1.274	0.983	0.983	0.336	2.014	-2.397	2	6	7
G_π	-0.362	0.438	0.282	176.9	-147.6	-46.85	3	4	5
H_π	0.747	3.030	-0.349	-3.182	-0.128	3.343	4	5	6

TABLE III. Parameters of the BSA in the QC model.

$\mathcal{F}_i$	Λ	U_1	U_2	U_3	U_4	σ_1	σ_2	σ_3	σ_4	n_1	n_2	n_3	n_4
E_π	1.267	2.571	-1.572	...	...	2.136	1.889	...	...	4	5	...	...
F_π	1.247	0.007	2.660	-2.454	0.309	-76.88	-0.367	-0.020	1.024	2	6	7	10
G_π	0.971	-0.874	3.458	-2.096	...	-3.289	-1.714	-1.741	...	3	4	5	...
H_π	1.156	-0.207	1.606	-1.125	...	-39.54	-8.158	-6.261	...	4	6	7	...

$$\rho_j(\alpha) = \frac{\Gamma(\frac{3}{2})}{\sqrt{\pi}\Gamma(1)} \left[C_0^{1/2}(\alpha) + \sigma_j C_2^{1/2}(\alpha) \right], \quad (38)$$

where $C_n^{1/2}(\alpha)$ are Gegenbauer polynomials of order $1/2$. Since the BSA in the CF model (27) is only known in the chiral limit, its angular dependence vanishes and the $k \cdot p$ term in the denominator of Eq. (37) is dispensable, while the integrals over the spectral functions $\rho_i(\alpha)$ are trivial. Therefore, the BSA of the CF model can be parametrized with a simpler expression,

$$\mathcal{F}_i(k^2) = \sum_{j=1}^N \frac{U_j \Lambda_j^{2n_j}}{(k^2 + \Lambda_j^2)^{n_j}}, \quad (39)$$

and the parameters of this CF-BSA are found in Table II.

On the other hand, the numerical solution of the BSA in Eq. (15) is obtained in the near-chiral limit, $m_\pi \approx 0.3$ MeV, and it turns out that even for such a small meson mass the angular dependence is, to a certain extent, relevant. We therefore expand the expression in Eq. (37) with $\Lambda_i = \Lambda$ up to order z_k^2 , where $z_k = k \cdot p / |k||p|$, which yields

$$\mathcal{F}_i(k^2, z_k) = \sum_{j=1}^N U_j \Lambda^{2n_j} \left[\frac{1}{(k^2 + \Lambda^2)^{n_j}} - \frac{k^2 m_\pi^2 z_k^2 n_j (n_j + 1) (5 + 2\sigma_j)}{30(k^2 + \Lambda^2)^{n_j+2}} \right], \quad (40)$$

and use the Chebyshev expansion of the scalar amplitudes E_π , F_π , G_π , and H_π ,

$$\mathcal{F}_i(k^2, k \cdot p) = \sum_{m=0}^{\infty} \mathcal{F}_{im}(k^2) U_m(z_k), \quad (41)$$

keeping terms up to z_k^2 of the Chebyshev polynomials $U_m(z_k)$ of the second kind. Fitting the parameters U_j , n_j ,

and σ_j for $N = 4$ in Eq. (40), we obtain the values listed in Table III.

We present our results obtained with the theoretical frameworks detailed in Sec. III for the PDF and FF in Figs. 2 and 3, respectively, from which we infer that the predictions of the semiperturbative CF model and of the QC model are qualitatively and even quantitatively comparable. The only notable differences occur for $x \gtrsim 0.5$, where the distributions peak at very similar momentum fractions but the PDF and FF of the CF model fall off more slowly. In Fig. 4, the full FF (8) is compared to the normalized and unnormalized elementary FF, $\hat{d}_u^\pi(z)$ and $d_u^\pi(z)$. Clearly, for values smaller than $z \approx 0.75$ the full fragmentation is similarly enhanced in both calculations and, as expected, one recovers the normalized elementary FF in the limit of $z \rightarrow 1$.

We obtain the moments of the full FF using Eq. (11). For both approaches, we find

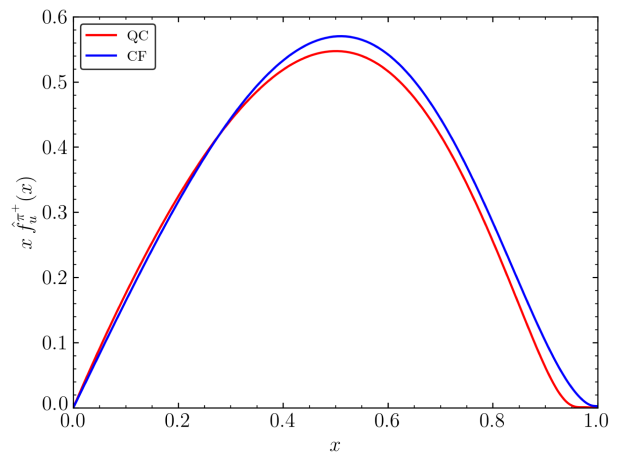


FIG. 2. Normalized PDF computed with Eq. (2), the QC and CF frameworks.

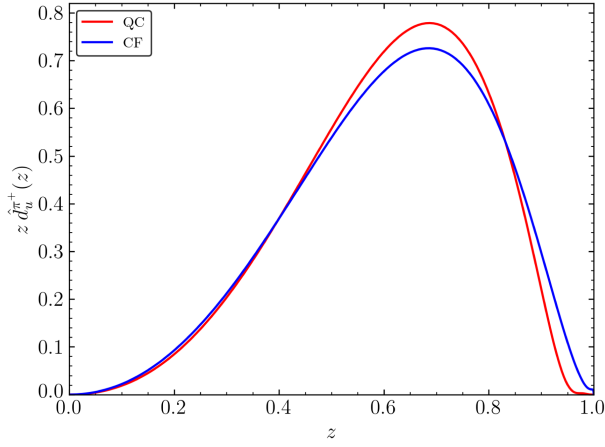


FIG. 3. Normalized FF computed with Eq. (6), the QC and CF frameworks.

$$\langle z \rangle_{D_0^\pi}^{\text{QC}} = 0.876, \quad \langle z \rangle_{D_1^\pi}^{\text{QC}} = 0.482, \quad \langle z \rangle_{D_u^{\pi^+}}^{\text{QC}} = 0.453, \quad (42)$$

$$\langle z \rangle_{D_0^\pi}^{\text{CF}} = 0.848, \quad \langle z \rangle_{D_1^\pi}^{\text{CF}} = 0.480, \quad \langle z \rangle_{D_u^{\pi^+}}^{\text{CF}} = 0.443. \quad (43)$$

Consistent with what we observe in Fig. 4, the CF model yields a full FF that is slightly suppressed compared with that in the QC model. This is reflected in the lower $\langle z \rangle_{D_u^{\pi^+}}$ value in Eq. (43).

The renormalization scale of our calculations is determined such that the momentum fraction carried by the valence quarks agrees with the first moment of a πN Drell-Yan analysis, $2\langle x \rangle_v = 0.47(2)$ [81,82] at a scale $Q = 2$ GeV. We do so by matching the moments using next-to-leading order Dokshitzer-Gribov-Lipatov-Altarelli-Parisi evolution [83] including strange- and charm-quark mass thresholds from the experimental scale Q to our model

scale Q_0 . We find for the QC model calculation $Q_0 = 0.77$ GeV and for the CF model $Q_0 = 0.73$ GeV.

V. CONCLUSIONS

We computed the PDF and FF of the pion using two different gluon dressing models within the same framework of gap and bound-state equations: the QC and the CF gluon models. We restricted ourselves to the massless pion, as the BSA in the CF approach is known in the chiral limit only [43]. Our main goal was to verify whether a calculation with the massive CF propagator reproduces the distribution and fragmentation functions obtained with the QC model. It turns out that both calculations are not only in good qualitative agreement, but also display remarkable quantitative consistency. Given that the pion PDF in Ref. [80] compares favorably with experiment, these findings support the validity of the CF model as an effective, dynamical chiral symmetry breaking approach that can be used to study the internal momentum distribution of mesons and possibly baryons in the future.

The present study can be extended beyond the chiral limit once the BSA of the physical pion in the CF model is known. This would allow for a direct comparison with a large body of work within the DSE approach. Moreover, the NJL model has been used to obtain the elementary FFs of the pion, kaon, and ρ mesons and to couple their full FFs [49,84,85]. A similar study in the CF framework requires the computation of the corresponding BSAs of these bound states, from which we can derive transverse-momentum-dependent distribution and fragmentation functions and further test the robustness of the CF model.

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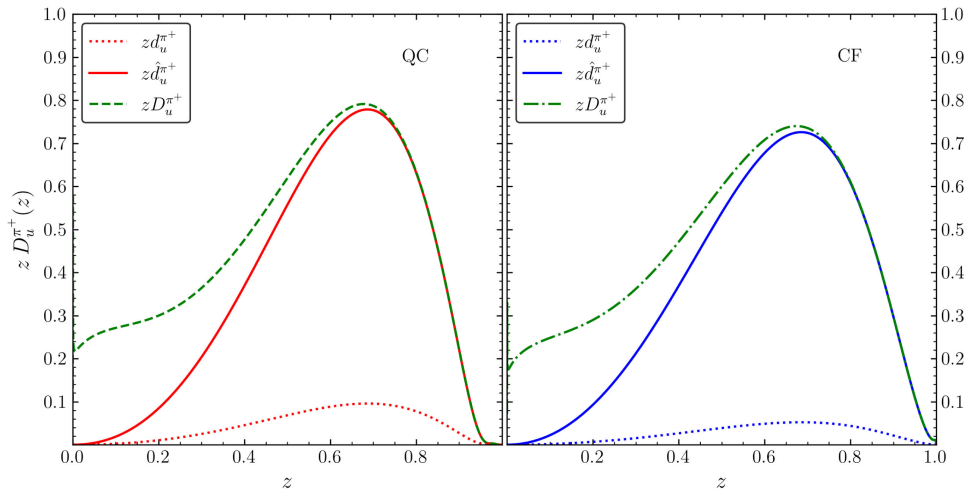


FIG. 4. Comparison of the (un)normalized elementary and full fragmentation functions, Eqs. (6) (dotted line), (7) (solid line), and (8) (dashed line). Left: predictions of the QC model III A. Right: predictions of the CF model, Sec. III B.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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