

Covariant extrinsic curvature expansion of the nonlocal effective action for a massless scalar field on a manifold with boundary

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(Received 19 May 2026; accepted 23 June 2026; published 15 July 2026)

We study the nonlocal effective action of a massless scalar field defined on a flat manifold with a curved boundary. Using a heat-kernel approach, we derive a covariant expansion of the nonlocal contribution to quadratic order in the extrinsic curvature tensor. Our construction provides a geometric framework that both reproduces earlier results obtained for Monge-patch embeddings and extends them to more general surfaces that need not admit a global Monge-patch description. The expansion is valid in the regime where gradients of the extrinsic curvature dominate over nonlinear curvature effects. As an application, we compute the particle-creation rate for an oscillating deformed ring in $2 + 1$ dimensions and an oscillating deformed sphere in $3 + 1$ dimensions.

DOI: [10.1103/ysg33-72z3](https://doi.org/10.1103/ysg33-72z3)

I. INTRODUCTION

The one-loop effective action of a quantum field encodes, in a single functional, two qualitatively different kinds of information. Its analytic, local part controls the ultraviolet divergences and is absorbed into the renormalization of the bare couplings of the theory. Its nonanalytic, nonlocal part is finite, calculable, and physically responsible for genuinely quantum phenomena: vacuum polarization, conformal anomalies, Hawking radiation, the Casimir force, particle creation by time-dependent backgrounds, and dissipative backreaction on classical sources [1–6]. While the local sector is by now textbook material, controlled covariantly through the heat-kernel or DeWitt-Schwinger expansion [7–9], the nonlocal sector is harder to extract because it is, by definition, invisible to any finite-order expansion in derivatives.

A useful strategy to access the nonlocal sector while remaining within the heat-kernel framework was developed by Vilkovisky and coworkers, and later by Barvinsky and Vilkovisky [10–16]. Rather than truncating the heat-kernel expansion at a fixed order in proper time, one truncates at a fixed order in some background invariant (typically curvature). The local divergent coefficients of the heat-kernel expansion, organized in this way, then fix (by analyticity) the full nonlocal kernel that multiplies the chosen invariant. The same logic was later refined and applied in a variety of

contexts: running of Newton’s constant and noninteger powers of the d’Alembertian [17–19], the conformal anomaly action [20–22], nonlocal cosmology and gravitational radiation [23–28], and most recently the construction of effective actions in D dimensions and in nontrivial cosmological and astrophysical backgrounds [29–31].

Once the field lives on a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, the heat-kernel expansion picks up an additional set of boundary contributions, with coefficients that encode the local boundary condition and depend only on the extrinsic geometry of $\partial\mathcal{M}$ and its covariant derivatives [7,8,32]. The corresponding effective action governs much of the physics of mirrors, films, and dielectrics. Its real part generates the static Casimir energy of arbitrary geometries [33–36]; its imaginary part, when the boundary is allowed to move, generates the dynamical Casimir effect, the spontaneous emission of correlated pairs of quanta out of the vacuum by an accelerated mirror, conjectured by Moore [37] and computed explicitly by Fulling and Davies for one-dimensional mirrors [38,39]. The dynamical Casimir effect has since been studied for a wide range of geometries, boundary conditions, and quantum statistics and has been reviewed extensively [40–43].

Two of us recently considered precisely this dynamical setting, working in a flat $(d + 1)$ -dimensional bulk with a single moving boundary supporting Dirichlet [44] or Neumann [45] conditions. The boundary in those works was parametrized as a Monge patch, that is, as a global graph $x_{d+1} = \psi(y^a)$ of a height function over a hyperplane. Within that parametrization, the nonlocal effective action could be reduced to a quadratic form in ψ whose kernel, for even d , contains a logarithm of the boundary Laplacian and, for odd d , a fractional power of it: exactly the nonanalytic structure responsible for the imaginary part and the radiation rate.

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The Monge-patch parametrization, however, cannot globally describe systems with closed boundaries (cylinders, spheres, tori), nor surfaces with overhang, multivalued sections, or nontrivial topology; and even when it does apply, the result, written in terms of ψ and flat Cartesian derivatives, conceals the geometric content of the answer. In the present work we lift the analysis to a manifestly covariant, geometric formulation in which the natural variable is the extrinsic curvature tensor K_{ab} of $\partial\mathcal{M}$ and the natural building blocks are its diffeomorphism-invariant scalars. Using the standard boundary heat-kernel coefficients of Vassilevich [7] and Branson *et al.* [7,32] together with the reconstruction procedure [10,17], we derive the unique nonlocal contribution to the effective action at quadratic order in K_{ab} , valid in the regime in which gradients of K_{ab} dominate over higher powers of K_{ab} itself, $\nabla\nabla K \gg K^3$, in close analogy with the Barvinsky-Vilkovisky expansion of the bulk effective action in semiclassical gravity. We show that the Monge-patch results [44,45] are recovered as a special case, providing both a nontrivial cross-check and, by comparison, the determination of the overall coefficient κ_d in odd d , a constant which the heat-kernel coefficients alone are not able to fix.

The covariant formulation has direct physical consequences. For an arbitrary oscillating boundary, the extrinsic curvature appears bilinearly in the action through a kernel of the form $(-\nabla^2)^{d/2-1} \log(-\nabla^2)$ in even d and $(-\nabla^2)^{d/2-1}$ in odd d , where ∇^2 is the Laplace-Beltrami operator of the worldtube $\partial\mathcal{M}$ itself, regardless of whether that worldtube admits a global Monge description. As applications, we compute the particle-creation rate for an oscillating deformed cylinder (“pulsating ring”) in $2+1$ dimensions and for an oscillating multipolar deformation of a sphere in $3+1$ dimensions. The latter exhibits a clean mode-by-mode threshold structure: the multipole ℓ radiates only above a characteristic gap frequency $\omega_\ell = \sqrt{\ell(\ell+1)}/R$ set by the spherical-harmonic eigenvalue, and the rates for Dirichlet and Neumann boundary conditions differ by a fixed numerical ratio.

The paper is organized as follows. In Sec. II we collect the heat-kernel ingredients and introduce our geometric conventions. In Sec. III we derive the nonlocal effective action at quadratic order in K_{ab} , separately for even and odd boundary dimensions. In Sec. IV we make contact with the Monge-patch results of [44,45] and use them to fix the overall coefficient of the odd- d result. In Sec. V we work out two examples in detail: the oscillating ring (Sec. VA) and the oscillating multipolar sphere (Sec. VB). Section VI contains our conclusions; the Appendix relates $\text{Im } \Gamma$ to the Bogoliubov-coefficient computation of [46] and provides a consistency check of our formulas.

II. HEAT KERNEL ON MANIFOLDS WITH BOUNDARY

In this section we review some basic aspects of the heat-kernel expansion of the effective action and introduce the

geometric notions we will use. We consider a free real scalar field ϕ with mass m on a $(d+1)$ -dimensional manifold $\mathcal{M}$. From now on, we will refer to $\mathcal{M}$ as the bulk, and we assume that $\mathcal{M}$ is a Riemannian manifold, i.e., with Euclidean signature. Although our main interest will be the massless case, we keep m at this stage, where it can also be regarded as an infrared regulator. The equation of motion for the scalar field in the bulk is

$$\underbrace{(-\nabla^\mu \nabla_\mu + m^2)}_{:=\hat{\mathcal{O}}} \phi = \left(-\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu) + m^2 \right) \phi = 0, \quad (1)$$

where $g_{\mu\nu} = g_{\mu\nu}(x)$ is the metric tensor on $\mathcal{M}$, $g = \det g_{\mu\nu}$, and ∇_μ is the associated covariant derivative.

We also assume that the bulk has a smooth, nonempty boundary $\partial\mathcal{M}$. This boundary is a codimension-one submanifold of $\mathcal{M}$, so that $\dim \partial\mathcal{M} = d$. Moreover, we take $\partial(\partial\mathcal{M}) = \emptyset$. On the boundary, the scalar field is required to satisfy one of the following local boundary conditions:

Dirichlet boundary condition (D): $\phi|_{\partial\mathcal{M}}(y) = 0$,

Neumann boundary condition (N): $n^\mu \nabla_\mu \phi|_{\partial\mathcal{M}}(y) = 0$

for all $y \in \partial\mathcal{M}$, where n^μ is the outward-pointing unit normal to $\partial\mathcal{M}$. We use x^μ , with $\mu = 1, \dots, d+1$, to denote coordinates on the bulk, and y^a , with $a = 1, \dots, d$, to denote coordinates on the boundary. Greek indices refer to bulk coordinates, whereas Latin indices refer to boundary coordinates.

Within the path-integral formalism, the one-loop effective action Γ of the theory can be written as

$$\Gamma^{(D,N)} = \frac{1}{2} \text{Tr}[\log \hat{\mathcal{O}}^{(D,N)}], \quad (2)$$

where the boundary conditions are encoded in the domain of the operator $\hat{\mathcal{O}}$ defined in Eq. (1). Equivalently, the trace is taken over the modes satisfying the chosen boundary condition. In what follows, we will omit the superscripts referring to the boundary conditions when no confusion can arise. The logarithm of the operator $\hat{\mathcal{O}}$ can also be expressed formally as an integral over proper time,

$$\log \hat{\mathcal{O}} = - \int_0^\infty \frac{ds}{s} e^{-s\hat{\mathcal{O}}}, \quad (3)$$

up to an additive constant independent of $\hat{\mathcal{O}}$. Therefore,

$$\begin{aligned} \Gamma &= \frac{1}{2} \text{Tr}[\log(-\nabla^\mu \nabla_\mu + m^2)] \\ &= -\frac{1}{2} \int_0^\infty \frac{ds}{s} e^{-sm^2} \text{Tr}[e^{s\nabla^\mu \nabla_\mu}]. \end{aligned} \quad (4)$$

The trace in Eq. (4) is the heat trace of the operator $-\nabla^\mu \nabla_\mu$. It can be computed from the coincidence limit of the heat

kernel $\langle x|e^{s\nabla^\mu\nabla_\mu}|x'\rangle$ [7], which satisfies the heat equation

$$(\partial_s - \nabla^\mu\nabla_\mu)\langle x|e^{s\nabla^\mu\nabla_\mu}|x'\rangle = 0, \quad (5)$$

where the covariant derivatives act on the coordinate x , with the initial condition

$$\lim_{s \rightarrow 0^+} \langle x|e^{s\nabla^\mu\nabla_\mu}|x'\rangle = \frac{1}{\sqrt{g(x)}} \delta^{d+1}(x - x'). \quad (6)$$

For $s \rightarrow 0^+$, the heat trace admits the asymptotic expansion [1,7,8]

$$\begin{aligned} \text{Tr}[e^{s\nabla^\mu\nabla_\mu}] &\approx (4\pi s)^{-\frac{d+1}{2}} \sum_{n=0}^{\infty} s^{n/2} \left(\int_{\mathcal{M}} a_{n/2}(x) \right. \\ &\quad \left. + \sqrt{4\pi} \int_{\partial\mathcal{M}} b_{n/2}(y) \right), \end{aligned} \quad (7)$$

where the symbol $\approx$ denotes equality in the sense of a short-proper-time asymptotic expansion. The quantities $a_{n/2}(x)$ and $b_{n/2}(y)$, with $n \in \mathbb{N}_0$, are the heat-kernel, or DeWitt-Schwinger, coefficients. The integral on $\mathcal{M}$ implicitly has the volume element of the bulk, $\sqrt{g}d^{d+1}x$, while the integral on $\partial\mathcal{M}$ has the volume element of the boundary, $\sqrt{h}d^d y$, with $h = \det h_{ab}$, where h_{ab} is the metric induced on $\partial\mathcal{M}$. The coefficients can be obtained, for example, by substituting the expansion equation (7) into the heat equation in Eq. (5) before taking $x = x'$, together with the corresponding boundary conditions, but we do not pursue that route here.

By inserting the short-proper-time expansion in Eq. (7) into Eq. (4) and exchanging the sum with the integrals at the formal level, we obtain the heat-kernel contribution to the effective action,

$$\begin{aligned} \Gamma^{\text{UV}} &= -\frac{(4\pi)^{-\frac{d+1}{2}}}{2} \int_{\mathcal{M}} \int_0^\infty ds s^{-\frac{d+3}{2}} e^{-sm^2} \left(\sum_{n=0}^{\infty} s^{\frac{n}{2}} a_{n/2}(x) \right) \\ &\quad - \frac{(4\pi)^{-\frac{d}{2}}}{2} \int_{\partial\mathcal{M}} \int_0^\infty ds s^{-\frac{d+3}{2}} e^{-sm^2} \left(\sum_{n=0}^{\infty} s^{\frac{n}{2}} b_{n/2}(y) \right). \end{aligned} \quad (8)$$

Equation (8) should not be understood as an exact expression for the full effective action over the whole range of proper time. Rather, it represents the contribution generated by the short-proper-time expansion, which controls the UV divergences and will be used in the next section to reconstruct the corresponding nonlocal terms. The coefficients $a_{n/2}$ and $b_{n/2}$ are local geometric quantities and do not depend on s or m . Within this formalism, for the boundary conditions considered here, the dependence on the boundary condition is encoded in the boundary coefficients $b_{n/2}$. These coefficients are local scalar quantities constructed from the geometry of $\partial\mathcal{M}$ and differ for Dirichlet and Neumann boundary conditions.

In what follows, we take the bulk to be flat. Then the relevant boundary invariants can be written in terms of the

extrinsic curvature tensor and its covariant derivatives. We define the extrinsic curvature as

$$K_{ab} = \frac{\partial x^\mu}{\partial y^a} \frac{\partial x^\nu}{\partial y^b} \nabla_\mu n_\nu, \quad (9)$$

where $x^\mu = x^\mu(y)$ is the embedding of the boundary in the bulk. Latin indices are raised and lowered with the induced metric on $\partial\mathcal{M}$, and ∇_a denotes the covariant derivative compatible with that metric. We also define the mean curvature K as the trace of the extrinsic curvature tensor,

$$K = h^{ab} K_{ab}. \quad (10)$$

For both Dirichlet and Neumann boundary conditions, the bulk coefficients of the heat-kernel expansion are

$$a_0^{(\text{D,N})} = 1, \quad a_{n/2}^{(\text{D,N})} = 0, \quad n \geq 1, \quad (11)$$

while the first boundary coefficients are, for the Dirichlet boundary condition [7,32],

$$\begin{aligned} b_0^{(\text{D})} &= 0, \\ b_{1/2}^{(\text{D})} &= -\frac{1}{4}, \\ b_1^{(\text{D})} &= \frac{1}{3\sqrt{4\pi}} K, \\ b_{3/2}^{(\text{D})} &= \frac{1}{384} (10K_{ab}K^{ab} - 7K^2), \\ b_2^{(\text{D})} &= \frac{1}{360\sqrt{4\pi}} \left[24\nabla^a \nabla_a K + \frac{1}{21} (40K^3 - 264KK^{ab}K_{ab} \right. \\ &\quad \left. + 320K_a{}^b K_b{}^c K_c{}^a) \right], \\ b_{5/2}^{(\text{D})} &= -\frac{1}{2048} \nabla^c K^{ab} \nabla_c K_{ab} + \mathcal{O}(K^4). \end{aligned} \quad (12)$$

For the Neumann boundary condition, they are

$$\begin{aligned} b_0^{(\text{N})} &= 0, \\ b_{1/2}^{(\text{N})} &= \frac{1}{4}, \\ b_1^{(\text{N})} &= \frac{1}{3\sqrt{4\pi}} K, \\ b_{3/2}^{(\text{N})} &= \frac{1}{384} (2K_{ab}K^{ab} + 13K^2), \\ b_2^{(\text{N})} &= \frac{1}{360\sqrt{4\pi}} \left[24\nabla^a \nabla_a K + \frac{1}{21} (280K^3 - 168KK^{ab}K_{ab} \right. \\ &\quad \left. + 224K_a{}^b K_b{}^c K_c{}^a) \right], \\ b_{5/2}^{(\text{N})} &= -\frac{19}{2048} \nabla^c K^{ab} \nabla_c K_{ab} + \mathcal{O}(K^4). \end{aligned} \quad (13)$$

III. NONLOCAL EFFECTIVE ACTION FOR A MASSLESS THEORY

Our goal is to obtain the nonlocal part of the effective action in Eq. (8) for a massless scalar field in a flat bulk, with either Dirichlet or Neumann boundary conditions. The nonlocal part of the effective action contains the contributions that cannot be expressed in terms of analytic functions of covariant derivatives. It therefore encodes the nonanalyticities of the action. In particular, after analytic continuation to the corresponding Lorentzian theory, the imaginary part of the effective action is contained in this nonlocal piece, making it relevant for the study of particle creation.

To this end, we employ a procedure introduced in previous works [10,17], in which the nonlocal part of the effective action is reconstructed from the divergences associated with a finite number of heat-kernel coefficients. In those works, the method was applied to curved bulks without boundary, and the resulting actions correspond to semiclassical gravity theories. Here, instead, we apply this technique to a theory in a flat bulk with a nontrivial boundary. The results obtained in this section can be used, for example, to study particle creation induced by the dynamical Casimir effect for a broad class of possible boundaries.

As mentioned above, the key idea in Refs. [10,17] is to obtain the nonlocal part of the effective action using only the information contained in the divergent coefficients of the heat-kernel expansion. Let us focus on the boundary coefficients $b_{n/2}$ in Eq. (8) and perform a dimensional analysis before proceeding. In units where energy has dimension +1, the relevant dimensions are

$$[\nabla_a] = [K_{ab}] = 1, \quad [b_{n/2}] = n - 1. \quad (14)$$

Since each coefficient $b_{n/2}$ is a local scalar constructed from covariant geometric quantities, at fixed n it may contain K , K_{ab} , covariant derivatives, and their contractions, with total dimension $n - 1$. No additional geometric objects, such as the normal vector to $\partial\mathcal{M}$, appear explicitly in the $b_{n/2}$'s, since any occurrence of them can be rewritten in terms of K_{ab} and covariant derivatives on $\partial\mathcal{M}$. In a flat bulk, the local geometric invariants of a smooth codimension-one boundary can therefore be written in terms of K_{ab} and its covariant derivatives.

To compute the effective action, one would have to sum the series in Eq. (8); however, in practice, the full series is not available. To proceed, instead of applying a direct truncation in the heat-kernel order, we organize the expansion according to the number of powers of the extrinsic curvature.¹ Let us analyze this point in more detail.

¹From now on, by powers of the extrinsic curvature we mean powers of K and/or K_{ab} .

Zeroth-order terms provide only contributions proportional to the volumes of $\mathcal{M}$ and $\partial\mathcal{M}$, and they are absorbed into the bare constants of the theory through renormalization. At first order in the extrinsic curvature, one obtains either K itself, integrated over $\partial\mathcal{M}$, which gives a local contribution to the effective action, or terms such as $\nabla^a \nabla_a \dots \nabla^b \nabla_b K$ and $\nabla^a \nabla_a \dots \nabla^b \nabla^c K_{bc}$, which are total derivatives and do not contribute after integration over $\partial\mathcal{M}$, since $\partial(\partial\mathcal{M}) = \emptyset$. We therefore focus on the quadratic contributions in the extrinsic curvature, which give the first nontrivial contribution to the nonlocal part of the effective action.

It is useful to note that the integer-indexed coefficients b_n do not contain quadratic terms in the extrinsic curvature. Indeed, before contractions with derivatives, a quadratic term contains 0, 2, or 4 free boundary indices carried by the two extrinsic curvatures. However, dimensional analysis implies that the quadratic contributions to the integer-indexed coefficients would contain an odd number of derivatives, and no scalar covariant term of this form can be constructed. Therefore, at quadratic order in the extrinsic curvature, the effective action can be written schematically as

$$\Gamma^{\text{UV}(2)} = -\frac{1}{2(4\pi)^{d/2}} \int_0^\infty \frac{ds}{s^{d/2}} e^{-sm^2} \sum_{k=0}^\infty s^k \sum_i \int_{\partial\mathcal{M}} \mathcal{Q}_{k,i}, \quad (15)$$

where we have used the fact that only heat-kernel coefficients with half-integer index, starting from $b_{3/2}$, contribute at this order. Here $\mathcal{Q}_{k,i}$ denotes a generic scalar term quadratic in K_{ab} with $2k$ covariant derivatives, together with a real coefficient determined by the corresponding heat-kernel coefficient $b_{k+3/2}$. For instance, from $b_{3/2}^{(D)}$ in Eq. (12), we can identify $\mathcal{Q}_{0,1}^{(D)} = \frac{10}{384} K^{ab} K_{ab}$ and $\mathcal{Q}_{0,2}^{(D)} = -\frac{7}{384} K^2$, while from $b_{5/2}^{(D)}$ we obtain $\mathcal{Q}_{1,1}^{(D)} = -\frac{1}{2048} \nabla^c K^{ab} \nabla_c K_{ab}$.

Keeping only the terms quadratic in the extrinsic curvature amounts to working in a regime where the curvature is perturbatively small, while its derivatives are not necessarily treated as small. More precisely, the approximation consists of neglecting cubic and higher powers of the extrinsic curvature in comparison with terms containing two powers of the curvature and an arbitrary number of covariant derivatives. Schematically, this requires

$$\nabla \nabla K \gg K^3. \quad (16)$$

In this sense, Eq. (15) represents the leading contribution to the heat-kernel effective action in an expansion in powers of the extrinsic curvature.

Despite the apparently complicated form of Eq. (15), all quadratic terms can be reduced to two simpler structures.

By repeated integration by parts on $\partial\mathcal{M}$, and using the Codazzi identity for a hypersurface embedded in a flat bulk,

$$\nabla_a K_{bc} - \nabla_b K_{ac} = 0, \quad (17)$$

all the quadratic structures $\mathcal{Q}_{k,i}$ can be written, under the integral over $\partial\mathcal{M}$, in terms of

$$K(-\nabla^2)^k K \quad \text{and} \quad K^{ab}(-\nabla^2)^k K_{ab}. \quad (18)$$

Here $\nabla^2 \equiv \nabla^a \nabla_a$ is the Laplace-Beltrami operator on the boundary.

Moreover, for terms with $k \geq 1$, the covariant derivatives can be commuted at the price of generating only higher-order terms in the extrinsic curvature. Indeed, when acting on a boundary tensor, the commutator of covariant derivatives produces the intrinsic Riemann tensor of the boundary, so that, schematically, $[\nabla, \nabla] \sim R$. On the other hand, for a hypersurface embedded in a flat bulk, the Gauss identity gives

$$R_{abcd} = K_{ac}K_{bd} - K_{ad}K_{bc}, \quad (19)$$

so that schematically $[\nabla, \nabla] \sim R \sim K^2$. Therefore, when such commutators appear in terms already quadratic in K_{ab} , they generate contributions of order $\mathcal{O}(K^4)$. Thus, up to terms of order $\mathcal{O}(K^4)$ and total derivatives, all the contributions with $k \geq 1$ can then be written in terms of a single scalar structure,

$$K(-\nabla^2)^k K. \quad (20)$$

To complete the reduction, we have to treat separately the derivative-free terms coming from the coefficient $b_{3/2}$, namely, K^2 and $K^{ab}K_{ab}$. By taking a covariant derivative of the Codazzi identity in Eq. (17) and contracting the corresponding indices, we obtain

$$\nabla^2 K_{ab} = \nabla_a \nabla_b K + \mathcal{O}(K^3). \quad (21)$$

Therefore,

$$K_{ab} = \frac{\nabla_a \nabla_b}{\nabla^2} K + \mathcal{O}(K^3), \quad (22)$$

up to the addition of a homogeneous contribution annihilated by ∇^2 . This homogeneous contribution is not fixed by the local equation, but by the prescription chosen for the inverse of the Laplace-Beltrami operator. In what follows, we choose this prescription so as to discard such homogeneous modes, in analogy with the treatment of nonlocal curvature invariants in Ref. [14]. Using this expression inside the integral, and integrating by parts while keeping the same prescription for the inverse Laplace-Beltrami operator, one obtains

$$\int_{\partial\mathcal{M}} K^{ab} K_{ab} = \int_{\partial\mathcal{M}} K^2 + \mathcal{O}(K^4). \quad (23)$$

Therefore, all the quadratic structures in Eq. (15) can be reduced to a single quadratic form, namely,

$$\Gamma^{\text{UV}(2)} = -\frac{1}{2(4\pi)^{d/2}} \int_{\partial\mathcal{M}} \int_0^\infty \frac{ds}{s^{d/2}} \times e^{-sm^2} \left[\sum_{k=0}^\infty \alpha_k K(-\nabla^2)^k K s^k \right], \quad (24)$$

where each coefficient α_k is obtained from the heat-kernel coefficient $b_{k+3/2}$, after the manipulations described above. This reduction allows us to read the first coefficients α_k directly from the boundary heat-kernel coefficients listed above. For instance, from Eq. (12) we obtain

$$\begin{aligned} \alpha_0^{(\text{D})} &= \frac{10}{384} - \frac{7}{384} = \frac{1}{128}, \\ \alpha_1^{(\text{D})} &= -\frac{1}{2048}; \end{aligned} \quad (25)$$

while from Eq. (13) we obtain

$$\begin{aligned} \alpha_0^{(\text{N})} &= \frac{2}{384} + \frac{13}{384} = \frac{5}{128}, \\ \alpha_1^{(\text{N})} &= -\frac{19}{2048}. \end{aligned} \quad (26)$$

The reduction of the generic form of the second-order effective action in Eq. (15) to a single scalar structure in Eq. (24) is crucial to capture the associated nonlocal contribution. Following the procedure presented in Refs. [10,17] to extract the nonlocal contribution to the effective action, we take the massless limit in Eq. (24) and, at least formally, sum the series. This gives

$$\Gamma^{\text{UV}(2)} = -\frac{1}{2(4\pi)^{d/2}} \int_{\partial\mathcal{M}} \int_0^\infty \frac{ds}{s^{d/2}} (K \alpha(-s\nabla^2) K), \quad (27)$$

where

$$\alpha(\xi) = \sum_{k=0}^\infty \alpha_k \xi^k. \quad (28)$$

In taking the limit $m \rightarrow 0$, we assume only that the integral

$$\int_0^\infty \frac{d\xi}{\xi^{d/2}} \alpha(\xi) \quad (29)$$

converges at the upper limit.

To extract the nonlocal contribution, or equivalently, the terms involving nonholomorphic functions of derivatives, we do not need to know the complete function $\alpha(\xi)$ in the

even-dimensional case. Instead, we can study the ultra-violet divergences in Eq. (27), which arise from the lower limit of the proper-time integral, and deduce the corresponding nonlocal contribution from them. We therefore introduce a cutoff Λ with dimensions of energy and write

$$\int_{\frac{1}{\Lambda^2}}^{\infty} \frac{ds}{s^{d/2}} \alpha(-s\nabla^2) = (-\nabla^2)^{\frac{d}{2}-1} \int_{\frac{-\nabla^2}{\Lambda^2}}^{\infty} \frac{d\xi}{\xi^{d/2}} \alpha(\xi), \quad (30)$$

where we have used the fact that $-\nabla^2$ is a non-negative operator. Possible zero modes are understood to be treated according to the prescription specified above. We identify two possible structures in the limit $\Lambda \rightarrow \infty$.

A. Even-dimensional boundary

For even d , we obtain

$$\begin{aligned} \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{ds}{s^{d/2}} \alpha(-s\nabla^2) &= \text{const} \times (-\nabla^2)^{\frac{d}{2}-1} \\ &\quad - \sum_{k=0}^{\frac{d}{2}-2} \frac{2\alpha_k}{2(k+1)-d} \Lambda^{d-2k-2} (-\nabla^2)^k \\ &\quad - \alpha_{\frac{d}{2}-1} (-\nabla^2)^{\frac{d}{2}-1} \log\left(\frac{-\nabla^2}{\Lambda^2}\right). \end{aligned} \quad (31)$$

Thus, all the nonlocal structure is logarithmic,

$$\begin{aligned} \int_0^{\infty} \frac{ds}{s^{d/2}} \alpha(-s\nabla^2) &= -\alpha_{\frac{d}{2}-1} (-\nabla^2)^{\frac{d}{2}-1} \log\left(\frac{-\nabla^2}{\mu^2}\right) \\ &\quad + \text{local divergent terms}, \end{aligned} \quad (32)$$

where we have added and subtracted the local term $\alpha_{\frac{d}{2}-1} \log(\mu^2) (-\nabla^2)^{\frac{d}{2}-1}$, with μ a finite energy scale. Therefore, the renormalized nonlocal effective action reads

$$\Gamma_{\text{NL}}^{(2)} = \frac{1}{2} \kappa_d \int_{\partial\mathcal{M}} K (-\nabla^2)^{\frac{d}{2}-1} \log\left(\frac{-\nabla^2}{\mu^2}\right) K, \quad (33)$$

where we have defined $\kappa_d := \alpha_{\frac{d}{2}-1}/(4\pi)^{d/2}$ for even d to streamline the notation.

B. Odd-dimensional boundary

For odd d , we obtain

$$\begin{aligned} \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{ds}{s^{d/2}} \alpha(-s\nabla^2) &= \text{const} \times (-\nabla^2)^{\frac{d}{2}-1} \\ &\quad - \sum_{k=0}^{\frac{d-3}{2}} \frac{2\alpha_k}{2(k+1)-d} \Lambda^{d-2k-2} (-\nabla^2)^k. \end{aligned} \quad (34)$$

Again, the divergent terms are proportional to integer powers of ∇^2 and therefore give local contributions to the unrenormalized effective action. However, in this case the nonlocal structure is encoded in the fractional power of the Laplace-Beltrami operator appearing in the first term on the right-hand side of Eq. (34). Its coefficient cannot be determined from the ultraviolet expansion alone, because it depends on the upper limit of the proper-time integral. To compute this constant, one needs to know the function $\alpha(\xi)$ for all values of its argument; the first terms of its small- ξ expansion, unlike in the even-dimensional case, are not sufficient. Thus, in this case we can write the renormalized nonlocal effective action as

$$\Gamma_{\text{NL}}^{(2)} = \frac{1}{2} \kappa_d \int_{\partial\mathcal{M}} K (-\nabla^2)^{\frac{d}{2}-1} K, \quad (35)$$

where the constant κ_d for odd d is given by

$$\kappa_d = -\frac{1}{(4\pi)^{d/2}} \int_0^{\infty} \frac{d\xi}{\xi^{d/2}} \left(\alpha(\xi) - \sum_{k=0}^{\frac{d-3}{2}} \alpha_k \xi^k \right). \quad (36)$$

In both even and odd dimensions, we obtained the nonlocal contributions to the effective action by introducing a UV cutoff in the unrenormalized action. In other words, in either case the form of the nonlocal operator is encoded in the divergent contributions, although the proportionality constant κ_d in Eq. (35) cannot be determined solely from the divergent terms in the heat-kernel expansion. Therefore, the expressions in Eqs. (33) and (35) encode the full nonlocal structure of the effective action at quadratic order in the extrinsic curvature, even though they were obtained from Eq. (27), which describes only the UV sector of the complete effective action.

With the coefficients in Eqs. (25) and (26), we can express the nonlocal part of the effective action for both Dirichlet and Neumann boundary conditions on boundaries of dimensions $d = 2$ and $d = 4$. Higher even-dimensional boundaries require additional heat-kernel coefficients, whose computation becomes increasingly involved. On the other hand, for odd-dimensional boundaries, the heat-kernel coefficients alone are not enough to determine the overall coefficient of the nonlocal effective action. However, in the next section we will show that the coefficients in arbitrary dimension can be determined by comparison with previous results [44,45], where the nonlocal part of the effective action was computed for a more restricted class of boundaries.

IV. MONGE PATCH

In this section we unify the results of this work with those of Refs. [44,45], showing that the coefficients in Eqs. (33) and (35) can be obtained from the nonlocal effective action of a massless scalar field in $d+1$

dimensions subject to Dirichlet or Neumann boundary conditions on a hypersurface parametrized as a Monge patch, with the field defined on both sides of the hypersurface. Comparing that setup to the present one, in which a massless scalar field on flat spacetime is restricted to one side of the same hypersurface, the only difference is the doubling of the available bulk volume, i.e., above and below the surface, which produces an overall factor of $\frac{1}{2}$ between the two effective actions, as we now show.

To compare the coefficients of the two effective actions for even d , we write (33) in the case that the boundary admits a single global Monge patch. Following the conventions of Refs. [44,45], we use the first d spacetime components of x^μ as parameters, $y = (x_1, \dots, x_d)$, and define the Monge patch by

$$\Sigma: x_{d+1} = \psi(y), \quad (37)$$

where $\psi(y)$ specifies the “height” of the surface, measured with respect to the hyperplane $x_{d+1} = 0$, and the y parameters are left as Cartesian coordinates. Since the target effective actions are of second order in the height function ψ , we adapt Eq. (33) to the same regime. The scalar extrinsic curvature for a Monge patch is

$$K = \partial_a \left(\frac{\partial^a \psi}{\sqrt{1 + \partial_b \psi \partial^b \psi}} \right), \quad (38)$$

where indices are contracted with the Euclidean metric δ_{ab} . We can expand Eq. (38) in powers of ψ as

$$K = \partial^2 \psi - \frac{1}{2} \partial^2 \psi \partial_a \psi \partial^a \psi - \partial^a \psi \partial^b \psi \partial_a \partial_b \psi + \dots, \quad (39)$$

where the leading term is first order in ψ , namely, $\partial^2 \psi$, with ∂^2 the Euclidean d -dimensional Laplacian in Cartesian coordinates.

Since Eq. (33) is of order 2 in K , and the leading term in K is of order 1 in ψ , it suffices to retain $K \approx \partial^2 \psi$ inside that expression to obtain the order- ψ^2 part of the effective action; equivalently, we identify $\nabla^2 \rightarrow \partial^2$. Using integration by parts, we then write $K(-\partial^2)^{\frac{d}{2}-1} \log(-\partial^2)K$ as $\psi(-\partial^2)^{\frac{d}{2}+1} \log(-\partial^2)\psi$, valid inside the integral. Therefore, for the Monge patch in Eq. (37) and up to second order in ψ , the effective action with even d in Eq. (33) reads

$$\Gamma_{\text{NL}}^{(2)} = \frac{1}{2} \kappa_d^{(\text{D,N})} \int d^d y \left[\psi(y) (-\partial^2)^{\frac{d}{2}+1} \log\left(\frac{-\partial^2}{\mu^2}\right) \psi(y) \right], \quad (40)$$

where the integral is over the full Cartesian range of the parameters y .

The target formulas are, for even values of d ,

$$\Gamma_{[\text{FG}]}^{(2)} = \frac{1}{2} \eta_d^{(\text{D,N})} \int \frac{d^d k}{(2\pi)^d} |\tilde{\psi}(k)|^2 (k^2)^{\frac{d}{2}+1} \log\left(\frac{k^2}{\mu^2}\right), \quad (41)$$

where we identify the correct functional structure using the Fourier transform convention $\tilde{\psi}(k) = \int d^d y e^{-ik \cdot y} \psi(y)$. Having

$$\eta_d^{(\text{D})} = \frac{(-1)^{\frac{d}{2}+1}}{(4\pi)^{\frac{d}{2}+1}} \frac{(\Gamma(\frac{d+1}{2}))^2}{(\frac{d}{2}+1)! d!}, \quad (42)$$

$$\eta_d^{(\text{N})} = \eta_d^{(\text{D})} + \frac{(-1)^{\frac{d}{2}+1}}{(d/2)!} (d-1) \frac{(\Gamma(\frac{d+1}{2}))^2}{2^{d+1} \pi^{\frac{d+2}{2}} \Gamma(d+1)}, \quad (43)$$

we verified, for $d = 2$ and $d = 4$, that

$$\kappa_d^{(\text{D,N})} = \frac{1}{2} \eta_d^{(\text{D,N})} \quad (44)$$

is satisfied, and the factor $\frac{1}{2}$ is due to the difference between defining the theory on both sides of the boundary and on a single side. Moreover, the relation in Eq. (44) holds for all even values of d , since the manipulation that takes Eq. (41) into Eq. (40) does not affect the prefactor of the integral, regardless of the value of d .

For an odd-dimensional boundary, we have

$$\Gamma_{[\text{FG}]}^{(2)} = \frac{1}{2} \eta_d^{(\text{D,N})} \int \frac{d^d k}{(2\pi)^d} |\tilde{\psi}(k)|^2 (k^2)^{\frac{d}{2}+1}, \quad (45)$$

with

$$\eta_d^{(\text{D})} = \frac{(-1)^{\frac{d+1}{2}}}{(2\pi)^{\frac{d+1}{2}}} \frac{((\frac{d-1}{2})!)^2}{d! (d+2)!}, \quad (46)$$

$$\eta_d^{(\text{N})} = \eta_d^{(\text{D})} + (d-1) \frac{(\Gamma(\frac{d+1}{2}))^2 \Gamma(-\frac{d}{2})}{2^{d+1} \pi^{\frac{d+2}{2}} \Gamma(d+1)}. \quad (47)$$

Although κ_d in Eq. (36) cannot be evaluated directly without the explicit form of $\alpha(\xi)$, its value can be fixed by using the Monge-patch case in the approximation $K \approx \partial^2 \psi$. This leads to the same coefficient relation as in Eq. (44). Therefore, Eq. (44) holds for both even and odd boundary dimensions.

V. EXAMPLES

In this section we consider an oscillating deformed ring in $2 + 1$ dimensions and an oscillating deformed sphere in $3 + 1$ dimensions. The two cases differ in an essential way. In Euclidean formulation, the unperturbed (static) ring corresponds to a cylinder, which has vanishing intrinsic curvature, and the structures K^2 and $K_{ab} K^{ab}$ are trivially

proportional at the background level. The sphere, on the other hand, has nontrivial intrinsic curvature already at the unperturbed level, so that the reduction of $K_{ab}K^{ab}$ to K^2 in Eq. (23) relies essentially on the presence of the nonlocal operator inside the integral.

A. Oscillating ring

Let us compute the nonlocal effective action for a cylindrical boundary in the case $d = 2$ for both Dirichlet and Neumann boundary conditions. Using cylindrical coordinates $\{r, \theta, z\}$ in the bulk $\mathbb{R}^3$, we consider the boundary surface Σ defined by

$$\Sigma: r = R(1 + \epsilon \cos(n\theta) \cos(mz)), \quad (48)$$

where $\epsilon \ll 1$ is a small dimensionless perturbative parameter, and R is the radius of the unperturbed cylinder. Choosing $\{\theta, z\}$, with $0 \leq \theta < 2\pi$ and $z \in \mathbb{R}$ as coordinates on Σ , the scalar extrinsic curvature is

$$K = K^{(0)} + \epsilon K^{(1)} + \mathcal{O}(\epsilon^2), \quad (49)$$

where the superscript indicates the order in the expansion in ϵ ,

$$K^{(0)} = -\frac{1}{R},$$

$$K^{(1)} = \frac{1}{R}(1 - n^2 - m^2 R^2) \cos(mz) \cos(n\theta), \quad (50)$$

and we have adopted here the convention that the normal points inward.

The contribution to the nonlocal effective action from Eq. (33) reads

$$\Gamma_{\text{NL}}^{(2)} = \frac{\kappa_2^{(\text{D,N})}}{2} \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz \sqrt{h} K \log\left(\frac{-\nabla^2}{\mu^2}\right) K, \quad (51)$$

where we expand in powers of ϵ by adding a second superscript denoting the order in ϵ ,

$$\Gamma_{\text{NL}}^{(2)} = \Gamma_{\text{NL}}^{(2,0)} + \Gamma_{\text{NL}}^{(2,1)} + \Gamma_{\text{NL}}^{(2,2)} + \dots \quad (52)$$

Using the convention $\log(-\nabla^2/\mu^2)\text{constant} = 0$, together with the self-adjointness of ∇^2 with respect to the usual inner product on Σ , so

$$\begin{aligned} & \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz \sqrt{h} \varphi \log\left(\frac{-\nabla^2}{\mu^2}\right) \psi \\ &= \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz \sqrt{h} \psi \log\left(\frac{-\nabla^2}{\mu^2}\right) \varphi, \end{aligned} \quad (53)$$

for any test functions φ and ψ , we find that the first nontrivial contribution to Eq. (51) comes from the term

quadratic in $K^{(1)}$, namely,

$$\Gamma_{\text{NL}}^{(2,2)} = \frac{\kappa_2^{(\text{D,N})}}{2} \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz \sqrt{h^{(0)}} K^{(1)} \log\left(\frac{-\nabla^{2(0)}}{\mu^2}\right) K^{(1)}, \quad (54)$$

where $\sqrt{h^{(0)}} = R$ and $\nabla^{2(0)} = R^{-2} \partial_\theta^2 + \partial_z^2$ are used to remain at fixed order in ϵ .

Observing that $K^{(1)}$ is an eigenfunction of $-\nabla^{2(0)}$ with eigenvalue $\frac{n^2}{R^2} + m^2$, we have, for $n \neq 0$ and $m \neq 0$,

$$\Gamma_{\text{NL}}^{(2,2)} = \frac{\kappa_2^{(\text{D,N})}}{2} \frac{\epsilon^2 \pi L (n^2 + m^2 R^2 - 1)^2}{2R} \log\left(\frac{\frac{n^2}{R^2} + m^2}{\mu^2}\right), \quad (55)$$

where $L = \int_{-\infty}^{\infty} dz$ is a total length factor. For the particular case $n = 0$ there is an additional factor 2 in Eq. (55). To pass to Lorentzian signature, we perform the Wick rotation $-\Gamma_{\text{NL}}^{(2,2)} \rightarrow i\Gamma_{\text{NL}}^{(2,2)}$, by setting $z = it$, with t the Lorentzian time and performing the replacement $m^2 \rightarrow -\omega^2 - i0^+$ with the corresponding Feynman prescription. Therefore,

$$\Gamma_{\text{NL}}^{(2,2)} = -\frac{\kappa_2^{(\text{D,N})}}{2} \frac{\epsilon^2 \pi T (n^2 - \omega^2 R^2 - 1)^2}{2R} \log\left(\frac{\frac{n^2}{R^2} - \omega^2 - i0^+}{\mu^2}\right), \quad (56)$$

where T denotes the total time interval. The effective action in Eq. (56) corresponds to the quantum field theory of a massless scalar field in flat $(2+1)$ -dimensional spacetime, living in the inner or outer region of a ring of radius R with n sinusoidal deformations oscillating at frequency ω , with Dirichlet or Neumann boundary conditions imposed at the boundary.

The range of validity of the approximation is given by $\epsilon \ll 1$ and $\nabla \nabla K \gg K^3$, which in terms of the physical parameters become $n \gg 1$ or $\omega R \gg 1$. Within this range, taking the imaginary part of Eq. (56) we get

$$\frac{\text{Im}[\Gamma_{\text{NL}}^{(2,2)}]}{T} = \frac{\kappa_2^{(\text{D,N})}}{2} \frac{\epsilon^2 \pi^2 (n^2 - \omega^2 R^2 - 1)^2}{2R} \Theta(\omega R - n), \quad (57)$$

where $\kappa_2^{(\text{D})} = \frac{1}{512\pi}$ and $\kappa_2^{(\text{N})} = \frac{5}{512\pi}$.

B. Oscillating multipolar sphere

We turn now to a closed boundary which cannot be described by a single Monge patch and on which the background extrinsic geometry is itself nontrivial: a two-sphere of radius R in $3+1$ dimensions, undergoing a small time-dependent deformation along a single spherical-harmonic mode,

$$\Sigma_t: r = \rho(t, \Omega) = R[1 + \epsilon f(t) Y_{\ell m}(\Omega)], \quad \epsilon \ll 1, \quad (58)$$

with $\Omega = (\theta, \phi)$ and $Y_{\ell m}$ a real spherical harmonic, taken as an eigenfunction of the unit-sphere Laplacian, $\nabla_{\Omega}^2 Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}$. The Euclidean worldtube is $\partial\mathcal{M} = S_R^2 \times \mathbb{R}_\tau$, so that $d = 3$ and we are in the odd-dimensional case described by Eq. (35). The case $\ell = 0$ corresponds to the breathing mode considered, by a different method, in Ref. [46] and reproduced in the Appendix; here we focus on the genuinely nontrivial multipolar modes $\ell \geq 1$.

A direct computation, performed using the level-set formula for the mean curvature of the worldtube embedded in flat Euclidean $\mathbb{R}^4$ yields, to first order in ϵ ,

$$K = \frac{2}{R} - \frac{\epsilon}{R} [R^2 \partial_t^2 h + 2h + \nabla_{\Omega}^2 h] + \mathcal{O}(\epsilon^2), \quad (59)$$

where $h(t, \Omega) := f(t)Y_{\ell m}(\Omega)$ is the (rescaled) deformation profile.² Specializing to a harmonic mode $h = f(t)Y_{\ell m}$,

$$K^{(1)}(t, \Omega) = -\frac{1}{R} [R^2 \ddot{f}(t) - (\ell-1)(\ell+2)f(t)] Y_{\ell m}(\Omega), \quad (60)$$

where we have used the algebraic identity $\ell(\ell+1) - 2 = (\ell-1)(\ell+2)$, which exhibits the well-known fact that the dipole modes ($\ell = 1$) carry no static contribution to $K^{(1)}$, since a rigid translation of the sphere leaves its extrinsic geometry unchanged, and that the lowest genuinely shape-changing oscillation is the quadrupole, $\ell = 2$, where $(\ell-1)(\ell+2) = 4$.

To find the nonlocal effective action, for $d = 3$, the relevant building block is Eq. (35), considering its first nontrivial contribution in orders of ϵ , namely,

$$\Gamma_{\text{NL}}^{(2,2)} = \frac{\kappa_3^{(\text{D,N})}}{2} \int_{\partial\mathcal{M}} \sqrt{h^{(0)}} K^{(1)} (-\nabla^{2(0)})^{1/2} K^{(1)}, \quad (61)$$

with the coefficients

$$\kappa_3^{(\text{D})} = \frac{1}{720\pi^2}, \quad \kappa_3^{(\text{N})} = \frac{11}{720\pi^2}, \quad (62)$$

read off from Eq. (44) (extended to odd d), and again we are using $\sqrt{h^{(0)}}$ and $\nabla^{2(0)}$ to fix the order in ϵ . The Laplace-Beltrami operator on the worldtube $\partial\mathcal{M} = S_R^2 \times \mathbb{R}_t$ is, in Lorentzian signature,

$$\square = \partial_t^2 - \frac{1}{R^2} \nabla_{\Omega}^2, \quad (63)$$

²We are using the sign convention in which a static sphere bounding the bulk from outside has $K^{(0)} = +2/R$; the action depends bilinearly on K , so the overall sign is irrelevant.

and its eigenvalues on plane waves $e^{-i\omega t} Y_{\ell m}(\Omega)$ are $\lambda_{\ell}(\omega) = -\omega^2 + \ell(\ell+1)/R^2$. With the Feynman $-i0^+$ prescription,

$$\square^{1/2} e^{-i\omega t} Y_{\ell m} = \begin{cases} +\sqrt{\omega_{\ell}^2 - \omega^2} e^{-i\omega t} Y_{\ell m}, & \omega^2 < \omega_{\ell}^2, \\ -i\sqrt{\omega^2 - \omega_{\ell}^2} e^{-i\omega t} Y_{\ell m}, & \omega^2 > \omega_{\ell}^2, \end{cases} \quad (64)$$

with the multipole-dependent gap

$$\omega_{\ell} := \frac{\sqrt{\ell(\ell+1)}}{R}. \quad (65)$$

The branch cut at $\omega = \omega_{\ell}$ is the geometric origin of the imaginary part of the effective action and, hence, of the radiation rate: the multipole ℓ radiates only when the oscillation frequency exceeds ω_{ℓ} .

Inserting Eq. (60) into the effective action and using Eq. (64) together with the orthogonality of the spherical harmonics, $\int_{S^2} |Y_{\ell m}|^2 d\Omega = 1$, and $\sqrt{h^{(0)}} d^3y = R^2 \sin\theta d\theta d\phi dt$, one finds, for a monochromatic deformation $f(t) = \cos(\omega t)$ over a window of duration $T \gg 1/\omega$,

$$\text{Im}[\Gamma_{\text{NL}}^{(2,2)}] = \frac{\kappa_3^{(\text{D,N})}}{4} \epsilon^2 T [\Lambda_{\ell}(\omega)]^2 \sqrt{\omega^2 - \omega_{\ell}^2} \Theta(\omega - \omega_{\ell}), \quad (66)$$

with

$$\Lambda_{\ell}(\omega) := R^2 \omega^2 - (\ell-1)(\ell+2). \quad (67)$$

Equivalently, using $\kappa_3^{(\text{D})}/4 = 1/(2880\pi^2)$ and $\kappa_3^{(\text{N})}/4 = 11/(2880\pi^2)$,

$$\text{Im}[\Gamma_{\text{NL}}^{(2,2)}] = \frac{c_{(\text{D,N})}}{\pi^2} \epsilon^2 T [R^2 \omega^2 - (\ell-1)(\ell+2)]^2 \sqrt{\omega^2 - \omega_{\ell}^2}, \quad \omega > \omega_{\ell}, \quad (68)$$

with $c_{\text{D}} = 1/2880$ and $c_{\text{N}} = 11/2880$.

The mean number of created particles can be computed using the relation $\text{Im} \Gamma \approx \langle N \rangle / 4$ (see the Appendix),

$$\frac{\langle N \rangle}{T} \approx \frac{4c_{(\text{D,N})}}{\pi^2} \epsilon^2 [R^2 \omega^2 - (\ell-1)(\ell+2)]^2 \sqrt{\omega^2 - \omega_{\ell}^2}, \quad \omega > \omega_{\ell}. \quad (69)$$

Several features of this example, summarized in Fig. 1, deserve comment.

- (1) The mode ℓ radiates only for $\omega > \omega_{\ell} = \sqrt{\ell(\ell+1)}/R$. The threshold is the angular eigenvalue of the worldtube Laplacian; geometrically, below it the boundary perturbation cannot supply enough on-shell momentum to the pair of emitted

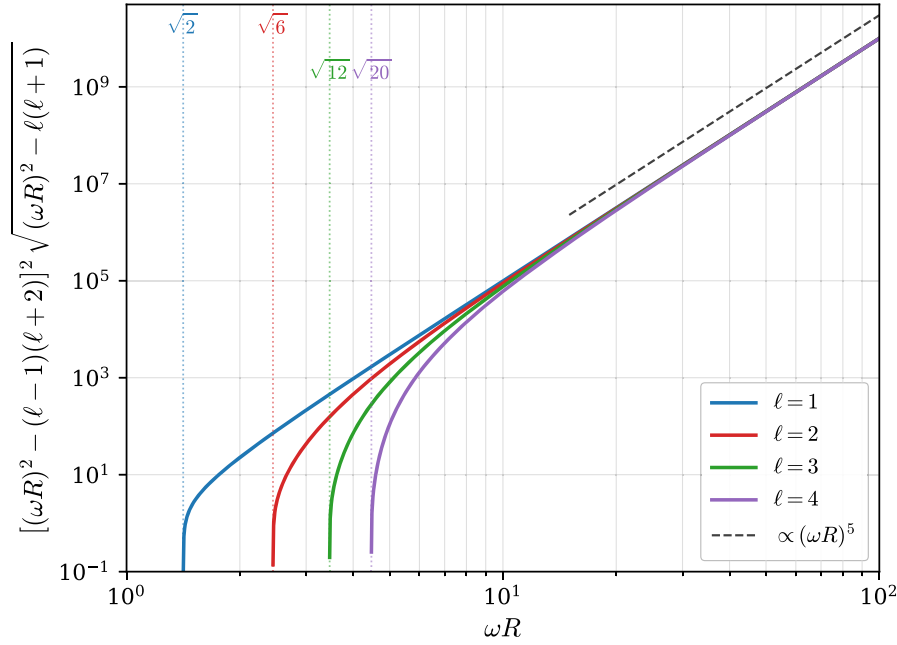


FIG. 1. Dimensionless mean number of created particles per unit time, $[(\omega R)^2 - (\ell - 1)(\ell + 2)]^2 \sqrt{(\omega R)^2 - \ell(\ell + 1)}$, as a function of ωR , for the multipoles $\ell = 1, 2, 3, 4$. The full rate is recovered by multiplication by $4c_{(D,N)}\epsilon^2/(\pi^2 R)$ with $c_D = 1/2880$ and $c_N = 11/2880$; the Neumann result is therefore obtained from the Dirichlet curve by an overall factor of 11, independent of ℓ . Colored dotted vertical lines mark the multipole-dependent gap frequencies $\omega_\ell R = \sqrt{\ell(\ell + 1)}$, below which the corresponding mode does not radiate. The dashed gray line indicates the universal high-frequency asymptote $\propto (\omega R)^5$ onto which all multipoles converge for $\omega R \gg \sqrt{\ell(\ell + 1)}$, recovering the scaling expected for a two-dimensional oscillating mirror in $3 + 1$ dimensions.

quanta. Note that $\omega_0 = 0$ (no threshold for the breathing mode), $\omega_1 = \sqrt{2}/R$, $\omega_2 = \sqrt{6}/R$, etc.

- (2) For $\ell = 1$, Eq. (60) reduces to $K^{(1)} = -R\ddot{f}(t)Y_{1m}(\Omega)$. The static piece vanishes identically because a rigid sphere translation preserves the extrinsic geometry, and only the kinematic $\ddot{f}$ piece contributes. Consistently, $\Lambda_1(\omega) = R^2\omega^2$ vanishes at $\omega = 0$.
- (3) The lowest genuine shape oscillation is $\ell = 2$, with $\omega_2 = \sqrt{6}/R$ and $\Lambda_2(\omega) = R^2\omega^2 - 4$. For Dirichlet and Neumann conditions,

$$\left.\frac{\langle N \rangle}{T}\right|_{\ell=2}^{(D)} = \frac{1}{720\pi^2}\epsilon^2(R^2\omega^2 - 4)^2\sqrt{\omega^2 - 6/R^2}, \quad (70)$$

$$\left.\frac{\langle N \rangle}{T}\right|_{\ell=2}^{(N)} = \frac{11}{720\pi^2}\epsilon^2(R^2\omega^2 - 4)^2\sqrt{\omega^2 - 6/R^2}. \quad (71)$$

The Neumann rate is 11 times the Dirichlet rate, the same universal ratio dictated by $\kappa_3^{(N)}/\kappa_3^{(D)}$ and therefore independent of ℓ .

- (4) For $\omega R \gg 1$, all multipoles reach the regime $\langle N \rangle/T \simeq (4c_{(D,N)}/\pi^2)\epsilon^2 R^4 \omega^5$, recovering the universal ω^5 scaling expected for emission from a two-dimensional oscillating mirror in $3 + 1$ dimensions.

The result above complements the breathing-sphere computation of the Appendix: in the bulk-mode language

of [46], only $\ell = 0$ contributes to the $\delta(W_{\ell nk} - \omega)$ at leading order in ϵ , in agreement with the threshold $\omega_0 = 0$ of our formula.

VI. CONCLUSIONS

We have constructed the nonlocal effective action of a free massless scalar field in flat $(d + 1)$ -dimensional spacetime, in the presence of either Dirichlet or Neumann boundary conditions on a smooth, codimension-one hypersurface $\partial\mathcal{M}$, working at quadratic order in the extrinsic curvature tensor K_{ab} . Two features distinguish this construction from previous treatments of the same physical setting. First, the result is manifestly covariant under diffeomorphisms of $\partial\mathcal{M}$: it is expressed entirely in terms of the geometric objects K_{ab} , the induced metric h_{ab} , the Laplace-Beltrami operator ∇^2 on the boundary, and no embedding-specific or coordinate-dependent quantity (such as a height function ψ) appears. Second, the construction does not require a global Monge-patch parametrization and, therefore, extends to surfaces with overhang or nontrivial topology that lie outside the regime of Refs. [44,45].

The structure of the result reflects the parity of the boundary manifold dimension. In even d , the kernel is $(-\nabla^2)^{d/2-1} \log(-\nabla^2/\mu^2)$, with an overall coefficient κ_d that is fully determined by the half-integer-indexed

boundary heat-kernel coefficient $b_{(d-1)/2+1}$. In odd d , the kernel is the fractional power $(-\nabla^2)^{d/2-1}$, and its overall coefficient is not accessible from the divergent terms of the heat-kernel expansion alone: it requires either a full knowledge of the function $\alpha(\xi)$ defined in Eq. (28), or, as we showed in Sec. IV, a comparison with the Monge-patch results of Refs. [44,45]. The latter route fixes κ_d in arbitrary d and provides a nontrivial bridge between the geometric and the height-function formulations: the Monge-patch result is recovered as a special case of our covariant expression, with the overall factor of $\frac{1}{2}$ accounting for whether the field lives on one or both sides of the boundary. The reduction $\int K_{ab} K^{ab} = \int K^2$ at order K^2 , derived through the Codazzi identity and a definite prescription for the inverse Laplacian, plays a central role in the unification.

Two explicit applications illustrate the formalism. The oscillating ring in $2 + 1$ dimensions is a case in which the relevant dynamics is encoded in the perturbation around a flat induced metric. The oscillating sphere in $3 + 1$ dimensions, by contrast, has nonvanishing intrinsic curvature already at the unperturbed level and provides a test of the covariant reduction of $K^{ab} K_{ab}$ to K^2 when the background is intrinsically curved. For a multipolar deformation $r = R[1 + \epsilon \cos(\omega t) Y_{\ell m}(\Omega)]$, the imaginary part of the effective action displays a mode-by-mode threshold $\omega_\ell = \sqrt{\ell(\ell+1)}/R$ inherited from the angular spectrum of the worldtube and a universal Neumann-to-Dirichlet ratio of the radiation rate equal to $\kappa_3^{(N)}/\kappa_3^{(D)} = 11$, independent of the multipole (the same ratio appears when computing the dynamical Casimir effect for transverse electric and transverse magnetic modes of the electromagnetic field in a plane cavity [47]). The dipole sector $\ell = 1$ is silent at static order in the deformation, consistent with the geometric fact that a rigid translation of the sphere leaves K_{ab} unchanged.

Several natural extensions might be within the reach of the same techniques. Pushing the geometric expansion to cubic order in K_{ab} is the immediate next step: the integer-indexed coefficient b_2 is fully tabulated, but introduces independent invariants (K^3 , $KK^{ab}K_{ab}$, and $K_a{}^b K_b{}^c K_c{}^a$) that mix nontrivially under integration by parts when the boundary has nonzero intrinsic curvature and require a careful generalization of the present reduction. Beyond cubic order, the heat-kernel route alone is insufficient and the analysis must be supplemented by a direct calculation of $\alpha(\xi)$, accessible in principle from explicit propagator computations in geometries of high symmetry. A second direction is the inclusion of richer field content like fermions and gauge fields, each with its own set of boundary heat-kernel coefficients, required for the dynamical Casimir effect of the electromagnetic field. A third is the combination of the present geometric setup with a curved bulk, relevant for accelerated mirrors in cosmological backgrounds and for boundary

contributions to semiclassical gravity. We hope to address some of these directions in future work.

ACKNOWLEDGMENTS

This research was supported by Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET).

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: RELATION BETWEEN $\text{Im } \Gamma$ AND BOGOLIUBOV COEFFICIENTS

As an explicit test of our result in Eq. (69) we compute the imaginary part of the effective action for an oscillating sphere, using the results of Ref. [46], where the authors obtained the mean number of created particles through Bogoliubov transformations.

First, we relate the mean number of created particles to the imaginary part of the effective action. Let α and β be the matrices of Bogoliubov coefficients relating the “in” and “out” Fock spaces, whose vacuum states are $|0_{\text{in}}\rangle$ and $|0_{\text{out}}\rangle$, respectively. If N_i is the number operator associated with the i th mode of the out Fock space, then the total number of particles evaluated in the in vacuum is [1]

$$\langle N \rangle \equiv \sum_i \langle 0_{\text{in}} | N_i | 0_{\text{in}} \rangle = \text{Tr}(\beta \beta^\dagger). \quad (\text{A1})$$

To relate this quantity to the imaginary part of the Lorentzian effective action Γ , we express the latter in terms of the Bogoliubov coefficients through the relation [6]

$$e^{i\Gamma} = (\det \alpha)^{-1/2}. \quad (\text{A2})$$

Thus, Eq. (A2) gives

$$\text{Im } \Gamma = \frac{1}{4} \text{Tr}[\log(1 + \beta \beta^\dagger)] \Rightarrow \text{Im } \Gamma \approx \frac{1}{4} \langle N \rangle, \quad (\text{A3})$$

where the approximation holds when the number of produced particles is perturbatively small.

Reference [46] considers a massless scalar quantum field theory in the interior of a sphere with radius $a(t)$ with Dirichlet boundary conditions imposed at the surface. The time-dependent radius is assumed to be

$$a(t) = \begin{cases} R[1 + \epsilon Y_{00}(\Omega) \sin(\omega t)], & 0 \leq t \leq T, \\ R, & t < 0, \quad t > T, \end{cases} \quad (\text{A4})$$

with $\omega > 0$, $\epsilon \ll 1$ the perturbative parameter, and $Y_{00}(\Omega) = 1/\sqrt{4\pi}$. Up to the first nontrivial order in ϵ , the mean number of particles created during the oscillation is

$$\begin{aligned} \langle N \rangle = & \frac{\epsilon^2}{\pi} \sum_{\ell=0}^{\infty} (2\ell+1) \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \sum_{\sigma=\pm} (v_{\ell nk}^{\sigma})^2 \frac{\sin^2[\frac{T}{2}(W_{\ell nk} - \sigma\omega)]}{(W_{\ell nk} - \sigma\omega)^2} \right. \\ & \left. + v_{\ell nk}^{-} v_{\ell nk}^{+} \frac{\cos(\omega T)}{W_{\ell nk}^2 - \omega^2} [\cos(\omega T) - \cos(W_{\ell nk} T)] \right\}, \end{aligned} \quad (\text{A5})$$

where

$$W_{\ell nk} := \omega_{\ell n} + \omega_{\ell k}, \quad \omega_{\ell n} = \frac{j_{\ell,n}}{R}, \quad (\text{A6})$$

with $j_{\ell,n}$ the n th zero of the spherical Bessel function of order ℓ . The quantities $v_{\ell nk}^{\pm}$ are functions of the oscillation frequency ω . See Ref. [46] for more details.

The case of interest is the limit $\omega R \gg 1$, together with long times, $T \gg 1/\omega$, since in the example of Sec. V we considered undamped oscillations. This case corresponds to taking the continuum limit of Eq. (A5), in which three different contributions can be identified. The first term, proportional to $(v_{\ell nk}^{+})^2$, contains a δ -function approximant, since, in the distributional sense,

$$\frac{\sin^2[\frac{T}{2}(W_{\ell nk} - \omega)]}{(W_{\ell nk} - \omega)^2} \xrightarrow{T \rightarrow \infty} \frac{\pi T}{2} \delta(W_{\ell nk} - \omega). \quad (\text{A7})$$

The second term, proportional to $(v_{\ell nk}^{-})^2$, gives an analogous δ function evaluated at $W_{\ell nk} + \omega$, but this contribution vanishes because $W_{\ell nk} > 0$ and $\omega > 0$. Finally, the third term, involving $v_{\ell nk}^{-} v_{\ell nk}^{+}$, also has an approximate δ function centered at $W_{\ell nk} - \omega$,

$$\frac{\cos(\omega T) - \cos(W_{\ell nk} T)}{W_{\ell nk}^2 - \omega^2} \xrightarrow[\omega \approx W_{\ell nk}]{T \rightarrow \infty} \frac{\pi}{\omega} \sin(\omega T) \delta(W_{\ell nk} - \omega). \quad (\text{A8})$$

However, unlike the first term, this contribution is not proportional to the timescale T . It is therefore negligible, in the large-time limit, compared with the contribution that grows linearly with T . Thus, in this regime, the leading contribution comes only from the first term in Eq. (A5).

To take the continuum limit in Eq. (A5), we replace the sums by integrals according to

$$\begin{aligned} \sum_{\ell=0}^{\infty} (2\ell+1) & \mapsto \int_0^{\infty} d\nu 2\nu, \quad \text{and} \\ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} & \mapsto \int_0^{\infty} d\omega_1 \frac{\partial n}{\partial \omega_1} \int_0^{\infty} d\omega_2 \frac{\partial k}{\partial \omega_2}, \end{aligned} \quad (\text{A9})$$

where $\omega_1 = \omega_{\ell n}$, $\omega_2 = \omega_{\ell k}$, and $\nu = \ell + 1/2$. Using $W_{\ell nk} = \omega$ to evaluate the coefficients $v_{\ell nk}^{\pm}$ in Ref. [46],

we obtain $v_{\ell nk}^{+} = \frac{1}{2} \sqrt{\omega_{\ell n} \omega_{\ell k}}$, thus

$$\begin{aligned} \langle N \rangle \approx & \frac{1}{4} \epsilon^2 T \int_0^{\infty} d\nu \nu \int_0^{\infty} d\omega_1 \frac{\partial n}{\partial \omega_1} \int_0^{\infty} d\omega_2 \\ & \times \frac{\partial k}{\partial \omega_2} \omega_1 \omega_2 \delta(\omega_1 + \omega_2 - \omega). \end{aligned} \quad (\text{A10})$$

We therefore need to approximate the zeros of the spherical Bessel functions in order to compute the densities $\partial n / \partial \omega_1$ and $\partial k / \partial \omega_2$. The condition $\omega R \gg 1$ implies that the relevant zeros have large radial quantum numbers. Moreover, because of the weight 2ν in the angular integral, large values of ν also contribute. We therefore need the asymptotic distribution of zeros in the regime where both n and ν are large. For this purpose, we use the Debye asymptotic expansion in the oscillatory region $x > \nu$ [48],

$$\begin{aligned} j_{\ell}(x) & \sim \frac{1}{x} \left(1 - \frac{\nu^2}{x^2}\right)^{-1/4} \cos\left(\xi(\nu, x) - \frac{\pi}{4}\right), \\ \xi(\nu, x) & = \sqrt{x^2 - \nu^2} - \nu \arccos\left(\frac{\nu}{x}\right), \end{aligned} \quad (\text{A11})$$

so the n th zero of the spherical Bessel function of order $\ell = \nu - 1/2$ satisfies, in this regime, $\xi(\nu, j_{\ell,n}) = n\pi$. Thus,

$$\frac{\partial n}{\partial \omega_1} \approx \frac{R}{\pi} \sqrt{1 - \frac{\nu^2}{R^2 \omega_1^2}}, \quad \omega_1 R \geq \nu, \quad (\text{A12})$$

and similarly for $\partial k / \partial \omega_2$. Therefore, after rescaling the integration variables to dimensionless ones, Eq. (A10) becomes

$$\begin{aligned} \langle N \rangle \approx & \frac{1}{2\pi^2} \epsilon^2 T \omega^5 R^4 \int_0^{1/2} dx x (1-x) \\ & \times \int_0^x dy y \sqrt{1 - \frac{y^2}{x^2}} \sqrt{1 - \frac{y^2}{(1-x)^2}}, \end{aligned} \quad (\text{A13})$$

which can be evaluated explicitly,

$$\frac{\langle N \rangle}{T} \approx \frac{1}{720\pi^2} \epsilon^2 \omega^5 R^4. \quad (\text{A14})$$

This agrees with the high-frequency limit of Eq. (69) for $\ell = 0$.

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