

Dark higher-form portals and duality

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Light scalar or vector particles are among the most studied dark matter candidates. Yet, those are always described as scalar or vector fields. In this paper, we explore instead the embedding of the scalar particle in an antisymmetric rank-three tensor field, and the dark photon into an antisymmetric rank-two tensor field (a so-called Kalb-Ramond field), and construct minimal bases of effective interactions with Standard Model fields. Then, keeping phenomenological applications as our main objective, a number of theoretical aspects are clarified, in particular related to the impact of existing dualities among the corresponding free theories, and concerning their Stueckelberg representations. Besides, for the rank-two field, we present for the first time its full propagator, accounting for the possible presence of a pseudoscalar mass term. Thanks to these results, and with their different kinematics, gauge-invariant limits, and Lorentz properties, we show that these higher-form fields provide genuine alternative frameworks, with different couplings and expected signatures at low-energy or at colliders.

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I. INTRODUCTION

Cosmological and astrophysical evidences for dark matter (DM) have been piling up for a long time now, but despite considerable efforts, its true nature still eludes us. The space of possibilities is *a priori* vast, but has been for quite some time dominated by the so-called weakly-interacting massive particle paradigm, not least thanks to its natural presence in supersymmetric versions of the Standard Model [1]. The absence of supersymmetric signals at the LHC has weakened this prime candidate, reopening the space of possibilities, but with to some extent the axion taking its place (for recent reviews, see, e.g., Refs. [2,3]). Still, many other light particles, sufficiently long-lived and weakly coupled with normal matter, could make up a sizeable fraction of the observed DM relic abundance. In addition, that particle could be accompanied by others, shorter-lived and of various spins, altogether populating a whole dark sector [4].

The present paper will concentrate on the simple scalar and vector DM candidates, the latter often referred to as a dark photon [5,6]. However, and contrary to other works, these states will not be introduced via scalar and vector

fields but will be embedded into so called higher form fields. Those are antisymmetric tensor fields of higher ranks, and in particular, our goal is to use the antisymmetric rank-two tensor $B_{\mu\nu}$ for the dark photon, and the antisymmetric rank-three tensor $C_{\mu\nu\rho}$ for the dark scalar. Theoretically, these fields have been known for a long time, especially in the context of string theory (see, e.g., Ref. [7]), but to our knowledge, they have never been explored in details as true embedding for those dark matter candidates. The only exception is the string theory version of the axion (for reviews, see, e.g., [7,8]), originating from a massless $B_{\mu\nu}$, that is most often called a Kalb-Ramond field [9,10] but sometimes also referred to as the notoph [11]. Antisymmetric rank-three fields, for their part, have to our knowledge first been described in Refs. [12,13].

There are two main phenomenological motivations to go to the hassle of using higher form fields. First, though not immediately apparent, the massive $B_{\mu\nu}$ and $C_{\mu\nu\rho}$ fields do indeed have the right number of physical degrees of freedom to match that of a massive vector field A_μ [11,14,15] and a massive scalar field ϕ [11,16], respectively. However, from a Lorentz invariant point of view, it is clear that having a different number of indices changes the way in which those fields can couple to SM matter fields. One immediate question is then to identify which operators exist, and among them, which are of the least mass dimension, the so-called portals.

A second motivation is that we do expect very different scaling behaviors for these portals. Let us take the dark photon to illustrate this point. For a massive vector field, one can always understand its mass as coming from an

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auxiliary scalar field, in the so-called Stueckelberg construction [17,18],

$$A_\mu \rightarrow A_\mu - \frac{1}{m_V} \partial_\mu \phi. \quad (1)$$

This means that whenever the dark photon is coupled to a nonconserved current, it is its longitudinal degree of freedom represented by ϕ that dominates when the typical energy of the process is large compared to the dark photon mass m_V , giving it a somewhat axionlike behavior. Now, we will see that the exact opposite happens when the dark photon is introduced via a rank two tensor. Its Stueckelberg construction takes the form

$$B_{\mu\nu} \rightarrow B_{\mu\nu} - \frac{1}{m_V} (\partial_\mu A_\nu - \partial_\nu A_\mu), \quad (2)$$

where A_μ represents the transverse polarization states. This reflects the fact that a massless vector is transverse, so it needs to receive one scalar longitudinal degree of freedom to be massive. This is the essence of the Higgs mechanism. On the contrary, a massless $B_{\mu\nu}$ is essentially an axionlike scalar field [9,19], so it needs to be given a whole transverse vector field to become massive, and those are the modes that get enhanced for small masses. From this, we do expect a quite different phenomenology for those two embeddings of the dark photon.

There is however one profound feature of these higher-rank tensor fields that complicates this program. It stems from the existence of dualities among higher rank tensor field theories (see, e.g., Refs. [7,20]). Those are of three types: algebraic, massless, and massive. The first comes from properties of differential forms, while the last two are generalizations of the well-known electromagnetic duality. For instance, the massive A and B fields of Eqs. (1) and (2) are related under [21,22]

$$B_{\mu\nu} \rightarrow -\frac{1}{2!m} \epsilon_{\mu\nu\rho\sigma} F^{A,\rho\sigma}, \quad (3)$$

with $F_{\mu\nu}^A = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the massive rank-three form field is related to the scalar field via [12],

$$C_{\mu\nu\rho} \rightarrow -\frac{1}{m} \epsilon_{\mu\nu\rho\sigma} F^{\phi,\sigma}, \quad (4)$$

with $F_\mu^\phi = \partial_\mu \phi$. Such dualities may explain why these fields have not received much phenomenological attention. Yet, there are three important issues to be addressed in an effective framework:

- (i) First, let us stress that strictly speaking, dualities are valid for free fields only. Typically, they exchange equation of motion and Bianchi identity, so the former better not involve any external current. In practice, duality transformations are still possible in

the presence of interactions, but their interpretation changes in that they relate theories that are no longer dynamically identical.

- (ii) A second caveat is the occurrence of mass scales in Eqs. (3) and (4). Once these dualities are applied to interacting theories, they mix up the effective operators arising at different orders, and what is a portal in one picture is in general no longer so once dualized.
- (iii) A third point is specific to the B field, for which besides the normal mass term $m^2 B_{\mu\nu} B^{\mu\nu}$, there can be a pseudoscalar term $\tilde{m}^2 B_{\mu\nu} \tilde{B}^{\mu\nu}$ where $\tilde{B}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} B^{\rho\sigma}/2$. Under the duality transformation of Eq. (3), this would seem to be equivalent to an irrelevant topological term $\theta F_{\mu\nu}^A \tilde{F}^{A,\mu\nu}$, but we will see this is not the case. Instead, the $\tilde{m}$ term alters even the dynamics of a free B field, changing its polarization states.
- (iv) On a practical level, though such dualities are very well known (see, e.g., Refs. [20,23] for some recent accounts), they appear scattered in the literature, and are often discussed in abstract field theoretic terms. Indeed, the natural representation of higher rank tensor fields is that of higher rank differential forms. Though it is to some extent necessary to adopt that language, one of our goals will be to review all these aspects and express them back in a form suitable for phenomenological applications.

At the end of the day, we will see that dualities do provide useful dynamical information, but do not make the higher-form embeddings of the dark scalar and photon trivially equivalent to the standard ones. Those turn out to be true alternative frameworks.

The paper is organized as follows. The first section is intended as an introduction to higher form fields, their action and equations of motion, and their degrees of freedom. The propagators in the massive and massless case are also derived. The only original result in that Section is the nonperturbative treatment of the dual mass term for the rank-two field, leading to a more general polarization sum for these states. In Sec. II, the effective interactions with all the SM particles are derived for tensor fields of rank between zero and four, with up to two external dark states. We include operators up to rather high dimensions there, to explore the properties of these bases. Then, in Secs. III, IV, and V are discussed the algebraic, massless, and massive dualities, respectively, with emphasis on their impact on the scaling of effective interactions. This is then used in the phenomenological analysis of Sec. VI, in conjunction with the generalized Stueckelberg procedure (which we describe in detail). Finally, our results are summarized in the Conclusion.

II. ABELIAN p -FORM FIELDS

Higher form gauge fields are particular tensor generalizations of the usual electromagnetic vector field.

To understand their particularities, let us start by recalling how the vector field arises in quantum electrodynamics (QED). First, the gauge field A_μ is introduced as a connection. It serves to define covariant derivatives, $\partial_\mu \rightarrow \mathcal{D}_\mu = \partial_\mu - ieA_\mu$ for a field of charge e , thereby allowing one to realize the $U(1)$ symmetry locally. Its kinetic term then uses the curvature $[\mathcal{D}_\mu, \mathcal{D}_\nu] = ieF_{\mu\nu}$, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. From this interpretation, it is natural to construct the Wilson line [24], which integrates the connection along a path starting at x and going to y ,

$$U(x, y) = \exp \left(-i \int_P dz^\mu A_\mu(z) \right). \quad (5)$$

The quantity $U(x, y)$ is such that $\phi(x)$ and $U(x, y)\phi(y)$ transform identically. If under the gauge transformation $A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x)$, the field undergoes $\phi(x) \rightarrow \exp(-i\Lambda(x))\phi(x)$, then $U(x, y) \rightarrow \exp(-i\Lambda(x))U(x, y) \times \exp(i\Lambda(y))$. Though $U(x, y)$ is not gauge invariant, it becomes so if the path closes into a Wilson loop. In that case, using Stokes theorem, it is expressible in terms of the flux of the field strength through the surface enclosed by the loop,

$$\begin{aligned} U(x, x) &= \exp \left(-i \int_{\partial\Sigma} dz^\mu A_\mu(z) \right) \\ &= \exp \left(-\frac{i}{2} \int_\Sigma d\eta^{\mu\nu} F_{\mu\nu}(z) \right). \end{aligned} \quad (6)$$

The idea of higher p -form gauge fields is to generalize the Wilson construction to higher dimensions, to Wilson p -dimensional loops enclosing $p+1$ dimensional surfaces. One peculiarity in this case is that only Abelian fields can be constructed once $p > 1$. Naively, this stems from the liberty in higher dimensions to move symmetry charge operators past one another, so that they end up commuting. Those constructions were first encountered in the context of string theory, with the $p=2$ Kalb-Ramond field as a

prototype [9,10]. Nowadays, there is a lot of renewed interest in these higher form symmetries, whether global or local (for reviews, see, e.g., Refs. [25,26]).

Mathematically, if we want to integrate p -dimensional gauge connections along p -dimensional loops, they should be represented by differential p -forms. Specifically, a generic p -form gauge field corresponds to an antisymmetric tensor field with p Lorentz indices $A_{\mu_1 \dots \mu_p}$, that is

$$A = \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \quad (7)$$

with $C_p^n = n!/p!(n-p)!$ degrees of freedom (DoF) in n dimensions, and $p \leq n$. This is the natural language to deal with these objects (a short summary of the main definitions is in the Appendix). Though it is not compulsory, we will often adopt some aspects of that language to streamline the notations and calculations. In particular, many developments can be done keeping p arbitrary instead of painstakingly deriving the results for each value of p . For example, the field strength is the exterior derivative of the gauge field, i.e., the $p+1$ -form $F = dA$. In components, the expression is less simple

$$F = dA \Leftrightarrow F_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}, \quad (8)$$

with the convention that [...] represents the normalized antisymmetrization. We also immediately get the Bianchi identities $dF = 0$ from the fact that $d^2 = 0$, which corresponds to

$$\partial_{[\mu_1} F_{\mu_2 \dots \mu_{p+2}]} = 0. \quad (9)$$

Nevertheless, in an attempt to provide results of practical phenomenological use, we will as much as possible fall back to the usual tensorial notation at important steps. In particular, the naming conventions we shall use are

$$p=0: \phi, \quad F_\mu^\phi = \partial_\mu \phi, \quad (10a)$$

$$p=1: A = A_\mu dx^\mu, \quad F_{\mu\nu}^A = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (10b)$$

$$p=2: B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu, \quad F_{\mu\nu\rho}^B = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \quad (10c)$$

$$p=3: C = \frac{1}{3!} C_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho, \quad F_{\mu\nu\rho\sigma}^C = \partial_\mu C_{\nu\rho\sigma} + \partial_\nu C_{\rho\mu\sigma} + \partial_\rho C_{\mu\nu\sigma} + \partial_\sigma C_{\nu\rho\mu}, \quad (10d)$$

$$p=4: D = \frac{1}{4!} D_{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma, \quad F_{\mu\nu\rho\sigma\lambda}^D = 0. \quad (10e)$$

Of note is the fact that we will also consider massive p -form fields, for which there is no gauge symmetry. Still, as long as they are represented by antisymmetric p -index tensors, they can be characterized in terms of differential forms. Said differently, in the present work, it is understood that free massive p -form fields would become p -form gauge fields if the mass term is removed. Specifically, the action for a massive p -form field is,

$$\mathcal{S}_{\text{p-form}} = (-1)^p \int \frac{1}{2} F \wedge \star F - \frac{1}{2} m^2 A \wedge \star A + A \wedge \star j, \quad (11)$$

where the current j is also a p -form. The $\wedge$ operator is the wedge product, and $\star$ stands for the Hodge dual (see the Appendix). The reason for the $(-1)^p$ factor comes from our metric signature of $(+1, -1, -1, -1)$, as will become clear below. In terms of components, this action corresponds to the Lagrangian

$$\mathcal{L}_{\text{p-form}} = \frac{(-1)^p}{p!} \left(\frac{1}{2} \frac{1}{p+1} F_{\mu_1 \dots \mu_{p+1}} F^{\mu_1 \dots \mu_{p+1}} - \frac{1}{2} m^2 A_{\mu_1 \dots \mu_p} A^{\mu_1 \dots \mu_p} + A_{\mu_1 \dots \mu_p} J^{\mu_1 \dots \mu_p} \right). \quad (12)$$

In both cases, the action for the massless case is simply obtained by setting $m = 0$, while the definitions of A and $F = dA$ stay identical. From this action, imposing that it is stationary against small variations of the field, the equation of motion (EoM) is found to be,

$$- \star d \star F + m^2 A = j \Leftrightarrow \partial^\alpha F_{\alpha\mu_1 \dots \mu_p} + m^2 A_{\mu_1 \dots \mu_p} = J_{\mu_1 \dots \mu_p}. \quad (13)$$

The similarity is evident with the usual Proca equation describing a massive vector field, or the inhomogeneous Maxwell equations when $m = 0$. To further explore the consequences, we need to discuss separately the massive and massless case.

A. Massive fields

When the field is massive, taking an additional derivative shows that $m^2 d \star A = d \star j$ since $d^2 \star F = 0$. In other words, $d \star A = 0$ for a free field or when the current is conserved, which in components means that the Lorenz condition $\partial^\alpha A_{\alpha\mu_1 \dots \mu_{p-1}} = 0$ has to be fulfilled. Under this condition, the EoM of the p -form field takes a simpler form. Plugging in $F = dA$,

$$j = - \star d \star dA + m^2 A = (-\Delta + m^2)A + d \star d \star A = (-\Delta + m^2)A, \quad (14)$$

where the differential Laplacian is given by $\Delta = \star d \star d + d \star d \star$, and in components matches the usual d'Alembertian $\Delta = -\square$ in flat Minkowski space. Despite the heavy use of the differential machinery, there is nothing special here, and the same result can be found starting from Eq. (12) and integrating by part,

$$\mathcal{L}_{\text{p-form}} = \frac{(-1)^p}{p!} \left(-\frac{1}{2} A_{\mu_1 \dots \mu_p} (\square + m^2) A^{\mu_1 \dots \mu_p} - \frac{p}{2} \partial^\alpha A_{\alpha\mu_2 \dots \mu_p} \partial_\beta A^{\beta\mu_2 \dots \mu_p} + A_{\mu_1 \dots \mu_p} J^{\mu_1 \dots \mu_p} \right), \quad (15)$$

where the middle term drops out upon imposing $\partial^\alpha A_{\alpha\mu_1 \dots \mu_{p-1}} = 0$. The EoM of Eq. (14) is directly obtained by varying with respect to $A_{\mu_1 \dots \mu_p}$.

Let us count the number of physical DoF (for an early derivation in four dimensions, see Ref. [15]). Starting from the $C_p^n = n!/p!(n-p)!$ DoF encoded into the totally antisymmetric tensor $A_{\mu_1 \dots \mu_p}$, we can immediately remove all those of the form $A_{0i_1 \dots i_{p-1}}$, where the i 's stand for spatial indices. Indeed, from the definition of the field strength in Eq. (8), it is clear that only the purely spatial components have time derivatives. In the Heisenberg picture, the C_{p-1}^{n-1} conjugate momenta of $A_{0i_1 \dots i_{p-1}}$ vanish, and these C_{p-1}^{n-1} temporal components are nondynamical. This is consistent since the Lorenz condition, which is built in the EoM, amounts to C_{p-1}^{n-1} constraints, which is sufficient to fix the C_{p-1}^{n-1} temporal components $A^{0i_1 \dots i_{p-1}}$, leaving no further constraint on the remaining spatial components. All in all, this leaves

$$C_p^n - C_{p-1}^{n-1} = C_p^{n-1}, \quad (16)$$

propagating DoF, corresponding to the number of ways to pick p spatial indices in n dimensions, see Table I.

We can now understand the origin of the $(-1)^p$ factor in Eq. (11). Going back to $\mathcal{L}_{\text{p-form}}$ in Eq. (15), consider the first term. The $1/p!$ is simply there to compensate for the summation over permutations of $\mu_1, \dots, \mu_p$. It disappears if we identify the C_p^n fields as $A_{\mu_1 \dots \mu_p}$ with $\mu_i > \mu_{i+1}$. Further, only the purely spatial degrees of freedom are physical, and bringing down the indices gives $A_{i_1 \dots i_p} = (-1)^p A^{i_1 \dots i_p}$ given our metric $(+1, -1, -1, -1)$. This cancels with the $(-1)^p$ factor in front, leaving the usual Klein-Gordon kinetic term for each of the C_p^{n-1} spatial DoF, $(-1/2) A_{i_1 \dots i_p} (\square + m^2) A_{i_1 \dots i_p}$.

B. Massless fields

Without the mass term, the theory has the gauge symmetry $A \rightarrow A + d\Lambda$ with Λ a $p-1$ -form since $F = dA$ is

TABLE I. Decomposition of the number of degrees of freedom for the higher form fields in four dimensions. Notice in particular that in the massive case, the 0- and 3-form fields both have one physical DoF, while the one- and 2-form fields have 3. In the massless case, the 0- and 2-form fields carry one physical DoF. These are the somewhat coincidental correspondences that will be formalized via dualities later on in Secs. V and VI.

$n = 4$	Degree of freedom			Gauge freedom		Massive	Massless
	Initial	Temporal	Spatial	Total	Spatial		
p	C_p^n	C_{p-1}^{n-1}	C_p^{n-1}	C_{p-1}^{n-1}	C_{p-1}^{n-2}	C_p^{n-1}	C_p^{n-2}
0	1	0	1	0	0	1	1
1	4	1	3	1	1	3	2
2	6	3	3	3	2	3	1
3	4	3	1	3	1	1	0
4	1	1	0	1	0	0	0

invariant. This symmetry is preserved by the $A \wedge \star j$ term provided the current is conserved since under partial integration, $d\Lambda \wedge \star j \rightarrow (-1)^{p-1} \Lambda \wedge d \star j$ and $d \star j = 0$, which is nothing but $\partial^{\mu_1} J_{\mu_1 \dots \mu_p} = 0$. A peculiarity for higher form gauge fields is that there are fewer gauge DoF than the C_{p-1}^n components of the $p-1$ -form Λ . Indeed, these gauge parameters themselves have gauge DoF since $\Lambda \rightarrow \Lambda + d\Lambda'$ with Λ' a $p-2$ -form gives the same gauge transformation $A \rightarrow A + d\Lambda$. This pattern repeats down to a 0-form gauge invariance. Thus, these recursive invariances mean that there are actually [27]

$$C_{p-1}^n - C_{p-2}^n + C_{p-3}^n - \dots = C_{p-1}^{n-1}, \quad (17)$$

gauge DoF for an Abelian p -form gauge field. Explicitly, these invariances are

$$\phi \rightarrow \phi, \quad (18a)$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \quad (18b)$$

$$B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \quad (18c)$$

$$C_{\mu\nu\rho} \rightarrow C_{\mu\nu\rho} + \partial_\mu \Lambda_{\nu\rho} + \partial_\nu \Lambda_{\rho\mu} + \partial_\rho \Lambda_{\mu\nu}, \quad (18d)$$

$$D_{\mu\nu\rho\sigma} \rightarrow D_{\mu\nu\rho\sigma} + \partial_\mu \Lambda_{\nu\rho\sigma} + \partial_\nu \Lambda_{\rho\mu\sigma} + \partial_\rho \Lambda_{\mu\nu\sigma} + \partial_\sigma \Lambda_{\mu\rho\nu}, \quad (18e)$$

where Λ_μ has the same gauge invariance as A_μ , $\Lambda_\mu \rightarrow \Lambda_\mu + \partial_\mu \Lambda'$, $\Lambda_{\mu\nu}$ the same as $B_{\mu\nu}$, and so on. To quantize these fields, we generalize the QED Lorenz condition and fix the gauge via $d \star A = 0$. Notice that this leaves some residual gauge invariance corresponding to those Λ that verify $d \star d\Lambda = 0$, i.e., $-\Delta\Lambda + d \star d \star \Lambda = 0$. But then, one needs to fix, e.g., $d \star \Lambda = 0$ to deal with the gauge-for-gauge invariance under $\Lambda \rightarrow \Lambda + d\Lambda'$, and so on. All in all, the residual invariances are harmonic at each level, as in QED. In practice, we will see later on how the propagator

for these fields is derived by adding the usual gauge-fixing term to the Lagrangian.

Let us now count the number of DoF. We cannot simply subtract the total gauge freedom out of the original C_p^n DoF since, for the same reason as in the massive case, the C_{p-1}^{n-1} temporal components are non-dynamical. Said differently, some of the gauge DoF are not relevant for the counting since they remove non-dynamical DoF. To disentangle the constraints from the gauge DoF, a simple strategy is to first adopt the temporal gauge and set $A_{0i_1 \dots i_{p-1}} = 0$, thereby removing the C_{p-1}^{n-1} temporal components. Then, the remaining gauge DoF are simply those of a p gauge field living in a $n-1$ dimensional space, since it has spatial indices only. From Eq. (17), this corresponds to C_{p-1}^{n-2} gauge DoF. Altogether then, the number of physical DoF is

$$C_p^n - C_{p-1}^{n-1} - C_{p-1}^{n-2} = C_p^{n-2}. \quad (19)$$

Without surprise, this corresponds to the number of ways to pick p spatial and transverse indices in n dimensions, see Table I.

C. Propagators

Besides the number of DoF, it is necessary to know how to sum over the polarization states to compute decay rates involving higher-form fields. In the massive case, these polarization sums can be identified with the numerator of the corresponding field propagators taken on shell. In turn, those are immediately obtained by inverting the kinetic terms of Eq. (15). The only delicate point here is that one must properly antisymmetrize these kinetic term to account for the fact that a p field is fully antisymmetric in its indices.

Let us define the kinetic kernel $\mathcal{K}$ by writing Eq. (15) as $(1/2)A_{\mu_1 \dots \mu_p} \mathcal{K}^{\mu_1 \dots \mu_p, \nu_1 \dots \nu_p} A_{\nu_1 \dots \nu_p}$. Once properly antisymmetrized and in momentum space, it becomes

$$\mathcal{K} = (-1)^p \left(\frac{1}{p!} \mathcal{I}_0(k^2 - m^2) - \frac{1}{(p-1)!} \mathcal{I}_2 \right), \quad (20)$$

where the fully antisymmetric invariants are

$$\begin{aligned} \mathcal{I}_0^{\mu_1 \dots \mu_p, \nu_1 \dots \nu_p} &= \frac{\delta_{\rho_1 \dots \rho_p}^{\mu_1 \dots \mu_p} \delta_{\sigma_1 \dots \sigma_p}^{\nu_1 \dots \nu_p}}{p! p!} g^{\rho_1 \sigma_1} \dots g^{\rho_p \sigma_p}, \\ \mathcal{I}_2^{\mu_1 \dots \mu_p, \nu_1 \dots \nu_p} &= \frac{\delta_{\rho_1 \dots \rho_p}^{\mu_1 \dots \mu_p} \delta_{\sigma_1 \dots \sigma_p}^{\nu_1 \dots \nu_p}}{p! p!} k^{\rho_1} k^{\sigma_1} g^{\rho_2 \sigma_2} \dots g^{\rho_p \sigma_p}. \end{aligned} \quad (21)$$

The first one is nothing but the identity, $(\mathcal{I}_0)_{\rho_1 \dots \rho_p}^{\mu_1 \dots \mu_p} = \delta_{\rho_1 \dots \rho_p}^{\mu_1 \dots \mu_p} / p!$. The propagator in momentum space satisfies $\mathcal{K}\mathcal{P} = \mathcal{I}_0$, hence

$$\mathcal{P} = i \frac{(-1)^p p!}{k^2 - m^2} \left(\mathcal{I}_0 - \frac{p}{m^2} \mathcal{I}_2 \right). \quad (22)$$

To arrive to that form is immediate using the identities $\mathcal{I}_0 \mathcal{I}_0 = \mathcal{I}_0$, $\mathcal{I}_0 \mathcal{I}_2 = \mathcal{I}_2 \mathcal{I}_0 = \mathcal{I}_2$, and $p \mathcal{I}_2 \mathcal{I}_2 = k^2 \mathcal{I}_2$. This last identity ensures that the numerator is indeed transverse on shell,

$$k \cdot \left(\mathcal{I}_0 - \frac{p}{m^2} \mathcal{I}_2 \right) \Big|_{k^2=m^2} = \left(\mathcal{I}_0 - \frac{p}{m^2} \mathcal{I}_2 \right) \cdot k \Big|_{k^2=m^2} = 0, \quad (23)$$

where $k \cdot \mathcal{I} = k_{\mu_1} \mathcal{I}^{\mu_1 \dots \mu_p, \nu_1 \dots \nu_p}$, $\mathcal{I} \cdot k = k_{\nu_1} \mathcal{I}^{\mu_1 \dots \mu_p, \nu_1 \dots \nu_p}$. Orthogonal invariants could be defined (see, e.g., Ref. [28] for those with $p = 2$), but this is not essential and further, we will see in the following that the $\mathcal{I}_0$ and $\mathcal{I}_2$ structure do carry dynamical information.

Explicitly, the invariant functions for $p = 0$ are $\mathcal{I}_0 = 1$ and $\mathcal{I}_2 = 0$, as they should be for a scalar field. For $p = 1$, we recover the usual $\mathcal{I}_0^{\mu, \nu} = g^{\mu \nu}$ and $\mathcal{I}_2^{\mu, \nu} = k^\mu k^\nu$. At higher orders, we get for $p = 2$ (in agreement with Ref. [29]),

$$\mathcal{I}_0^{\mu \nu, \alpha \beta} = \frac{1}{2} g^{\mu \alpha} g^{\nu \beta} - \frac{1}{2} g^{\mu \beta} g^{\nu \alpha}, \quad (24)$$

$$\mathcal{I}_2^{\mu \nu, \alpha \beta} = \frac{1}{4} g^{\nu \beta} k^\mu k^\alpha - \frac{1}{4} g^{\mu \beta} k^\nu k^\alpha - \frac{1}{4} g^{\nu \alpha} k^\mu k^\beta + \frac{1}{4} g^{\mu \alpha} k^\nu k^\beta, \quad (25)$$

in terms of which, for $p = 3$,

$$\mathcal{I}_0^{\mu \nu, \rho, \alpha \beta \gamma} = \frac{1}{3} \mathcal{I}_0^{\mu \alpha, \nu \beta} g^{\rho \gamma} - \frac{1}{3} \mathcal{I}_0^{\rho \alpha, \nu \beta} g^{\mu \gamma} - \frac{1}{3} \mathcal{I}_0^{\mu \alpha, \rho \beta} g^{\nu \gamma}, \quad (26)$$

$$\begin{aligned} \mathcal{I}_2^{\mu \nu, \rho, \alpha \beta \gamma} &= \frac{1}{9} \mathcal{I}_0^{\nu \beta, \rho \gamma} k^\mu k^\alpha - \frac{1}{9} \mathcal{I}_0^{\mu \beta, \rho \gamma} k^\nu k^\alpha + \frac{1}{9} \mathcal{I}_0^{\mu \alpha, \rho \gamma} k^\nu k^\beta \\ &\quad - \frac{1}{9} \mathcal{I}_0^{\nu \alpha, \rho \gamma} k^\mu k^\beta + \frac{1}{9} \mathcal{I}_0^{\mu \beta, \nu \gamma} k^\rho k^\alpha - \frac{1}{9} \mathcal{I}_0^{\mu \alpha, \nu \gamma} k^\rho k^\beta \\ &\quad - \frac{1}{9} \mathcal{I}_0^{\nu \beta, \rho \alpha} k^\mu k^\gamma + \frac{1}{9} \mathcal{I}_0^{\mu \beta, \rho \alpha} k^\nu k^\gamma + \frac{1}{9} \mathcal{I}_0^{\mu \alpha, \nu \beta} k^\rho k^\gamma. \end{aligned} \quad (27)$$

Finally, for $p = 4$, the propagator is a trivial contact term proportional to $\mathcal{I}_0/m^2$ since the field strength vanishes, with $(\mathcal{I}_0)_{\nu_1 \dots \nu_4}^{\mu_1 \dots \mu_4} = -\epsilon_{\nu_1 \dots \nu_4} \epsilon^{\mu_1 \dots \mu_4} / 4!$.

For massless states, the kinetic kernel is a projector and cannot be inverted. This situation is well-known in QED: the gauge has to be fixed to quantize the theory. This can be done by adding to the Lagrangian a term quadratic in the Lorenz condition,

$$\mathcal{S}_{\text{p-form}} = (-1)^p \int \frac{1}{2} F \wedge \star F - \frac{1}{2\xi} d \star A \wedge \star d \star A, \quad (28)$$

in which case the gauge-dependent propagator becomes

$$\mathcal{P} = i \frac{(-1)^p p!}{k^2} \left(\mathcal{I}_0 - (1 - \xi) \frac{p}{k^2} \mathcal{I}_2 \right). \quad (29)$$

The cancelation of the $\mathcal{I}_2$ term is then ensured by the conservation of the currents to which the propagator couples.

Finally, there are a few peculiar one- and two-point vertices specific to four dimensions: $dA \wedge dA$ for $p = 1$, $A \wedge A$, $d \star A \wedge \star dA$ for $p = 2$, and dA for $p = 3$. All of them are 4-forms and can thus enter in the action. Now, the former is the well-known theta term $F_{\mu\nu}^A \tilde{F}^{A, \mu\nu}$ that can be discarded for a topologically trivial $U(1)$, while the latter is a pure boundary term that can also be discarded. This leaves only the $p = 2$ terms: the pseudoscalar mass term $\tilde{m}^2 B \wedge B \rightarrow \tilde{m}^2 B_{\mu\nu} \tilde{B}^{\mu\nu}$ and the mixed kinetic term $d \star B \wedge \star dB \rightarrow \epsilon_{\mu\nu\rho\sigma} \partial_\alpha B^{\alpha\mu} F^{B, \nu\rho\sigma}$. For simplicity here, we will keep the standard kinetic term only (except briefly in Sec. IV). Let us thus concentrate on $\tilde{m}^2 B_{\mu\nu} \tilde{B}^{\mu\nu}$. It could be dealt with perturbatively, but since $\tilde{m}$ could be as large as m , a better way to proceed is to immediately resum all $\tilde{m}$ mass insertions by encoding the $\tilde{m}$ term directly into the B propagator. This requires one to extend the basis of antisymmetric invariants to include

$$\mathcal{I}_3^{\mu_1 \mu_2, \nu_1 \nu_2} = \frac{1}{2} \epsilon^{\mu_1 \mu_2 \nu_1 \nu_2}, \quad (30)$$

$$\mathcal{I}_{41}^{\mu_1 \mu_2, \nu_1 \nu_2} = \frac{1}{4} (k^{\mu_1} k_\alpha \epsilon^{\alpha \mu_2 \nu_1 \nu_2} - k^{\mu_2} k_\alpha \epsilon^{\alpha \mu_1 \nu_1 \nu_2}), \quad (31)$$

$$\mathcal{I}_{42}^{\mu_1 \mu_2, \nu_1 \nu_2} = \frac{1}{4} (k^{\nu_1} k_\alpha \epsilon^{\alpha \mu_1 \mu_2 \nu_2} - k^{\nu_2} k_\alpha \epsilon^{\alpha \mu_1 \mu_2 \nu_1}), \quad (32)$$

which obey simple relations with the other ones like, e.g., $\mathcal{I}_3 \mathcal{I}_3 = -\mathcal{I}_0$, $\mathcal{I}_3 \mathcal{I}_2 = -\mathcal{I}_{42}$, $\mathcal{I}_2 \mathcal{I}_3 = -\mathcal{I}_{41}$, $\mathcal{I}_2 \mathcal{I}_{42} = \mathcal{I}_{41} \mathcal{I}_2 = 0$, $2\mathcal{I}_{42} \mathcal{I}_2 = k^2 \mathcal{I}_{42}$, $2\mathcal{I}_2 \mathcal{I}_{41} = k^2 \mathcal{I}_{41}$, and so on. Inverting the kinetic term, we then find

$$\begin{aligned} \mathcal{P}^{2\text{-form}} &= \frac{2}{k^2 - m^2 (1 + \tilde{m}^4 / m^4)} \left(\mathcal{I}_0 - \frac{2}{m^2} \mathcal{I}_2 - \frac{\tilde{m}^2}{k^2 - m^2} \right. \\ &\quad \left. \times \left(\mathcal{I}_3 + \frac{2}{m^2} (\mathcal{I}_{41} + \mathcal{I}_{42}) \right) \right), \end{aligned} \quad (33)$$

where it is understood that the Lagrangian terms are normalized as $-\tilde{m}^2 B_{\mu\nu} B^{\mu\nu}/4 + \tilde{m}^2 B_{\mu\nu} \tilde{B}^{\mu\nu}/4$, with $\tilde{B}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} B^{\rho\sigma}/2$. Notice that if $m = 0$ but $\tilde{m} \neq 0$, the kinetic term is not invertible, so the situation is a bit pathological in that case. Here, taking both mass terms as nonzero, the true pole mass of the B field becomes¹

$$m_B^2 = \frac{m^4 + \tilde{m}^4}{m^2}. \quad (34)$$

Yet, given this mass, the presence of a second pole at $k^2 = m^2$ in $\mathcal{P}^{2\text{-form}}$ cannot be physical. To see that it is spurious, one should remember that $\mathcal{P}^{2\text{-form}}$ will always be sandwiched as $J_{1,\mu_1\mu_2} \mathcal{P}_{2\text{-form}}^{\mu_1\mu_2,\nu_1\nu_2} J_{2,\nu_1\nu_2}$ for some vertices J_1 and J_2 that are antisymmetric in their indices. As a result, one can check that $J_1 \cdot (\mathcal{I}_{41} + \mathcal{I}_{42}) \cdot J_2 = -(k^2/2) J_1 \cdot \mathcal{I}_3 \cdot J_2$, such that effectively, the full propagator can be taken as

$$\mathcal{P}^{2\text{-form}} = \frac{2}{k^2 - (m^4 + \tilde{m}^4)/m^2} \left(\mathcal{I}_0 - \frac{2}{m^2} \mathcal{I}_2 + \frac{\tilde{m}^2}{m^2} \mathcal{I}_3 \right), \quad (35)$$

which now has its pole at the physical mass $k^2 = m_B^2$. The singularity as $m \rightarrow 0$ has the same physical interpretation as that in Eq. (23). In both cases, the physics abruptly changes when $m \rightarrow 0$ since the kinetic term ceased to be invertible. We will see later on how to rederive these results using duality, along with the polarization sum in the numerator of the full propagator. At this stage, one may be surprised to notice that this polarization sum is no longer transverse on shell. However, this should be expected. In the presence of the $\tilde{m}$ term, the EoM becomes

$$\partial^\alpha F_{\alpha\mu_1\mu_2}^B + m^2 B_{\mu_1\mu_2} - \tilde{m}^2 \tilde{B}_{\mu_1\mu_2} = J_{\mu_1\mu_2}. \quad (36)$$

Thus, for free fields, the Lorenz condition should read (why a term proportional to the field strength appears will become obvious in Sec. IV),

$$m^2 \partial^\nu B_{\mu\nu} - \frac{1}{3!} \tilde{m}^2 \epsilon_{\mu\nu\rho\sigma} F^{B,\nu\rho\sigma} = 0, \quad (37)$$

and the polarization matrices thus have to satisfy $k_\mu (m^2 \mathcal{I}_0^{\mu\nu,\rho\sigma} - \tilde{m}^2 \mathcal{I}_3^{\mu\nu,\rho\sigma}) \epsilon_{\rho\sigma}^{(\lambda)} = 0$. One can check that this

¹A similar result was obtained in Ref. [27], but in the context of compact QED with monopole condensates.

is consistent with the sum in the numerator of Eq. (35),

$$k \cdot (m^2 \mathcal{I}_0 - \tilde{m}^2 \mathcal{I}_3) \cdot \left(\mathcal{I}_0 - \frac{2}{m^2} \mathcal{I}_2 + \frac{\tilde{m}^2}{m^2} \mathcal{I}_3 \right) \Big|_{k^2=m_B^2} = 0. \quad (38)$$

III. EFFECTIVE COUPLINGS FOR HIGHER FIELDS

The goal here is to construct the leading operators coupling p -form fields to SM matter fields, with $p = 0, \dots, 4$. Neither the Lorenz condition $\partial^\mu A_{\mu\nu\dots} = 0$ nor the free EoM $\partial^\mu F_{\mu\nu\dots} = 0$ (or $\partial^2 A_{\mu\nu\dots} = 0$) are imposed. Only the Bianchi identities for $p = 0, 1, 2$ are enforced. Importantly, no dark gauge symmetry (or shift symmetry for $p = 0$) will be imposed here. Instead, invariant and noninvariant operators will be constructed and put in separate classes. This will permit one to compare phenomenological scenarios in which the dark gauge symmetry is totally broken at the low scale, or only broken by the dark state mass term.

Also, operators with three or more p fields will not be constructed. Though one could introduce some quantum number to allow no fewer than n of them to be produced, for n any given integer, one rarely encounters n larger than 2 since this would conflict with the kinetic terms. Phenomenologically, producing more than two dark states is strongly but trivially phase-space suppressed, and thus less likely to end up as observable. Another argument is that one of our goals will be to analyze the relationships between these bases, comparing the situation when a gauge (or shift) symmetry is active or not. For three or more p fields, the leading symmetric operators would involve three or more field strengths, and thus be of prohibitively high dimension.

The language of differential form is not well suited to the construction of all these effective interactions. Instead, we will follow the pedestrian path of taking products of antisymmetric tensor fields together with SM fields, contracting all the Lorentz indices to form invariants. Mathematically, this corresponds to taking various products of interior products of differential forms with suitably constructed vectors made of combinations of SM fields, but there appears to be no advantage in adopting that formalism. The same is true concerning higher form field self-interactions. Generic renormalizable self-interactions could be constructed from $A \wedge A \wedge \star A$, $A \wedge \star A \wedge \star (d \star A)$, $A \wedge \star A \wedge \star dA$, $(A \wedge \star A) \wedge \star (A \wedge \star A)$, etc., but this is not particularly efficient since only some of these combinations correspond to 4-form for a given p . It is actually much easier to directly construct the suitable self-interactions for each p . Though neither those renormalizable couplings nor their extensions to higher orders will concern us here (see Ref. [30] for a discussion in the $p = 1$ and $p = 2$ case), for completeness, the full list up to dimension four can easily be written down and is actually quite limited,

$$p = 0: \phi^3, \phi^4, \quad (39a)$$

$$p = 1: A_\mu A^\mu \partial^\nu A_\nu, \quad A_\mu A^\mu A_\nu A^\nu, \quad (39b)$$

$$p = 2: B_{\mu\nu} B^{\mu\nu} B_{\rho\sigma} B^{\rho\sigma}, \quad B_{\mu\nu} B^{\mu\nu} B_{\rho\sigma} \tilde{B}^{\rho\sigma}, \\ B_{\mu\nu} B^{\mu\rho} B_{\rho\sigma} B^{\sigma\nu}, \quad B_{\mu\nu} B^{\mu\rho} B_{\rho\sigma} \tilde{B}^{\sigma\nu}, \quad (39c)$$

$$p = 3: C_{\mu\nu\rho} C^{\mu\nu\rho} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta}^C, \quad C_{\mu\nu\rho} C^{\mu\nu\rho} C_{\alpha\beta\gamma} C^{\alpha\beta\gamma}, \quad (39d)$$

$$p = 4: D_{\mu\nu\rho\sigma} D^{\mu\nu\rho\sigma} D_{\alpha\beta\gamma\delta} \epsilon^{\alpha\beta\gamma\delta}, \quad D_{\mu\nu\rho\sigma} D^{\mu\nu\rho\sigma} D_{\alpha\beta\gamma\delta} D^{\alpha\beta\gamma\delta}. \quad (39e)$$

All these extra operators break the dark gauge symmetry. Their presence or impact could thus be restricted phenomenologically by assuming that this symmetry remains active in the dark sector, except of course for the dark state mass term. Notice also that only ϕ and $D_{\mu\nu\rho\sigma}$ can have a dimensionful self-interaction, while none of the others can.

In the following, most of the work will concentrate on fermionic operators, and for them, various identities are useful. Those are totally standard, but it is worth collecting them here because they are not that often encountered in practice. Indeed, a specificity of higher form fields is to have many indices, and thus the effective operators involve many contractions, including with the epsilon tensor. Further, since in this paper we are concerned by the properties of these bases, we will push the construction to effective operators involving up to two extra derivatives, further extending the Lorentz index counts. Yet, the properties of the Dirac matrices makes reducing these operators nearly always possible, drastically reducing the number of independent operators at each order.

First, we define $2\sigma^{\mu\nu} \equiv i[\gamma^\mu, \gamma^\nu]$, and reduce any string of more than two Dirac matrices using the Chisholm identity,

$$\gamma^\mu \gamma^\nu \gamma^\rho = g^{\mu\nu} \gamma^\rho - g^{\mu\rho} \gamma^\nu + g^{\nu\rho} \gamma^\mu + i\epsilon^{\mu\nu\rho\alpha} \gamma_\alpha \gamma_5 \\ \Rightarrow \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho = -(3!) i \gamma_\sigma \gamma_5. \quad (40)$$

This also shows that whenever a $\sigma^{\mu\nu}$ appears, it cannot be accompanied by a epsilon tensor since $2\sigma^{\mu\nu} \gamma_5 = i\epsilon^{\mu\nu\alpha\beta} \sigma_{\alpha\beta}$ implies

$$\epsilon^{\alpha\beta\gamma\delta} \sigma_{\mu\nu} \psi_R = \frac{i}{2} \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} \sigma^{\rho\sigma} \psi_R = -\frac{i}{2} \delta_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta} \sigma^{\rho\sigma} \psi_R, \quad (41)$$

and similarly with ψ_L . For fermionic fields, we do use the fermion equation of motion whenever possible,² $i\gamma^\mu \mathcal{D}_\mu \psi_{L,R} \rightarrow m\psi_{L,R}$. This implies in particular

²Covariant derivatives are always meant to include the SM gauge fields only, and not any of the dark fields.

$$\sigma^{\mu\nu} \mathcal{D}_\nu \psi_R = -i(g^{\mu\nu} - \gamma^\mu \gamma^\nu) \mathcal{D}_\nu \psi_R = -i\mathcal{D}^\mu \psi_R + m\gamma^\mu \psi_L. \quad (42)$$

Within a spinor contraction, derivative can be assumed to always act on the right, since this is equivalent to $\overleftrightarrow{\mathcal{D}}$ up to a total derivative of the whole spinor contraction, which can then be made to act on the other fields by partial integration. By consistency once operators with two derivatives are included, those with SM field strengths have to be present since $[\mathcal{D}_\mu, \mathcal{D}_\nu] = ieF_{\mu\nu}$ when acting on a charged fermion field, leading to identities like

$$\mathcal{D}_\mu \mathcal{D}^\mu \psi_L = m^2 \psi_L + \frac{i}{2} \sigma^{\mu\nu} [\mathcal{D}_\mu, \mathcal{D}_\nu] \psi_L \\ = m^2 \psi_L - \frac{e}{2} \sigma^{\mu\nu} F_{\mu\nu} \psi_L. \quad (43)$$

Finally, additional identities that prove useful are

$$\partial_\mu (\bar{\psi}_L \gamma^\mu \psi_L) = m\bar{\psi}_L \psi_R - m\bar{\psi}_R \psi_L, \quad (44a)$$

$$\partial_\mu (\bar{\psi}_L \sigma^{\mu\nu} \psi_R) = i\bar{\psi}_L \overleftrightarrow{\mathcal{D}}^\nu \psi_R - m\bar{\psi}_L \gamma^\nu \psi_L - m\bar{\psi}_R \gamma^\nu \psi_R, \quad (44b)$$

$$\partial_\mu (\bar{\psi} \sigma^{\mu\nu} \mathcal{D}^\rho \psi) = -i\partial^\nu (\bar{\psi} \mathcal{D}^\rho \psi) + 2i\bar{\psi}_L \mathcal{D}^\nu \mathcal{D}^\rho \psi_R \\ + e\bar{\psi}_L \gamma^\nu \gamma^\mu \psi_R F_\mu^\rho - m\bar{\psi}_R \gamma^\nu \mathcal{D}^\rho \psi_R \\ - m\bar{\psi}_L \gamma^\nu \mathcal{D}^\rho \psi_L. \quad (44c)$$

By convention in the present section, F always denotes SM field strengths, while those associated with higher form fields receive that field as a superscript, as F^A , F^B , and so on, see Eq. (10).

Finally, it should be noted that in the SM before symmetry breaking, the fermion EoM would involve the Higgs boson and not simply the fermion mass. Yet, generalizing our bases to that case amounts to add Higgs fields for all $\psi_L \Gamma \psi_R$ contractions, plus operators involving derivatives of the Higgs fields, and that can easily be done separately.

A. 0-form field effective operators

These operators are well-known (see, e.g., Ref. [31]) since a 0-form field is simply a generic neutral scalar field. The leading operators to SM fermion fields are

d	Type	
4	II	$\bar{\psi}_L \psi_R \phi$
5	I	$(\bar{\psi}_L \gamma^\mu \psi_L F_\mu^\phi)$
	II	$\dots$
6	I	$\bar{\psi}_L \psi_R \partial^\mu F_\mu^\phi$
	II	$\dots$
	III	$\bar{\psi}_L \sigma_{\mu\nu} \psi_R \phi F^{\mu\nu}$

for one dark field, and

d	Type	
5	II	$\bar{\psi}_L \psi_R \phi^2$
6	II	$\bar{\psi}_L \gamma^\mu \psi_L \phi F_\mu^\phi$
7	I	$\bar{\psi}_L \psi_R F_\mu^\phi F^{\phi,\mu}$
	II	$\bar{\psi}_L \psi_R \phi \partial^2 \phi \quad \bar{\psi}_L \mathcal{D}^\mu \psi_R \phi F_\mu^\phi$
	III	$\bar{\psi}_L \sigma_{\mu\nu} \psi_R \phi^2 F^{\mu\nu}$

(46)

for two dark fields. In those Tables, type I and II distinguish shift-symmetric and non-shift-symmetric operators, while type III involves an extra photon field. For simplicity, all the operators with $L \leftrightarrow R$ are understood. Those are simply the Hermitian conjugate operators for those involving scalar or tensor spinor structures. Remember that $F_\mu^\phi = \partial_\mu \phi$, and the Bianchi identity holds (so an operator involving $F^{\mu\nu} \partial_\mu F_\nu^\phi \rightarrow 0$ is absent). The operator $\bar{\psi}_L \gamma^\mu \psi_L F_\mu^\phi$ is in parentheses because it is reducible, $\bar{\psi}_L \gamma^\mu \psi_L F_\mu^\phi \rightarrow \partial_\mu (\bar{\psi}_L \gamma^\mu \psi_L) \phi$ using the Dirac equation for on-shell fermions. Yet, it is important for $\bar{\psi}_L \gamma^\mu \psi_L F_\mu^\phi$ to appear explicitly since it is the leading shift-invariant effective interaction. This situation is ubiquitous in axion effective theories. By contrast, all the other operators that could be eliminated by partial integration and the use of the Dirac equation do not bring anything special and have not been kept.

If the effective operators are constructed at the SM scale, we must add a Higgs field for all the LR operators of Eqs. (45) and (46) to make them symmetric under $SU(2)_L \otimes U(1)_Y$ since ψ_R and ψ_L have different gauge charges. This increases their dimension by one. Then, restricting our attention to operators up to dimension five, the non-fermionic operators that can be constructed are (omitting Wilson coefficients for clarity)

$$\begin{aligned} \mathcal{L}_{\text{Int}}^{0\text{-Form}} = & \Lambda \Phi^\dagger \Phi \phi + \Phi^\dagger \Phi \phi^2 + \frac{1}{\Lambda} \Phi^\dagger \overleftrightarrow{\mathcal{D}}_\mu \Phi F^{\phi,\mu} \\ & + \frac{1}{\Lambda} \phi \mathcal{D}^\mu \Phi^\dagger \mathcal{D}_\mu \Phi + \frac{1}{\Lambda} \phi (\Phi^\dagger \Phi)^2 + \frac{1}{\Lambda} \phi F_{\mu\nu} F^{\mu\nu} \\ & + \frac{1}{\Lambda} \phi F_{\mu\nu} \tilde{F}^{\mu\nu} + \mathcal{O}(\Lambda^{-2}), \end{aligned} \quad (47)$$

where Φ stands for the Higgs boson doublet and F for either the electroweak or gluon field strength, $B_{\mu\nu}$, $W_{\mu\nu}^i$, or $G_{\mu\nu}^a$.

B. 1-form field effective operators

The operators for a Proca field together with a fermion and up to two derivatives are (operators with $L \leftrightarrow R$ are understood),

d	Type	
4	II	$\bar{\psi}_L \gamma^\mu \psi_L A_\mu$
5	I	$\bar{\psi}_L \sigma^{\mu\nu} \psi_R F_{\mu\nu}^A$
	IV	$\bar{\psi}_L \psi_R \partial^\mu A_\mu$
6	I	$\bar{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_L F_{\mu\nu}^A \quad \bar{\psi}_L \gamma^\nu \psi_L \partial^\mu F_{\mu\nu}^A$
	II	$\dots$
	III	$\bar{\psi}_L \gamma_\nu \psi_L A_\mu F^{\mu\nu} \quad \bar{\psi}_L \gamma_\nu \psi_L A_\mu \tilde{F}^{\mu\nu}$

(48)

Remember here that the covariant derivative acting on ψ may include the SM gauge fields only. For reasons that will become clear later on, we introduce as a fourth class of operators all those that vanish upon enforcing the Lorenz condition. Notice that no other operator appears at $\mathcal{O}(\Lambda^{-1})$ because Eq. (44) is used to express $\bar{\psi}_L \mathcal{D}^\nu \psi_R A_\nu$ in terms of $\bar{\psi}_L \sigma^{\mu\nu} \psi_R F_{\mu\nu}^A$. Similarly, at the dimension-six level, $\bar{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_L \partial_\nu A_\mu$ is reduced to $\bar{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_L F_{\mu\nu}^A$ by first subtracting $\bar{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_L \partial_\mu A_\nu$, reducible via the Dirac equation, and then by using Eq. (40) and the antisymmetry under $\mu \leftrightarrow \nu$. Also, rewriting Eq. (40) as

$$\gamma^\mu \mathcal{D}^\nu - \gamma^\nu \mathcal{D}^\mu = \gamma^\mu \gamma^\nu \gamma^\rho \mathcal{D}_\rho - g^{\mu\nu} \gamma^\rho \mathcal{D}_\rho - i \epsilon^{\mu\nu\rho\alpha} \gamma_\alpha \gamma_5 \mathcal{D}_\rho, \quad (49)$$

allows one to relate $\bar{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_L F_{\mu\nu}^A$ and $\bar{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_L \tilde{F}_{\mu\nu}^A$, up to $\mathcal{O}(m)$ corrections to $\bar{\psi}_L \sigma^{\mu\nu} \psi_R F_{\mu\nu}^A$. So, in the absence of the photon field, there are no gauge-dependent dimension-six operators because they all collapse to $\mathcal{O}(m)$ contributions to the dimension-five $\bar{\psi}_L \psi_R \partial^\mu A_\mu$ operator and $\mathcal{O}(m^2)$ contribution to the dimension four $\bar{\psi}_L \gamma^\mu \psi_L A_\mu$ operator. Since in the absence of the gauge symmetry those should be present anyway, there is no need to include separately the gauge-dependent dimension-six ones. Finally, concerning the leading nongauge invariant couplings $\bar{\psi}_L \gamma^\mu \psi_L A_\mu$ and $\bar{\psi}_R \gamma^\mu \psi_R A_\mu$, it should be remembered that the vector combination is actually gauge invariant since the vector current is conserved.

With two Proca fields, the leading operators are

d	Type				
5	II	$\bar{\psi}_L \psi_R A_\mu A^\mu$			
6	II	$\bar{\psi}_L \gamma^\nu \psi_L A^\mu F_{\mu\nu}^A$	$\bar{\psi}_L \gamma^\nu \psi_L A^\mu \tilde{F}_{\mu\nu}^A$	$\bar{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_L A_\mu A_\nu$	
	IV	$\bar{\psi}_L \gamma^\nu \psi_L A_\nu \partial^\mu A_\mu$			
7	I	$\bar{\psi}_L \psi_R F_{\mu\nu}^A F^{A,\mu\nu}$	$\bar{\psi}_L \psi_R F_{\mu\nu}^A \tilde{F}^{A,\mu\nu}$		
	II	$\bar{\psi}_L \psi_R \partial_\nu A_\mu \partial^\nu A^\mu$	$\bar{\psi}_L \psi_R A^\mu \partial^\nu F_{\mu\nu}$	$\bar{\psi}_L \sigma^{\mu\nu} \psi_R A_\mu \partial^\rho F_{\nu\rho}$	$\bar{\psi}_L \sigma^{\mu\nu} \psi_R A_\rho \partial^\rho F_{\mu\nu}^A$
		$\bar{\psi}_L \sigma^{\mu\nu} \mathcal{D}^\rho \psi_R A_\mu \partial_\rho A_\nu$	$\bar{\psi}_L \sigma^{\nu\rho} \mathcal{D}^\mu \psi_R A_\mu F_{\nu\rho}^A$	$\bar{\psi}_L \mathcal{D}^\mu \mathcal{D}^\nu \psi_R A_\mu A_\nu$	
	III	$\bar{\psi}_L \sigma^{\mu\nu} \psi_R A_\rho A^\rho F_{\mu\nu}$	$\bar{\psi}_L \sigma^{\mu\nu} \psi_R A_\mu A^\rho F_{\rho\nu}$		
IV		$\bar{\psi}_L \psi_R A_\mu \partial^\mu \partial^\nu A_\nu$	$\bar{\psi}_L \psi_R \partial^\mu A_\mu \partial^\nu A_\nu$	$\bar{\psi}_L \sigma^{\mu\nu} \psi_R F_{\mu\nu}^A \partial^\rho A_\rho$	

Again, many identities have been exploited to reduce the number of independent operators, and this basis is minimal.

Applied at the electroweak scale, the same provision as in the scalar case applies for adding Higgs boson fields for all LR operators. Then, the operators involving only SM bosonic fields are, up to dimension five,

$$\mathcal{L}_{\text{Int}}^{1\text{-Form}} = F_{\mu\nu}^A F_Y^{\mu\nu} + \Phi^\dagger \Phi \partial^\mu A_\mu + \Phi^\dagger \overleftrightarrow{\mathcal{D}}_\mu \Phi A^\mu + \Phi^\dagger \Phi A_\mu A^\mu + \mathcal{O}(\Lambda^{-2}), \quad (51)$$

where Φ stands for the Higgs boson doublet and we denote as $F_Y^{\mu\nu}$ the $U(1)_Y$ field strength. The first operator is the well-known kinetic mixing operator. We do not include the topological $F_{\mu\nu}^A \tilde{F}_Y^{\mu\nu}$ coupling, which vanishes upon partial integration thanks to the Bianchi identity. Notice that a specific combination of the last two operators is gauge invariant. It can be absorbed into the covariant derivative acting on the Higgs field, thereby giving it a dark charge. Once diagonalized, the kinetic mixing has a similar impact. This is described in detail in many places, see, e.g., Refs. [6,32], but for completeness, let us briefly summarize the main feature. We take the case of a low-energy mixing with the photon field alone, which we denote as A_μ^γ and $F_{\mu\nu}^\gamma$, and start from

$$L_{\text{kin}} = -\frac{1}{4} F_{\mu\nu}^\gamma F^{\gamma,\mu\nu} - \frac{1}{4} F_{\mu\nu}^A F^{A,\mu\nu} + m_A^2 A_\mu A^\mu + \frac{\chi}{2} F_{\mu\nu}^A F^{\gamma,\mu\nu}. \quad (52)$$

To diagonalize the kinetic terms, one performs

$$A_\mu^\gamma \rightarrow A_\mu^\gamma + \sinh \eta A_\mu, \quad A_\mu \rightarrow \cosh \eta A_\mu, \quad \tanh \eta = \chi. \quad (53)$$

This is really a reparametrization of the fields, not a mere rotation, and as such it need not be unitary. All that matters is to produce canonical kinetic terms. Phenomenologically, the two-point effective coupling rescales the dark vector mass as $m_A \rightarrow m_A \cosh \eta$, and adds gauge invariant couplings with

strength $q \times \sinh \eta$ for all the fermions of electric charge q to the dark vector. Those do not alter the basis constructed above and can be absorbed either directly in its gauge-invariant effective couplings or into gauge-invariant combinations of its effective couplings like $\bar{\psi}_L \gamma^\mu \psi_L A_\mu + \bar{\psi}_R \gamma^\mu \psi_R A_\mu$ for a charged fermion, or $\Phi^\dagger \overleftrightarrow{\mathcal{D}}_\mu \Phi A^\mu + \Phi^\dagger \Phi A_\mu A^\mu$ for a charged scalar. The description of the full $SU(2)_L \otimes U(1)$ case, including the mixing with the Z boson, does not alter this picture, see Refs. [6,32].

C. 2-form field effective operators

For the 2-form field, the dominant operators are found to be,

d	Type				
4	II	$\bar{\psi}_L \sigma^{\mu\nu} \psi_R B_{\mu\nu}$			
5	I	$\bar{\psi}_L \gamma_\sigma \psi_L \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^B$			
	II	$\bar{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_L B_{\mu\nu}$			
	IV	$\bar{\psi}_L \gamma^\nu \psi_L \partial^\mu B_{\mu\nu}$			
6	I	$\bar{\psi}_L \sigma^{\mu\nu} \psi_R \partial^\rho F_{\mu\nu}^B$			
	II	...			
	III	$\bar{\psi}_L \psi_R B_{\mu\nu} F^{\mu\nu}$	$\bar{\psi}_L \psi_R B_{\mu\nu} \tilde{F}^{\mu\nu}$	$\bar{\psi}_L \sigma^{\mu\rho} \psi_R B_{\mu\nu} F_{\rho}^\nu$	
	IV	$\bar{\psi}_L \sigma^{\mu\nu} \psi_R \partial_\mu \partial^\rho B_{\nu\rho}$			

The dimension-four and the first dimension-five operator were already identified in Ref. [28], while the leading tensor interaction was considered in Refs. [33,29] in the context of DM searches and Bhabha scattering, respectively. To derive all the others, partial integration was used as much as possible and the Bianchi identity $\epsilon^{\alpha\mu\nu\rho} \partial_\alpha F_{\mu\nu}^B = 0$ was enforced. Besides the fermionic identities discussed before, we also used Eq. (A4) to simplify any partial contraction between an epsilon tensor and B or F^B , e.g., as

$$\epsilon^{\alpha\beta\gamma\mu} B_{\mu\nu} = -\frac{1}{4} \epsilon^{\alpha\beta\gamma\mu} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\lambda\kappa\rho\sigma} B_{\lambda\kappa} = -\frac{1}{4} \delta_{\nu\rho\sigma}^{\alpha\beta\gamma} \epsilon^{\lambda\kappa\rho\sigma} B_{\lambda\kappa} = -\frac{1}{2} (\delta_\nu^\alpha \tilde{B}^{\sigma\rho} + \delta_\nu^\beta \tilde{B}^{\rho\alpha} + \delta_\nu^\gamma \tilde{B}^{\alpha\sigma}), \quad (55)$$

where $\tilde{B}_{\mu\nu} = (1/2)\epsilon_{\mu\nu\rho\sigma} B^{\rho\sigma}$ is the dual of the B field. For two B fields, the leading operators are

d	Type			
5	II	$\bar{\psi}_L \psi_R B_{\mu\nu} B^{\mu\nu}$	$\bar{\psi}_L \psi_R B_{\mu\nu} \tilde{B}^{\mu\nu}$	
6	II	$\bar{\psi} \gamma_\rho \mathcal{D}^\mu \psi B_{\mu\nu} B^{\nu\rho}$	$\bar{\psi} \gamma_\rho \mathcal{D}^\mu \psi B_{\mu\nu} \tilde{B}^{\nu\rho}$	$\bar{\psi} \gamma_\rho \mathcal{D}^\mu \psi \tilde{B}_{\mu\nu} B^{\nu\rho}$
		$\bar{\psi}_L \gamma^\mu \psi_L B_{\mu\nu} \partial_\rho \tilde{B}^{\nu\rho}$	$\bar{\psi}_L \gamma^\mu \psi_L \tilde{B}_{\mu\nu} \partial_\rho \tilde{B}^{\nu\rho}$	
	IV	$\bar{\psi}_L \gamma^\mu \psi_L \tilde{B}_{\mu\nu} \partial_\rho B^{\nu\rho}$	$\bar{\psi}_L \gamma^\mu \psi_L B_{\mu\nu} \partial_\rho B^{\nu\rho}$	
7	I	$\bar{\psi}_L \psi_R F_{\mu\nu\rho}^B F^{B,\mu\nu\rho}$		
	II, III, IV	...		

(56)

At the dimension-seven level, we keep only the gauge-invariant operator because it is the leading one being so. The gauge-dependent operators are too numerous to be useful, so their quite tricky reduction to a minimal basis does not appear worth the effort.

The couplings with the other SM fields are

$$\begin{aligned} \mathcal{L}_{\text{Int}}^{2\text{-form}} = & \Lambda B_{\mu\nu} F_Y^{\mu\nu} + \Lambda B_{\mu\nu} \tilde{F}_Y^{\mu\nu} + \Phi^\dagger \Phi B_{\mu\nu} B^{\mu\nu} + \Phi^\dagger \Phi B_{\mu\nu} \tilde{B}^{\mu\nu} \\ & + \frac{1}{\Lambda} \Phi^\dagger \overleftrightarrow{D}_\alpha \Phi \epsilon^{\mu\nu\rho\alpha} F_{\mu\nu\rho}^B + \frac{1}{\Lambda} \partial^\alpha F_{\alpha\beta} \epsilon^{\mu\nu\rho\beta} F_{\mu\nu\rho}^B + \frac{1}{\Lambda} H^\dagger H F^{\mu\nu} B_{\mu\nu} \\ & + \frac{1}{\Lambda} \Phi^\dagger \overleftrightarrow{D}_\alpha \Phi \partial_\mu B^{\mu\alpha} + \frac{1}{\Lambda} \partial^\alpha F_{\alpha\mu} \partial_\mu B^{\mu\nu} + \frac{1}{\Lambda} H^\dagger H \tilde{F}^{\mu\nu} B_{\mu\nu} + \mathcal{O}(\Lambda^{-2}). \end{aligned} \quad (57)$$

Notice that the $B_{\mu\nu} F_Y^{\mu\nu}$ term at $\mathcal{O}(\Lambda)$ is actually a surface term upon enforcing the Lorentz condition $\partial^\mu B_{\mu\nu} = 0$ since

$$B_{\mu\nu} F_Y^{\mu\nu} = 2B_{\mu\nu} \partial^\mu A_Y^\nu = 2\partial^\mu (B_{\mu\nu} A_Y^\nu) - \partial^\mu B_{\mu\nu} A_Y^\nu. \quad (58)$$

It is thus essentially topological (in the absence of kinetic terms for A and B , this term alone is known as the topological BF theory [34]) and can be discarded. The second term is not topological since $B_{\mu\nu} \tilde{F}_Y^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} \partial_\rho A_{Y,\sigma} = \epsilon^{\mu\nu\rho\sigma} A_{Y,\sigma} F_{\mu\nu\rho}^B$. As for the kinetic mixing, this two-point vertex must be eliminated to get canonical kinetic terms, and this can be done again via a reparametrization.

Specifically, let us consider the low-energy situation of a mixing with the photon field strength

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{2\text{-Form}} \supset & \frac{1}{12} F_{\mu\nu\rho}^B F^{B,\mu\nu\rho} - \frac{1}{4} m^2 B_{\mu\nu} B^{\mu\nu} + \frac{1}{4} \tilde{m}^2 B_{\mu\nu} \tilde{B}^{\mu\nu} \\ & - \frac{1}{4} F_{\mu\nu}^\gamma F^{\gamma,\mu\nu} + \frac{\Lambda_\gamma}{2} B_{\mu\nu} \tilde{F}^{\gamma,\mu\nu}. \end{aligned} \quad (59)$$

We can perform the reparametrization

$$A_\mu^\gamma \rightarrow A_\mu^\gamma (1 - \eta^2)^{-1/2}, \quad B_{\mu\nu} \rightarrow B_{\mu\nu} - \eta F_{\mu\nu}^\gamma, \quad \eta = \frac{\Lambda_\gamma}{\tilde{m}^2}, \quad (60)$$

to eliminate the $B_{\mu\nu} \tilde{F}^{\gamma,\mu\nu}$ coupling. This leaves the field strength $F^B = dB$ invariant, but regenerates a topological $B_{\mu\nu} F_Y^{\mu\nu}$ coupling that needs again to be discarded (along with a topologically trivial $F_Y^\gamma \tilde{F}^{\gamma,\mu\nu}$ term).

A crucial difference with the dark vector kinetic mixing though is that it is now the dark field that is shifted, not the photon field. As a result, the B field does not inherit any new coupling. Instead, it is the photon that does if B is coupled to SM fields. For instance, shifting the $\bar{\psi}_L \sigma^{\mu\nu} \psi_R B_{\mu\nu}$ operator generates new contributions to the electric and/or magnetic dipole operators (EDM and MDM, respectively),

$$c_1 \bar{\psi}_L \sigma^{\mu\nu} \psi_R B_{\mu\nu} \rightarrow c_1 \bar{\psi}_L \sigma^{\mu\nu} \psi_R B_{\mu\nu} - c_1 \eta \bar{\psi}_L \sigma^{\mu\nu} \psi_R F_{\mu\nu}^\gamma, \quad (61)$$

which are tightly bounded. Specifically, adding the $L \leftrightarrow R$ operator, which is simply the Hermitian conjugate, this becomes

$$\begin{aligned} c_1 \bar{\psi}_L \sigma^{\mu\nu} \psi_R B_{\mu\nu} \rightarrow & -\frac{\text{Re}(c_1 \eta)}{2} \bar{\psi} \sigma^{\mu\nu} \psi F_{\mu\nu}^\gamma \\ & + i \frac{\text{Im}(c_1 \eta)}{2} \bar{\psi} \sigma^{\mu\nu} \gamma^5 \psi F_{\mu\nu}^\gamma, \end{aligned} \quad (62)$$

with then the ψ MDM and EDM given by $\text{Re}(c_1 \eta) = e a_\psi / 2 m_\psi$ and $d_\psi = \text{Im}(c_1 \eta)$, respectively. Overall, assuming c_1 is $\mathcal{O}(1)$ and real,

$$a_\psi \sim \frac{m_\psi \operatorname{Re} \Lambda_\gamma}{m \tilde{m}}, \quad d_\psi \sim \frac{\operatorname{Im} \Lambda_\gamma}{\tilde{m}^2}. \quad (63)$$

If we ask that the contribution to a_μ is less than 10^{-12} , and that to d_e less than 10^{-30} ecm (based on the PDG data [35]), this imposes $\operatorname{Re} \Lambda_\gamma / \tilde{m}^2 < 10^{-20}$ eV and $\operatorname{Im} \Lambda_\gamma / \tilde{m}^2 < 10^{-25}$ eV. If $\tilde{m}_B$ is to be below the electroweak scale, then Λ_γ needs to be extremely suppressed. If $\tilde{m}_B$ is larger, these LR operators have to first involve the Higgs boson, and $c_0 \sim \mathcal{O}(v_{ew}/\Lambda)$, with v_{ew} the electroweak vacuum. Yet, in this case also, a large scaling $\Lambda_\gamma \ll \Lambda$ is required if $\tilde{m}_B$ is to be relatively light since otherwise, with $\Lambda_\gamma \approx \Lambda$, a_ψ and d_ψ would depend only on $\tilde{m}_B^2$, which would then have to be well above the TeV scale.

For completeness, had we kept the topological $\Lambda' B_{\mu\nu} F_Y^{\mu\nu}$ term, or in the absence of the $\tilde{m}_B$ term, an additional $\eta' \tilde{F}_{\mu\nu}$ shift of $B_{\mu\nu}$ would be needed, with η' either $\Lambda'/\tilde{m}$ or Λ/m , respectively. Under this shift, besides the EDM and MDM operators, new corrections arise because the field strength $F^{B,\mu\nu\rho}$ ceased to be invariant. The shift then produces a contribution to the $\epsilon_{\alpha\beta\gamma\sigma} F^{B,\alpha\beta\gamma} \partial_\mu F^{\gamma,\mu\sigma}$ operator of Eq. (57), along with a Uehling correction in $\eta'^2 F_{\mu\nu} \partial^2 F^{\mu\nu}$. These corrections are much less strict than that coming from EDM and MDM, with, e.g., $|\Lambda|/m^2 < 10^{-10}$ eV if we require the Uehling correction to be less than a thousandth of the QED contribution of $\alpha/(60\pi m_e^2)$.

To close this Section, it should be stressed that having a strong suppression of the $B_{\mu\nu} \tilde{F}^{\gamma,\mu\nu}$ mixing term is not unrealistic. Indeed, contrary to the dark vector kinetic mixing, $B_{\mu\nu} \tilde{F}^{\gamma,\mu\nu}$ and $B_{\mu\nu} F^{\gamma,\mu\nu}$ break the B gauge invariance. That is a crucial feature since one could imagine

models in which a breaking of that symmetry first generates the B mass term, and only subsequently induces $B_{\mu\nu} \tilde{F}^{\gamma,\mu\nu}$, $B_{\mu\nu} F^{\gamma,\mu\nu}$, $\tilde{\psi}_L \sigma^{\mu\nu} \psi_R B_{\mu\nu}$, ... Further, a strong scaling $m \gg \Lambda$ could be relatively stable as the coupling $\tilde{\psi}_L \sigma^{\mu\nu} \psi_R B_{\mu\nu}$ cannot induce a $B_{\mu\nu} \tilde{F}^{\gamma,\mu\nu}$ interaction at one loop, and the contribution from the subleading coupling $\tilde{\psi}_L \gamma_\sigma \psi_L \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^B$ is protected by gauge invariance. So, in our opinion, a scenario with a light B state that does not significantly mix with the photon, and is dominantly coupled to SM matter via either $\tilde{\psi}_L \sigma^{\mu\nu} \psi_R B_{\mu\nu}$ or $\tilde{\psi}_L \gamma_\sigma \psi_L \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^B$, is viable and phenomenologically relevant. We will come back to this point in Sec. VII B.

D. 3-form field effective operators

For a 3-form field, the leading fermionic operators are

d	Type	
4	II	$\tilde{\psi}_L \gamma_\sigma \psi_L \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho}$
5	I	$\tilde{\psi}_L \psi_R \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^C$
	IV	$\tilde{\psi}_L \sigma^{\mu\nu} \psi_R \partial^\rho C_{\mu\nu\rho}$
6	I	...
	II	...
	III	$\tilde{\psi}_L \gamma^\rho \psi_L C_{\mu\nu\rho} F^{\mu\nu}$ $\tilde{\psi}_L \gamma^\rho \psi_L C_{\mu\nu\rho} \tilde{F}^{\mu\nu}$
	IV	$\tilde{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_R \partial^\rho C_{\mu\nu\rho}$ $\tilde{\psi}_L \gamma_\sigma \psi_R \epsilon^{\sigma\alpha\mu\nu} \partial_\alpha \partial^\rho C_{\mu\nu\rho}$

while for two 3-form fields, we find

d	Type	
5	II	$\tilde{\psi}_L \psi_R C^{\mu\nu\rho} C_{\mu\nu\rho}$
6	II	$\tilde{\psi}_L \gamma^\mu \psi_L C^{\nu\rho\sigma} F_{\mu\nu\rho}^C$ $\tilde{\psi}_L \gamma^\mu \mathcal{D}_\nu \psi_L C_{\mu\rho\sigma} C^{\nu\rho\sigma}$
	IV	$\tilde{\psi}_L \gamma^\sigma \psi_L C_{\sigma\mu\nu} \partial_\rho C^{\mu\nu\rho}$ $\tilde{\psi}_L \gamma^\mu \psi_L \epsilon^{\nu\alpha\beta\gamma} C_{\alpha\beta\gamma} \partial^\rho C_{\mu\nu\rho}$
7	I	$\tilde{\psi}_L \psi_R F_{\mu\nu\rho}^C F^{C,\mu\nu\rho\sigma}$
	II	$\tilde{\psi}_L \psi_R \partial_\sigma C^{\mu\nu\rho} \partial^\sigma C_{\mu\nu\rho}$ $\tilde{\psi}_L \psi_C \epsilon^{\mu\nu\rho\sigma} \partial^\sigma F_{\mu\nu\rho}^C$
		$\tilde{\psi}_L \sigma^{\mu\nu} \mathcal{D}^\rho \psi_R C^{\alpha\beta}{}_\nu \partial_\rho C_{\mu\alpha\beta}$ $\tilde{\psi}_L \mathcal{D}^\mu \mathcal{D}_\nu \psi_R C_{\mu\rho\sigma} C^{\nu\rho\sigma}$
	III	$\tilde{\psi}_L \sigma^{\alpha\beta} \psi_R C^{\mu\nu\rho} C_{\mu\nu\rho} F_{\alpha\beta}$ $\tilde{\psi}_L \sigma^{\mu\nu} \psi_R C_{\mu\alpha\beta} C^{\rho\alpha\beta} F_{\rho\nu}$
	IV	$\tilde{\psi}_L \psi_R C^{\mu\nu\sigma} \partial_\sigma \partial^\rho C_{\mu\nu\rho}$ $\tilde{\psi}_L \sigma^{\mu\nu} \psi_R C^{\alpha\beta}{}_\nu \partial_\alpha \partial^\rho C_{\mu\beta\rho}$ $\tilde{\psi}_L \sigma^{\alpha\beta} \psi_R \partial_\rho C^{\mu\nu\rho} F_{\mu\nu\alpha\beta}^C$
		$\tilde{\psi}_L \psi_R \epsilon^{\mu\nu\alpha\beta} \partial^\rho C_{\alpha\beta\rho} \partial^\sigma C_{\mu\nu\sigma}$ $\tilde{\psi}_L \sigma_{\alpha\beta} \psi_R C^{\mu\nu\alpha} \partial^\beta \partial^\rho C_{\mu\nu\rho}$
		$\tilde{\psi}_L \psi_R \partial_\rho C^{\mu\nu\rho} \partial^\sigma C_{\mu\nu\sigma}$ $\tilde{\psi}_L \sigma^{\nu\rho} \mathcal{D}^\mu \psi_R C_{\rho\nu}^\alpha \partial^\sigma C_{\mu\alpha\sigma}$

Many reductions are quite subtle and require not only the various spinor identities, but also exploiting the antisymmetry of $C_{\mu\nu\rho}$ or $F_{\mu\nu\rho\sigma}^C$ together with Eq. (A4), in a way similar to Eq. (55). For example, the operator

$\tilde{\psi}_L \gamma_\alpha \psi_L \epsilon^{\mu\nu\rho\alpha} \partial^\sigma F_{\mu\nu\rho\sigma}^C$ is absent because it can be written as $\tilde{\psi}_L \gamma_\alpha \psi_L \partial^\alpha (\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}^C)$, which is reducible by integrating by part and using the Dirac equation. The situation is to be contrasted to that of the $\tilde{\psi}_L \gamma^\mu \psi_L F_\mu^\phi$ operator in the scalar

basis. In that case, we choose to add it in parentheses because it gets reduced to $\bar{\psi}_L \psi_R \phi$, hiding the shift symmetry. By contrast here, $\bar{\psi}_L \gamma_\alpha \psi_L \epsilon^{\mu\nu\rho\sigma} \partial^\sigma F_{\mu\nu\rho}^C$ sum up to $\mathcal{O}(m)$ contributions to $\bar{\psi}_L \psi_R \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^C$, which is still gauge invariant, so there is no need to keep track of $\bar{\psi}_L \gamma_\alpha \psi_L \epsilon^{\mu\nu\rho\sigma} \partial^\sigma F_{\mu\nu\rho}^C$.

All in all, it is quite remarkable that so few operators survive. At first glance, with many Lorentz indices at our disposal hence many alternative ways to contract them, one could have expected the number of operators to be quite large, especially with two C fields. The reason why this is not the case resides in the existence of Hodge dualities relating this basis to that for the $p = 1$ field. This will be explored in detailed in Sec. IV, but we can already state that provided all the operators vanishing under the Lorenz conditions are included, there are precisely as many operators for a one and a 3-form field. In practice, we nevertheless derived all the above operators from scratch except for those of dimension seven, which were directly constructed from their Proca field counterparts. Indeed, at that level, the number of ways to contract all the indices is simply too large for a brute force method.

With SM fields, we can construct

$$\begin{aligned} \mathcal{L}_{\text{Int}}^{3\text{-Form}} = & \partial^\rho C_{\rho\mu\nu} \tilde{F}_Y^{\mu\nu} + \Phi^\dagger \Phi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^C + \Phi^\dagger \overleftrightarrow{D}_\alpha \Phi \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho} \\ & + \Phi^\dagger \Phi C^{\mu\nu\rho} C_{\mu\nu\rho} + \mathcal{O}(\Lambda^{-2}). \end{aligned} \quad (66)$$

Notice that $\partial^\rho C_{\rho\mu\nu} \tilde{F}_Y^{\mu\nu} \sim \partial^\rho F_{Y,\tau\sigma} \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho}$, while $\partial^\rho C_{\rho\mu\nu} F_Y^{\mu\nu}$ is not included because it vanishes upon partial integration thanks to the Bianchi identity for $F_Y^{\mu\nu}$.

E. 4-form field effective operators

With four indices, the number of possible contractions becomes very limited since no more than two Dirac matrices can appear. In practice, a Levi-Civita tensor is always needed to bring down the number of indices. At the same time, for a 4-form field, there is a unique way to contract its indices with the antisymmetric tensor because

$$\begin{aligned} \epsilon^{\alpha\beta\gamma\delta} D_{\mu\nu\rho\sigma} &= -\frac{1}{4!} \epsilon^{\alpha\beta\gamma\delta} D_{\lambda\kappa\pi\sigma} \epsilon^{\lambda\kappa\pi\sigma} \epsilon_{\mu\nu\rho\sigma} \\ &= \frac{1}{4!} \delta_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta} (\epsilon^{\lambda\kappa\pi\sigma} D_{\lambda\kappa\pi\sigma}), \end{aligned} \quad (67)$$

from which identities for $\epsilon^{\alpha\beta\gamma\sigma} D_{\mu\nu\rho\sigma}$, $\epsilon^{\alpha\beta\rho\sigma} D_{\mu\nu\rho\sigma}$, and $\epsilon^{\alpha\nu\rho\sigma} D_{\mu\nu\rho\sigma}$ can be deduced. As a result, only operators involving the scalar combination $\epsilon^{\mu\nu\rho\sigma} D_{\mu\nu\rho\sigma}$ need to be considered. In practice, the construction of all the possible operators then follows exactly the same step as for the scalar fields since $\epsilon^{\mu\nu\rho\sigma} D_{\mu\nu\rho\sigma}$ and ϕ have the same Lorentz properties. The final list of operators need not be repeated here; it is simply obtained from that in Eqs. (45) and (46) by substituting $\phi \rightarrow \epsilon^{\mu\nu\rho\sigma} D_{\mu\nu\rho\sigma}$.

IV. EQUIVALENCES VIA ALGEBRAIC DUALITIES

When constructing the basis for the p -form fields in the previous Section, we only paid attention to the Lorentz structure of the fields, not to their dynamics. In Sec. II, we describe how kinetic terms can be constructed for p -form fields by generalizing the Proca or Maxwell Lagrangian. We also show how this dynamics, supplemented or not by a gauge symmetry, reduces the number of degrees of freedom down to the physical ones.

Once dealing with higher-form fields, there is another route to prescribe the dynamics, thereby leading to a different number of physical degrees of freedom. Indeed, a gauge-fixing Lagrangian term built on the Lorenz condition is nothing but a kinetic term for the Hodge dual field,

$$\mathcal{L}_{gf}(A) = \frac{1}{2} d \star A \wedge \star d \star A = \frac{1}{2} F^{\star A} \wedge \star F^{\star A} = \mathcal{L}_{\text{kin}}(\star A). \quad (68)$$

The converse is obviously true: the usual kinetic term for a field A can be written as a gauge fixing term for the dual field $\star A$. To be more explicit, consider a p -form field A . By definition, its Hodge dual $\star A$ is the $n - p$ -form field given by

$$(\star A)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \epsilon_{\mu_1 \dots \mu_{n-p} \nu_1 \dots \nu_p} A^{\nu_1 \dots \nu_p}, \quad (69)$$

and one can check that the Lorenz condition for $\star A$ gives back the field strength for A ,

$$\partial^{\mu_1} (\star A)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{(p+1)!} \epsilon_{\mu_1 \dots \mu_{n-p} \nu_1 \dots \nu_p} F^{\mu_1 \nu_1 \dots \nu_p}. \quad (70)$$

In practice, this means that the kinetic terms of p -form field theories have two algebraically equivalent realizations,

$$\begin{aligned} \text{3-form theory: } \mathcal{L}_{\text{kin}} &= \frac{1}{2} \partial_\mu (\star C)^\mu \partial_\nu (\star C)^\nu \\ &= -\frac{1}{2 \cdot 4!} F_{\mu\nu\rho\sigma}^C F^{C,\mu\nu\rho\sigma}, \end{aligned} \quad (71a)$$

$$\begin{aligned} \text{2-form theory: } \mathcal{L}_{\text{kin}} &= -\frac{1}{2} \partial_\mu (\star B)^{\mu\nu} \partial^\rho (\star B)_{\rho\nu} \\ &= \frac{1}{2 \cdot 3!} F_{\mu\nu\rho}^B F^{B,\mu\nu\rho}, \end{aligned} \quad (71b)$$

$$\begin{aligned} \text{1-form theory: } \mathcal{L}_{\text{kin}} &= \frac{1}{2 \cdot 2!} \partial_\mu (\star A)^{\mu\nu\rho} \partial^\sigma (\star A)_{\sigma\nu\rho} \\ &= -\frac{1}{2 \cdot 2!} F_{\mu\nu}^A F^{A,\mu\nu}, \end{aligned} \quad (71c)$$

$$\begin{aligned} \text{0-form theory: } \mathcal{L}_{\text{kin}} &= -\frac{1}{2 \cdot 4!} \partial_\mu (\star \phi)^{\nu\rho\sigma\lambda} \partial^\mu (\star \phi)_{\nu\rho\sigma\lambda} \\ &= \frac{1}{2} F_\mu^\phi F^{\phi,\mu}. \end{aligned} \quad (71d)$$

Adding mass terms for either A or $\star A$ is totally equivalent since mass terms are invariant (up to the sign) under dualization, $A \wedge \star A = -(\star A) \wedge \star (\star A)$. Let us stress that these relationships are purely mathematical. They do not truly represent dualities between theories but rather alternative notational representations. It is however important to keep these in mind as in the literature, the 3-form theory is often prescribed via its dual vector field as in Eq. (71a), or the 2-form field via Eq. (71b).

In the absence of mass terms, it must be remarked that the Lorenz kinetic term does have precisely the required gauge symmetry to match that manifest in the dual field-strength form. For example, imagine starting with a 1-form field A^μ , with its four DoF. If we prescribe the usual Maxwell kinetic term, only two physical DoFs survive. This is not the case if we instead set its kinetic term as,

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} \partial_\mu A^\mu \partial_\nu A^\nu. \quad (72)$$

It is obviously not invariant under the usual gauge invariance $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ for generic Λ , but rather under

$$A_\mu \rightarrow A_\mu + \frac{1}{2!} \epsilon_{\mu\nu\rho\sigma} \partial^\nu \Lambda^{\rho\sigma}, \quad (73)$$

for any 2-form Λ . This is precisely the larger gauge invariance expected for a 3-form field since

$$\begin{aligned} C_{\mu\nu\rho} &\rightarrow C_{\mu\nu\rho} + \partial_\mu \Lambda_{\nu\rho} + \partial_\nu \Lambda_{\rho\mu} + \partial_\rho \Lambda_{\mu\nu} \rightarrow (\star C)_\mu \\ &\rightarrow (\star C)_\mu + \frac{1}{2!} \epsilon_{\nu\rho\sigma\mu} \partial^\nu \Lambda^{\rho\sigma}. \end{aligned} \quad (74)$$

Thus, the Lorenz kinetic term for a vector field actually describes a 3-form gauge field [12], which has no remaining physical DoF. The converse is of course also true: a Lorenz kinetic term for a 3-form field has the $C_{\mu\nu\rho} \rightarrow C_{\mu\nu\rho} + \epsilon_{\sigma\mu\nu\rho} \partial^\sigma \Lambda$ symmetry. This is a smaller invariance, matching that of the 1-form gauge field $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$ with the usual Maxwell kinetic term. Since A^μ and $C^{\mu\nu\rho}$ both start with four degrees of freedom, this invariance leaves precisely the usual two transverse degrees of freedom, so the 3-form theory with the Lorenz kinetic term indeed corresponds to the 1-form theory with the usual Maxwell kinetic term.

Notice that this dualization is equally valid at the level of the propagator. Under

$$\begin{aligned} \mathcal{I}_i^{\mu_1 \dots \mu_p, \alpha_1 \dots \alpha_p} &\rightarrow \mathcal{I}_{i, \rho_1 \dots \rho_{n-p}, \gamma_1 \dots \gamma_{n-p}} \\ &= \frac{\epsilon_{\mu_1 \dots \mu_p \rho_1 \dots \rho_{n-p}}}{p!} \frac{\epsilon_{\alpha_1 \dots \alpha_p \gamma_1 \dots \gamma_{n-p}}}{p!} \mathcal{I}_i^{\mu_1 \dots \mu_p, \alpha_1 \dots \alpha_p}, \end{aligned} \quad (75)$$

we get

$$\begin{aligned} \mathcal{I}_0^p &\rightarrow -\frac{(n-p)!}{p!} \mathcal{I}_0^{n-p}, \\ \mathcal{I}_2^p &\rightarrow \frac{(n-p)!}{p!} \left(\frac{n-p}{p} \mathcal{I}_2^{n-p} - \frac{k^2}{p} \mathcal{I}_0^{n-p} \right), \end{aligned} \quad (76)$$

where the superscripts on the $\mathcal{I}_i$ invariants indicate their dimensionality. With this, the propagator of a field is dualized as

$$\begin{aligned} \mathcal{P}(A) &= i \frac{(-1)^p p!}{k^2 - m^2} \left(\mathcal{I}_0^p - \frac{p}{m^2} \mathcal{I}_2^p \right) \rightarrow \mathcal{P}(\star A) \\ &= i \frac{(-1)^p (n-p)!}{m^2} \left(\mathcal{I}_0^{n-p} - \frac{n-p}{k^2 - m^2} \mathcal{I}_2^{n-p} \right), \end{aligned} \quad (77)$$

which is precisely what one could derive directly starting from the Lorenz kinetic term. Similarly for the massless propagator, dualizing the invariants in Eq. (29) reproduces that of the Lorenz kinetic term gauge-fixed via a $F^{\star A} \wedge \star F^{\star A} / 2\xi$ term.³

Returning to our effective operator bases, it is now clear that they are not all independent of each other. For the scalar field, dualization is kind of automatic, and we already showed that the basis for $D^{\mu\nu\rho\sigma}$ is in one-to-one correspondence with that for ϕ . The situation is more interesting for 3-form fields. For instance, if we add to the operators involving $C^{\mu\nu\rho}$ a Lorenz kinetic term $\partial_\mu C^{\mu\nu\rho} \partial^\sigma C_{\sigma\nu\rho}$, and dualize the C field into a 1-form field $C_{\mu\nu\rho} \rightarrow \epsilon_{\mu\nu\rho\sigma} A^\sigma$, the whole effective theory matches onto that of the vector field. This provides a powerful check of these operator bases, provided of course that all the operators involving the Lorenz condition are kept. That is the reason why we did so in the previous Section. For example, at the dimension five level,

$$\bar{\psi}_L \psi_R \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}^C \leftrightarrow \bar{\psi}_L \psi_R \partial^\mu A_\mu, \quad (78)$$

$$\bar{\psi}_L \sigma^{\mu\nu} \psi_R \partial^\rho C_{\mu\nu\rho} \leftrightarrow \bar{\psi}_L \sigma^{\mu\nu} \psi_R F_{\mu\nu}^A, \quad (79)$$

and similarly for the other couplings to SM fields in Eqs. (51) and (66). This also explains why the C field has no dimension-six gauge invariant operator (apart from an extra photon field). It is a manifestation of the fact that only gauge invariant operators exist for the 1-form field, and those all get mapped onto Lorenz operators for the 3-form field.

Similarly, for the 2-form field, the basis must be self-dual under $B^{\mu\nu} \leftrightarrow \tilde{B}^{\mu\nu}$ in the sense that the operators of a given dimension can only get reorganized. Most operators are by themselves self-dual thanks to Dirac matrices identities like Eqs. (41) or (44), while field strength and

³In the earliest work on $p = 2$ fields [11] (see also Ref. [36]), the kinetic term is actually written in the dual form $\partial_\mu B^{\mu\rho} \partial^\nu B_{\nu\rho}$, and the gauge is fixed by enforcing $\epsilon^{\mu\nu\rho\sigma} F_{\nu\rho\sigma}^B = 0$.

Lorenz condition operators get interchanged,

$$\bar{\psi}_L \gamma_\sigma \psi_L \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^B \leftrightarrow \bar{\psi}_L \gamma^\nu \psi_L \partial^\mu B_{\mu\nu}, \quad (80)$$

$$\bar{\psi}_L \sigma^{\mu\nu} \psi_R \partial^\rho F_{\mu\nu\rho}^B \leftrightarrow \bar{\psi}_L \sigma^{\mu\nu} \psi_R \partial_\mu \partial^\rho B_{\nu\rho}. \quad (81)$$

A peculiarity of the B field is to have both $m^2 B_{\mu\nu} B^{\mu\nu}$ and a $\tilde{m}^2 B_{\mu\nu} \tilde{B}^{\mu\nu}$ mass terms, and we have seen before that when both are present, the Lorenz condition must be generalized to include a term proportional to the dual of the field strength, see Eq. (37). This can be easily understood on the basis of Eq. (70). Indeed, the $\tilde{m}^2$ term could be absorbed entirely into the m^2 term upon a reparametrization $B \rightarrow B + \lambda \star B$ for some λ , but this would split the kinetic term into a combination of $F^B \wedge \star F^B$ and $d \star B \wedge \star d \star B$, from which the generalized Lorenz condition Eq. (37) would emerge.

While these dualities are very useful as cross-checks for the operator bases, they do not bring much phenomenologically. In the following, we will always assume that the operators are accompanied by the usual field strength kinetic terms.

V. EQUIVALENCES VIA MASSLESS DUALITIES

For massless theories, dualities between a p -gauge field A and a $n - p - 2$ gauge field $A^\star$ can be obtained by dualizing their field strengths. Indeed, using the properties of the wedge product, the usual kinetic term can be rewritten as

$$\begin{aligned} \mathcal{L}_{\text{kin}}(F = dA) &= F \wedge \star F = -(\star F) \wedge \star (\star F) \\ &= \mathcal{L}_{\text{kin}}^\star(\star F = dA^\star). \end{aligned} \quad (82)$$

Beware that here, $A^\star$ is not the dual to A , which would be a $n - p$ gauge field. Instead, dualizing the field strength in the absence of sources interchanges EoM and Bianchi identity. Specifically, the EoM derived from $\mathcal{L}_{\text{kin}}$ is $d \star F = 0$ and the Bianchi identity is $dF = 0$, while the dual theory $\mathcal{L}_{\text{kin}}^\star$ has the EoM $d \star (\star F) = dF = 0$, and the Bianchi identity $d(\star F) = d \star F = 0$. This ensures that there exists a gauge field $A^\star$ such that $\star F = dA^\star$, but it does not tell us how it is related to A . In practice, the only nontrivial dualizations in $n = 4$ dimensions are that interchanging $F_{\mu\nu}^A$ and $\tilde{F}_{\mu\nu}^A$, which corresponds to the well-known electromagnetic duality of the Maxwell theory in vacuum, and that relating the massless scalar F_μ^ϕ to the tensor $F_{\mu\nu\rho}^B$, which has the same number of degrees of freedom, see Table I. These cases are detailed in Secs. VA and VB below, while in Sec. VC, we show what happens if one tries to dualize the 3-form field strength.

A. Equivalence in the massless 1-form model

To set the stage, let us discuss the duality $F_{\mu\nu}^A \leftrightarrow \tilde{F}_{\mu\nu}^A$ in a way that can immediately be generalized to the 0- and 2-form duality. The idea is to start from a parent Lagrangian (see, e.g., App. B.4 in Ref. [7])

$$\mathcal{L}_{\text{parent}}(F, \tilde{A}) = -\frac{1}{2} \frac{1}{2!} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2!} \frac{1}{2!} \epsilon_{\mu\nu\rho\sigma} \tilde{A}^\sigma \partial^\mu F^{\nu\rho}, \quad (83)$$

where F and $\tilde{A}$ are fundamental fields of mass dimension 2 and 1, respectively (the tilde on A has no particular meaning, it is just a notation). The key feature is the mixing term, which can make either $\tilde{A}$ or F auxiliary under partial integration since $\epsilon_{\mu\nu\rho\sigma} \tilde{A}^\sigma \partial^\mu F^{\nu\rho} \leftrightarrow -\tilde{F}_{\mu\nu}^{\tilde{A}} F^{\mu\nu}$.

A first point of view on this Lagrangian is that $\tilde{A}$ is a Lagrange multiplier. It is not propagating since it has no kinetic term. All it does is imposes $\epsilon_{\mu\nu\rho\sigma} \partial^\mu F^{\nu\rho} = 0$ via its EoM, that is, a Bianchi identity for F . Barring topological obstructions, if F is closed it is also exact, and F must be the field strength of some vector field, $F = F^A = dA$. Then,

$$\mathcal{L}_{\text{parent}}(F = F^A, 0) = -\frac{1}{2} \frac{1}{2!} F_{\mu\nu}^A F^{A,\mu\nu}, \quad F_{\mu\nu}^A = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (84)$$

which is the usual Maxwell Lagrangian for the gauge field A . Alternatively, treating F as auxiliary after integrating by part, its EoM is $F = -\tilde{F}^{\tilde{A}}$. With this, we find again the Maxwell Lagrangian, but in terms of $F^{\tilde{A}}$,

$$\mathcal{L}_{\text{parent}}(F(\tilde{A}), \tilde{A}) = -\frac{1}{2} \frac{1}{2!} F_{\mu\nu}^{\tilde{A}} F^{\tilde{A},\mu\nu}, \quad F_{\mu\nu}^{\tilde{A}} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu. \quad (85)$$

The A and $\tilde{A}$ formulations are totally equivalent. In this case, their Lagrangian even have the same form because both A and $\tilde{A}$ are 1-form fields. All this is nothing but the usual electromagnetic duality, i.e., the fact that Maxwell's equations in vacuum are symmetric under the exchange of electric and magnetic fields, $E \rightarrow B$, $B \rightarrow -E$, that is, under $F_{\mu\nu}^\gamma \leftrightarrow \tilde{F}_{\mu\nu}^\gamma$ for the photon field strength.

In the presence of interactions, the situation is more complicated. Duality interchanges Bianchi identity and EoM, with $\partial_\mu F^{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} \partial^\nu \tilde{F}^{\rho\sigma} = 0$ and $\partial_\mu \tilde{F}^{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} \partial^\rho F^{\sigma\mu} = 0$. Interactions break this pattern since $\partial_\mu F^{\mu\nu} = J^\nu$. Yet, the above formulation can accommodate for some effective interactions, so let us see what happens in that case. First, since it is F that starts as fundamental, only interactions of the form $F_{\mu\nu} J^{\mu\nu}/2!$ can be added, with $J^{\mu\nu}$ encoding effective interactions like $F^{\gamma,\mu\nu}$, $\bar{\psi}_L \sigma^{\mu\nu} \psi_R$, $\bar{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_L$, $\partial^\mu (\bar{\psi}_L \gamma^\nu \psi_L)$, etc., i.e., all those present in the basis of Eq. (54). This is in accordance with the fact that SM fields

would all be neutral under the dark gauge symmetry.⁴ Then, eliminating G proceeds as before, simply replacing $F_{\mu\nu}J^{\mu\nu} \rightarrow F_{\mu\nu}^A J^{\mu\nu}$. Eliminating F , on the other hand, is affected by the presence of J . The EoM becomes $F = -\tilde{F}^{\tilde{A}} + J$, which when plugged back into $\mathcal{L}_{\text{parent}}(F = -\tilde{F}^{\tilde{A}} + J, \tilde{A})$, generates the dual interactions $\tilde{F}_{\mu\nu}^A J^{\mu\nu}$ together with a whole series of contact terms $J_{\mu\nu}J^{\mu\nu}$ [37].

Explicitly, the effective interactions when $F = F^A$, $\tilde{A} = 0$ and those when $F = -\tilde{F}^{\tilde{A}} + J$ are related as

$$F_{\mu\nu}^A F_Y^{\mu\nu} \rightarrow F_{\mu\nu}^{\tilde{A}} \tilde{F}_Y^{\mu\nu}, \quad (86a)$$

$$F_{\mu\nu}^A \tilde{F}_Y^{\mu\nu} \rightarrow F_{\mu\nu}^{\tilde{A}} F_Y^{\mu\nu}, \quad (86b)$$

$$\bar{\psi}_L \sigma^{\mu\nu} \psi_R F_{\mu\nu}^A \rightarrow i \bar{\psi}_L \sigma^{\mu\nu} \psi_R \tilde{F}_{\mu\nu}^{\tilde{A}}, \quad (86c)$$

$$\bar{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_L F_{\mu\nu}^A \rightarrow \bar{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_L \tilde{F}_{\mu\nu}^{\tilde{A}}, \quad (86d)$$

$$\bar{\psi}_L \gamma^\nu \psi_L \partial^\mu \tilde{F}_{\mu\nu}^A \rightarrow \bar{\psi}_L \gamma^\nu \psi_L \partial^\mu \tilde{F}_{\mu\nu}^{\tilde{A}}, \quad (86e)$$

$$\bar{\psi}_L \gamma^\nu \psi_L \partial^\mu F_{\mu\nu}^A \rightarrow \bar{\psi}_L \gamma^\nu \psi_L \partial^\mu \tilde{F}_{\mu\nu}^{\tilde{A}}. \quad (86f)$$

For the third operator, the appearance of the i factor comes from spinor identities, see Eqs. (40) to (44). In practice, this swaps magnetic and electric interactions, including the magnetic and electric dipole operators. For the other operators, remember that $\partial^\mu \tilde{F}_{\mu\nu} = 0$ when $F = dA$ and $\partial^\mu \tilde{F}_{\mu\nu}^{\tilde{A}} = 0$ when $F^{\tilde{A}} = d\tilde{A}$, while $\partial^\mu F_{\mu\nu} = \partial^\mu F_{\mu\nu}^{\tilde{A}} = 0$ on shell. This means in particular that the kinetic mixings are unrelated in those two formulations.

In the effective basis, we also derived many operators involving pairs of dark states. Those operators cannot be put in the form $F_{\mu\nu}J^{\mu\nu}/2!$. Yet, when the dark vector field is external, whether one computes physical observables in terms of A or $\tilde{A}$ always gives the same results. For example, consider the process $A \rightarrow J$ derived from $F_{\mu\nu}^A J^{\mu\nu}$ and $\tilde{A} \rightarrow J$ from $\tilde{F}_{\mu\nu}^{\tilde{A}} J^{\mu\nu}$. The amplitudes have the forms

$$\mathcal{M}(A \rightarrow J) = \epsilon_\alpha^{(\lambda)} (k^\mu g^{\nu\alpha} - k^\nu g^{\mu\alpha}) J_{\mu\nu}, \quad (87)$$

$$\mathcal{M}(\tilde{A} \rightarrow J) = \tilde{\epsilon}_\alpha^{(\lambda)} k_\gamma \epsilon^{\alpha\gamma\mu\nu} J_{\mu\nu}, \quad (88)$$

and, using $\sum_\lambda \epsilon_\alpha^{(\lambda)} \epsilon_\beta^{*(\lambda)} = \sum_\lambda \tilde{\epsilon}_\alpha^{(\lambda)} \tilde{\epsilon}_\beta^{*(\lambda)} = -g_{\alpha\beta}$ since the $k_\alpha k_\beta$ part cancels out,

$$\sum_\lambda |\mathcal{M}(\tilde{A} \rightarrow J)|^2 = \sum_\lambda |\mathcal{M}(A \rightarrow J)|^2 + 2k^2 J_{\mu\nu} J^{\mu\nu}. \quad (89)$$

⁴Consequently, remember that the covariant derivative acting on ψ may contain only SM gauge fields.

On shell, $k^2 = 0$ and both expressions coincide. Clearly, this remains true even if more than one dark photon is present. However, dividing this equation by $k^2 + i\epsilon$, it relates the $J \rightarrow \tilde{A} \rightarrow J$ and $J \rightarrow A \rightarrow J$ amplitudes since the ξ dependent part of the propagator cancels out when the couplings to $J^{\mu\nu}$ are gauge invariant. Off shell, both then differ by the $J_{\mu\nu}J^{\mu\nu}$ contact term, in accordance with our earlier finding.

It should be understood that duality no longer produces equivalent theories in the presence of interactions. Phenomenologically, this has several consequences. First, whether the effective interactions are understood in terms of F^A or its dual does not change their forms, but mixes up their CP properties. This is particularly relevant for the kinetic mixing term. Indeed, if one could justify that $F_{\mu\nu} \tilde{F}_Y^{\mu\nu}$ is initially absent by imposing some CP properties on $F_{\mu\nu}J^{\mu\nu}$, while still allowing for the antisymmetric mixing term, then the dual theory ends up with no kinetic mixing at all. This offers an alternative realization of the dark photon scenario in which the dominant couplings would be the $\bar{\psi}_L \sigma^{\mu\nu} \psi_R F_{\mu\nu}^A$ couplings. Notice that the kinetic mixing is by essence off shell, so it is consistent for it to be intrinsically different in both realizations.

A second consequence is that once nonrenormalizable effective interactions with SM fields are present, consistency requires the presence of effective interactions among SM fields only. For example, if the dimension-five $\bar{\psi}_L \sigma^{\mu\nu} \psi_R F_{\mu\nu}^A / \Lambda$ coupling is present, the fact that we do not know whether it should instead be interpreted as $\bar{\psi}_L \sigma^{\mu\nu} \psi_R \tilde{F}_{\mu\nu}^{\tilde{A}} / \Lambda$ means that we should also include the dimension-six contact interaction $\bar{\psi}_L \sigma^{\mu\nu} \psi_R \bar{\psi}_L \sigma_{\mu\nu} \psi_R / \Lambda^2$. In some sense, this is expected since once a complete UV theory at some scale Λ is able to produce $\bar{\psi}_L \sigma^{\mu\nu} \psi_R \tilde{F}_{\mu\nu}^{\tilde{A}}$ interactions, there is no reason not to expect it to also generate this contact term.

B. Equivalence between the massless 0- and 2-form models

There are many ways to express the massless $p = 2$ gauge theory in terms of a scalar field. Here, let us derive it following the same steps as in the previous Section, starting with the parent Lagrangian

$$\mathcal{L}_{\text{parent}}(\phi, F) = \frac{1}{2} \frac{1}{3!} F_{\mu\nu\rho} F^{\mu\nu\rho} - \frac{1}{3!} \epsilon_{\mu\nu\rho\sigma} \phi \partial^\mu F^{\nu\rho\sigma}. \quad (90)$$

At this level, $F_{\mu\nu\rho}$ is a fundamental antisymmetric tensor field of mass dimension 2, while ϕ is a scalar field. Treating ϕ as a Lagrange multiplier, its equation of motion enforces $\epsilon_{\mu\nu\rho\sigma} \partial^\mu F^{\nu\rho\sigma} = 0$, i.e., that F has to satisfy the Bianchi identity $dF = 0$. Being closed and barring topological issues, F is also exact and $F \rightarrow F^B = dB$ with B a 2-form field. Thus, after eliminating ϕ ,

$$\mathcal{L}_{\text{parent}}(0, F = dB) = \frac{1}{2} \frac{1}{3!} F_{\mu\nu\rho}^B F^{B,\mu\nu\rho},$$

$$F_{\mu\nu\rho}^B = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \quad (91)$$

which is the massless $p = 2$ gauge theory. The opposite situation follows from first integrating by part the second term, upon which $F^{\nu\rho\sigma}$ becomes an auxiliary field. Integrating it out gives

$$\mathcal{L}_{\text{parent}}(\phi, F(\phi)) = \frac{1}{2} F_\mu^\phi F^{\phi,\mu}, \quad F_\mu^\phi = \partial_\mu \phi, \quad (92)$$

which is the usual massless scalar theory. This proves the duality between the massless 0- and 2-form theories, in the same sense as the duality between Eqs. (84) and (85).

Let us proceed by adding effective operators to the parent Lagrangian. As discussed in the previous section, the EoM and the Bianchi identity can be dual only provided the fields are massless and free. Yet, there is no real obstruction to simply adding effective operators to the parent Lagrangian of Eq. (90), provided we work to a given order, stick to operators involving only one dark field, and integrate all these effective interactions by part to put them in a suitable algebraic form. Further, at the parent level, $F_{\mu\nu\rho}$ is a generic 3-form field (of mass dimension two) and not the B field strength yet. Thus, the relevant operators to add are those of the C field basis, with $C_{\mu\nu\rho} \rightarrow F_{\mu\nu\rho}$, so that

$$\mathcal{L}_{\text{parent}}^{\text{eff}}(\phi, F) = \mathcal{L}_{\text{parent}}(\phi, F) + \frac{1}{3!} F_{\mu\nu\rho} J^{\mu\nu\rho}, \quad (93)$$

with (in the present Section, for simplicity, we keep only the operators appearing in Eq. (64), and leave out all those involving $L \leftrightarrow R$ fields)

$$F_{\mu\nu\rho} J^{\mu\nu\rho} = \frac{c_1}{\Lambda} \bar{\psi}_L \gamma_\sigma \psi_L \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho} - \frac{3c_2}{\Lambda^2} \bar{\psi}_L \sigma^{\mu\nu} \psi_R \partial^\rho F_{\mu\nu\rho}$$

$$- \frac{c_3}{\Lambda^2} \bar{\psi}_L \psi_R \epsilon^{\sigma\mu\nu\rho} \partial_\sigma F_{\mu\nu\rho} + \mathcal{O}(\Lambda^{-3}). \quad (94)$$

The last two operators have to be integrated by part to keep F auxiliary. Notice that compared to simply taking the operators involving F^B in the B field basis, there is the additional operator with c_3 , which can be written $\bar{\psi}_L \psi_R \epsilon^{\sigma\mu\nu\rho} \partial_\sigma F_{\mu\nu\rho}$. This is irrelevant if we integrate ϕ out since we then get back the Bianchi identity $\epsilon_{\mu\nu\rho\sigma} \partial^\mu F^{\nu\rho\sigma} = 0$ and $F \rightarrow F^B$. This extra operator cancels out and we recover exactly the type I operators of Eq. (54),

$$\mathcal{L}_{\text{parent}}^{\text{eff}}(0, F = dB) = \frac{1}{2} \frac{1}{3!} F_{\mu\nu\rho}^B F^{B,\mu\nu\rho}$$

$$+ \frac{1}{3!} \frac{c_1}{\Lambda} \bar{\psi}_L \gamma_\sigma \psi_L \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^B$$

$$- \frac{1}{2!} \frac{c_2}{\Lambda^2} \bar{\psi}_L \sigma^{\mu\nu} \psi_R \partial^\rho F_{\mu\nu\rho}^B. \quad (95)$$

If instead we integrate $F^{\alpha\beta\gamma}$ out, the EoM in Eq. (92) receives an extra term $J^{\alpha\beta\gamma}$, and the effective Lagrangian becomes

$$\mathcal{L}_{\text{parent}}^{\text{eff}}(\phi, F(\phi)) = \frac{1}{2} \partial^\alpha \phi \partial_\alpha \phi + \frac{c_1}{\Lambda} \bar{\psi}_L \gamma^\mu \psi_L F_\mu^\phi$$

$$+ \frac{c_3}{\Lambda^2} \bar{\psi}_L \psi_R \partial^\mu F_\mu^\phi$$

$$+ \frac{1}{2} \frac{c_1^2}{\Lambda^2} \bar{\psi}_L \gamma_\mu \psi_L \bar{\psi}_L \gamma^\mu \psi_L + \mathcal{O}(\Lambda^{-3}). \quad (96)$$

This time, it is the c_2 operator that cancels out trivially because it ends up proportional to $\bar{\psi}_L \sigma^{\mu\nu} \psi_R \partial_\mu \partial_\nu \phi$, and we recover the same two type I effective operators as in the ϕ basis of Eq. (45). The presence of the contact term is to be noted though: Duality cannot work without a complete basis of operators, including those that do not involve the dark state. Notice finally that c_2 and c_3 disappear on shell since ϕ and B are massless, leaving only c_1 in both cases.

To describe the opposite situation of starting with the scalar effective operators, we must consider the parent Lagrangian

$$\mathcal{L}_{\text{parent}}^{\text{eff}}(F, B) = \frac{1}{2} F^\mu F_\mu - \frac{1}{3!} \epsilon_{\mu\nu\rho\sigma} F^\mu F^{B,\nu\rho\sigma} + F_\mu J^\mu, \quad (97)$$

where F^μ is a generic vector field of mass dimension two, thus with the effective interactions taken from the A field basis,

$$F_\mu J^\mu = \frac{c_1}{\Lambda} \bar{\psi}_L \gamma^\mu \psi_L F_\mu + \frac{c_2}{\Lambda^2} \bar{\psi}_L \sigma^{\mu\nu} \psi_R (\partial_\mu F_\nu - \partial_\nu F_\mu)$$

$$+ \frac{c_3}{\Lambda^2} \bar{\psi}_L \psi_R \partial^\mu F_\mu + \mathcal{O}(\Lambda^{-3}). \quad (98)$$

Compared to the scalar effective operators, we again have one more operator in the form of c_2 . Integrating B out, its EoM implies $\epsilon_{\mu\nu\rho\sigma} \partial^\mu F^\nu = 0$, hence $F_\mu = \partial_\mu \phi = F_\mu^\phi$ (barring topological obstruction). The extra operator c_2 disappears thanks to the Bianchi identity $\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_\mu^\phi$, and the 0-form field effective theory is correctly reproduced

$$\mathcal{L}_{\text{parent}}(F = d\phi, 0) = \frac{1}{2} \partial^\alpha \phi \partial_\alpha \phi + \frac{c_1}{\Lambda} \bar{\psi}_L \gamma^\mu \psi_L F_\mu^\phi$$

$$+ \frac{c_3}{\Lambda^2} \bar{\psi}_L \psi_R \partial^\mu F_\mu^\phi + \mathcal{O}(\Lambda^{-3}). \quad (99)$$

If instead we integrate F_μ out, its EoM is algebraic and once plugged back in the Lagrangian, we find

$$\begin{aligned}\mathcal{L}_{\text{parent}}^{\text{eff}}(F(B), B) = & \frac{1}{23!} F_{\mu\nu\rho}^B F^{B,\mu\nu\rho} + \frac{1}{3!} \frac{c_1}{\Lambda} \bar{\psi}_L \gamma_\mu \psi_L \epsilon^{\mu\nu\rho\sigma} F_{\nu\rho\sigma}^B \\ & - \frac{1}{2!} \frac{c_2}{\Lambda^2} \bar{\psi}_L \sigma^{\mu\nu} \psi_R \partial^\rho F_{\mu\nu\rho}^B \\ & - \frac{1}{2} \frac{c_1^2}{\Lambda^2} \bar{\psi}_L \gamma_\mu \psi_L \bar{\psi}_L \gamma^\mu \psi_L + \mathcal{O}(\Lambda^{-3}),\end{aligned}\quad (100)$$

where the c_3 operator disappears upon enforcing the Bianchi identity $\epsilon^{\mu\nu\rho\sigma} \partial_\mu F_{\nu\rho\sigma}^B$. The results Eqs. (95) and (96) are manifestly consistent with Eqs. (99) and (100), as could have been expected since the effective currents of the parent Lagrangian Eq. (93) and (97) are dual to each other, $J_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma} J^\sigma$. Also, if we imagine that the four-fermion operator is initially already present in either Lagrangian, both treatments shift its coupling strength in opposite directions, making the final realizations consistent with each other.

As for the dark photon, the duality between the scalar and 2-form fields holds more generally provided these states are kept on shell and external since they are then essentially free. For instance, if we dualize $F_{\mu\nu\rho}^B J^{\mu\nu\rho} = 3 \partial_\mu B_{\nu\rho} J^{\mu\nu\rho}$ into $F^{\phi,\mu} \epsilon_{\mu\nu\rho\sigma} J^{\nu\rho\sigma}$, the $B \rightarrow J$ and $\phi \rightarrow J$ amplitudes

$$\mathcal{M}(B \rightarrow J) = 3k^\mu \epsilon_{(\lambda)}^{\nu\rho} J_{\mu\nu\rho}, \quad (101)$$

$$\mathcal{M}(\phi \rightarrow J) = -k^\sigma \epsilon_{\mu\nu\rho\sigma} J^{\mu\nu\rho}, \quad (102)$$

are related as

$$\sum_\lambda |\mathcal{M}(B \rightarrow J)|^2 = |\mathcal{M}(\phi \rightarrow J)|^2 + 3! k^2 J_{\mu\nu\rho} J^{\mu\nu\rho}, \quad (103)$$

where we used that $\sum_\lambda \epsilon_{(\lambda)}^{\alpha\beta} \epsilon_{(\lambda)}^{\mu\nu} = 2\mathcal{I}_0^{\alpha\beta,\mu\nu}$ since the $\mathcal{I}_2^{\alpha\beta,\mu\nu}$ component cancels out by gauge invariance. Exactly like in the previous Section, the scalar and 2-form duality $\epsilon_{\mu\nu\rho\sigma} F^{\phi,\sigma} \leftrightarrow F_{\mu\nu\rho}^B$ holds on shell, where $k^2 = 0$, even for interactions involving more than one dark state, but contact terms are necessary off shell.

These contact terms also take on a new role compared to that in the previous Section. There, the current was defined at the level of the field strength, with $F_{\mu\nu}^A J^{\mu\nu}$ associated to $F_{\mu\nu}^{\tilde{A}} \tilde{J}^{\mu\nu}$. These currents need not be conserved for gauge invariance. Instead, after partial integration, we get $A_\nu \partial_\mu J^{\mu\nu}$ associated to $\tilde{A}_\nu \partial_\mu \tilde{J}^{\mu\nu}$. It is these divergences $\partial_\mu J^{\mu\nu}$ and $\partial_\mu \tilde{J}^{\mu\nu}$ that need to be conserved to preserve gauge invariance, and they trivially are since $\partial_\mu \partial_\nu J^{\mu\nu} = \partial_\mu \partial_\nu \tilde{J}^{\mu\nu} = 0$. The same happens starting with the $F_{\mu\nu\rho}^B J^{\mu\nu\rho}$ coupling, which is equivalent to $-3B_{\nu\rho} \partial_\mu J^{\mu\nu\rho}$ after partial integration, but the scalar field breaks this pattern. While p -form fields are built on gauge-invariance, all that remains for $p = 0$ is a

constant shift symmetry. But dualizing as $J_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma} J^\sigma$, the shift symmetry of $F_\mu^\phi \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho\sigma}$ is lost after partial integration into $\phi \epsilon^{\mu\nu\rho\sigma} \partial_\mu J_{\nu\rho\sigma}$. Intuitively, a constant shift of ϕ prevents it from vanishing at infinity, so it is not really surprising that partial integration can be problematic.

The only way for the shift symmetry to remain active is for $\epsilon^{\mu\nu\rho\sigma} \partial_\mu J_{\nu\rho\sigma}$ to vanish, in which case ϕ decouples entirely. Then, setting $\mathcal{M}(\phi \rightarrow J) = 0$ in Eq. (103), it predicts that B must also decouple entirely at $k^2 = 0$ even though the coupling $B_{\nu\rho} \partial_\mu J^{\mu\nu\rho}$ does not vanish. Let us illustrate this phenomenon with an example. Imagine that $\bar{\psi}_L \gamma^\mu \psi_L F_\mu^\phi$ is accompanied by $\bar{\psi}_R \gamma^\mu \psi_R F_\mu^\phi$, summing up in the vectorial combination $\bar{\psi} \gamma^\mu \psi F_\mu^\phi$. Such a coupling is spurious since after partial integration, ϕ decouples as $\partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0$. Yet, the dual B amplitude is induced by $\bar{\psi} \gamma_\mu \psi \epsilon^{\mu\nu\rho\sigma} F_{\nu\rho\sigma}^B$, which does not vanish. An explicit calculation shows that Eq. (103) still holds, with the amplitude squared for $B \rightarrow J$ proportional to k^2 . In other words, the occurrence of the contact term times k^2 in Eq. (103) is here needed to maintain consistency when the $\phi \rightarrow J$ process is trivially absent.

C. Triviality of the massless 3-form model

As said at the beginning, dualizing field strengths relates p and $n - p - 2$ -gauge field models. In $d = 4$, neither could be 3-form fields, so one may think that model is simply independent of all the others. In fact, this impossibility is a manifestation of the triviality of massless 3-form fields. To see this, notice that it is actually possible to write down a parent Lagrangian for a 3-form field as

$$\mathcal{L}_{\text{parent}}^{\text{eff}}(F, C) = -\frac{1}{2} FF + \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} F F^C{}_{\mu\nu\rho\sigma}, \quad (104)$$

where F is of mass dimension two, and $F^C = dC$ with C a 3-form field of mass dimension 1. Integrating F out gives back the 3-form kinetic term

$$\mathcal{L}_{\text{parent}}^{\text{eff}}(F(C), C) = -\frac{1}{24!} F_{\mu\nu\rho\sigma}^C F^{C,\mu\nu\rho\sigma}. \quad (105)$$

However, writing $\epsilon_{\mu\nu\rho\sigma} F F^{C,\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma} \partial^\mu F C^{\nu\rho\sigma}$, C becomes a Lagrange multiplier and imposes $\partial^\mu F = 0$, i.e., $F = F_0$ must be constant,

$$\mathcal{L}_{\text{parent}}^{\text{eff}}(F = F_0, 0) = -\frac{1}{2} F_0^2. \quad (106)$$

Thus, the fact that a 3-form field is not paired with any other field under field-strength dualization implies that it does not have any dynamics at all, in agreement with the counting done in Table I.

This conclusion remains true if we add interactions in the form of FJ . Integrating C out still imposes $F = F_0$,

but integrating F out now produces

$$\mathcal{L}_{\text{parent}}^{\text{eff}}(F(C), C) = -\frac{1}{24!} F_{\mu\nu\rho\sigma}^C F^{C,\mu\nu\rho\sigma} + \frac{1}{4!} J \epsilon_{\mu\nu\rho\sigma} F^{C,\mu\nu\rho\sigma} + \frac{J^2}{2}. \quad (107)$$

Notice that the interaction term is actually fully general for a gauge-invariant C field, because any effective interaction of the form $J_{\mu\nu\rho\sigma} F^{C,\mu\nu\rho\sigma}$ can be rewritten as $J \epsilon_{\mu\nu\rho\sigma} F^{C,\mu\nu\rho\sigma}$ with $J = \epsilon_{\mu\nu\rho\sigma} J^{\mu\nu\rho\sigma}$ using Eq. (A4). In practice, the $C \rightarrow J$ amplitude is thus

$$\begin{aligned} \mathcal{M}(C \rightarrow J) &= \frac{1}{3!} \epsilon_{\mu\nu\rho\sigma} \epsilon_{(\lambda)}^{\mu\nu\rho} k^\sigma J \rightarrow \sum_\lambda |\mathcal{M}(C \rightarrow J)|^2 \\ &= -k^2 J^2. \end{aligned} \quad (108)$$

The amplitude squared vanishes on shell, while off shell, only a trivial contact term survives.

VI. EQUIVALENCES VIA MASSIVE DUALITIES

For massive states, dualities have to be established at the level of fields, not field strengths. Generically, those arise between a p -form field A and a $n-p-1$ -form field A' , and follow from the parent Lagrangian [38]

$$\begin{aligned} \mathcal{L}_{\text{parent}}(A, A') &= -(-1)^p \frac{m_1^2}{2} A \wedge \star A + m_2 (-1)^p A \wedge dA' \\ &\quad - (-1)^{n-p-1} \frac{m_3^2}{2} A' \wedge \star A'. \end{aligned} \quad (109)$$

Fundamentally, this parent Lagrangian includes those discussed for massless states as special cases. For instance, if we set one of the mass term to zero, say $m_1 = 0$, then the EoM of A asks for $dA' = 0$, which means that A' is not a field but rather the field strength of a $n-p-2$ field, $A' = dA''$. It is then massless since the m_3 term becomes the usual kinetic term for A'' .

Let us thus assume that neither m_1 nor m_3 vanishes. We also need $m_2 \neq 0$ otherwise there is no dynamics. Then, either A or A' can be made algebraic since $dA \wedge A' = (-1)^p A \wedge dA'$, up to a total derivative, giving

$$\begin{aligned} A &= (-1)^{p(n-p)+1} \frac{m_2}{m_1^2} \star F^{A'} : \mathcal{L}_{\text{parent}}(A') \\ &= \frac{(-1)^{p-1}}{2} F^{A'} \wedge \star F^{A'} - (-1)^{n-p-1} \frac{m_1^2 m_3^2}{2m_2^2} A' \wedge \star A', \\ A' &= (-1)^{n-p} \frac{m_2}{m_3^2} \star F^A : \mathcal{L}_{\text{parent}}(A) \\ &= \frac{(-1)^{n-p}}{2} F^A \wedge \star F^A - (-1)^p \frac{m_1^2 m_3^2}{2m_2^2} A \wedge \star A, \end{aligned} \quad (110)$$

where the fields have been rescaled as either $A' \rightarrow A' \times m_1/m_2$ or $A \rightarrow A \times m_3/m_2$ to bring the kinetic term in its canonical form. The signs are consistent with our Lorentzian metric, but could be adapted to other cases. In practice, we get a nontrivial duality if $p = 0$, between a massive scalar and a massive 3-form tensor field, or if $p = 1$, between a massive vector and a massive 2-form tensor field, see Table I. Dualities can also be obtained starting with a $m_2 \star A \wedge dA'$ mixing term, but those produce Lorenz-type kinetic terms and are simply the Hodge dual of those obtained from $m_2 A \wedge dA'$, see Sec. IV.

A. Equivalence between the massive 0- and 3-form models

Let us start with the duality between the $p = 0$ - and $p = 3$ -form fields with parent Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{parent}}(\phi, C) &= -\frac{m_1^2}{2} \phi^2 + \frac{m_2}{3!} \epsilon_{\mu\nu\rho\sigma} C^{\mu\nu\rho} \partial^\sigma \phi \\ &\quad + \frac{1}{2} \frac{m_3^2}{3!} C_{\mu\nu\rho} C^{\mu\nu\rho}. \end{aligned} \quad (111)$$

By partial integration, either the C or ϕ field can be made algebraic and integrated out,

$$\begin{aligned} C_{\mu\nu\rho} &= -\frac{m_2}{m_3^2} \epsilon_{\mu\nu\rho\sigma} \partial^\sigma \phi : \mathcal{L}_{\text{parent}}(\phi, C(\phi)) \\ &= \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{m_1^2 m_3^2}{2m_2^2} \phi^2, \\ \phi &= \frac{1}{4!} \frac{m_2}{m_1^2} \epsilon_{\mu\nu\rho\sigma} F^{C,\mu\nu\rho\sigma} : \mathcal{L}_{\text{parent}}(\phi(C), C) \\ &= -\frac{1}{24!} F_{\mu\nu\rho\sigma}^C F^{C,\mu\nu\rho\sigma} + \frac{1}{2} \frac{1}{3!} \frac{m_1^2 m_3^2}{m_2^2} C_{\mu\nu\rho} C^{\mu\nu\rho}. \end{aligned} \quad (112)$$

After the rescaling $\phi \rightarrow \phi \times m_3/m_2$ or $C \rightarrow C \times m_1/m_2$, these states end up with the same mass $m_S = m_1 m_3/m_2$.

As in the massless case, if we add generic effective couplings, the EoM for the selected auxiliary field may no longer be algebraic, and even if it is, it is in general impossible to solve. If only effective operators linear in the dark field are kept, then that field EoM can remain algebraic, but contact interactions will necessarily appear.

For phenomenological purposes, it is not that interesting to express the scalar theory in terms of a 3-form field theory, but the converse offers a genuine alternative description for a dark scalar field. So, let us add the effective operators that are linear in the 3-form C by defining an effective $J_{\text{eff}}^{\mu\nu\rho}$ current, and take the parent effective Lagrangian as $\mathcal{L}_{\text{parent}}^{\text{eff}}(\phi, C) = \mathcal{L}_{\text{parent}}(\phi, C) + C_{\mu\nu\rho} J_{\text{eff}}^{\mu\nu\rho}/3!$. If we integrate ϕ out, we simply get back the effective Lagrangian for the massive C field, but with all the effective interactions rescaled by m_1/m_2 . Integrating C out instead, $J_{\text{eff}}^{\mu\nu\rho}$ now appears in its EoM in Eq. (112), and we find

$$\begin{aligned}\mathcal{L}_{\text{parent}}^{\text{eff}}(\phi, C(\phi)) &= \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi - \frac{m_1^2 m_3^2}{2m_2^2} \phi^2 \\ &\quad - \frac{1}{3!} \frac{1}{m_3} \epsilon_{\mu\nu\rho\alpha} \partial^\alpha \phi J_{\text{eff}}^{\mu\nu\rho} - \frac{1}{2} \frac{1}{3!} \frac{1}{m_3^2} J_{\text{eff},\mu\nu\rho} J_{\text{eff}}^{\mu\nu\rho},\end{aligned}\quad (113)$$

where we have rescaled $\phi \rightarrow \phi \times m_3/m_2$. In going from the effective interactions in terms of C to that in terms of ϕ , they all increase by one dimension, with m_3 acting as a compensating scale. This parameter is essentially free since the values of m_1 and m_2 can be adapted to keep $m_S = m_1 m_3/m_2$ fixed at some chosen value. Explicitly, the effective C and ϕ couplings of dimensions up to six are then related as

$$\frac{m_1}{m_2} \bar{\psi}_L \gamma_\sigma \psi_L \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho} \leftrightarrow \frac{1}{m_3} \bar{\psi}_L \gamma_\alpha \psi_L \partial^\alpha \phi, \quad (114)$$

$$\frac{m_1}{m_2} \bar{\psi}_L \psi_R \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}^C \leftrightarrow \frac{1}{m_3} (\bar{\psi}_L \psi_R) \partial^2 \phi, \quad (115)$$

$$\frac{m_1}{m_2} \bar{\psi}_L \sigma^{\mu\nu} \psi_R \partial^\rho C_{\mu\nu\rho} \leftrightarrow 0, \quad (116)$$

plus the same relations for $L \leftrightarrow R$. Note that these effective interactions could have been obtained directly by substituting $C_{\mu\nu\rho} \rightarrow (m_1/m_2) C_{\mu\nu\rho}$ on one hand, and $C_{\mu\nu\rho} = (1/m_3) \epsilon_{\mu\nu\rho\sigma} \partial^\sigma \phi$ on the other. In particular, the effective couplings initially involving the C field Lorenz condition do not have an equivalent in the ϕ description because of the ϕ Bianchi identity, $\partial^\rho C_{\mu\nu\rho} \rightarrow \epsilon_{\mu\nu\rho\sigma} \partial^\rho \partial^\sigma \phi = 0$. This is expected since in the C description, the Lorenz condition holds so they would also disappear. Another feature is to generate only shift-invariant effective scalar interactions, all involving the ϕ field strength $F_\mu^\phi = \partial_\mu \phi$. If this pattern holds also for operators involving pairs of C field (and we will see below when it indeed does), the leading operator would end up being the dimension-seven operator

$$\frac{m_1^2}{m_2^2} \bar{\psi}_L \psi_R C^{\mu\nu\rho} C_{\mu\nu\rho} \leftrightarrow \frac{1}{m_3^2} \bar{\psi}_L \psi_R F_\mu^\phi F^{\phi,\mu}. \quad (117)$$

For the contact interactions, there are only four-fermion couplings at the dimension-six level,

$$\begin{aligned}\frac{1}{m_3^2} J_{\text{eff},\mu\nu\rho} J_{\text{eff}}^{\mu\nu\rho} &= \frac{c_{1,L}^2}{m_3^2} \bar{\psi}_L \gamma_\mu \psi_L \bar{\psi}_L \gamma^\mu \psi_L \\ &\quad + \frac{c_{1,R}^2}{m_3^2} \bar{\psi}_L \gamma_\mu \psi_L \bar{\psi}_R \gamma^\mu \psi_R \\ &\quad + \frac{c_{1,L} c_{1,R}}{m_3^2} \bar{\psi}_L \gamma_\mu \psi_L \bar{\psi}_R \gamma^\mu \psi_R.\end{aligned}\quad (118)$$

For consistency, duality thus requires all these effective operators to be present. Though dimension-seven operators

are not relevant phenomenologically, one feature is worth mentioning. At that level, contact interactions arising from operators that would vanish upon enforcing the Lorenz condition start to appear, with for example

$$\begin{aligned}c_1 \bar{\psi}_L \gamma_\sigma \psi_L \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho} \otimes \frac{c_3}{\Lambda} \bar{\psi}_L \sigma^{\mu\nu} \psi_R \partial^\rho C_{\mu\nu\rho} \\ \rightarrow \frac{c_1 c_3}{m_3^2 \Lambda} \bar{\psi}_L \gamma_\mu \psi_L \partial_\nu (\bar{\psi}_L \sigma^{\mu\nu} \psi_R).\end{aligned}\quad (119)$$

Had we set $\partial^\rho C_{\mu\nu\rho} \rightarrow 0$ too soon, this kind of contact interaction would be missed.

Including now the SM operators of Eq. (66), we find

$$\partial^\rho C_{\rho\mu\nu} \tilde{F}_Y^{\mu\nu} \leftrightarrow 0, \quad (120)$$

$$\frac{m_1}{m_2} \Phi^\dagger \Phi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}^C \leftrightarrow \frac{1}{m_3} \Phi^\dagger \Phi \partial^2 \phi, \quad (121)$$

$$\frac{m_1}{m_2} \Phi^\dagger \overleftrightarrow{\mathcal{D}}_\alpha \Phi \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho} \leftrightarrow \frac{1}{m_3} \Phi^\dagger \overleftrightarrow{\mathcal{D}}_\mu \Phi \partial^\mu \phi, \quad (122)$$

$$\frac{m_1^2}{m_2^2} \Phi^\dagger \Phi C^{\mu\nu\rho} C_{\mu\nu\rho} \leftrightarrow \frac{1}{m_3^2} \Phi^\dagger \Phi \partial_\mu \phi \partial^\mu \phi. \quad (123)$$

From the point of view of the stability of the scalar theory, introducing it via the parent Lagrangian thus appears quite desirable. First, we can suppress all the couplings with the Higgs doublet by setting m_3 and m_2 to rather large values, with m_1 alone setting the scale of the dark scalar mass via $m_S = m_1 \times m_3/m_2$. Second, all the ϕ operators have to be shift invariant, and one avoids the dangerous renormalizable $\phi \Phi^\dagger \Phi$ or $\phi^2 \Phi^\dagger \Phi$ operators that often have to be severely fine-tuned to maintain a separation between the electroweak scale and that of the dark operators.

Finally, though our derivation of the duality severely restricts the form of the effective interactions, its validity is more general and holds whenever those states are external. This can be demonstrated as in the massless case before, by comparing a process $C \rightarrow J$ and $\phi \rightarrow J$. We start by adding the vertex $C_{\mu\nu\rho} J^{\mu\nu\rho}$ to the parent Lagrangian. The corresponding amplitudes are

$$\mathcal{M}(C \rightarrow J) = \frac{m_1}{m_2} \epsilon_{(\lambda)}^{\mu\nu\rho} J_{\mu\nu\rho},$$

$$\sum_\lambda \epsilon_{(\lambda)}^{*\alpha\beta\gamma} \epsilon_{(\lambda)}^{\mu\nu\rho} = -3! \left(\mathcal{I}_0^{\alpha\beta\gamma,\mu\nu\rho} - \frac{3}{m_V^2} \mathcal{I}_2^{\alpha\beta\gamma,\mu\nu\rho} \right), \quad (124a)$$

$$\mathcal{M}(\phi \rightarrow J) = -\frac{1}{m_3} k^\sigma \epsilon_{\mu\nu\rho\sigma} J^{\mu\nu\rho}. \quad (124b)$$

For $C \rightarrow J$, the m_1/m_2 comes from the rescaling $C \rightarrow C \times m_1/m_2$, necessary after eliminating ϕ , while for $\phi \rightarrow J$, the amplitude is derived by expressing $C_{\mu\nu\rho} J^{\mu\nu\rho}$ in terms of $C_{\mu\nu\rho} = -(m_2/m_3^2) \epsilon_{\mu\nu\rho\sigma} \partial^\sigma \phi$ of Eq. (112), followed by the

rescaling $\phi \rightarrow \phi \times m_3/m_2$. With this, summing over the polarizations,

$$\sum_{\lambda} |\mathcal{M}(C \rightarrow J)|^2 = |\mathcal{M}(\phi \rightarrow J)|^2 + 3! \frac{k^2 - m_S^2}{m_3^2} J_{\mu\nu\rho} J^{\mu\nu\rho}, \quad (125)$$

where $m_S = m_1 m_3 / m_2$ is the dark field mass. Thus, on shell, duality is expected to hold even for multiple external dark states. Notice that this identity predicts that the squared amplitude for $\bar{\psi} \gamma_\sigma \psi \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho}$ vanishes identically when $k^2 = m_S^2$ since that of $\bar{\psi} \gamma_\alpha \psi \partial^\alpha \phi$ does (a similar prediction was obtained in the massless case in Sec. VB). This is a nontrivial result that could not have been obtained from the symmetry properties of the coupling alone. This can have striking phenomenological implications whenever C and ϕ are exchanged between such conserved currents, since only the C could produce visible effects.

B. Equivalence between the massive 1- and 2-form models

The other massive duality is that between the $p = 1$ and $p = 2$ fields, derived from the parent Lagrangian

$$\mathcal{L}_{\text{parent}}(A, B) = \frac{1}{2} \frac{m_1^2}{1!} A_\mu A^\mu - \frac{m_2}{3!} \epsilon^{\mu\nu\rho\sigma} A_\mu F_{\nu\rho\sigma}^B - \frac{1}{2} \frac{m_3^2}{2!} B_{\mu\nu} B^{\mu\nu}. \quad (126)$$

Again, A or B can be made algebraic and eliminated since under partial integration,

$$\frac{m_2}{3!} \epsilon^{\mu\nu\rho\sigma} A_\mu F_{\nu\rho\sigma}^B = \frac{1}{2!} \frac{m_2}{2!} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^A B_{\mu\nu}. \quad (127)$$

This generates either the massive vector model or the massive 2-form tensor model,

$$\begin{aligned} A_\mu &= \frac{m_2}{3! m_1^2} \epsilon_{\mu\nu\rho\sigma} F^{B,\nu\rho\sigma} : \mathcal{L}_{\text{parent}}(A(B), B) = \frac{1}{23!} F_{\mu\nu\rho}^B F^{B,\mu\nu\rho} - \frac{1}{22!} \frac{m_1^2 m_3^2}{m_2^2} B_{\mu\nu} B^{\mu\nu}, \\ B_{\mu\nu} &= -\frac{m_2}{2! m_3^2} \epsilon_{\mu\nu\rho\sigma} F^{A,\rho\sigma} : \mathcal{L}_{\text{parent}}(A, B(A)) = -\frac{1}{22!} F_{\mu\nu}^A F^{A,\mu\nu} + \frac{1}{2} \frac{m_1^2 m_3^2}{m_2^2} A_\mu A^\mu. \end{aligned} \quad (128)$$

Again, after their appropriate rescalings $A \rightarrow A \times m_3/m_2$ and $B \rightarrow B \times m_1/m_2$, both fields end up with the same mass $m_V = m_1 m_3 / m_2$. As mentioned earlier, had we put the BF mixing term $B_{\mu\nu} F^{A,\mu\nu}$, the same duality would arise but with the Lorenz kinetic term $\partial^\nu B_{\mu\nu} \partial_\rho B^{\rho\mu}$, so we do not consider that case here. For simplicity, initially, we do not include the pseudoscalar mass term $B_{\mu\nu} \tilde{B}^{\mu\nu}$, but we will correct for that in a second step below.

Following the same logic as in the previous Section, our goal is to see how introducing a massive vector via a 2-form field affects the effective theory once expressed back in terms of the usual vector field. Again, we cannot simply replace $B_{\mu\nu} \rightarrow \epsilon_{\mu\nu\rho\sigma} F^{A,\rho\sigma} / m_3$ because the presence of effective interactions affects the EoM. Instead, let us again keep only the effective interactions linear in the B field, and encode them into a current $\mathcal{L}_{\text{parent}}^{\text{eff}}(A, B) = \mathcal{L}_{\text{parent}}(A, B) + B_{\mu\nu} J_{\text{eff}}^{\mu\nu} / 2!$. Integrating A out gives back the 2-form field, but with its interactions rescaled by $(m_1/m_2) B_{\mu\nu} J_{\text{eff}}^{\mu\nu} / 2!$. Integrating B out gives the Proca Lagrangian in terms of A , effective interactions for the Proca field rescaled by $1/m_3$, and contact interactions,

$$\begin{aligned} \mathcal{L}_{\text{parent}}^{\text{eff}}(A, B(A)) &= -\frac{1}{2} \frac{1}{2!} F_{\rho\sigma}^A F^{A,\rho\sigma} + \frac{1}{2} \frac{m_1^2 m_3^2}{m_2^2} A_\mu A^\mu \\ &+ \frac{1}{2!} \frac{1}{2!} \frac{1}{m_3} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^A J_{\mu\nu} - \frac{1}{2} \frac{1}{2!} \frac{1}{m_3^2} J_{\mu\nu} J^{\mu\nu}. \end{aligned} \quad (129)$$

The effective interactions are thus related to those of the B operators as

$$\frac{m_1}{m_2} \bar{\psi}_L \sigma^{\mu\nu} \psi_R B_{\mu\nu} \leftrightarrow \frac{1}{m_3} i \bar{\psi}_L \sigma^{\mu\nu} \psi_R F_{\mu\nu}^A, \quad (130a)$$

$$\frac{m_1}{m_2} \bar{\psi}_L \gamma_\sigma \psi_L \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^B \rightarrow \frac{1}{m_3} \bar{\psi}_L \gamma_\nu \psi_L \partial^\mu F_{\mu\nu}^A, \quad (130b)$$

$$\frac{m_1}{m_2} \bar{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_L B_{\mu\nu} \rightarrow \frac{1}{m_3} \bar{\psi}_L \gamma^\mu \mathcal{D}^\nu \psi_L \tilde{F}_{\mu\nu}^A, \quad (130c)$$

$$\frac{m_1}{m_2} \bar{\psi}_L \gamma^\nu \psi_L \partial^\mu B_{\mu\nu} \rightarrow 0, \quad (130d)$$

while the dimension-six effective B operators generate dimension-seven A operators that we do not keep. As before, operators with $L \leftrightarrow R$ are understood. Obviously, only gauge invariant operators arise since B is replaced by the dual field strength $\star F^A$. Concerning the couplings to the other SM fields, concentrating on operators of dimensions less than six, we have to include the mass mixing with $F_Y^{\mu\nu}$ since the reparametrization used in Sec. III C to rotate it away would mess up the terms in the parent Lagrangian. Thus, those operators become

$$\Lambda_\gamma \frac{m_1}{m_2} B_{\mu\nu} F_Y^{\mu\nu} \leftrightarrow 0, \quad (131)$$

$$\Lambda_\gamma \frac{m_1}{m_2} B_{\mu\nu} \tilde{F}_Y^{\mu\nu} \leftrightarrow \frac{\Lambda_\gamma}{m_3} F_{\mu\nu}^A F_Y^{\mu\nu}. \quad (132)$$

Provided m_2 and m_3 are sufficiently larger than Λ_γ , the kinetic mixing can be strongly suppressed. Further, this does not prevent the vector field from being light since it suffices for m_1 to be small to compensate.

The operators involving the B field Lorenz condition have no counterparts in the A picture, but they do contribute to the contact terms. Altogether, the operators of dimensions up to six are (setting all the Wilson coefficients to one for clarity, and introducing the scale $\Lambda \neq \Lambda_\gamma$ for nonrenormalizable effective interactions),

$$\frac{1}{m_3^2} J_{\mu\nu} J^{\mu\nu} = \frac{\Lambda_\gamma}{m_3^2} \bar{\psi}_L \sigma_{\mu\nu} \psi_R F_Y^{\mu\nu} + \frac{\Lambda_\gamma}{m_3^2} \bar{\psi}_L \sigma_{\mu\nu} \psi_R \tilde{F}_Y^{\mu\nu} + \text{H.c.} \quad (133)$$

$$\begin{aligned} & + \frac{\Lambda_\gamma}{\Lambda} \frac{1}{m_3^2} \bar{\psi}_{L,R} \gamma_\mu \mathcal{D}_\nu \psi_{L,R} F_Y^{\mu\nu} \\ & + \frac{\Lambda_\gamma}{\Lambda} \frac{1}{m_3^2} \bar{\psi}_{L,R} \gamma_\nu \psi_{L,R} \partial_\mu F_Y^{\mu\nu} \end{aligned} \quad (134)$$

$$\begin{aligned} & + \frac{1}{m_3^2} \bar{\psi}_L \sigma_{\mu\nu} \psi_R \bar{\psi}_R \sigma^{\mu\nu} \psi_L \\ & + \frac{1}{m_3^2} \bar{\psi}_L \sigma_{\mu\nu} \psi_R \bar{\psi}_L \sigma^{\mu\nu} \psi_R + \text{H.c.} \end{aligned} \quad (135)$$

If Higgs fields are included, the operators in the third line become dimension eight and can be discarded while all the others are dimension six. The presence of the operators in the first line is quite striking. When coupling the B field to fermions, those have to be included for consistency. At low energy, barring fine-tuned scenarios, the existence of a dark vector could then signal itself via shifts in the magnetic and/or electric dipole operators. Said differently, any fundamental theory leading to this set of effective B interaction has to also induce dipole operators. Though increasing m_3 could make these EDM and MDM corrections small, that would also suppress all the direct couplings in Eq. (130). So, it is Λ_γ alone that needs to be small to allow for observable effects.

As for the other scenarios, the above correspondences under duality hold more generally for external dark states. If we consider a $B_{\mu\nu} J^{\mu\nu}$ coupling in the parent Lagrangian, the amplitudes are

$$\mathcal{M}(B \rightarrow J) = \frac{m_1}{m_2} \epsilon_{(\lambda)}^{\mu\nu} J_{\mu\nu}, \quad \sum_\lambda \epsilon_{(\lambda)}^{*\alpha\beta} \epsilon_{(\lambda)}^{\mu\nu} = 2! \left(\mathcal{I}_0^{\alpha\beta,\mu\nu} - \frac{2}{m_V^2} \mathcal{I}_2^{\alpha\beta,\mu\nu} \right), \quad (136a)$$

$$\mathcal{M}(A \rightarrow J) = \frac{1}{m_3} \epsilon_{\mu\nu\rho\sigma} k^\rho \epsilon_{(\lambda)}^\sigma J^{\mu\nu}, \quad \sum_\lambda \epsilon_{(\lambda)}^{*\alpha} \epsilon_{(\lambda)}^\mu = - \left(\mathcal{I}_0^{\alpha,\mu} - \frac{2}{m_V^2} \mathcal{I}_2^{\alpha,\mu} \right), \quad (136b)$$

and

$$\sum_\lambda |\mathcal{M}(B \rightarrow J)|^2 = \sum_\lambda |\mathcal{M}(A \rightarrow J)|^2 - 2! \frac{k^2 - m_V^2}{m_3^2} J_{\mu\nu} J^{\mu\nu}, \quad (137)$$

where $m_V = m_1 m_3 / m_2$ is the dark vector mass. This equation is the exact analog of Eq. (125), and shows that on shell, duality is expected to hold even for multiple external dark states (a similar relation was derived already a long time ago in Ref. [39]).

It is time now to show how duality accommodates the presence of a pseudoscalar mass term for the B field. For that, let us simply add it to the parent Lagrangian

$$\mathcal{L}_{\text{parent}}(A, B) = \frac{1}{2} \frac{m_1^2}{1!} A_\mu A^\mu + \frac{1}{2!} \frac{m_2}{2!} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^A B_{\mu\nu} - \frac{1}{2} \frac{m_3^2}{2!} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} \frac{\tilde{m}_3^2}{2!} B_{\mu\nu} \tilde{B}^{\mu\nu}. \quad (138)$$

Eliminating A proceeds as before, and after the rescaling $B \rightarrow B \times m_1 / m_2$, Eq. (34) predicts the B mass to be

$$m_V^2 = m_1^2 \frac{m_3^4 + \tilde{m}_3^4}{m_3^2 m_2^2}. \quad (139)$$

Treating instead B as the auxiliary field, the EoM becomes

$$(m_3^2 \mathcal{I}_0^{\mu\nu,\rho\sigma} - \tilde{m}_3^2 \mathcal{I}_3^{\mu\nu,\rho\sigma}) B_{\rho\sigma} = \frac{m_2}{2!} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^A \Leftrightarrow B^{\mu\nu} = m_2 \frac{m_3^2 \tilde{F}^{A,\mu\nu} - \tilde{m}_3^2 F^{A,\mu\nu}}{m_3^4 + \tilde{m}_3^4}. \quad (140)$$

Plugged back in the Lagrangian, this gives, after the appropriate rescaling of the A field

$$\mathcal{L}_{\text{parent}}(A, B(A)) = -\frac{1}{4}F_{\mu\nu}^A F^{A,\mu\nu} - \frac{1}{4}\frac{\tilde{m}_3^2}{m_3^2}F_{\mu\nu}^A \tilde{F}^{A,\mu\nu} + \frac{1}{2}m_1^2 \frac{m_3^4 + \tilde{m}_3^4}{m_2^2 m_3^2} A_\mu A^\mu, \quad (141)$$

which is the Proca Lagrangian for a massive vector field with the same mass as in Eq. (139). This confirms the validity of Eq. (34) in an independent and somewhat simpler way. The second term in $\mathcal{L}_{\text{parent}}$ is an irrelevant Abelian theta term that can be safely discarded.

From here, we can repeat the matching between the effective operators by adding a $B_{\mu\nu} J^{\mu\nu}$ term in the parent Lagrangian. With now $B^{\mu\nu}$ having both a $\tilde{F}^{A,\mu\nu}$ and $F^{A,\mu\nu}$ component, the CP properties of all the operators get mixed up. The relative scaling between their B and A representations is also parametrically more complicated, being dependent on all the mass parameters, but the main features discussed previously remain valid so we will not go into those details here. However, before closing this Section, let us use duality to confirm the polarization sum appearing in the numerator of the full B propagator in the presence of the $\tilde{m}_3$ parameter, i.e., Eq. (35) with $m = m_3 m_1 / m_2$ and $\tilde{m} = \tilde{m}_3 m_1 / m_2$. To this end, we compare

$$\begin{aligned} \mathcal{M}(B \rightarrow J) &= \frac{m_1}{m_2} \epsilon_{(\lambda)}^{\mu\nu} J_{\mu\nu}, \sum_{\lambda} \epsilon_{(\lambda)}^{*\alpha\beta} \epsilon_{(\lambda)}^{\mu\nu} \\ &= 2 \left(\mathcal{I}_0^{\alpha\beta,\mu\nu} - \frac{2m_2^2}{m_3^2 m_1^2} \mathcal{I}_2^{\alpha\beta,\mu\nu} + \frac{\tilde{m}_3^2}{m_3^2} \mathcal{I}_3^{\alpha\beta,\mu\nu} \right), \end{aligned} \quad (142a)$$

$$\mathcal{M}(A \rightarrow J) = \frac{1}{m_3} \left(\frac{m_3^2 \epsilon_{\mu\nu\rho\sigma} k^\rho \epsilon_{(\lambda)}^\sigma}{\sqrt{m_3^4 + \tilde{m}_3^4}} - \frac{\tilde{m}_3^2 (k_\mu \epsilon_{\nu}^{(\lambda)} - k_\nu \epsilon_{\mu}^{(\lambda)})}{\sqrt{m_3^4 + \tilde{m}_3^4}} \right) J^{\mu\nu}. \quad (142b)$$

As before, these equations account for the rescaling necessary to have canonical kinetic terms. From them, it is immediate to check that

$$\begin{aligned} \sum_{\lambda} |\mathcal{M}(B \rightarrow J)|^2 &= \sum_{\lambda} |\mathcal{M}(A \rightarrow J)|^2 - 2! \frac{k^2 - m_V^2}{m_3^4 + \tilde{m}_3^4} \\ &\times (m_3^2 J_{\mu\nu} J^{\mu\nu} + \tilde{m}_3^2 \epsilon^{\mu\nu\rho\sigma} J_{\mu\nu} J_{\rho\sigma}), \end{aligned} \quad (143)$$

with m_V given in Eq. (139). These descriptions are equivalent on shell, and their difference off shell precisely matches the contact terms one could derive by adding the $J^{\mu\nu}$ term in the EoM of Eq. (140). Finally, it is worth remarking that equating $\mathcal{M}(A \rightarrow J)$ and $\mathcal{M}(B \rightarrow J)$ on shell provides a representation of the three polarization matrices $\epsilon_{(\lambda)}^{\mu\nu}$ in terms of that of a massive vector $\epsilon_{\mu}^{(\lambda)}$ satisfying the modified transversality constraint of Eq. (37).

VII. EQUIVALENCES À LA STUECKELBERG

The massless and massive dualities discussed in the previous sections permit one to match a parent Lagrangian either into a higher-form effective theory, or onto its corresponding scalar or vector effective theory. One should realize though that these are truly different realizations of the dynamics, not a mere change of variables. This is well illustrated in the case of the Maxwell theory, in which imposing either $\partial_\mu F^{\mu\nu} = J^\nu$ or $\partial_\mu \tilde{F}^{\mu\nu} = J^\nu$ switches electric and magnetic charges. To some extent, an exact equivalence is recovered when the dark fields stay external since the free equations of motion are satisfied. In practice, equivalence then follows from algebraic identities among polarization sums, and effective operators for two dual scenarios can be matched onto each other, up to various rescalings. Those are important, not least because they allow for suppressing even the renormalizable dimension 3 and 4 couplings.

At the same time, looking at Table I, we can see that the dynamics itself should be sufficient to relate fields having the same number of degrees of freedom, without recourse to parity-violating dualization. For example, it is well known that a vector field of mass m_V is essentially made of a transverse massless vector field together with a longitudinal scalar field. This is the essence of the Higgs mechanism. In the Stueckelberg picture, the vector field becomes massive without breaking gauge invariance thanks to the presence of a scalar field, in the combination $A_\mu - \partial_\mu \phi / m_V$. Whenever m_V is much smaller than the typical energy scale of a given process, this $1/m_V$ factor has important phenomenological consequences: an effective interaction in which the ϕ component does not decouple is strongly enhanced and can become a prime target for experiments.

The Stueckelberg picture is straightforwardly extended to p -form fields, and offers an alternative to duality to explore the relationships between p -form effective interactions (the $p = 2$ case has been considered recently in Ref. [40]). In practice, comparing the gauge transformations in Eq. (18) with the definitions of the field strengths in Eq. (10), one notices that the gauge variation of a p -form field $A \rightarrow A + d\Lambda$ can be compensated by the shift $F^{B+\Lambda} \rightarrow F^B + d\Lambda$ with $B \rightarrow B + \Lambda$ a $p - 1$ -form field. Provided the gauge and shift transformations are made coherently, $A - F^B$ becomes invariant. For dimensional reasons, the F component should actually involve a $1/m$ factor, such that the A mass term becomes

$$m^2 A \wedge \star A \rightarrow m^2 \left(A - \frac{1}{m} F^B \right) \wedge \star \left(A - \frac{1}{m} F^B \right) = m^2 A \wedge \star A - 2m F^B \wedge \star A + F^B \wedge \star F^B. \quad (144)$$

Applied to $d = 4$, the possible constructions are then (setting $m = 1$ for clarity)

$$(A_\mu - F_\mu^\phi): \phi \rightarrow \phi + \Lambda, \quad A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \quad (145a)$$

$$(B_{\mu\nu} - F_{\mu\nu}^A): A_\mu \rightarrow A_\mu + \Lambda_\mu, \quad B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \quad (145b)$$

$$(C_{\mu\nu\rho} - F_{\mu\nu\rho}^B): B_{\mu\nu} \rightarrow B_{\mu\nu} + \Lambda_{\mu\nu}, \quad C_{\mu\nu\rho} \rightarrow C_{\mu\nu\rho} + \partial_\mu \Lambda_{\nu\rho} + \partial_\nu \Lambda_{\rho\mu} + \partial_\rho \Lambda_{\mu\nu}. \quad (145c)$$

A fourth possibility involves $D_{\mu\nu\rho\sigma} - F_{\mu\nu\rho\sigma}^C$, but it is rather trivial since $D_{\mu\nu\rho\sigma}$ has no dynamics, and will not be of any use in the following.

In the next Subsection, we explore in some detail the relationship between the Stueckelberg construction and the equivalence theorem. We also show how it provides for another way of organizing the physical and unphysical degrees of freedom. Then, this will be put to phenomenological use in the following Subsection.

A. Stueckelberg construction and equivalence theorem

Imagine a coupling $A \wedge \star J \rightarrow A_{\mu_1 \dots \mu_p} J^{\mu_1 \dots \mu_p}$, to which we can schematically associate the squared amplitude

$$\sum_\lambda |\mathcal{M}(A \rightarrow J)|^2 = J_{\mu_1 \dots \mu_p} J_{\nu_1 \dots \nu_p}^\dagger \sum_\lambda \mathcal{E}_{(\lambda)}^{*\mu_1 \dots \mu_p} \mathcal{E}_{(\lambda)}^{\nu_1 \dots \nu_p}, \quad (146)$$

with

$$\begin{aligned} & \sum_\lambda \mathcal{E}_{(\lambda)}^{*\mu_1 \dots \mu_p} \mathcal{E}_{(\lambda)}^{\nu_1 \dots \nu_p} \\ &= (-1)^p p! \left(\mathcal{I}_0^{A, \mu_1 \dots \mu_p, \nu_1 \dots \nu_p} - \frac{p}{m^2} \mathcal{I}_2^{A, \mu_1 \dots \mu_p, \nu_1 \dots \nu_p} \right). \end{aligned} \quad (147)$$

For a vector field getting its mass through the Stueckelberg mechanism, it is well known that the $\mathcal{I}_2^A$ term above can be interpreted as that coming from its scalar longitudinal degree of freedom. That part dominates whenever the energy of the process is large compared to the vector boson mass m . Since the Stueckelberg construction is closely related to the Abelian Higgs model, this is nothing but a reformulation of the equivalence theorem for Goldstone bosons. Our goal here is to generalize this to higher form fields.

Consider thus a generic Stueckelberg model written in terms of $A - F^B/m$. The Lorenz condition required to make the Hamiltonian positive definite is modified to $d \star A + m \star B = 0$ (see, e.g., Ref. [18]). In turn, this condition can only be fulfilled provided B satisfies the usual Lorenz condition $d \star B = 0$. In a path integral formalism, these conditions are enforced by adding gauge fixing terms in the Lagrangian (along with fermionic ghosts, which we shall

not discuss here). In that context, it is customary to modify them slightly and adopt the R_ξ gauge-fixing term,

$$\begin{aligned} \mathcal{S}_{A,B} &= \frac{(-1)^p}{2} \int \frac{1}{2} F^A \wedge \star F^A \\ &\quad - \frac{m^2}{2} \left(A - \frac{1}{m} F^B \right) \wedge \star \left(A - \frac{1}{m} F^B \right) \\ &\quad - \frac{1}{2\xi} (d \star A + (-1)^{p-1} \xi m \star B) \\ &\quad \wedge \star (d \star A + (-1)^{p-1} \xi m \star B) \\ &\quad + \frac{1}{2\zeta} d \star B \wedge \star (d \star B). \end{aligned} \quad (148)$$

The advantage of this gauge fixing is to immediately remove the mixing between the A gauge boson and its B partner, $\mathcal{S}_{A,B} = \mathcal{S}_A + \mathcal{S}_B$ with

$$\begin{aligned} \mathcal{S}_A &= (-1)^p \int \frac{1}{2} F^A \wedge \star F^A - \frac{1}{2} m^2 A \wedge \star A \\ &\quad - \frac{1}{2\xi} (d \star A) \wedge \star (d \star A), \end{aligned} \quad (149)$$

$$\begin{aligned} \mathcal{S}_B &= (-1)^{p-1} \int \frac{1}{2} F^B \wedge \star F^B - \frac{1}{2} \xi m^2 B \wedge \star B \\ &\quad - \frac{1}{2\zeta} d \star B \wedge \star (d \star B). \end{aligned} \quad (150)$$

Thanks to this separation, the full propagators can be read off the kinetic terms without the need for resummations,

$$\mathcal{P}^A = i \frac{(-1)^p p!}{k^2 - m^2} \left(\mathcal{I}_0^A - (1 - \xi) \frac{p}{k^2 - \xi m^2} \mathcal{I}_2^A \right), \quad (151)$$

$$\mathcal{P}^B = i \frac{(-1)^{p-1} (p-1)!}{k^2 - \xi m^2} \left(\mathcal{I}_0^B - (1 - \zeta) \frac{p-1}{k^2 - \zeta \xi m^2} \mathcal{I}_2^B \right). \quad (152)$$

The ζ gauge parameter is needed to deal with the $\xi = 0$ gauge. When $\xi \neq 0$, the B field is massive, the Lorenz condition is automatic, and we should actually move to the

TABLE II. Stueckelberg separation of the propagating number of degrees of freedom for a p -form field A and its auxiliary $(p-1)$ -form field B in four dimensions for the unitary, Landau, and Feynman gauges.

Gauge:	$\xi \rightarrow \infty$			$\xi = 0$			$\xi = 1$		
p	Massive A	=	Massless A	+	Massless B	=	Unconstrained A	-	Massive B
0	1	=	1	+	0	=	1	-	0
1	3	=	2	+	1	=	4	-	1
2	3	=	1	+	2	=	6	-	3
3	1	=	0	+	1	=	4	-	3
4	0	=	0	+	0	=	1	-	1

unitary gauge $\xi \rightarrow \infty$. Alternatively, one could introduce a $p-2$ Stueckelberg field C to regularize the B mass term, and then maybe a $p-3$ field for the C mass term, and so on down to a scalar field. In practice, this tower of Stueckelberg fields is not needed because the B field always couples in a gauge-invariant way via the $(A - F^B/m)$ combination, and the $\mathcal{I}_2^B$ part never contributes. The only interest of introducing the ξ parameter is for counting the DoF in various gauges, as we now discuss.

First, in the unitary gauge $\xi \rightarrow \infty$, only A propagates and we recover the massive propagator of Eq. (22). For any $\xi < \infty$, however, the number of DoF does not match the expected physical number, and this has to be compensated

by the B field. To see this, consider again the $A \wedge \star J$ interaction, which we make gauge invariant by adding the B field as $(A - F^B/m) \wedge \star J$. The associated Feynman rules are

$$\mathcal{V}_{JA} = \frac{1}{p!} \mathcal{I}_0^A, \quad \mathcal{V}_{JB} = \frac{i}{(p-1)! m} (\mathcal{I}_0^A \cdot k) \mathcal{I}_0^B, \quad (153)$$

where the dot notation means $((\mathcal{I}_0^A \cdot k) \mathcal{I}_0^B)_{\rho_1 \dots \rho_{p-1}}^{\mu_1 \dots \mu_p} = (\mathcal{I}_0^A)^{\mu_1 \dots \mu_p, \nu_1 \dots \nu_p} k_{\nu_1} (\mathcal{I}_0^B)_{\nu_2 \dots \nu_p, \rho_1 \dots \rho_{p-1}}$. An amplitude for $J \rightarrow A \rightarrow J$ is then accompanied by $J \rightarrow B \rightarrow J$, and their sum gives

$$\begin{aligned} \langle J(p) J(-p) \rangle &= \mathcal{V}_{JA} \mathcal{P}^A \mathcal{V}_{AJ} + \mathcal{V}_{JB} \mathcal{P}^B \mathcal{V}_{BJ} \\ &= \frac{1}{p!} \frac{1}{p!} \left(\mathcal{P}^A + \frac{p^2}{m^2} (\mathcal{I}_0^A \cdot k) \mathcal{P}^B (k \cdot \mathcal{I}_0^A) \right) = \frac{1}{p!} \frac{i(-1)^p}{k^2 - m^2} \left(\mathcal{I}_0^A - \frac{p}{m^2} \mathcal{I}_2^A \right). \end{aligned} \quad (154)$$

This result is gauge independent, and in particular, corresponds to that in the unitary gauge where only A propagates. Notice that $\mathcal{P}^B$ contributes only to the $\mathcal{I}_2^A$ term, and this only via its $\mathcal{I}_0^B$ component. This follows from the identity $(\mathcal{I}_0^A \cdot k) \mathcal{I}_0^B (k \cdot \mathcal{I}_0^A) = \mathcal{I}_2^A$, which is essentially a rewriting of the definition Eq. (21), but $(\mathcal{I}_0^A \cdot k) \mathcal{I}_2^B (k \cdot \mathcal{I}_0^A) = 0$ by antisymmetry. As said earlier, this explains why there is no need to regularize the B mass term.

The unitary gauge can be compared to the counting in the Landau gauge $\xi = 0$ and Feynman gauge $\xi = 1$. In the former case, both A and B are purely transverse, even off shell, with a decomposition

$$C_p^{n-1} \{\text{massive } p \text{ field}\} = C_p^{n-2} \{\text{transverse } p \text{ field}\} + C_{p-1}^{n-2} \{\text{transverse } p-1 \text{ field}\}. \quad (155)$$

By contrast, in the Feynman gauge $\xi = 1$, A is allowed to propagate all its C_p^n degrees of freedom, but the massive B field is tasked with canceling the spurious ones, so that

$$C_p^{n-1} \{\text{massive } p \text{ field}\} = C_p^n \{\text{unconstrained } p \text{ field}\} - C_{p-1}^{n-1} \{\text{massive } p-1 \text{ field}\}. \quad (156)$$

The situation of Eqs. (155) and (156) is summarized in Table II, to be compared to Table I.

Returning to the equivalence theorem, the polarization sum in Eq. (146) clearly matches that of the full propagator in Eq. (154). This is best interpreted in terms of the propagator numerators in the Feynman gauge, where A

and B have the same mass, so that A and B can be put on shell via $(x - i\epsilon)^{-1} = P(1/x) - i\pi\delta(x)$. In that case, the $\mathcal{I}_2^A$ component is entirely generated by the B field, as a consequence of the $(\mathcal{I}_0^A \cdot k) \mathcal{I}_0^B (k \cdot \mathcal{I}_0^A) = \mathcal{I}_2^A$ identity. This proves that the $1/m$ terms of the polarization sum can indeed be extracted either by using the Stueckelberg

TABLE III. Dominant operators for one or two dark states to photons, scalars, and fermions, without or with dark gauge invariance (a and b , respectively). For the first two, only renormalizable interactions are kept, while for fermions the leading operators in each class are given.

		ϕ	A	B	C
I	a	...	...	$F_{\mu\nu}^\gamma \tilde{B}^{\mu\nu}$	...
	b	...	$F_{\mu\nu}^\gamma F^{A,\mu\nu}$	...	...
II	a	$\Lambda \Phi^\dagger \Phi \phi$	$\Phi^\dagger \overleftrightarrow{D}_\mu \Phi A^\mu$	...	$\Phi^\dagger \overleftrightarrow{D}_\alpha \Phi \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho}$
		$\Phi^\dagger \Phi \phi^2$	$\Phi^\dagger \Phi A_\mu A^\mu$	$\Phi^\dagger \Phi B_{\mu\nu} B^{\mu\nu}, \Phi^\dagger \Phi B_{\mu\nu} \tilde{B}^{\mu\nu}$	$\Phi^\dagger \Phi C^{\mu\nu\rho} C_{\mu\nu\rho}$
	b	...	...	...	$\Phi^\dagger \Phi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}^C$
III	a	$\bar{\psi} \psi \phi$	$\bar{\psi} \gamma^\mu \psi A_\mu$	$\bar{\psi} \sigma^{\mu\nu} \psi B_{\mu\nu}$	$\bar{\psi} \gamma_\sigma \psi \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho}$
		$\bar{\psi} \gamma_5 \psi \phi$	$\bar{\psi} \gamma^\mu \gamma_5 \psi A_\mu$	$\bar{\psi} \sigma^{\mu\nu} \psi \tilde{B}_{\mu\nu}$	$\bar{\psi} \gamma_\sigma \gamma_5 \psi \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho}$
	b	$\bar{\psi} \gamma^\mu \psi F_\mu^\phi$	$\bar{\psi} \sigma^{\mu\nu} \psi F_{\mu\nu}^A$	$\bar{\psi} \gamma_\sigma \gamma_5 \psi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^B$	$\bar{\psi} \gamma_\sigma \psi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}^C$
		$\bar{\psi} \gamma^\mu \gamma_5 \psi F_\mu^\phi$	$\bar{\psi} \sigma^{\mu\nu} \psi \tilde{F}_{\mu\nu}^A$	$\bar{\psi} \gamma_\sigma \psi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^B$	$\bar{\psi} \gamma_\sigma \psi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}^C$
IV	a	$\bar{\psi} \psi \phi^2$	$\bar{\psi} \psi A_\mu A^\mu$	$\bar{\psi} \psi B_{\mu\nu} B^{\mu\nu}, \bar{\psi} \psi B_{\mu\nu} \tilde{B}^{\mu\nu}$	$\bar{\psi} \psi C^{\mu\nu\rho} C_{\mu\nu\rho}$
		$\bar{\psi} \gamma_5 \psi \phi^2$	$\bar{\psi} \gamma_5 \psi A_\mu A^\mu$	$\bar{\psi} \gamma_5 \psi B_{\mu\nu} B^{\mu\nu}, \bar{\psi} \gamma_5 \psi B_{\mu\nu} \tilde{B}^{\mu\nu}$	$\bar{\psi} \gamma_5 \psi C^{\mu\nu\rho} C_{\mu\nu\rho}$
	b	$\bar{\psi} \psi F_\mu^\phi F^{\phi,\mu}$	$\bar{\psi} \psi F_{\mu\nu}^A F^{A,\mu\nu}$	$\bar{\psi} \psi F_{\mu\nu\rho}^B F^{B,\mu\nu\rho}$	$\bar{\psi} \psi F_{\mu\nu\rho\sigma}^C F^{C,\mu\nu\rho\sigma}$
		$\bar{\psi} \gamma_5 \psi F_\mu^\phi F^{\phi,\mu}$	$\bar{\psi} \gamma_5 \psi F_{\mu\nu}^A F^{A,\mu\nu}$	$\bar{\psi} \gamma_5 \psi F_{\mu\nu\rho}^B F^{B,\mu\nu\rho}$	$\bar{\psi} \gamma_5 \psi F_{\mu\nu\rho\sigma}^C F^{C,\mu\nu\rho\sigma}$

construction or from the equivalence theorem. The two pictures are totally equivalent.⁵

B. Phenomenological comparisons

The goal of this Section is to identify the main phenomenological differences between a dark photon embedded as a 1- or 2-form field, with or without gauge invariance, and between a dark scalar embedded as a 0- or 3-form field. For ease of reference, we repeat in Table III the most relevant effective operators for each scenario, now adopting the fermion mass eigenstate basis instead of the chiral basis of Secs. III A to III D. Though the purpose of the present section is phenomenological, we will not attempt to draw experimental constraints on the coefficients of the various operators, but simply identify the main portals through which the dark states could be looked for. A detailed numerical study of the impact on the exclusion plot for dark matter searches is certainly called for but would require a dedicated study that we leave for a future work.

1. Dark photon gauge-breaking couplings

When embedded into a vector field, and if gauge invariance does not hold, the dominant operator among those that are renormalizable is $\bar{\psi} \gamma^\mu \gamma_5 \psi A_\mu$ because its

longitudinal component is enhanced. Under $A \rightarrow A - F^{\phi_A}/m_V$, it becomes

$$g_A \bar{\psi} \gamma^\mu \gamma_5 \psi A_\mu \rightarrow \frac{g_A}{m_V} \bar{\psi} \gamma^\mu \gamma_5 \psi \partial_\mu \phi_A. \quad (157)$$

This is an axionlike effective interaction, but at the scale m_V instead of the typical scale in the 10^9 GeV region. It thus requires a strong suppression of its coupling if m_V is light compared to the typical energy scale of the considered process. By contrast, the $\bar{\psi} \gamma^\mu \psi A_\mu$ operator, which also encodes the effect of the kinetic mixing after the reparametrization of Eq. (53), is purely transverse thanks to the conservation of the current and is not enhanced in the $m_V \rightarrow 0$ limit. The same holds for the $\Phi^\dagger \overleftrightarrow{D}_\mu \Phi A^\mu$ coupling, whether Φ is a fundamental scalar or a low-energy meson (see, e.g., Ref. [41]).

The exact opposite happens in the B case. In the absence of gauge invariance, the leading operator behaves under $B \rightarrow B - F^{A_B}/m_V$ as

$$g_T \bar{\psi} \sigma^{\mu\nu} \psi B_{\mu\nu} + g_{\tilde{T}} \bar{\psi} \sigma^{\mu\nu} \psi \tilde{B}_{\mu\nu} \rightarrow \frac{g_T}{m_V} \bar{\psi} \sigma^{\mu\nu} \psi F_{\mu\nu}^{A_B} + \frac{g_{\tilde{T}}}{m_V} \bar{\psi} \sigma^{\mu\nu} \psi \tilde{F}_{\mu\nu}^{A_B}. \quad (158)$$

Again, the starting operator being renormalizable, a strong suppression of $g_{T,\tilde{T}}$ is required if m_V is small. Yet, notice that it is now the transverse component of B , encoded into F^{A_B} , that entirely dominates in the $m_V \rightarrow 0$ limit (if A_B is treated as a massive vector, its longitudinal component cancels out). Also, though similar transverse operators are

⁵There is a caveat here for the B field, because of the $\tilde{m}^2 B \wedge B$ mass term. Under the Stueckelberg substitution $B \rightarrow B - F^A/m$, the R_ξ trick no longer decouples the B field from F^A , but leaves a $B \wedge F^A$ coupling quite analog to the $B \wedge F^\gamma$ coupling discussed in Sec. III C. Some form of Dyson resummation appears necessary to prove the equivalence, but we leave this for a further study.

present in the 1-form picture, they are necessarily suppressed by some scale Λ , with in general $\Lambda \gg m_V$.

The consequence of a mixing with the photon is also opposite for the A and B : while purely transverse for the A , it get enhanced in the $m_V \rightarrow 0$ limit for the B . As discussed before, adding $e\Lambda_\gamma F_{\mu\nu}^{\gamma} \tilde{B}^{\mu\nu}$ to the $g_{T,\tilde{T}}$ tensor operators, a tree-level exchange of the B meson generates EDM and MDM operators $e\tilde{\psi}_L \sigma^{\mu\nu} \psi_R F_{\mu\nu}^{\gamma}$ and $e\tilde{\psi}_L \sigma^{\mu\nu} \psi_R \tilde{F}_{\mu\nu}^{\gamma}$, which are tightly bounded. Those scale as $g_{T,\tilde{T}} \Lambda_\gamma / m_V^2$, so if $\Lambda_\gamma \approx m_V$, the bound on $g_{T,\tilde{T}}$ would be so strict that a direct B interaction is unlikely to be ever seen. We thus arrive at the same conclusion as using duality: Λ_γ needs be very suppressed.

All in all, both scenarios require some level of fine-tuning of their parameters to be viable. In both cases, the nongauge invariant couplings have to be suppressed. This makes the situation for the B field slightly less appealing since both its coupling to fermions and to the photon break gauge invariance. Yet, from a phenomenological perspective, it is possible that $e\Lambda F_{\mu\nu}^{\gamma} \tilde{B}^{\mu\nu}$ is absent because of its odd parity. If that is the case, it would be worth to search for the dark photon not only via its vector coupling to matter, but to also probe for tensor interactions.

There is yet another feature to analyze. The B field is unique in that it can have both a parity conserving and parity-violating mass term, $m^2 B_{\mu\nu} B^{\mu\nu}$ and $\tilde{m}^2 B_{\mu\nu} \tilde{B}^{\mu\nu}$, with the physical mass $m_V^2 = (m^4 + \tilde{m}^4)/m^2$. As discussed earlier, this does not help with MDM and EDM constraints, and Λ_γ must still be extremely suppressed even though parity is no longer of much help since $B_{\mu\nu} \tilde{B}^{\mu\nu}$ and $F_{\mu\nu}^{\gamma} \tilde{B}^{\mu\nu}$ are both parity odd. Nevertheless, assuming that is the case, the phenomenology is then different and the Stueckelberg substitution fails to capture all the $1/m_V$ terms. To be more specific, we know from Eq. (35) that if B is coupled to some current J , then

$$\begin{aligned} \mathcal{M}(B \rightarrow J) &= \epsilon_{(\lambda)}^{\mu\nu} J_{\mu\nu} \cdot \sum_{\lambda} \epsilon_{(\lambda)}^{*\alpha\beta} \epsilon_{(\lambda)}^{\mu\nu} \\ &= 2 \left(\mathcal{I}_0^{\alpha\beta,\mu\nu} - \frac{2}{m^2} \mathcal{I}_2^{\alpha\beta,\mu\nu} - \frac{\tilde{m}^2}{m^2} \mathcal{I}_3^{\alpha\beta,\mu\nu} \right). \end{aligned} \quad (159)$$

There is no substitution $B \rightarrow B - xF^{A_B} - y\tilde{F}^{A_B}$ for some x and y that would alone generate only the $1/m^2$ term [instead, there exist x, y such that $xF^{A_B} + y\tilde{F}^{A_B}$ alone reproduces the whole polarization sum, see Eq. (142)].

At first sight, there is no manifest pole as $m_V \rightarrow 0$, but remember that $m_V \rightarrow 0$ is attainable only if both $m \rightarrow 0$ and $\tilde{m} \rightarrow 0$. For a given m_V , the maximum $m \rightarrow m_V$ is attained when $\tilde{m} \rightarrow 0$, while the maximum $\tilde{m} \rightarrow m_V/\sqrt{2}$ is reached when $m \rightarrow m_V/\sqrt{2}$, but both m and $\tilde{m}$ can be as small as one wishes. Actually, for a given m_V and a value for $m \leq m_V$, it suffices to take

$$\tilde{m}^2 = m^2 \sqrt{m_V^2/m^2 - 1}. \quad (160)$$

Thus, the $\mathcal{I}_3$ contribution is always smaller than that of $\mathcal{I}_2$, but both get enhanced when $m \rightarrow 0$. One should understand though that this singularity is totally similar to the $1/m_V$ singularities in Eq. (158) since $m_V = m$ when $\tilde{m} = 0$. Further, one should remember that it is m that tames the singularity of the propagator, i.e., which ensures the B kinetic term can be inverted. In both cases, the underlying physics abruptly changes when $m \rightarrow 0$, which is thus not attainable, and this shows up phenomenologically as a boost in the rate when m becomes small compared to the typical energy scale of the process. All that happens in the presence of the $\tilde{m}$ term is to decorrelate the physical mass m_V from the parameter m , but the underlying physics stay identical. In practice, for the tensor operators of Eq. (158), the $\mathcal{I}_3$ component cancels out completely (we will encounter later on a case where the $\mathcal{I}_3$ component does contribute), but the presence of $\tilde{m}$ can make m reach values much smaller than m_V , thereby strengthening the bounds on $g_{T,\tilde{T}}$.

2. Dark photon gauge-invariant couplings

If the A and B gauge-invariance hold for the effective interactions, but still assuming the dark photon gets a small mass term, the situations are again opposite.⁶ When all the effective interactions are written in terms of field strengths, the extra piece in the Stueckelberg substitution disappears. In effect, only the degrees of freedom that would exist if the field was massless can contribute. Thus, if we look at the effective couplings to fermions, the dominant interactions are always of dimension five, with the purely transverse $\tilde{\psi} \sigma^{\mu\nu} \psi F_{\mu\nu}^A$, $\tilde{\psi} \sigma^{\mu\nu} \psi \tilde{F}_{\mu\nu}^A$ for A , and the purely longitudinal $\tilde{\psi} \gamma_\sigma \psi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^B$, $\tilde{\psi} \gamma_\sigma \gamma_5 \psi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^B$ for B . Whether $\tilde{m}$ is zero or not is irrelevant here since the $\mathcal{I}_{2,3}$ components of the polarization sum Eq. (159) cancel out, leaving $\mathcal{I}_0$ only.

A major difference though is the fact that the mixing of the photon with A is gauge invariant, but not that with B , which is thus now forbidden. Starting with $\epsilon F_{\mu\nu}^A F^{\gamma,\mu\nu}$, there will then remain dimension-four couplings of the form $\epsilon \tilde{\psi} \gamma^\mu \psi A_\mu$ (and $\epsilon \tilde{\psi} \vec{\gamma} \cdot \vec{D}_\mu \Phi A^\mu$), from which ϵ is experimentally constrained to be small. In this case, it is the B embedding that appears more natural, requiring no fine-tuning at all.

The $\tilde{\psi} \gamma_\sigma \psi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^B$ and $\tilde{\psi} \gamma_\sigma \gamma_5 \psi \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^B$ couplings produce only longitudinally polarized B , but are not equivalent to axionlike couplings. We cannot use massless dualities to represent $\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho}^B \rightarrow \partial^\sigma \phi_B$ since contact terms

⁶Notice that it is not sufficient to break gauge invariance softly, as this would still allow for the $B - \gamma$ mixing terms. Instead, we assume that breaking occurs in a secluded sector, and does not directly affect the B couplings to SM particles.

would contribute for $k^2 \neq 0$. To be more specific, let us compare the off-shell squared amplitude obtained from either

$$\mathcal{L}_{\text{eff}} \supset \frac{g_V}{\Lambda} \bar{\psi}_1 \gamma_\mu \psi_2 \partial^\mu \phi + \frac{g_A}{\Lambda} \bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2 \partial^\mu \phi + \text{H.c.}, \quad (161)$$

$$\mathcal{L}_{\text{eff}} \supset \frac{g_V}{\Lambda} \bar{\psi}_1 \gamma_\mu \psi_2 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^B + \frac{g_A}{\Lambda} \bar{\psi}_1 \gamma_\mu \gamma_5 \psi_2 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^B + \text{H.c.}, \quad (162)$$

which are for ϕ ,

$$\begin{aligned} |\mathcal{M}(\phi(k) \rightarrow \bar{\psi}_1 \psi_2)|^2 &= 2g_A^2 (m_1 + m_2)^2 (k^2 - (m_1 - m_2)^2) \\ &\quad + 2g_V^2 (m_1 - m_2)^2 (k^2 - (m_1 + m_2)^2), \end{aligned} \quad (163)$$

and for B ,

$$\begin{aligned} \sum_\lambda |\mathcal{M}(B(k) \rightarrow \bar{\psi}_1 \psi_2)|^2 &= 2g_A^2 (k^2 - (m_1 + m_2)^2) (2k^2 + (m_1 - m_2)^2) \\ &\quad + 2g_V^2 (k^2 - (m_1 - m_2)^2) (2k^2 + (m_1 + m_2)^2). \end{aligned} \quad (164)$$

Clearly, the vector current contributes even for $m_1 = m_2$, when it is conserved, for the B field but not for the ϕ field, showing that these two scenarios are intrinsically different. Notice that these two amplitudes match at $k^2 = 0$ though, as they should from Eq. (103). Finally, it should be stressed that introducing direct couplings to B is not the same as introducing them via the Stueckelberg component of a 3-form, as in Refs. [42,43]. Starting with the couplings $\bar{\psi} \gamma_\mu \psi \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho}$ and $\bar{\psi} \gamma_\mu \gamma_5 \psi \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho}$, substituting $C \rightarrow C - F^{Bc}/m$, the F^{Bc} accounts for the $\mathcal{I}_2$ component in Eq. (147). On shell, the squared amplitude for C is the same as that of ϕ . In particular, the current being conserved for $\bar{\psi} \gamma_\mu \psi \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho}$, both the $\phi \rightarrow \bar{\psi} \psi$ and $C \rightarrow \bar{\psi} \psi$ squared amplitude vanish. In terms of components, the $\mathcal{I}_2$ contribution in Eq. (147) becomes proportional to k^2 , allowing it to cancel exactly with the $\mathcal{I}_0$ part at $k^2 = m^2$. Yet, that $\mathcal{I}_2$ part does not vanish by itself, even on shell. This means that whenever the B field is fundamentally a vector field with three true degrees of freedom, its longitudinal component does couple to conserved fermionic currents.

3. Dark scalar couplings

Let us now compare the ϕ and C pictures for a scalar field of mass m_S . Notice first that neither of these states mixes with the photon, simplifying the analysis compared to the dark vector scenario. On the other hand, the Stueckelberg substitution is not particularly interesting here since both ϕ and C carry a unique degree of freedom. As a result, it does not exist for ϕ , and would substitute C

by $C - F^B/m_S$ with F^B not related to ϕ (as discussed in the previous Section). So, this cannot help to single out and characterize possible $1/m_S$ enhancements. What we can use instead are the dualities discussed previously, which relate algebraically the on-shell squared amplitudes for ϕ and C .

Let us start with the fermionic couplings, which for ϕ and C are either to the vector and axial currents $V, A = \bar{\psi} \gamma_\sigma \psi, \bar{\psi} \gamma_\sigma \gamma_5 \psi$ or to the scalar and pseudoscalar currents $S, P = \bar{\psi} \psi, \bar{\psi} \gamma_5 \psi$. Dimensionally, the situation is inverted for the C and the ϕ ,

$$\mathcal{L}_{\text{eff}} \supset g_S \bar{\psi}_1 \psi_2 \phi + g_P \bar{\psi}_1 \gamma_5 \psi_2 \phi \quad (165)$$

$$+ \frac{g_V}{\Lambda_\phi} \bar{\psi}_1 \gamma^\mu \psi_2 F_\mu^\phi + \frac{g_A}{\Lambda_\phi} \bar{\psi}_1 \gamma^\mu \gamma_5 \psi_2 F_\mu^\phi + \text{H.c.}, \quad (166)$$

$$\mathcal{L}_{\text{eff}} \supset g_V \bar{\psi}_1 \gamma_\sigma \psi_2 \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho} + g_A \bar{\psi}_1 \gamma_\sigma \gamma_5 \psi_2 \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho} \quad (167)$$

$$+ \frac{g_S}{\Lambda_C} \bar{\psi}_1 \psi_2 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^C + \frac{g_P}{\Lambda_C} \bar{\psi}_1 \gamma_5 \psi_2 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^C + \text{H.c.} \quad (168)$$

From Eq. (125), the S and P interactions are dominant for ϕ , and related to the corresponding subdominant C interactions via

$$|\mathcal{M}_{S,P}(C(k) \rightarrow \bar{\psi}_1 \psi_2)|^2 = \frac{k^2}{\Lambda_C^2} |\mathcal{M}_{S,P}(\phi(k) \rightarrow \bar{\psi}_1 \psi_2)|^2. \quad (169)$$

The k^2 factor can be understood by noting that the contact term satisfies $|\mathcal{M}_{S,P}(\phi(k) \rightarrow \bar{\psi}_1 \psi_2)|^2 = J^2$ when $J = \bar{\psi}_1 \psi_2$ or $\bar{\psi}_1 \gamma_5 \psi_2$, which is quite evident from a Feynman rule perspective. Phenomenologically, this means that the gauge-invariant scalar and pseudoscalar interactions for C are significantly more suppressed than expected when C is light and on shell. The opposite holds for the V and A interactions, which are dominant for the C and suppressed for ϕ , but related as,

$$\begin{aligned} |\mathcal{M}_{V,A}(\phi(k) \rightarrow \bar{\psi}_1 \psi_2)|^2 &= \frac{m_S^2}{\Lambda_\phi^2} |\mathcal{M}_{V,A}(C(k) \rightarrow \bar{\psi}_1 \psi_2)|^2 \\ &\quad + O\left(\frac{k^2 - m_S^2}{\Lambda_\phi^2}\right). \end{aligned} \quad (170)$$

Notice though that for C off shell, the squared amplitudes are intrinsically different functions of k^2 , the mass squared, and $g_{A,V}$ couplings.

An important peculiarity though is that $\mathcal{M}_{V,A}(\phi(k) \rightarrow \bar{\psi} \psi)$ is itself related to $\mathcal{M}_{S,P}(\phi(k) \rightarrow \bar{\psi} \psi)$ since the couplings are by partial integration and use of the equation of motion. Let us take $\psi_1 = \psi_2$ for simplicity, in which case

$$\begin{aligned}\mathcal{M}_V(\phi(k) \rightarrow \bar{\psi}\psi) &= 0, \\ \mathcal{M}_A(\phi(k) \rightarrow \bar{\psi}\psi) &= \frac{2m}{\Lambda_\phi} \mathcal{M}_P(\phi(k) \rightarrow \bar{\psi}\psi).\end{aligned}\quad (171)$$

This means that the dominant C interactions must satisfy

$$|\mathcal{M}_V(C(k) \rightarrow \bar{\psi}\psi)|^2 = O\left(\frac{k^2 - m_S^2}{m_S^2}\right), \quad (172)$$

$$\begin{aligned}|\mathcal{M}_A(C(k) \rightarrow \bar{\psi}\psi)|^2 \\ = 4 \frac{m_\psi^2 \Lambda_C^2}{m_S^2 k^2} |\mathcal{M}_P(C(k) \rightarrow \bar{\psi}\psi)|^2 + O\left(\frac{k^2 - m_S^2}{m_S^2}\right).\end{aligned}\quad (173)$$

These are two striking predictions: the on-shell vector current coupling gives no contribution on shell, while the axial one must be proportional to the fermion mass squared (it is actually equal to $8g_A^2 m_\psi^2$). This could not have been expected on the basis of the operators alone.

Concerning the couplings to scalar fields, the main difference with the dark vector case is that both ϕ and C do have linear renormalizable couplings. Their properties are quite different though. In the ϕ picture, the presence of $\Lambda\Phi^\dagger\Phi\phi$ generates tadpoles if Φ is the Higgs doublet, which is then required to shift the ϕ field by a large constant. In that case, all the other effective interactions better be shift invariant, otherwise SM particles could all receive large corrections to their masses and couplings, in particular from the (also renormalizable) $\bar{\psi}\psi\phi$ and $\bar{\psi}\gamma_5\psi\phi$ couplings. So, either $\Phi^\dagger\Phi\phi$ is severely fine-tuned, or these fermionic interactions are forbidden. By contrast, in the C picture, the tadpole interaction from $\Phi^\dagger\Phi\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}^C$ automatically drops out since it involves a derivative. All that remains then is effective interactions with the Z boson and the physical Higgs field from

$$g_\Phi\Phi^\dagger\overleftrightarrow{D}_\alpha\Phi\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho} \rightarrow g_\Phi g(v_{ew} + h)^2 Z_\mu \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho}, \quad (174)$$

with g the electroweak coupling and v_{ew} the electroweak vacuum expectation value. This includes a parity-odd mixing of C with the Z boson. A detailed analysis is left for a future work. Here, sticking to a low energy perspective and integrating the Z boson out, this mixing term combined with the Z couplings to light fermions simply generates $\mathcal{O}(g_\Phi)$ corrections to the renormalizable $\bar{\psi}\gamma_\sigma\psi\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho}$ and $\bar{\psi}\gamma_\sigma\gamma_5\psi\epsilon^{\mu\nu\rho\sigma}C_{\mu\nu\rho}$ interactions. As such, since the former decouples on shell, constraints on g_A immediately translate into similar constraints on g_Φ .

4. Two dark field couplings and differential rates

Looking at the class IV operators in Table III, the dominant interactions to pairs of dark states share essentially the same structure. For gauge-breaking operators, the contraction $A \wedge \star A \rightarrow A_{\mu_1 \dots \mu_p} A^{\mu_1 \dots \mu_p}$ is coupled to either

fermions via $\bar{\psi}\psi$ and $\bar{\psi}\gamma_5\psi$, or to scalars via $\Phi^\dagger\Phi$. The only exception is the B field, for which the dark state can also occur in the pseudoscalar combination $B \wedge B \rightarrow B_{\mu\nu}\tilde{B}^{\mu\nu}$. Gauge-invariant operators are similar, with $dA \wedge \star dA \rightarrow F_{\mu_1 \dots \mu_{p+1}} F^{\mu_1 \dots \mu_{p+1}}$ in place of $A \wedge \star A$, and now the exceptional case of the vector that also couples via $dA \wedge dA \rightarrow F_{\mu\nu}\tilde{F}^{A,\mu\nu}$.

Though all these couplings are very similar, they do not lead to identical predictions. To illustrate this, we will consider three-body processes like $\psi_1 \rightarrow \psi_2 XX$ or $\Phi_1 \rightarrow \Phi_2 XX$, with X a given dark state, and compare the differential rates as functions of the XX invariant mass $q^2 = (k_1 + k_2)^2$,

$$\begin{aligned}\frac{d\Gamma}{dq^2} &= \frac{1}{2m_a} \frac{\lambda(m_a^2, m_b^2, q^2)\lambda(q^2, m_X^2, m_X^2)}{128\pi^3} \frac{1}{s_a} \\ &\times \sum_{\lambda, s_a, s_b} |\mathcal{M}(a \rightarrow bXX)|^2,\end{aligned}\quad (175)$$

where $\lambda^2(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)$, which ranges between $q_{\min}^2 = 4m_X^2$ and $q_{\max}^2 = (m_a - m_b)^2$. For fermions ($s_a = 2$), this requires some flavor violation, but our purpose here is illustrative. For scalars ($s_a = 1$), similarly, we think of Φ_1 and Φ_2 as low-energy pseudo-scalar mesons. In this respect, it should be stressed that if the combination $\Phi^\dagger\Phi$ involves the Higgs doublet, then all the $\Phi^\dagger\Phi(A \wedge \star A)$ operators generate electroweak mass terms for the dark states. As usual, maintaining those states of light necessarily requires some level of fine-tuning there, or one needs to assume the gauge (or shift) symmetry holds.

Though completely straightforward, it is instructive to perform the calculation explicitly. The Feynman rules associated to $A \wedge \star A$ and $dA \wedge \star dA$, upon proper normalization, are

$$\frac{A_{\mu_1 \dots \mu_p} A^{\mu_1 \dots \mu_p}}{2p!} \rightarrow \mathcal{M}_{AA} = \frac{1}{p!} \mathcal{I}_0^{\mu_1 \dots \mu_p, \nu_1 \dots \nu_p} \epsilon_{\mu_1 \dots \mu_p}^{\lambda_1} \epsilon_{\nu_1 \dots \nu_p}^{\lambda_2}, \quad (176a)$$

$$\begin{aligned}\frac{F_{\mu_1 \dots \mu_{p+1}} F^{\mu_1 \dots \mu_{p+1}}}{2(p+1)!} &\rightarrow \mathcal{M}_{FF} \\ &= \frac{p+1}{p!} \mathcal{I}_0^{\mu_1 \dots \mu_{p+1}, \nu_1 \dots \nu_{p+1}} k_{1, \mu_1} k_{2, \nu_1} \epsilon_{\mu_2 \dots \mu_{p+1}}^{\lambda_1} \epsilon_{\nu_2 \dots \nu_{p+1}}^{\lambda_2}.\end{aligned}\quad (176b)$$

Given the factorized form of the full $a \rightarrow bXX$ amplitudes $\mathcal{M}_{a \rightarrow bXX} = \mathcal{M}_{a \rightarrow B} \mathcal{M}_{XX}$, these vertices can be separately squared and summed over polarizations, and we find

$$\sum_{\lambda_1, \lambda_2} |\mathcal{M}_{CC}|^2 = \frac{(0!)^2}{m_S^4} |\mathcal{M}_{F\phi F\phi}|^2 = 1 - \frac{q^2}{m_S^2} + \frac{q^4}{4m_S^4}, \quad (177a)$$

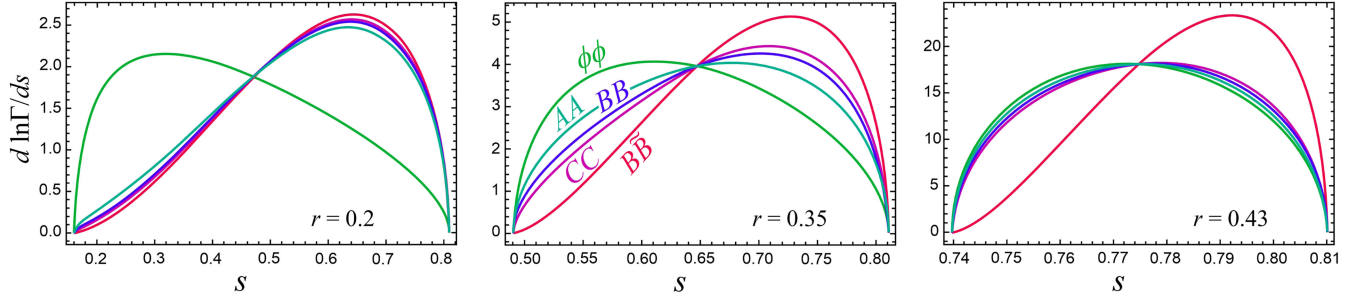


FIG. 1. Normalized differential rates for $a \rightarrow bXX$, $X = \phi, A, B, C$ as produced via the $\bar{\psi}_a \psi_b X \wedge \star X$ and $\bar{\psi}_a \psi_b B \wedge B$ couplings, see Eq. (175). We take arbitrary units and set $s = q^2/m_a^2$, $m_b/m_a = 0.1$, $r = m_X/m_a$.

$$\sum_{\lambda_1, \lambda_2} |\mathcal{M}_{BB}|^2 = \frac{(1!)^2}{m_V^4} \sum_{\lambda_1, \lambda_2} |\mathcal{M}_{F^A F^A}|^2 = 3 - \frac{2q^2}{m_V^2} + \frac{q^4}{2m_V^4}, \quad (177b)$$

$$\sum_{\lambda_1, \lambda_2} |\mathcal{M}_{AA}|^2 = \frac{(2!)^2}{m_V^4} \sum_{\lambda_1, \lambda_2} |\mathcal{M}_{F^B F^B}|^2 = 3 - \frac{q^2}{m_V^2} + \frac{q^4}{4m_V^4}, \quad (177c)$$

$$|\mathcal{M}_{\phi\phi}|^2 = \frac{(3!)^2}{m_S^4} \sum_{\lambda_1, \lambda_2} |\mathcal{M}_{F^C F^C}|^2 = 1. \quad (177d)$$

The vertices $B_{\mu\nu} \tilde{B}^{\mu\nu}$ and $F_{\mu\nu}^A \tilde{F}^{A, \mu\nu}$ can be treated similarly. Their Feynman rules are identical to that in Eq. (176) but for $\mathcal{I}_0 \rightarrow \mathcal{I}_3$, and we find

$$\sum_{\lambda_1, \lambda_2} |\mathcal{M}_{B\tilde{B}}|^2 = \frac{(1!)^2}{m_V^4} \sum_{\lambda_1, \lambda_2} |\mathcal{M}_{F^A \tilde{F}^A}|^2 = -\frac{2q^2}{m_V^2} + \frac{q^4}{2m_V^4}. \quad (178)$$

In principle, the $B_{\mu\nu} \tilde{B}^{\mu\nu}$ and $F_{\mu\nu}^A \tilde{F}^{A, \mu\nu}$ contributions should be added to that of $B_{\mu\nu} B^{\mu\nu}$ and $F_{\mu\nu}^A F^{A, \mu\nu}$ at the amplitude level, but producing the dark pairs in different states, they do not interfere.

The $\mathcal{M}_{a \rightarrow b}$ amplitude, once squared and appropriately summed and averaged, is $((m_1 + m_2)^2 - T^2)$, $((m_1 - m_2)^2 - T^2)$, and 1 for $\bar{\psi}_1 \psi_2$, $\bar{\psi}_1 \gamma_5 \psi_2$, and $\Phi_1^\dagger \Phi_2$, respectively. The different mass dimensionality comes from that of the vertices, with $\Phi^\dagger \Phi (A \wedge \star A)$ of dimension four, but $\bar{\psi} \psi (A \wedge \star A)$ of dimension five, so there is an implicit Λ^{-2} factor involved for fermions. The same holds for relating the $A \wedge \star A$ and $F^A \wedge \star F^A$ squared amplitudes.

Clearly, the massive dualities discussed before are at play to explain the relationships between $|\mathcal{M}_{AA}|^2$ and $|\mathcal{M}_{FF}|^2$, see in particular Eqs. (112) and (125) for $\phi - C$, and Eqs. (128) and (137) for $A - B$. Yet, importantly, the four $|\mathcal{M}_{AA}|^2$ do predict different kinematics, and this is then reflected in the corresponding differential rates, see Fig. 1. It is not the same to produce two dark scalars via either their ϕ or C representations, or to produce two dark photons via

either their A or B representations. As apparent in Fig. 1, all but the ϕ normalized differential rates tend to the same curve when $m_V \rightarrow 0$ because the q^4 component then dominates, and all but the $B\tilde{B}$ normalized differential rates coincide when $m_V \rightarrow (m_a - m_b)/2$ because the constant term then dominates.

For B , there is also the possibility to turn on the $\tilde{m}^2 B_{\mu\nu} \tilde{B}^{\mu\nu}$ mass term, bringing a $\mathcal{I}_3$ component in the polarization sum, Eq. (159). In its presence, the $a \rightarrow bBB$ and $a \rightarrow bB\tilde{B}$ amplitudes start to interfere. Introducing $g_{A,V}$ couplings for the operators involving $B_{\mu\nu} B^{\mu\nu}$ and $B_{\mu\nu} \tilde{B}^{\mu\nu}$, respectively, we find

$$\begin{aligned} \sum_{\lambda_1, \lambda_2} |\mathcal{M}_{g_V BB + g_A B\tilde{B}}|^2 &= g_V^2 \left(12 - 12 \frac{m_V^2}{m^2} + 3 \frac{m_V^4}{m^4} - 2 \frac{m_V^2}{m^2} \frac{q^2}{m^2} + \frac{q^4}{2m^4} \right) \\ &+ g_A^2 \left(-12 + 12 \frac{m_V^2}{m^2} - 2 \frac{m_V^2}{m^2} \frac{q^2}{m^2} + \frac{q^4}{2m^4} \right) \\ &+ 12 g_V g_A \sqrt{\frac{m_V^2}{m^2} - 1} \left(\frac{m_V^2}{m^2} - 2 \right), \end{aligned} \quad (179)$$

where $\tilde{m}$ has been expressed as in Eq. (160). We recover Eqs. (177b) and (178) if $m \rightarrow m_V$, i.e., when $\tilde{m} \rightarrow 0$. In this expression, m is essentially free, so the total rate becomes singular if $m \rightarrow 0$. As explained after Eq. (160), the nature of that singularity is totally similar to the $1/m_V$ singularities in Eqs. (177b) and (178).

Though not immediately apparent, the massive duality of Eq. (140) is still satisfied, and one can check that

$$\begin{aligned} \sum_{\lambda_1, \lambda_2} |\mathcal{M}_{g_V BB + g_A B\tilde{B}}|^2 &= \left(\frac{g_V(m^4 - \tilde{m}^4) - 2g_A m^2 \tilde{m}^2}{m^4} \right)^2 \sum_{\lambda_1, \lambda_2} |\mathcal{M}_{BB}|^2 \\ &+ \left(\frac{g_A(m^4 - \tilde{m}^4) + 2g_V m^2 \tilde{m}^2}{m^4} \right)^2 \sum_{\lambda_1, \lambda_2} |\mathcal{M}_{B\tilde{B}}|^2, \end{aligned} \quad (180)$$

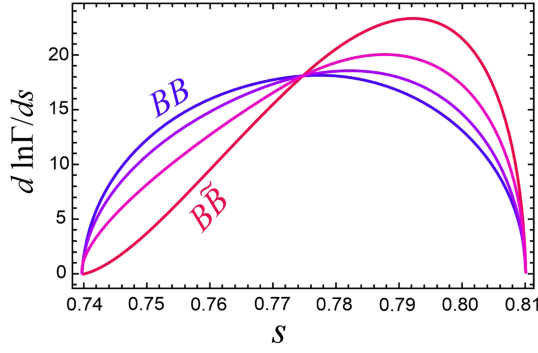


FIG. 2. Normalized differential rates for $a \rightarrow bBB$ via $g_V \bar{\psi}_a \psi_b B_{\mu\nu} B^{\mu\nu} + g_A \bar{\psi}_a \psi_b B_{\mu\nu} \tilde{B}^{\mu\nu}$, with $(g_V, g_A) = (1, 0), (1, 2), (1, 9),$ and $(0, 1)$, for the kinematical situation depicted in Fig. 1 for $r = 0.43$.

provided $m_V^2 = (m^4 + \tilde{m}^4)/m^2$, with the BB and $B\tilde{B}$ squared amplitudes in Eqs. (177b) and (178). Thus, the $a \rightarrow b(BB + B\tilde{B})$ differential rate is the sum of the $a \rightarrow bBB$ and $a \rightarrow bB\tilde{B}$ differential rates with just the right coefficients to eliminate all the $1/m_V$ factors, leaving only the $1/m$ factors apparent in Eq. (179). Phenomenologically though, the range of shapes of the normalized differential rate simply runs from that of pure BB to pure $B\tilde{B}$, as one could already obtain with arbitrary g_A and g_V but $\tilde{m} = 0$, see Fig. 2. All one needs to remember is that effectively, a $B\tilde{B}$ component in the rate can come either from the direct $a \rightarrow bB\tilde{B}$ coupling, or from the $\tilde{m}$ mass term.

To close this Section, let us remark that the present analysis could easily be adapted to the pair-creation processes $\psi_1 \psi_2 \rightarrow XX$ or $\Phi_1 \Phi_2 \rightarrow XX$ for $X = \phi, A, B, C$. As a function of the center of mass energy, their cross section would again behave differently because of Eqs. (177) and (179).

VIII. CONCLUSION

In this paper, we have analyzed in details the theoretical frameworks in which a dark scalar ϕ is represented by a rank-three antisymmetric field $C_{\mu\nu\rho}$, and the dark photon A_μ by a rank-two antisymmetric field $B_{\mu\nu}$. Though well-known dualities relate ϕ to $C_{\mu\nu\rho}$, and A_μ to $B_{\mu\nu}$, those are not equivalent phenomenologically once interactions are turned on. Starting with a more theoretical note, our main findings are

- (i) Minimal bases of effective operators involving ϕ , A_μ , $B_{\mu\nu}$, $C_{\mu\nu\rho}$, or even the fourth rank tensor $D_{\mu\nu\rho\sigma}$, singly or in pairs, and SM particles have been constructed, including operators with up to two extra derivatives. Those for ϕ and A_μ were known, but the others have never been systematically derived. Though it is not phenomenologically useful to go to such high orders, several interesting features emerged. First, surprisingly, these bases end up

involving relatively few operators. Even though the number of ways to contract the Lorentz indices quickly becomes huge, the antisymmetry of these states permits one to relate many operators, sometimes through quite intricate reductions. As a result, one peculiar feature is that neither B nor C can have dimension-six couplings with fermions only. A second feature is that the C basis is actually related to that for A . Provided all the effective operators involving the Lorenz conditions are kept, their operators are in one-to-one correspondence. For B , under the same condition, the operator basis has to be self-dual, i.e., operators at each order must only get reorganized under $B_{\mu\nu} \rightarrow \epsilon_{\mu\nu\rho\sigma} B^{\rho\sigma}$.

- (ii) We have systematically studied the impact of adding to the mass term $m^2 B_{\mu\nu} B^{\mu\nu}$ a pseudoscalar component $\tilde{m}^2 B_{\mu\nu} \tilde{B}^{\mu\nu}$. By treating this term nonperturbatively, we derived the modified physical mass of the B field, $m_B^2 = (m^4 + \tilde{m}^4)/m^2$, its full propagator, and its polarization sum. Also, these results were checked by rederiving them independently via an extension of the duality formalism, see Eqs. (140) and (141). Phenomenologically, this mass term not only alters the physical mass, but it also impacts observables, as we showed explicitly for a generic pair production process, see Fig. 2. A peculiar feature though is that the polarization sum keeps its pole in $1/m^2$ because even when $\tilde{m} \neq 0$, the B kinetic term remains noninvertible when $m \rightarrow 0$. This means that even a not-so-light B field could see its rate strongly enhanced if m becomes very small.
- (iii) Even if strictly speaking, dualities do not hold for interacting theories, we found that they remain as a powerful tool at the phenomenological level, where they show up as sum rules for polarization vectors and tensors, see Eqs. (125) and (137). As long as the dark states stay external, these sum rules are universally valid thanks to their algebraic nature. They can even accommodate for the pseudoscalar mass term for the B field, see Eq. (142).
- (iv) All these higher-rank tensor fields have an Abelian gauge symmetry when massless. As such, they appear particularly suited to Stueckelberg representations, and this opens the way to phenomenological interpretations on the basis of the equivalence theorem, even though we did not try to implement true Higgs mechanisms for these fields. Though these constructions have appeared before, it seems a systematic study along that line has not. Yet, we found that introducing R_ξ gauge-fixing for a generic antisymmetric tensor field sheds new light on their physical degrees of freedom, and their polarization sums.

Phenomenologically, once allowing for higher rank fields, we actually identified two different points of view

to proceed, depending on the assumed status of dualities. Specifically, these effective frameworks are

- (i) The parent Lagrangian formalism can be promoted to a construction mechanism to derive specific effective theories. In this case, the equivalence holds between the $\phi - C$ or $A - B$ embeddings but in a very special way. First, starting with fully generic effective interactions for the higher rank fields, only specific effective interactions can be present in the lower-rank forms. In particular, the shift symmetry is built in for the dark scalar, while the dark vector necessarily couples in a gauge-invariant way. Second, the parent Lagrangian formalism somewhat decouples the scales of the effective operators in each picture. Technically, it introduces three scales m_1 , m_2 , and m_3 , with the operators with higher-rank (lower-rank) fields tuned by m_1/m_2 ($1/m_3$), respectively, while its mass remains set at $m_{\text{dark}} = m_1 m_2 / m_3$. For the scalar theory, these two features permit one to circumvent stability problems from mixing of ϕ with the Higgs field. For the dark photon, it provides a natural mechanism to suppress the renormalizable and gauge-invariant kinetic mixing. Another consequence is the need for contact terms to relate these effective theories. These take the form of effective interactions among SM fields only, scaling as $1/m_3^2$, which can be looked for in totally different settings like, e.g., at colliders or in low-energy observables.
- (ii) Instead of the parent Lagrangian approach, one can also start directly with an effective theory written in terms of $B_{\mu\nu}$ for the dark photon, or $C_{\mu\nu\rho}$ for the dark scalar, without any recourse to duality arguments. The leading effective operators are then simply different than when using the A_μ or ϕ picture, with some advantages and disadvantages. Specifically, whatever the picture, some effective interactions need to be tiny to pass experimental bounds. This in general requires some additional assumptions on the unknown UV physics. In this context, imposing a B gauge invariance on the effective operators (but allowing for a mass term) proves particularly powerful. In this approach, even if duality is no longer called in to somehow translate the effective theory back into the usual A or ϕ picture, it is still present in the form of the polarization sum relations. Those imply in particular that processes involving B and C must have different momentum dependences compared to that involving A and ϕ . Since an external B field behaves essentially as a vector field strength, and C essentially as a derivatively coupled scalar, extra enhancements in the form of energy scale over dark mass are expected. To illustrate this effect in a simple setting, we compared the generic pair production processes for each scenario, where this

enhancement shows up in the differential rates in terms of the dark state invariant mass, see Fig. 1.

The stage is set for further theoretical and phenomenological studies. For the former, some questions remain on possible renormalizable UV completion for the higher form effective theories. In particular, the presence of some gauge invariance, and then the mechanism at the origin of the mass term(s), would need to be elucidated. For the latter, all the tools are ready for detailed analyses. First, in a dark matter context and low-energy searches, it would be very interesting to see how interpreting a dark scalar as a C field, or a dark photon as a B field, would alter the available parameter space. Second, more generally, B and C fields could show up in unexpected places. For instance, throughout this work we took the point of view that the B or C field should be light, but nothing in the formalism prevents them from arising at the TeV scale or above. In that case, they could be looked for at colliders, where their different kinematical behaviors would provide rather unique signatures.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: DIFFERENTIAL FORMS

The language of differential forms is particularly well suited to the formulation of gauge theories. The present appendix collects in the next Section all the relevant definitions. It is intended more as a repository of useful relations and conventions rather than a pedagogical introduction. For that, we refer, e.g., to Ref. [44] or any other book on differential geometry. In the following Section, the formalism is applied to the vector field, showing explicitly how the usual equations are recovered.

1. Definitions and conventions

Differential 1-forms are in essence gradients, in the sense that to $\partial f / \partial x = f'(x)$, one can associate the 1-form $\omega(x) = f'(x)dx$ such that integrating ω gives back the function f . Including the integration measure makes this definition independent of the coordinate system. 0-forms are just functions that can be evaluated (“integrated”) at any given point within the range of f .

To generalize this definition to higher dimensions while taking care of possible orientations of the integration region, the basic operation is the *wedge product*. For a p -form ω and a q -form η , this product is the $p + q$ -form

constructed from the antisymmetric product,

$$(\omega \wedge \eta)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!q!} \omega_{[\mu_1 \dots \mu_p} \eta_{\mu_{p+1} \dots \mu_{p+q}]}, \quad (\text{A1})$$

where [...] denotes the *normalized* antisymmetrization, hence the $(p+q)!$ prefactor. From the definition, one can show the graded commutativity $\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$ and distributivity over addition $\omega \wedge (\eta + \varphi) = \omega \wedge \eta + \omega \wedge \varphi$. Recursively, this permits one to define higher forms out of 1-forms, with, e.g., the wedge product of k 1-forms being a k -form. A natural basis for k -forms is then $dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}$, in which a generic k -form is

$$\omega = \frac{1}{k!} \omega_{\mu_1 \dots \mu_k} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k}. \quad (\text{A2})$$

In n -dimensional spacetime, a k -form ω has $C_k^n = n!/k!(n-k)!$ independent components. A k -form can be integrated over a k -dimensional space, while the n -form in n dimensions has one component since it is necessarily proportional to the *volume form*,

$$\text{vol} = \frac{1}{n!} \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} \equiv d^n x. \quad (\text{A3})$$

We will work in flat space throughout, so the tensor $\epsilon_{\mu_1 \dots \mu_n} = \epsilon_{\mu_1 \dots \mu_n}$ with $\epsilon_{\mu_1 \dots \mu_n}$ the usual constant algebraic antisymmetric symbol for which we set $\epsilon_{0, \dots, n-1} = +1$, while $\epsilon^{\mu_1 \dots \mu_n} = -\epsilon_{\mu_1 \dots \mu_n}$ given our metric signature $(+1, -1, -1, -1)$. This implies that

$$\epsilon_{\rho_1 \dots \rho_k \mu_1 \dots \mu_{n-k}} \epsilon^{\rho_1 \dots \rho_k \nu_1 \dots \nu_{n-k}} = -k! \delta_{\mu_1 \dots \mu_{n-k}}^{\nu_1 \dots \nu_{n-k}}, \quad (\text{A4})$$

with the *generalized Kronecker symbol* defined as

$$\delta_{\mu_1 \dots \mu_k}^{\nu_1 \dots \nu_k} = \sum_{\sigma} \text{sign}(\sigma) \delta_{\mu_1}^{\sigma(\nu_1)} \dots \delta_{\mu_k}^{\sigma(\nu_k)}, \quad (\text{A5})$$

where summation is over all the σ permutations of the k indices ν_1 to ν_k . Also, $\delta_{\mu_1 \dots \mu_k}^{\nu_1 \dots \nu_k} \omega_{\nu_1 \dots \nu_k} = k! \omega_{\mu_1 \dots \mu_k}$ and $\delta_{\mu_1 \dots \mu_k}^{\rho_1 \dots \rho_k} \delta_{\rho_1 \dots \rho_k}^{\nu_1 \dots \nu_k} = k! \delta_{\mu_1 \dots \mu_k}^{\nu_1 \dots \nu_k}$. With this, we can also express the volume integration measure as $dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} = \epsilon^{\mu_1 \dots \mu_n} d^n x$.

The *Hodge dual* of a k -form is defined as

$$\omega_{\nu_1 \dots \nu_k} \rightarrow (\star \omega)_{\mu_1 \dots \mu_{n-k}} = \frac{1}{k!} \epsilon_{\nu_1 \dots \nu_k \mu_1 \dots \mu_{n-k}} \omega^{\nu_1 \dots \nu_k}, \quad (\text{A6})$$

satisfying $\star(\star \omega) = (-1)^{k(n-k)-1} \omega$. With this, one can introduce the *inner product* $\langle \omega, \eta \rangle$ of two k -forms ω and η , as the volume form

$$\langle \omega, \eta \rangle \equiv \omega \wedge \star \eta = \frac{1}{k!} \omega_{\mu_1 \dots \mu_k} \eta^{\mu_1 \dots \mu_k} d^n x. \quad (\text{A7})$$

By symmetry, $\langle \omega, \eta \rangle = \langle \eta, \omega \rangle$, and $\langle \omega, \eta \rangle = -\langle \star \eta, \star \omega \rangle$ for a negative-signature metric.

Another way to recursively construct higher forms is by differentiation. The *exterior derivative* of a k -form ω is the $k+1$ -form obtained after properly antisymmetrizing the partial derivative

$$d\omega = \frac{1}{k!} \partial_{\mu_1} \omega_{\mu_2 \dots \mu_{k+1}} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_{k+1}}, \quad (\text{A8})$$

or $(d\omega)_{\mu_1 \dots \mu_{k+1}} = (k+1) \partial_{[\mu_1} \omega_{\mu_2 \dots \mu_{k+1}]}$, where [...] is normalized. The important properties of d are

$$\begin{aligned} d(\omega + \eta) &= d\omega + d\eta, d(\omega \wedge \eta) \\ &= d\omega \wedge \eta + (-1)^p \omega \wedge d\eta, d^2 \omega = 0. \end{aligned} \quad (\text{A9})$$

A form such that $d\omega = 0$ is said to be a *closed form*, while one such that $\omega = d\eta$ is said to be an *exact form*. All exact forms are closed, but the converse is true only locally. In terms of k -forms, *Stokes's theorem* takes the form

$$\int_M d\omega = \int_{\partial M} \omega, \quad (\text{A10})$$

for some $k+1$ dimensional space with k -dimensional boundary ∂M .

To combine exterior derivative with the Hodge dual, one defines the *CODIFFERENTIAL* of a k -form as the $k-1$ -form obtained via

$$\delta = (-1)^{n(k+1)} \star d \star \rightarrow (\delta \omega)_{\mu_1 \dots \mu_{k-1}} = -\partial^\alpha \omega_{\alpha \mu_1 \dots \mu_{k-1}}. \quad (\text{A11})$$

It has fewer properties than d but still $\delta^2 \omega = 0$. The codifferential is the adjoint of the exterior derivative, since from the definition we have $\star \delta = (-1)^k d \star$ and $\delta \star = (-1)^{k+1} \star d$ when acting on a k -form. Explicitly, if ω is a $k-1$ -form and η a k -form such that $\omega \wedge \star \eta$ vanishes on the integration volume boundary, then

$$\int d(\omega \wedge \star \eta) = 0 = \int d\omega \wedge \star \eta - \int \omega \wedge \star \delta \eta. \quad (\text{A12})$$

The exterior derivative and codifferential can be combined as $\delta d = (-1)^{n(k+2)} \star d \star d$ and $d\delta = (-1)^{n(k+1)} d \star d \star$, both producing k -forms when acting on a k -form. Of special interest is their sum, called the *Laplace-Beltrami operator* $\Delta = (\delta + d)^2 = \delta d + d\delta$. It is positive-definite and such that $\Delta \omega = 0$ is attained for $d\omega = \delta \omega = 0$, in which case ω is said to be a **harmonic form**. Notice finally that Δ commutes with the Hodge dual, $\star \Delta = \Delta \star$.

2. Application to Maxwell and Proca theories

To see all the definitions in action in a simple setting, it may be useful to rederive known results for vector fields. In that case, the field $A = A_\mu dx^\mu$ and the current $j = J_\mu dx^\mu$ are 1-forms. The field strength is the 2-form defined as $F = dA$,

$$F = \frac{1}{2!} F_{\mu\nu} dx^\mu \wedge dx^\nu = \partial_\mu A_\nu dx^\mu \wedge dx^\nu. \quad (\text{A13})$$

The Hodge dual $\star F$ is then

$$\star F = \frac{1}{2} \left(\frac{1}{2} \epsilon_{\alpha\beta\mu\nu} F^{\alpha\beta} \right) dx^\mu \wedge dx^\nu, \quad (\text{A14})$$

and is often denoted $\star F = \tilde{F}$. The homogeneous Maxwell's equations immediately follow from $dF = d^2 A = 0$ since $d^2 = 0$. Equivalently, $\star dF = 0$ translates into the Bianchi identity $\epsilon^{\beta\mu\nu\rho} \partial_\mu F_{\nu\rho} = 0$. The inhomogeneous equations correspond to the EoM derived from the Maxwell action,

$$\begin{aligned} S_{\text{EM}} &= \int -\frac{1}{2} F \wedge \star F - A \wedge \star j \\ &= \int \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu \right) d^n x. \end{aligned} \quad (\text{A15})$$

Notice that $F \wedge \star F$ is quadratic in F since the wedge product is symmetric. The action must be stationary against small variations $A \rightarrow A + \delta A$, which to linear order imposes the vanishing of

$$\begin{aligned} 0 &= \delta S_{\text{EM}} = \int -d\delta A \wedge \star F - \delta A \wedge \star j \\ &= \int \delta A \wedge (-d\star F - \star j). \end{aligned} \quad (\text{A16})$$

The total derivative is discarded since F is assumed to vanish at infinity, and $d(\delta A \wedge \star F) = d\delta A \wedge F - \delta A \wedge d\star F = 0$ since δA is a 1-form. Thus, the equation of motion is

$$d\star F + \star j = 0 \Leftrightarrow \delta F + j = 0 \Leftrightarrow (\partial^\mu F_{\mu\nu} - J_\nu) dx^\nu = 0, \quad (\text{A17})$$

where $\star(\star j) = j$ while $\delta F = -\partial^\mu F_{\mu\nu} dx^\nu$ from Eq. (A11). Acting with d on the EoM also implies $\delta j = 0$ since $d^2 \star F = 0$ automatically, i.e., the current must be conserved $\partial^\mu J_\mu = 0$. With this, the action S_{EM} is invariant under the gauge transformations $A \rightarrow A + d\Lambda$ for Λ a 0-form. Indeed, $F = dA$ is automatically invariant, while the source term varies as

$$\delta S_{\text{EM}} = \int -d\Lambda \wedge \star j = \int \Lambda \wedge d\star j = 0, \quad (\text{A18})$$

upon integrating by part and discarding the surface term, over which j is supposed to vanish. In terms of gauge fields, the EoM takes the form

$$(\Delta - d\delta)A = -j \rightarrow (\square g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu = 0. \quad (\text{A19})$$

Here, it is customary to enforce the Lorenz condition to fix the gauge, $\delta A = 0$, that is, $\partial^\mu A_\mu = 0$, so that the EoM collapses to $\Delta A = 0$. Notice that this leaves a residual gauge freedom, $A \rightarrow A + d\Lambda'$ with Λ' such that $\delta d\Lambda' = 0$. This can again be written as $(\Delta - d\delta)\Lambda' = 0$, but since Λ' is a 0-form, we immediately get $\Delta\Lambda' = 0$ and the residual gauge freedom is harmonic.

For the Proca Lagrangian, $F = dA$ and $dF = 0$ still hold, but the equation of motion is modified by the presence of the mass term,

$$S_{\text{Proca}} = S_{\text{EM}} + \int \frac{1}{2} m^2 A \wedge \star A = S_{\text{EM}} + \int \frac{m^2}{2} A_\mu A^\mu d^n x, \quad (\text{A20})$$

from which the EoM $-\star d\star F + m^2 A = j$ is derived. In this case, if the current is conserved, $m^2 d\star A = d\star j$, and the Lorenz condition $\delta A = 0$ emerges automatically. In terms of field, the EoM is then the usual Klein-Gordon equation, $(-\Delta + m^2)A = j$.

To close this appendix, let us stress that once written in differential form, all the equations of the present section remain essentially valid for higher form fields. The definition of the fields and field strengths, the action, EoM, gauge freedom, or mass term are, up to trivial signs, identical.

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