

Light baryon static properties in a dispersive approach

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We extend our dispersive analyses on meson static properties to those of light baryons. The formalism treats the dispersion relation, which a baryonic correlation function obeys, as an inverse problem, solves for the involved spectral density with available operator-product-expansion (OPE) inputs directly, and extracts baryon static properties from the spectral density. We observe that the simultaneous implementation of the chiral-even and chiral-odd dispersive constraints unambiguously determines baryon masses and pole residues. A common set of quark and gluon condensates, which appear in OPE factorization and are universal, is found to accommodate the masses of a ρ meson, a proton, and a $\Delta(1232)$ baryon. The advantage of our approach over the conventional handling of QCD sum rules is advocated. This work encourages broad applications of our nonperturbative analytical method to baryon systems.

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I. INTRODUCTION

It has been recognized that QCD sum rules [1], as a popular nonperturbative method, suffer theoretical uncertainties from the assumption of quark-hadron duality for parametrizing a spectral density and from the discretionary prescription for designating a stability window in a Borel mass [2–7]. To remedy the above drawbacks, we proposed to handle QCD sum rules as an inverse problem [8]; the spectral density on the hadron side of a dispersion relation, including both resonance and continuum contributions, is regarded as an unknown in an integral equation, which is solved with standard operator-product-expansion (OPE) inputs on the quark side. Once a dispersion relation is solved directly, there is no need to presume the existence of a resonance, to introduce a free continuum threshold, to implement the quark-hadron duality for excited-state contributions, or to assign discretionary weights to continuum contributions and power corrections in an OPE. In other words, our approach relies solely on analyticity of physical observables and positivity of spectral densities. Moreover, the precision of predictions can be improved systematically by adding subleading contributions to an OPE [9].

Our formalism is similar to the one developed in [10–13], which also inputs high-energy OPEs with quark and gluon condensates to bootstrap low-energy dynamics. The idea behind both frameworks stems from the S -matrix bootstrap conjecture postulated by Chew in the 1960s [14]; a well-defined infinite set of self-consistency conditions (analyticity, unitarity, crossing, etc.) determines uniquely the aspects of particles in nature [15,16]. The difference between our approach and [10–12] to, say, a two-current correlator, is that constraints on other observables like form factors, in addition to those on a spectral density, were incorporated in the latter. The inclusion of form factors is based on the assumption that the spectral density associated with the correlator is saturated by two pion states in an intermediate energy region. However, Refs. [10–12] considered only the constraints imposed by the first few moments of a spectral density through sum rules, while we concentrate on a spectral density and attempt to retrieve full information from it. We mention an alternative application of analyticity [17], where various meson resonance poles were extracted from dispersion relations with the inputs of $\pi\pi$ scattering data.

The dispersive analyses on nonperturbative quantities of several mesonic states have been conducted, covering masses and decay constants [8,9,18–20] for series of ρ resonances and glueballs and the pion light-cone distribution amplitude [21]. It was extended to investigations of hadronic properties in nuclear matter recently [22]. Besides, the perspective of treating a dispersion relation as an inverse problem was generalized to the understanding of neutral meson mixing parameters [23–25] and to the estimation of the hadronic vacuum-polarization

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contribution to the muon anomalous magnetic moment [26]. Here we will further extend the formalism for meson static properties to those of light baryons. A challenge arises immediately from the splitting of a correlation function defined by baryonic currents into a chiral-even part and a chiral-odd part. The OPE for the former (latter) contains only dimension-even (dimension-odd) condensates. It has been observed [8] that a sufficient number of power corrections are necessary for establishing a physical resonance solution. After the condensates of different dimensions are separated, the number of power corrections associated with each chirality is not enough for identifying a physical solution.

The difficulty can be overcome by the strategy below. We gather baryon masses and pole residues allowed by each dispersion relation and find the common ones between the two sets of solutions. It turns out that baryon masses and pole residues can be specified unambiguously in this way. The reason for the success of the strategy is that the simultaneous implementation of the chiral-even and chiral-odd dispersive constraints takes into account all available power corrections in some sense. To highlight the strength of our approach, we explicate that no stability window in a Borel mass exists in the conventional handling of QCD sum rules for the proton mass, so a definite determination requests additional prescriptions. Efforts and progresses of improving the conventional sum rule approach have been made continuously (see, for instance, Refs. [27,28]). Since the OPE formulation abides by short-distance factorization, involved condensates of quark and gluon fields are universal. We demonstrate that the same set of quark and gluon condensates can account for the masses of a ρ meson, a proton, and a $\Delta(1232)$ baryon collected in [29]. This work encourages intensive applications of our non-perturbative analytical method to baryon systems.

The rest of the paper is organized as follows. We recapture the derivation of the ρ meson mass and decay constant in Sec. II, focusing on the inverse matrix method for solving a dispersion relation [9]. The framework is applied to the proton and $\Delta(1232)$ baryon cases in Sec. III. The aforementioned strategy to establish physical solutions from chiral-even and chiral-odd dispersion relations is elaborated. The advantage of our approach over the conventional QCD sum rules is emphasized. We then conclude in Sec. IV.

II. INVERSE MATRIX METHOD

The key ingredients of our theoretical setup have been elucidated in [9], including the construction of the dispersion relation from a correlation function, which connects the dispersive integral of a spectral density to the corresponding OPE. Taking the simple ρ meson case as an example, we explain how to deduce the ρ meson mass and decay constant from the dispersion relation. The two-point correlator

$$\begin{aligned}\Pi^{\mu\nu}(q^2) &= \int d^4x e^{iq \cdot x} \langle 0 | T[J^\dagger(x) J^\nu(0)] | 0 \rangle \\ &= (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi(q^2),\end{aligned}\quad (1)$$

defines the vacuum-polarization function $\Pi(q^2)$, where T denotes the time-ordered product, and the momentum q is injected into the interpolating vector current $J^\mu = (\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)/\sqrt{2}$. The OPE for $\Pi(q^2)$ is reliable in the deep Euclidean region of q^2 and available up to the dimension-six condensate, i.e., to the power correction of $1/(q^2)^3$ as shown below [1].

The analyticity of the function $\Pi(q^2)$ leads to the dispersion relation [9]

$$\begin{aligned}\int_0^R ds \frac{\rho(s)}{s - q^2} &= \frac{1}{\pi} \int_0^R ds \frac{\text{Im} \Pi^{\text{pert}}(s)}{s - q^2} + \frac{1}{12\pi} \frac{\langle \alpha_s G^2 \rangle}{(q^2)^2} \\ &\quad + \frac{2\langle m_q \bar{q}q \rangle}{(q^2)^2} + \frac{224 \kappa \alpha_s \langle \bar{q}q \rangle^2}{81 (q^2)^3},\end{aligned}\quad (2)$$

where $\rho(s) \equiv \text{Im} \Pi(s)/\pi$ is the spectral density, $\text{Im} \Pi^{\text{pert}}(s) = (1 + \alpha_s/\pi)/(4\pi) \equiv c\pi$ is the imaginary part of the perturbative piece in the OPE, and R is an arbitrary ultraviolet cutoff of the integration variable s . Precisely speaking, R is the radius of the big circle as part of the contour on a complex plane, along which the integration of $\Pi(s)$ is performed. In Eq. (2) $\langle \alpha_s G^2 \rangle \equiv \langle \alpha_s G_{\mu\nu}^a G^{a\mu\nu} \rangle$ is the gluon condensate, m_q is a light quark mass, and the parameter $\kappa = 2-4$ [30–32] quantifies the violation in the factorization of the four-quark condensate $\langle (\bar{q}q)^2 \rangle$ into the square of the quark condensate $\langle \bar{q}q \rangle$. Condensates of higher dimensions can be added systematically, which are currently unknown. The only requirement for the spectral density $\rho(s)$ is that it approaches the perturbative one $\text{Im} \Pi^{\text{pert}}(s)$ as $s \rightarrow \infty$, which is the basis for constructing Eq. (2). We stress that it differs from the quark-hadron duality assumed in conventional sum rules for parametrizing excited-state contributions.

Working on the subtracted spectral density

$$\Delta\rho(s, \Lambda) = \rho(s) - \frac{1}{\pi} \text{Im} \Pi^{\text{pert}}(s) [1 - \exp(-s/\Lambda)], \quad (3)$$

we can push R toward the infinity, arriving at

$$\begin{aligned}\int_0^\infty dy \frac{\Delta\rho(y)}{x - y} &= \int_0^\infty dy \frac{c e^{-y}}{x - y} - \frac{1}{12\pi} \frac{\langle \alpha_s G^2 \rangle}{x^2 \Lambda^2} \\ &\quad - \frac{2\langle m_q \bar{q}q \rangle}{x^2 \Lambda^2} - \frac{224 \kappa \alpha_s \langle \bar{q}q \rangle^2}{81 x^3 \Lambda^3},\end{aligned}\quad (4)$$

where the variable changes $x = q^2/\Lambda$ and $y = s/\Lambda$ have been exerted. Because $\Delta\rho(s, \Lambda)$ is a dimensionless quantity, it has been cast into the form $\Delta\rho(y = s/\Lambda)$. It is seen that the arbitrary cutoff R is replaced by the arbitrary

scale Λ , which moves into the power corrections after the variable changes. The subtraction function $1 - \exp(-s/\Lambda)$ decreases like s at small s , and approaches unity at large $s \gg \Lambda$, such that $\Delta\rho(s, \Lambda)$ retains the behavior of $\rho(s) \sim s$ near the threshold $s \rightarrow 0$ [33] and diminishes quickly as $s > \Lambda$. We have confirmed that other subtraction functions with the similar boundary and asymptotic behaviors yield basically identical solutions for $\rho(s)$. The arbitrariness of Λ , to which a physical observable should be insensitive, will help identify the physical solution to the dispersion relation.

Equation (4) is classified as the first kind of Fredholm integral equations in the typical expression

$$\int_0^\infty dy \frac{\Delta\rho(y)}{x-y} = \omega(x), \quad (5)$$

where the unknown function $\Delta\rho(y)$ will be solved with the given input function $\omega(x)$. It has been proved [24] that a solution to Eq. (5), if existing, is unique. Techniques are available for solving the above potentially ill-posed inverse problem. Here we adopt the inverse matrix method proposed in [9]. If $\Delta\rho(y)$ decreases rapidly with the variable y , the kernel $1/(x-y)$ in Eq. (5) can be expanded into a power series of $1/x$. This argument also holds for the dispersive integral on the right-hand side of Eq. (5), for the numerator ce^{-y} descends fast in y . The function $\omega(x)$, representing the OPE on the right-hand side of Eq. (4), can be expanded into a power series of $1/x$ too. Hence, we write

$$\frac{1}{x-y} = \sum_{m=1}^N \frac{y^{m-1}}{x^m}, \quad \omega(x) = \sum_{n=1}^N \frac{b_n}{x^n}, \quad (6)$$

up to some integer N . The two sides of Eq. (5) can then be matched by equating the coefficients at each power of $1/x$.

To facilitate the matching, we reformulate Eq. (5) into a matrix form by expanding $\Delta\rho(y)$ in terms of a complete set of orthogonal polynomials. As discussed in [9], the Laguerre polynomials in the support $[0, +\infty)$, with a weight factor e^{-y} that suppresses the contribution from large y , serve the purpose. The expansion

$$\Delta\rho(y) = \sum_{n=1}^N a_n y^\alpha e^{-y} L_{n-1}^{(\alpha)}(y) \quad (7)$$

and Eq. (6) define the matrix elements

$$M_{nm} = \int_0^\infty dy y^{m-1+\alpha} e^{-y} L_{n-1}^{(\alpha)}(y), \quad (8)$$

with m and n running from 1 to N , and transform Eq. (5) into $Ma = b$. The vector $a = (a_1, a_2, \dots, a_N)$ groups the unknown coefficients in Eq. (7), and the vector $b = (b_1, b_2, \dots, b_N)$ comes from the known input $\omega(x)$

in Eq. (6). The integer N will be fixed later, and the choice of the index α depends on the behavior of $\Delta\rho(y)$ adjacent to the boundary $y = 0$. Simply inverting the matrix equation, we obtain the vector $a = M^{-1}b$, which constructs the function $\Delta\rho(y)$ according to Eq. (7).

We solve for the unknown $a = M^{-1}b$ with the input b from Eq. (4) in the inverse matrix method, the subtracted spectral density $\Delta\rho(s, \Lambda)$, and then the spectral density $\rho(s)$ in Eq. (3). In principle, the true solution is approached by increasing the number N of the Laguerre polynomials in Eq. (7). Nevertheless, both m and n have to be terminated at a finite value in a practical operation, because the determinant of the matrix M diminishes with its dimension. If N keeps growing, a tiny fluctuation of the input b will be amplified by the huge elements of the inverse matrix M^{-1} , and positivity of a spectral density is lost. This is a generic feature of an ill-posed inverse problem. Therefore, the optimal choice of N is set to the maximal integer, around which a solution is stable against the variation of N . We take the input parameters

$$\begin{aligned} \langle \alpha_s G^2 \rangle &= 0.028 \text{ GeV}^4, \quad \langle m_q \bar{q}q \rangle = 0.007 \times (-0.215)^3 \text{ GeV}^4, \\ \alpha_s \langle \bar{q}q \rangle^2 &= 6.64 \times 10^{-5} \text{ GeV}^6 \quad \alpha_s = 0.5, \quad \kappa = 2, \end{aligned} \quad (9)$$

for a numerical illustration. These values are similar to those in [9], except the gluon condensate $\langle \alpha_s G^2 \rangle$, which is much lower than 0.08 GeV^4 in [9]. We point out that there exists a range for this condensate [34–37] as summarized in Eq. (11) of [38]. The choice of the gluon condensate in Eq. (9) is still above the lower bound 0.01 GeV^4 , while the choice 0.08 GeV^4 is close to the upper bound. The adjustment of the parameters aims at a simultaneous accommodation of ρ meson, proton, and $\Delta(1232)$ static properties in this work.

Figure 1 displays the curves of the spectral density $\rho(s)$ for several N 's with $\Lambda = 2 \text{ GeV}^2$. It is obvious that the curves labeled by $N = 15, 16$, and 17 are pretty similar, exhibiting the stability in N . To substantiate such stability, we resort to a simplified version of the Frechet distance between two curves corresponding to adjacent N 's (with the same Λ),

$$F_{\Lambda, i}[\rho(s)] = \max\{D(\rho_i(s), \rho_{i+1}(s))\}_{s \in [0, 6 \text{ GeV}^2]}. \quad (10)$$

In the above definition the argument s takes a value in the interval $[0, 6 \text{ GeV}^2]$, and $D(\rho_i(s), \rho_{i+1}(s))$ is the distance between the points on the two curves associated with $N = i$ and $N = i + 1$. For two identical curves, the Frechet distance vanishes; for two distinct curves, the smaller the Frechet distance is, the more similar the two curves are. We attain the minimal Frechet distance $F_{\Lambda=2 \text{ GeV}^2, i=16}$ by scanning the range $[12, 20]$ of N , which suggests that $N = 16$ is the best choice to stabilize $\rho(s)$ in N for $\Lambda = 2 \text{ GeV}^2$. The best N 's for other Λ values can be acquired in the same

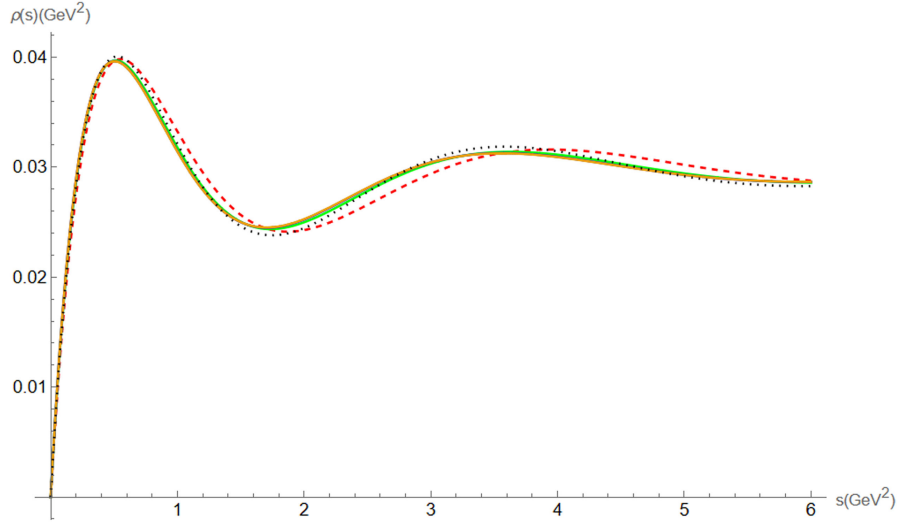


FIG. 1. Dependencies of $\rho(s)$ on s for $\Lambda = 2 \text{ GeV}^2$ and $N = 13$ (dashed line), $N = 15, 16, 17$ (solid lines), and $N = 19$ (dotted line).

manner. The first (major) peak of $\rho(s)$ for $\Lambda = 2 \text{ GeV}^2$ located at $s \approx 0.60 \text{ GeV}^2$ in Fig. 1 coincides with the $\rho(770)$ meson mass. The oscillatory tail at $s > 2 \text{ GeV}^2$ manifests that the continuum contribution to $\rho(s)$ deviates from the perturbative one $c = 0.029$. Namely, the local quark-hadron duality does not hold actually.

As stressed before, a physical observable should be insensitive to the variation of the arbitrary scale Λ . Equation (4) shows that the condensate effects decrease with Λ and diminish in the limit $\Lambda \rightarrow \infty$. It implies that $\rho(s)$ depends only on the scaling variable $y = s/\Lambda$ at large Λ , so a peak of $\rho(s)$ would drift in s with Λ . Then the peak location, failing the stability criterion, cannot be interpreted as a physical state. On the other hand, Λ cannot be too small, otherwise the enhanced condensate contributions break the OPE. A proper window in Λ is expected to exist,

within which the OPE, i.e., the reliability of solutions, is justified, and the stability of the ρ meson mass sustains. Figure 2, depicting the Λ dependence of the ρ meson mass M_ρ , reflects the aforementioned features; the curve climbs from $\Lambda = 1.0 \text{ GeV}^2$, goes up and down mildly about $M_\rho = 0.74 \text{ GeV}$ for $3.0 \text{ GeV}^2 \leq \Lambda \leq 7.0 \text{ GeV}^2$, and then ascends monotonically with Λ as $\Lambda > 8.0 \text{ GeV}^2$. We read off the ρ meson mass from the flat segment of the curve between $\Lambda = 4$ and $\Lambda = 6 \text{ GeV}^2$ in Fig. 2,

$$M_\rho = (0.737 \pm 0.004) \text{ GeV}, \quad (11)$$

whose uncertainty evinces the salient stability of our result in Λ . The ρ meson mass depends weakly on the OPE parameters [9]; the representative variation of the gluon

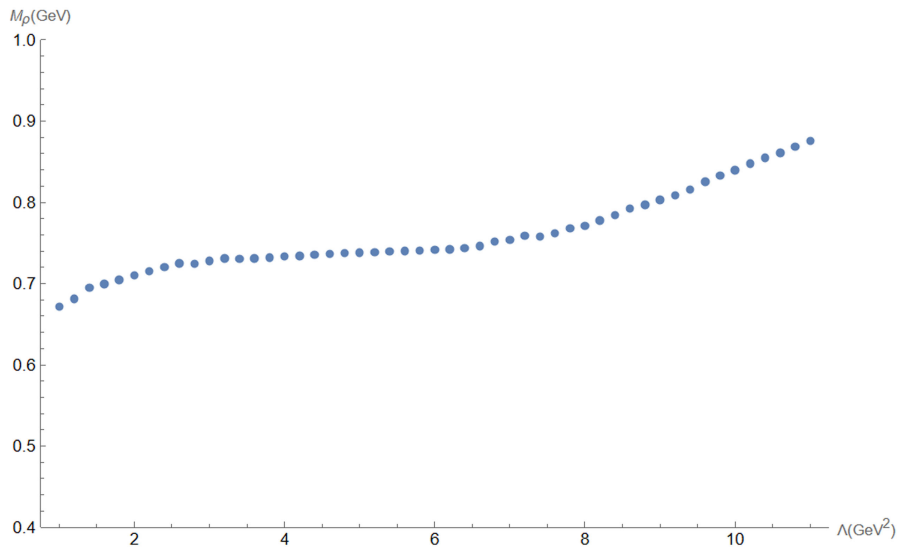


FIG. 2. Dependence of the ρ meson mass M_ρ on Λ . The specific N is taken for each Λ that minimizes the Frechet distance in Eq. (10).

condensate $\langle \alpha_s G^2 \rangle$ by $\pm 20\%$ changes M_p by $\mp 5\%$, which is not included in Eq. (11). The above outcome is close to (0.78 ± 0.03) GeV in [9], though the inputted gluon condensate in Eq. (9) is lower; the reduction of the gluon condensate has been more or less compensated by that of the quark condensate in the present analysis.

At last, we estimate the ρ meson decay constant following [9]

$$f_\rho^2 \approx \int_0^a ds \Delta\rho(s, \Lambda), \quad (12)$$

where the upper bound a of the integration variable s is set to the location of the first valley in $\Delta\rho(s, \Lambda)$. Compared to [9] which set $a = \infty$, Eq. (12) excludes potential minor contributions from excited states mostly. We get, for $\Lambda \in [4 \text{ GeV}^2, 6 \text{ GeV}^2]$, the decay constant $f_\rho \approx 0.218$ GeV, consistent with 0.20 GeV in [9].

III. BARYON STATIC PROPERTIES

A. Proton mass and pole residue

We begin with the two-point correlation function [39–41]

$$\Pi(k) = i \int d^4x e^{ik \cdot x} \langle 0 | T[\eta(x) \bar{\eta}(0)] | 0 \rangle = A(k^2)k + B(k^2), \quad (13)$$

where $\eta(x) = \epsilon^{abc} [u^{aT}(x) C \gamma^\mu u^b(x)] \gamma_5 \gamma_\mu d^c(x)$ is a baryonic interpolating current, and $A(k^2)$ [$B(k^2)$] denotes the chiral-even (chiral-odd) part of the correlator. Equation (13) describes a modified proton propagator at hadron level,

$$\Pi(k) = \lambda_p^2 \frac{k + m_p}{m_p^2 - k^2} + \dots, \quad (14)$$

with the proton mass m_p and the pole residue λ_p . Hence, the functions $A(k^2)$ and $B(k^2)$ correspond to

$$A(k^2) = \frac{\lambda_p^2}{M_p^2 - k^2}, \quad B(k^2) = \frac{\lambda_p^2 M_p}{M_p^2 - k^2}. \quad (15)$$

The OPEs for the functions $A(k^2)$ and $B(k^2)$ are given by [39]

$$\begin{aligned} \Pi_A(k^2) &= \frac{k^4}{64\pi^4} \ln\left(\frac{-k^2}{\mu^2}\right) + \frac{\langle \alpha_s G^2 \rangle}{32\pi^3} \ln\left(\frac{-k^2}{\mu^2}\right) - \frac{2}{3k^2} \kappa \langle \bar{q}q \rangle^2, \\ \Pi_B(k^2) &= -\left(\frac{k^2}{4\pi^2} \langle \bar{q}q \rangle + \frac{1}{8\pi^2} \langle \bar{q}g\sigma \cdot Gq \rangle\right) \ln\left(\frac{-k^2}{\mu^2}\right), \end{aligned} \quad (16)$$

which contain dimension-even and dimension-odd condensates, respectively. It is noticed that an additional quark-gluon condensate $\langle \bar{q}g\sigma \cdot Gq \rangle$ appears, compared to the ρ meson case. We derive the dispersion relations straightforwardly

$$\begin{aligned} \frac{1}{\pi} \int_0^R ds \frac{\rho_A(s)}{s - k^2} &= \frac{1}{\pi} \int_0^R ds \frac{\text{Im} \Pi_A(s)}{s - k^2} - \frac{2}{3k^2} \kappa \langle \bar{q}q \rangle^2, \\ \frac{1}{\pi} \int_0^R ds \frac{\rho_B(s)}{s - k^2} &= \frac{1}{\pi} \int_0^R ds \frac{\text{Im} \Pi_B(s)}{s - k^2}, \end{aligned} \quad (17)$$

with the spectral densities $\rho_A(s)$ and $\rho_B(s)$, and

$$\begin{aligned} \text{Im} \Pi_A(s) &= \pi \left(\frac{s^2}{64\pi^4} + \frac{\langle \alpha_s G^2 \rangle}{32\pi^3} \right), \\ \text{Im} \Pi_B(s) &= -\pi \left(\frac{s}{4\pi^2} \langle \bar{q}q \rangle + \frac{1}{8\pi^2} \langle \bar{q}g\sigma \cdot Gq \rangle \right). \end{aligned} \quad (18)$$

We define the subtracted spectral densities,

$$\begin{aligned} \Delta\rho_A(s, \Lambda) &= \rho_A(s) - \frac{s^2}{64\pi^4} [1 - \exp(-s/\Lambda)] - \frac{\langle \alpha_s G^2 \rangle}{32\pi^3} [1 - \exp(-s^2/\Lambda^2)], \\ \Delta\rho_B(s, \Lambda) &= \rho_B(s) + \frac{s}{4\pi^2} \langle \bar{q}q \rangle [1 - \exp(-s/\Lambda)] + \frac{1}{8\pi^2} \langle \bar{q}g\sigma \cdot Gq \rangle [1 - \exp(-s^2/\Lambda^2)], \end{aligned} \quad (19)$$

where the subtraction factor $1 - \exp(-s^2/\Lambda^2)$ for the last pieces goes down like s^2 as $s \rightarrow 0$. It is introduced in view that the condensates are constant, so stronger suppression is necessary for alleviating their impact on the spectral densities $\rho_{A,B}(s)$ in the small s region. The variable changes $x = k^2/\Lambda$ and $y = s/\Lambda$ lead to the dispersion relations for the subtracted spectral densities

$$\begin{aligned} \int_0^\infty dy \frac{\Delta\rho_A(y)}{x - y} &= \frac{1}{64\pi^4} \int_0^\infty dy \frac{y^2 e^{-y}}{x - y} + \frac{\langle \alpha_s G^2 \rangle}{32\pi^3 \Lambda^2} \int_0^\infty dy \frac{e^{-y^2}}{x - y} + \frac{2}{3x\Lambda^3} \kappa \langle \bar{q}q \rangle^2, \\ \int_0^\infty dy \frac{\Delta\rho_B(y)}{x - y} &= -\frac{\langle \bar{q}q \rangle}{4\pi^2 \Lambda^{3/2}} \int_0^\infty dy \frac{y e^{-y}}{x - y} - \frac{\langle \bar{q}g\sigma \cdot Gq \rangle}{8\pi^2 \Lambda^{5/2}} \int_0^\infty dy \frac{e^{-y^2}}{x - y}, \end{aligned} \quad (20)$$

where the arbitrary scale Λ , substituted for the arbitrary cutoff R in Eq. (17), has migrated into the condensate terms to make them dimensionless. Note that the dimensionless unknown functions $\Delta\rho_A(y)$ and $\Delta\rho_B(y)$ stand for $\Delta\rho_A(s, \Lambda)/\Lambda^2$ and $\Delta\rho_B(s, \Lambda)/\Lambda^{5/2}$, respectively.

We then solve for $\Delta\rho_{A,B}(y)$ from Eq. (20) with the inputs

$$\begin{aligned} \langle \bar{q}q \rangle &= -(0.215)^3 \text{ GeV}^3, & \langle \alpha_s G^2 \rangle &= 0.028 \text{ GeV}^2, & \langle \bar{q}g\sigma \cdot Gq \rangle &= m_0^2 \langle \bar{q}q \rangle, \\ m_0^2 &= 1.05 \text{ GeV}^2, & \kappa &= 2. \end{aligned} \quad (21)$$

The quark, gluon, and four-quark condensates, being universal, take the same values as in Eq. (9). The additional parameter m_0^2 associated with the quark-gluon condensate $\langle \bar{q}g\sigma \cdot Gq \rangle$ is set to a value around the upper bound of the range quoted in [39]. However, difficulty is encountered immediately; no solutions, which are stable against the variation of the polynomial number N , are found in a baryon case, indicating nontrivial disparity between the dispersive analyses on meson and baryon systems.

The reason why the procedure outlined in the previous section does not work here will be elaborated. There are two corrections with different powers of $1/x$ in the OPE for the ρ meson case on the right-hand side of Eq. (4), i.e., the $1/x^2$ piece proportional to the quark and gluon condensates and the $1/x^3$ piece proportional to the four-quark condensate. Their effects on the subtracted spectral density $\Delta\rho(y)$ are opposite; the combination of the quark and gluon condensates has a negative coefficient, and the inverse matrix elements $(M^{-1})_{n2}$ are negative, such that their products contribute to $\Delta\rho(y)$ constructively. The four-quark condensate has a negative coefficient and contributes destructively, after being multiplied by the positive inverse matrix elements $(M^{-1})_{n3}$. We have $|(M^{-1})_{n2}| > |(M^{-1})_{n3}|$ for small n , so the $1/x^2$ effect dominates. As n increases, $|(M^{-1})_{n3}|$ rises much faster than $|(M^{-1})_{n2}|$, and the $1/x^3$ term becomes important gradually. When the two pieces are comparable to each other, results for $\Delta\rho(y)$ are stabilized. This observation is in line with the one in [8] that both the $1/x^2$ and $1/x^3$ contributions are necessary for the emergence of the ρ resonance. In the proton case there is only one power correction, the $1/x$ term, in the first formula of Eq. (20). Therefore, its effect cannot be moderated by others, and no stable solutions in N for the subtracted spectral density exist.

We switch to a new strategy. Given a Λ value, we obtain a solution for $\rho_A(s)$ for a polynomial number N , extract the proton mass squared M_p^2 from the location of the first peak in s , and compute the pole residue squared λ_p^2 according to Eq. (12), i.e., the area under the first peak of $\Delta\rho_A(s, \Lambda)$. The spectral density $\rho_A(s)$ corresponding to the considered Λ and N must respect the requirement of positivity, of course. The resultant set of $\{M_p, \lambda_p\}$ formed by scanning the number N represents a curve on the $M_p - \lambda_p$ plane as shown in Fig. 3, where the discrete points correspond to various integers N . The solutions for $\rho_B(s)$ produce another curve on the $M_p - \lambda_p$ plane. Note that Eq. (12) for $\Delta\rho_B(s, \Lambda)$ gives rise to $\lambda_p^2 M_p$ according to Eq. (15). The above two curves intersect each other, specifying unique

values of M_p and λ_p , which are regarded as the solution for the given Λ . The intersections of the curves from $\rho_A(s)$ and $\rho_B(s)$ for $\Lambda = 2, 3$, and 4 GeV^2 are evident in Fig. 3. We remark that the intersection, i.e., the solution is labeled by a pair of noninteger N 's, strictly speaking, one from $\rho_A(s)$ and the other from $\rho_B(s)$. It differs from the ρ meson case, where a solution is labeled by a single N .

It is encouraging that the intersections of the two curves from the chiral-even and chiral-odd dispersion relations about $M_p = 0.93 \text{ GeV}$ and $\lambda_p = 0.023 \text{ GeV}^3$ are insensitive to the variation of the arbitrary scale Λ . That is, a stability window in Λ does exist for the proton static properties, which is indicated by the plateaus of M_p and λ_p in the interval $[2 \text{ GeV}^2, 4 \text{ GeV}^2]$ of Λ in Fig. 4. We conclude the proton mass and pole residue

$$\begin{aligned} M_p &= (0.936 \pm 0.005) \text{ GeV}, \\ \lambda_p &= (0.0232 \pm 0.0013) \text{ GeV}^3, \end{aligned} \quad (22)$$

where the former matches the data in [29], and the latter is lower than $\lambda_p = 0.0346 \text{ GeV}^3$ in [39]. The pole residue, changing by about 10% within the stability window, discloses a slight dependence on Λ . The similar trend of the ρ meson decay constant f_ρ derived from Eq. (12) has been also observed. This Λ dependence is not unexpected [9], since Eq. (12) is not a rigorous definition. For instance, truncating the integration variable s at the first valley of the subtracted spectral density is somewhat discretionary. In summary, imposing the dispersive constraints from the chiral-even and chiral-odd parts of the correlator at the same time enforces the balance among all the available condensate contributions, such that a physical solution can be established.

At last, we compare our approach to the conventional treatment of QCD sum rules. One applies the Borel transformation to the chiral-even and chiral-odd dispersion relations, and then takes their ratio to cancel the pole residue, yielding the proton mass [39]

$$M_p(M_B^2) = \frac{-\frac{\langle \bar{q}q \rangle}{4\pi^2} M_B^4 - \frac{\langle \bar{q}g\sigma \cdot Gq \rangle}{8\pi^2} M_B^2}{\frac{M_B^6}{32\pi^4} + \frac{2}{3}\kappa \langle \bar{q}q \rangle^2}, \quad (23)$$

with the Borel mass M_B . In principle, one ought to look for a stability window in M_B , within which the proton mass M_p remains constant roughly; a physical quantity should not depend on the arbitrary Borel mass. However, it is apparent that M_p changes with M_B dramatically, as

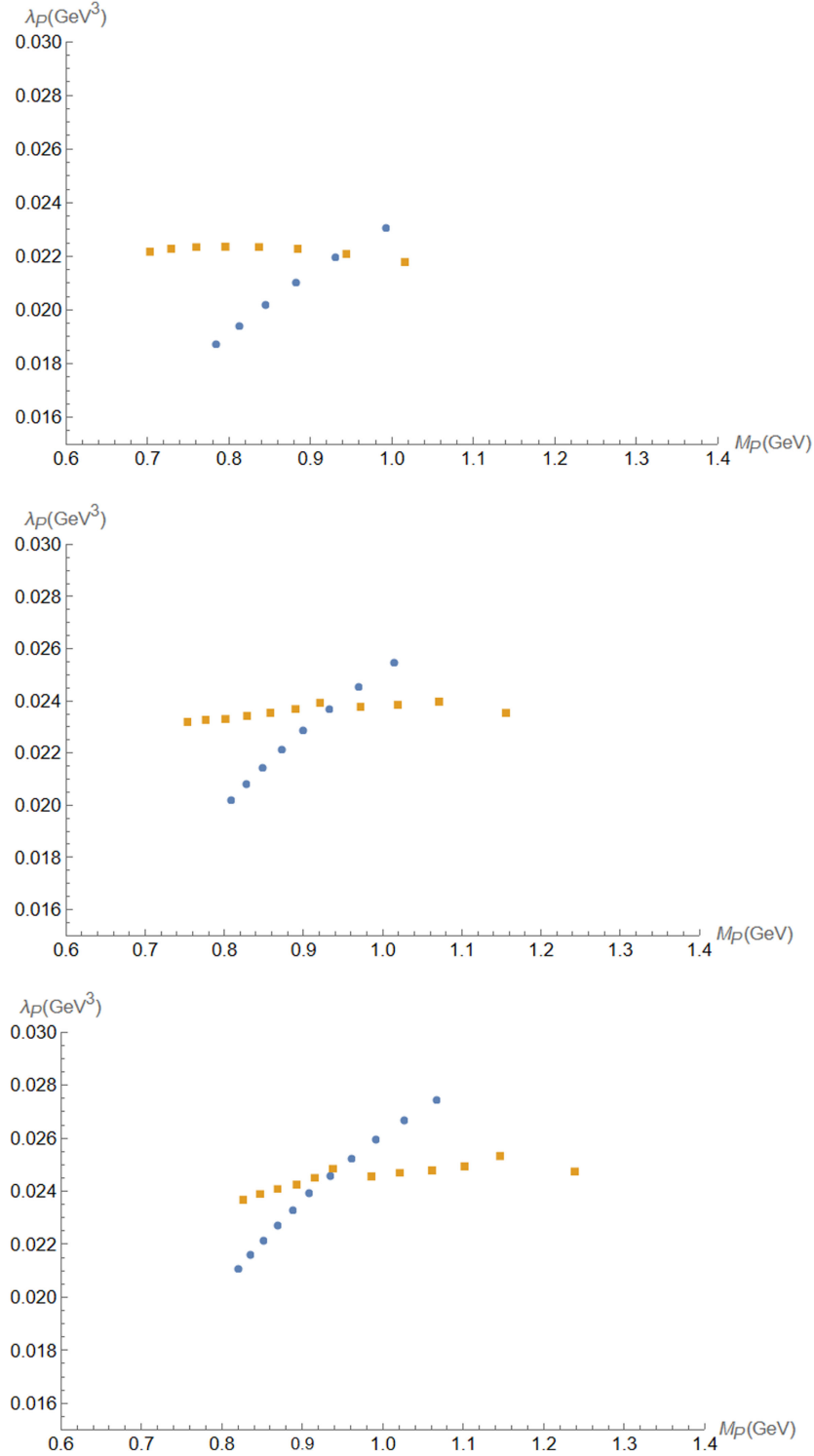
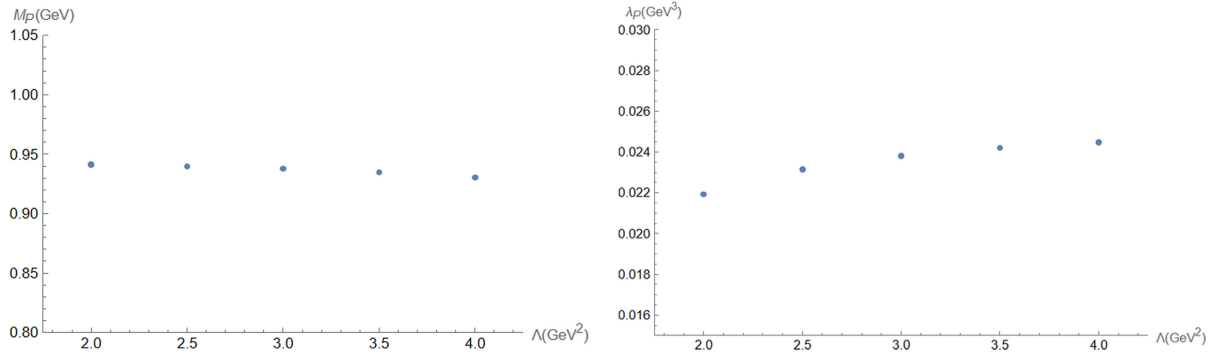
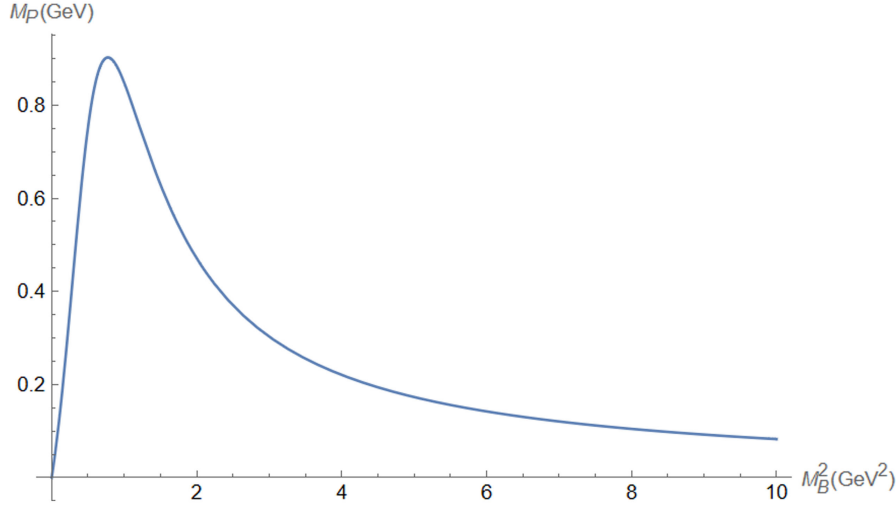


FIG. 3. Proton masses M_p and pole residues λ_p derived from the chiral-even dispersion relation (dots) and the chiral-odd dispersion relation (squares) for $\Lambda = 2, 3$, and 4 GeV^2 from top to bottom.

revealed in Fig. 5 with the inputs in Eq. (21). The author of [39] had to assume the accidental equality of the Borel mass and the proton mass in order to get an appropriate result. Alternatively, one selects a window in the Borel mass by allocating certain weights to the OPE continuum

contribution and condensates [42,43]. As a contrast, our predictions for the proton mass and pole residue manifest satisfactory stability against the variation of Λ in Fig. 4. That is, there is little ambiguity in making predictions in our formalism.

FIG. 4. Dependencies of the proton mass M_P and pole residue λ_P on Λ .FIG. 5. Dependence of the proton mass M_P on the Borel mass M_B in conventional QCD sum rules with the inputs in Eq. (21).

B. $\Delta(1232)$ mass and pole residue

The generalization to the investigation of a $\Delta(1232)$ state in the decuplet is straightforward. We just need to consider the $g^{\mu\nu}$ structure in the corresponding two-point correlation function

$$\begin{aligned} \Pi^{\mu\nu}(k^2) &= i \int d^4x e^{ik \cdot x} \langle 0 | T \{ \eta^\mu(x) \bar{\eta}^\nu(0) \} | 0 \rangle \\ &= -g^{\mu\nu} [A(k^2)k + B(k^2)] + \dots, \end{aligned} \quad (24)$$

with the current $\eta^\mu(x) = \epsilon^{abc} [u^{aT}(x) C \gamma^\mu u^b(x)] u^c(x)$, the chiral-even part $A(k^2)$, and the chiral-odd part $B(k^2)$. We adopt Ioffe's results for the OPEs [39]

$$\begin{aligned} \Pi_A(k^2) &= \frac{k^4}{160\pi^4} \ln\left(\frac{-k^2}{\mu^2}\right) - \frac{4}{3k^2} \kappa \langle \bar{q}q \rangle^2, \\ \Pi_B(k^2) &= -\left[\frac{k^2 \langle \bar{q}q \rangle}{3\pi^2} - \frac{\langle \bar{q}g\sigma Gq \rangle}{24\pi^2} \right] \ln\left(\frac{-k^2}{\mu^2}\right). \end{aligned} \quad (25)$$

Lee's expressions [44] agree with Ioffe's except the coefficient of the quark-gluon condensate,

$$\begin{aligned} \Pi'_A(k^2) &= \frac{k^4}{160\pi^4} \ln\left(\frac{-k^2}{\mu^2}\right) - \frac{4}{3k^2} \kappa \langle \bar{q}q \rangle^2, \\ \Pi'_B(k^2) &= -\left[\frac{k^2 \langle \bar{q}q \rangle}{3\pi^2} - \frac{\langle \bar{q}g\sigma Gq \rangle}{6\pi^2} \right] \ln\left(\frac{-k^2}{\mu^2}\right). \end{aligned} \quad (26)$$

We repeat the OPE calculation and find that Eq. (25) is favored. In fact, we cannot construct physical solutions to the spectral densities from Eq. (26) under the requirement of positivity, because the quark-gluon condensate gives a sizable destructive contribution.

We define the subtracted spectral densities

$$\begin{aligned} \Delta\rho_A(s, \Lambda) &= \rho_A(s) - \frac{s^2}{160\pi^4} [1 - \exp(-s/\Lambda)], \\ \Delta\rho_B(s, \Lambda) &= \rho_B(s) + \frac{s}{3\pi^2} \langle \bar{q}q \rangle [1 - \exp(-s^2/\Lambda^2)] \\ &\quad - \frac{1}{24\pi^2} \langle \bar{q}g\sigma \cdot Gq \rangle [1 - \exp(-s^3/\Lambda^3)]. \end{aligned} \quad (27)$$

The spectral densities associated with a heavier Δ are expected to have softer behaviors near the origin $s = 0$.

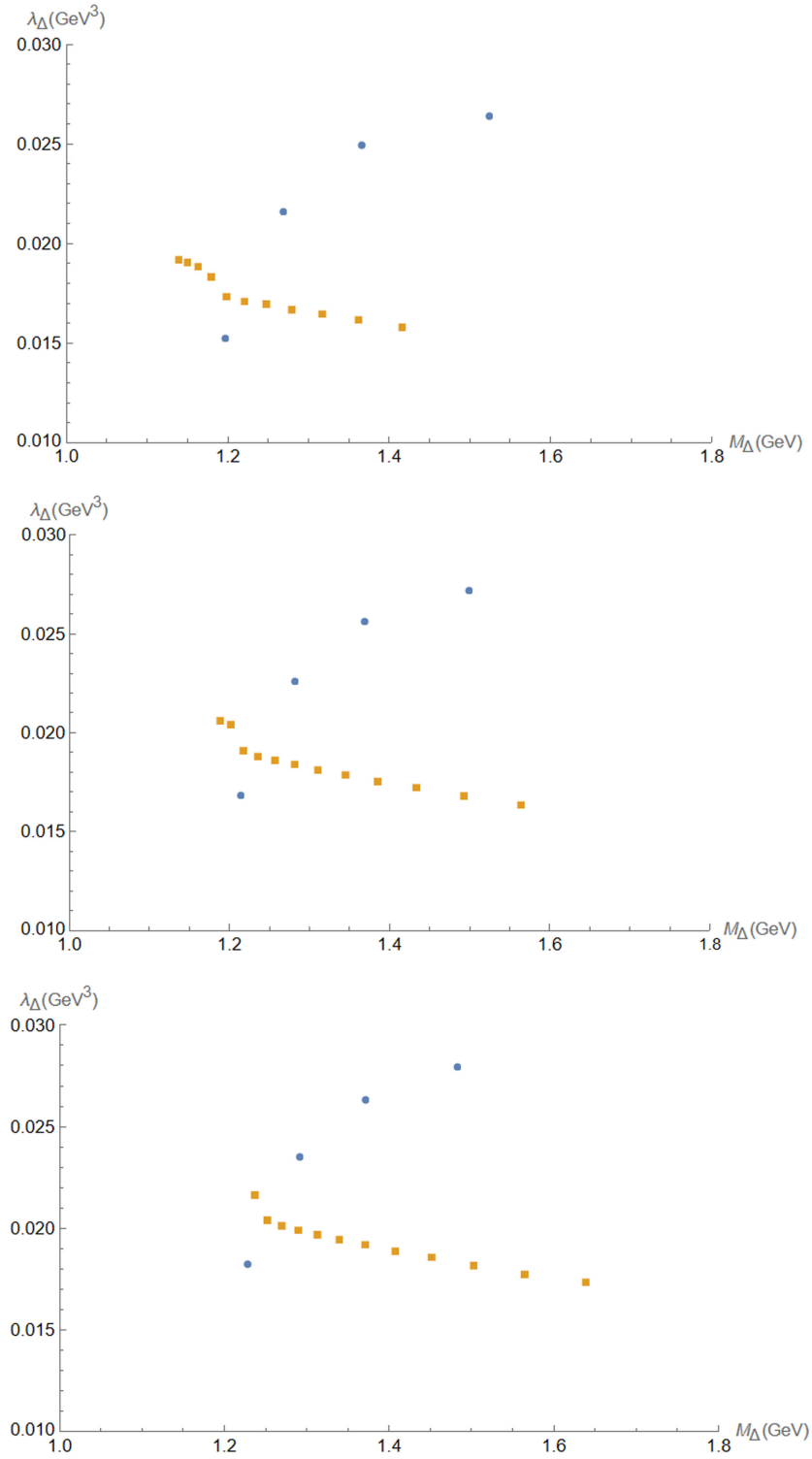


FIG. 6. $\Delta(1232)$ masses M_Δ and pole residues λ_Δ derived from the chiral-even dispersion relation (dots) and the chiral-odd dispersion relation (squares) for $\Lambda = 1.8, 2.0$, and 2.2 GeV^2 from top to bottom.

Hence, we employ the subtraction factors that fall faster as $s \rightarrow 0$ than in the proton case. For example, the constant quark-gluon condensate is suppressed by $1 - \exp(-s^3/\Lambda^3) \propto s^3$ to lessen its impact on the spectral density $\rho_B(s)$ at low s . The dispersion relations for the subtracted spectral densities read

$$\begin{aligned} \int_0^\infty dy \frac{\Delta\rho_A(y)}{x-y} &= \frac{1}{160\pi^4} \int_0^\infty dy \frac{y^2 e^{-y}}{x-y} + \frac{4}{3x\Lambda^3} \kappa \langle \bar{q}q \rangle^2, \\ \int_0^\infty dy \frac{\Delta\rho_B(y)}{x-y} &= -\frac{\langle \bar{q}q \rangle}{3\pi^2 \Lambda^{3/2}} \int_0^\infty dy \frac{y e^{-y^2}}{x-y} + \frac{\langle \bar{q}g\sigma \cdot Gq \rangle}{24\pi^2 \Lambda^{5/2}} \int_0^\infty dy \frac{e^{-y^3}}{x-y}. \end{aligned} \quad (28)$$

Inputting the same parameters as in Eq. (21) for the proton case, we present in Fig. 6 the curves from $\rho_A(s)$ and $\rho_B(s)$ on the $M_\Delta - \lambda_\Delta$ plane for $\Lambda = 1.8, 2.0$, and 2.2 GeV^2 . Note that we infer a resonance mass through the subtracted spectral density $\Delta\rho_B(s, \Lambda)$ actually, because its peaks are more prominent than those of $\rho_B(s)$ after the subtraction in Eq. (27). A similar situation has occurred in the dispersive analyses on glueball masses [9]. This alternative affects only a little bit the results for the $\Delta(1232)$ baryon. Again, the two curves in Fig. 6 intersect each other, designating unique values of M_Δ and λ_Δ for each Λ . The dependencies of M_Δ and λ_Δ on Λ are similar to those in Fig. 4 with acceptable stability, so they are not displayed for simplicity. The difference is that the stability window in Λ is narrower in the $\Delta(1232)$ case than in the proton one. We deduce the $\Delta(1232)$ mass and pole residue from Fig. 6,

$$M_\Delta = (1.238 \pm 0.018) \text{ GeV}, \quad \lambda_\Delta = (0.019 \pm 0.002) \text{ GeV}^3, \quad (29)$$

where the former is consistent with the data in [29], and the latter is lower than but of the same order as $\lambda_\Delta = 0.05 \text{ GeV}^3$ in [39].

After completing all the formulas, we are ready to explain how to choose the inputs for the condensates involved in this work. We fix the factorization violation parameter for the four-quark condensate to $\kappa = 2$ for an illustration. The OPEs associated with a $\Delta(1232)$ baryon are insensitive to the quark-gluon condensate owing to the tiny coefficient $1/(24\pi^2)$ and independent of the gluon condensate. The quark condensate is the most crucial for $\Delta(1232)$ properties and thus is set first. Second, the quark-gluon condensate does not contribute in the ρ meson case, so the gluon condensate is tuned to match the ρ meson data without altering the $\Delta(1232)$ outcomes. Finally, the parameter m_0^2 in the parametrization of the quark-gluon condensate is adjusted to account for proton observables with negligible modification of the $\Delta(1232)$ results. All the values for the above condensates are within their allowed ranges known in the literature. Therefore, the point of our analysis resides in the possibility of establishing unambiguous and stable solutions to the dispersion relations for meson and baryon systems with universal and reasonable condensate inputs.

IV. CONCLUSION

We have extended the dispersive studies on meson static properties to those of light baryons. The challenge arises from the structure of a baryonic correlation function distinct from a mesonic one; the former is split into a chiral-even part and a chiral-odd part, such that the number of power corrections for each chirality is insufficient, and a physical solution cannot be identified from each dispersion relation. To surmount the difficulty, we implemented the chiral-even and chiral-odd dispersive constraints simultaneously, and found the common solutions for the baryon masses and pole residues. It has been corroborated that the obtained solutions are stable against the variation of the arbitrary scale Λ , which traces back to the ultraviolet cutoff for the dispersive integrals. We have also commented on the conventional handling of QCD sum rules; it is hard to identify a stability window in the Borel mass in the proton mass sum rule. Our approach is thus superior to the conventional one. Motivated by the universality of condensates, we have demonstrated that the same set of quark and gluon condensates can accommodate the masses of a ρ meson, a proton, and a $\Delta(1232)$ baryon.

This work will stimulate broad applications of our nonperturbative analytical framework not only to mesonic but baryonic quantities. We will apply it to the determination of more complicated baryon light-cone distribution amplitudes, which are the essential nonperturbative inputs to the factorization theorem for high-energy baryonic processes and heavy baryon decays, but remain quite uncertain [45]. Besides, a global fit of various quark and gluon condensates to relevant data can be done, and fit results, based on their universality, can be employed to make predictions for other hadronic observables. The agreement of fit results with those evaluated in lattice QCD will offer a convincing support to our approach.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [29].

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