Letter

# Inequality for relativistic local quantum measurements

Riccardo Falcone and Claudio Conti

*Department of Physics, University of Sapienza, Piazzale Aldo Moro 5, 00185 Rome, Italy*

(Received 20 September 2025; accepted 3 March 2026; published 23 March 2026)

We investigate the trade-off between vacuum insensitivity and sensitivity to excitations in finite-size detectors, taking measurement locality as a fundamental constraint. We derive an upper bound on the detectability of vacuum excitation, given a small but nonzero probability of false positives in the vacuum state. The result is independent of the specific details of the measurement or the underlying physical mechanisms of the detector and relies only on the assumption of locality. Experimental confirmation or violation of the inequality would provide a test of the axioms of algebraic quantum field theory, offer new insights into the measurement problem in relativistic quantum physics, and establish a fundamental technological limit in local particle detection.

DOI: 10.1103/ydmg-5vjm

*Introduction.* In relativistic quantum field theory, the vacuum is globally defined as the absence of particles. Yet, when examined locally, it exhibits intense activity, with fluctuations and correlations that make it far from empty [1–7]. A key consequence is that no local measurement can yield a strictly positive outcome in the vacuum [4]: one cannot construct a pair of local projectors that would correspond to a simple yes/no answer to the question, "Is there no excitation from the vacuum in this region?" This clashes with the intuitive view that detectors register only local excitations while remaining inert on the vacuum.

In this work, we explore the hypothesis that real finite-size detectors deviate from ideal behavior with respect to the vacuum, in the sense that their vacuum response is small but nonzero. We show that suppressing false positives in the vacuum necessarily imposes constraints on the measurement outcomes for nonvacuum states. Physically, this trade-off can be understood as a consequence of vacuum fluctuations: a detector finely tuned to suppress these fluctuations will inevitably suppress signals from genuine excitations. We thus unveil a fundamental correlation between responsivity (i.e., the efficiency in converting particles into signals) and dark counts (or false positives), stemming directly from the requirement of locality in the underlying quantum measurement.

Concretely, we derive an explicit upper bound on the expectation value of a local measurement in an arbitrary excited state, given its expectation value in the vacuum. As an application, we consider scalar-field coherent states localized within the causal completion of the detector region. We obtain a closed-form expression for the bound and compare it with the idealized case of detectors extended over the whole spacetime.

It is widely acknowledged in the literature that any local detector inevitably exhibits dark counts and that a fundamental trade-off exists between detector reliability and its localizability [8]. What is novel in the present analysis is the quantitative assessment of *how small* one can make the vacuum click rate at the cost of weakening detection of other states.

The bound we obtain follows directly from the Reeh-Schlieder theorem [1,9], which is grounded in the axioms of algebraic quantum field theory (AQFT) and the principle of locality in relativistic quantum theory [9]. Our results therefore offer a quantitative and falsifiable test of these foundations.

To date, AQFT provides the most rigorous formulation of localization, fully incorporating the principles of quantum mechanics, locality, and relativistic causality. Yet ideal measurements (projective onto an eigenbasis) within finite spacetime regions remain problematic: local algebras lack minimal projections and normal pure states [10], and such idealized measurements have been shown to enable superluminal signaling [11–13]. While several alternative schemes exist [14–18], we pursue a simpler hypothesis: local measurements are modeled by infinite-rank projections or by local positive operator-valued measures (POVMs). The inequality we derive provides an experimental test of this assumption.

Our results may also help clarify the degree of locality involved in laboratory quantum measurements. In particular, they address the question of whether the observable associated with a given measurement is confined to a

*Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.*

localized region at the detector or represents a more distributed process involving the entire experimental setup. Since the derived bounds depend explicitly on the support of the measurement operator, they provide a means to empirically probe the operational scale at which quantum measurements occur.

No detailed detector modeling is assumed. The result is fully general, relying only on the measurement operator being a local POVM element, and it culminates in a single inequality derived under minimal assumptions. Such inequalities are particularly valued in quantum foundations [19–23] because they offer a clear and testable criterion: any empirical violation indicates a breakdown in the underlying theoretical model. If the inequality we present is violated, it could indicate either that the AQFT description of local measurements via localized projections and POVMs is incorrect, or that the presumed localization region must be enlarged, potentially to the entire apparatus or laboratory.

*Ideal and quasi-ideal detectors.* Consider a detector that clicks in response to excitations above the vacuum. We model its measurement by a projection-valued measure, or more generally, a POVM of the form $\{\hat{E}_{\rm click}, \mathbb{I} - \hat{E}_{\rm click}\}$, where a click corresponds to a positive outcome associated with the operator $\hat{E}_{\rm click}$. We assume the detector operates within a fixed spacetime region $\mathcal{O}_{\rm det}$. In accordance with the principles of AQFT, the measurement operator $\hat{E}_{\rm click}$ must be an element of the local algebra associated with this region, i.e., $\hat{E}_{\rm click} \in \mathfrak{A}(\mathcal{O}_{\rm det})$.

Hereafter, we denote $P_{\rm click} = \langle\psi|\hat{E}_{\rm click}|\psi\rangle$ and $P_{\rm dark} = \langle\Omega|\hat{E}_{\rm click}|\Omega\rangle$ as the probability that the detector registers a click in an arbitrary state $|\psi\rangle$ and in the vacuum state $|\Omega\rangle$, respectively. In line with the conventional understanding of particle detectors, an idealized detector is expected to satisfy the following two properties:

(i) Vacuum insensitivity: The detector should not respond when the system is in the vacuum state, meaning the probability of a dark count should be zero: $P_{\rm dark} = 0$.

(ii) Sensitivity to some state: There exists at least one state $|\psi\rangle$ for which the detector registers a click with nonzero probability, i.e., $P_{\rm click} > 0$.

We define a detector as *ideal* if it satisfies both properties (i) and (ii) described above [24]. In practice, realistic detectors with finite spatial extent cannot simultaneously fulfill both conditions. This limitation arises from the locality of the operator $\hat{E}_{\rm click}$ and a corollary of the Reeh-Schlieder theorem, which states that any positive semidefinite local operator with vanishing vacuum expectation value must be identically zero (i.e., the vacuum is separating for any local algebra) [4]. Therefore, if a detector satisfies property (i)—that is, it never clicks in the vacuum—then it must be trivial and cannot satisfy property (ii), which requires a nonzero response for some state.

This no-go result relies on the locality assumption for $\hat{E}_{\rm click}$. If this locality condition is lifted, then ideal measurements satisfying both properties (i) and (ii) can indeed exist. A notable example is the excitation projector $\hat{P}^{\perp}_{\Omega} = \hat{\mathbb{I}} - |\Omega\rangle\langle\Omega|$, which projects onto the subspace orthogonal to the vacuum. By construction, this operator has zero expectation value in the vacuum and unit expectation value in any state orthogonal to it, thereby satisfying both ideality properties. However, $\hat{P}^{\perp}_{\Omega}$ is not an element of any local algebra: it cannot represent a measurement performed by a finite-size detector.

We conjecture that the detectors employed in real experiments are not truly ideal, but rather quasi-ideal, in the sense that they only approximately fulfill properties (i) and (ii). Their response to the vacuum is small but not strictly zero, meaning that $0 < P_{\rm dark} \ll 1$. The smaller the value of $P_{\rm dark}$ is, the closer the detector is to being effectively ideal with respect to property (i). However, property (i) is fundamentally incompatible with property (ii). This leads to a natural trade-off: as a detector becomes more vacuum-insensitive (i.e., as $P_{\rm dark}$ decreases), its ability to detect excitations ($P_{\rm click}$) also diminishes. In other words, a detector that responds less to the vacuum tends to be less responsive to all states. In what follows, we quantify this trade-off by deriving an upper bound on $P_{\rm click}$, under the assumption that $P_{\rm dark} \neq 0$.

*Upper bound for $P_{\rm click}$.* The Reeh-Schlieder theorem asserts that any state $|\psi\rangle$ can be approximated arbitrarily well by applying a local operator to the vacuum state $|\Omega\rangle$ (i.e., the vacuum is cyclic for any local algebra) [1,9]. That is, for every $|\psi\rangle$ and spacetime region $\mathcal{O}$, there exists a family of operators $\hat{A}_\zeta$ localized in $\mathcal{O}$ such that the approximation error

$$\mathcal{E}_\zeta = \left\| |\psi\rangle - \hat{A}_\zeta|\Omega\rangle \right\| \tag{1}$$

tends to zero as $\zeta \to 0$.

We make use of this property of the vacuum by selecting $\mathcal{O} = \mathcal{O}'_{\rm det}$, where $\mathcal{O}'_{\rm det}$ denotes the causal complement of the detector region $\mathcal{O}_{\rm det}$. This choice ensures that $\hat{A}_\zeta$ is an element of the local algebra $\mathfrak{A}(\mathcal{O}'_{\rm det})$, and by microcausality [25], it follows that $\hat{A}_\zeta$ commutes with $\hat{E}_{\rm click}$.

Since $\hat{E}_{\rm click}$ is a positive operator, it admits a well-defined square root $\hat{E}^{1/2}_{\rm click}$ such that $P_{\rm click} = \|\hat{E}^{1/2}_{\rm click}|\psi\rangle\|^2$. By applying the triangle inequality to this norm, we obtain

$$P_{\rm click} \leq \left[ \left\| \hat{E}^{1/2}_{\rm click} \left( |\psi\rangle - \hat{A}_\zeta|\Omega\rangle d) \right\| + \left\| \hat{E}^{1/2}_{\rm click} \hat{A}_\zeta |\Omega\rangle \right\| \right]^2. \tag{2}$$

Since $\hat{E}^{1/2}_{\rm click}$ and $\hat{A}_\zeta$ commute, we can further write

$$P_{\rm click} \leq \left[ \left\| \hat{E}^{1/2}_{\rm click} \left( |\psi\rangle - \hat{A}_\zeta|\Omega\rangle \right) \right\| + \left\| \hat{A}_\zeta \hat{E}^{1/2}_{\rm click} |\Omega\rangle \right\| \right]^2. \tag{3}$$

We now invoke the definition of the operator norm, $\|\hat{O}\| = \sup_{|\psi\rangle\neq 0}\|\hat{O}|\psi\rangle\|/\||\psi\rangle\|$, which holds for any bounded operator $\hat{O}$. This gives the following inequalities:

$$\left\|\hat{E}^{1/2}_{\text{click}}\left(|\psi\rangle - \hat{A}_\zeta|\Omega\rangle\right)\right\| \leq \left\|\hat{E}^{1/2}_{\text{click}}\right\|\left\||\psi\rangle - \hat{A}_\zeta|\Omega\rangle\right\| \quad (4)$$

and

$$\left\|\hat{A}_\zeta\hat{E}^{1/2}_{\text{click}}|\Omega\rangle\right\| \leq \left\|\hat{A}_\zeta\right\|\left\|\hat{E}^{1/2}_{\text{click}}|\Omega\rangle\right\|. \quad (5)$$

Since $\hat{E}_{\text{click}}$ is a POVM element, we have $\|\hat{E}^{1/2}_{\text{click}}\| \leq 1$. Additionally, by using the definitions $\mathcal{E}_\zeta = \||\psi\rangle - \hat{A}_\zeta|\Omega\rangle\|$ and $P_{\text{dark}} = \langle\Omega|\hat{E}_{\text{click}}|\Omega\rangle = \|\hat{E}^{1/2}_{\text{click}}|\Omega\rangle\|^2$, we obtain

$$\left\|\hat{E}^{1/2}_{\text{click}}\left(|\psi\rangle - \hat{A}_\zeta|\Omega\rangle\right)\right\| \leq \mathcal{E}_\zeta \quad (6)$$

and

$$\left\|\hat{A}_\zeta\hat{E}^{1/2}_{\text{click}}|\Omega\rangle\right\| \leq \left\|\hat{A}_\zeta\right\|\sqrt{P_{\text{dark}}}. \quad (7)$$

By applying the bounds from Eqs. (6) and (7) to Eq. (3), we obtain the following inequality:

$$P_{\text{click}} \leq \left(\mathcal{E}_\zeta + \left\|\hat{A}_\zeta\right\|\sqrt{P_{\text{dark}}}\right)^2. \quad (8)$$

To gain further insight into this bound, consider a state $|\psi\rangle$ prepared by applying a unitary operator $\hat{U}$ to the vacuum $|\Omega\rangle$, i.e., $|\psi\rangle = \hat{U}|\Omega\rangle$. If $\hat{U}$ is localized in the causal complement of the detector region, i.e., $\hat{U} \in \mathfrak{A}(\mathcal{O}'_{\text{det}})$, then, in accordance with relativistic causality, the detector cannot distinguish the state $|\psi\rangle$ from the vacuum, and thus $P_{\text{click}} = P_{\text{dark}}$ [26]. In this case, the bound in Eq. (8) becomes uninformative, offering no further constraint on the relationship between $P_{\text{click}}$ and $P_{\text{dark}}$.

To obtain a nontrivial bound, we instead consider the case where $\hat{U}$ is localized in the causal completion of the detector region, i.e., $\hat{U} \in \mathfrak{A}(\mathcal{O}''_{\text{det}})$, with $\mathcal{O}''_{\text{det}} = (\mathcal{O}'_{\text{det}})'$ denoting the double causal complement of $\mathcal{O}_{\text{det}}$. This setup describes a scenario in which an agent, confined to the causal completion of the detector, locally prepares the excitation $|\psi\rangle$. Even though $\hat{A}_\zeta$ and $\hat{U}$ are localized in spacelike-separated regions, the state $\hat{A}_\zeta|\Omega\rangle$ can still approximate $\hat{U}|\Omega\rangle$ to arbitrary precision $\mathcal{E}_\zeta$. As Haag pointed out [9], this is due to the vacuum exhibiting weak but nonzero correlations across arbitrarily large spacelike separations. Improving the approximation (i.e., smaller $\mathcal{E}_\zeta$) comes at the cost of increasing the operator norm of $\hat{A}_\zeta$ [28]. As a result, in Eq. (8), a small value of $\mathcal{E}_\zeta$ is necessarily compensated by a large norm $\|\hat{A}_\zeta\|$.

Crucially, while $P_{\text{dark}}$ is fixed by the physical properties of the detector, the parameter $\zeta$ remains free to choose. This flexibility allows one to optimize the bound in Eq. (8) by minimizing its right-hand side over all positive values of $\zeta$, leading to

$$P_{\text{click}} \leq \min_{\zeta>0}\left(\mathcal{E}_\zeta + \left\|\hat{A}_\zeta\right\|\sqrt{P_{\text{dark}}}\right)^2. \quad (9)$$

Because the Reeh-Schlieder family $\{\hat{A}_\zeta\}$ is not unique, this bound is suboptimal rather than globally optimal. Nonetheless, it already provides a concrete experimentally accessible inequality. A further optimization over all possible Reeh-Schlieder families would merely sharpen the bound; for many purposes, however, an explicit suboptimal estimate is sufficient and remains of clear physical relevance.

This inequality clearly shows that fulfilling the ideality condition (i) (vacuum insensitivity) inevitably leads to a violation of the ideality condition (ii) (sensitivity to certain states). If $P_{\text{dark}}$ is set to zero, meaning the detector is entirely insensitive to the vacuum, then the minimum occurs at $\zeta = 0$, where $\mathcal{E}_\zeta = 0$. This leads to the strongest possible constraint on $P_{\text{click}}$, namely $P_{\text{click}} = 0$, which implies that the detector is also completely insensitive to the state $|\psi\rangle$.

In the nonideal case where $P_{\text{dark}}$ is different from zero, the term $\mathcal{E}_\zeta$ in the sum $\mathcal{E}_\zeta + \|\hat{A}_\zeta\|\sqrt{P_{\text{dark}}}$ is always compensated by the second term, which diverges as $\zeta \to 0$. As a result, the minimum of the sum is achieved through an optimal trade-off between these terms, leading to a bound that remains strictly greater than zero.

We remark that inequality (9) is experimentally testable. In a hypothetical experimental setup, one only needs to measure the click probability in a nonvacuum state, $P_{\text{click}}$, and compare it with the dark count probability, $P_{\text{dark}}$. The experimental data can then be confronted with the theoretical bound (9), provided that the detector spacetime region $\mathcal{O}_{\text{det}}$ and the state $|\psi\rangle$ are properly modeled.

*Scalar-field coherent states.* To illustrate with a concrete example, we consider a Klein-Gordon field $\hat{\phi}$ and a coherent state of the form $|f\rangle = \hat{W}(f)|\Omega\rangle$, where $\hat{W}(f) = \exp[i\hat{\phi}(f)]$ is the Weyl operator associated with the test function $f$, and $\hat{\phi}(f)$ denotes the field operator smeared with $f$. We assume that $\hat{W}(f)$ is localized within the region $\mathcal{O}''_{\text{det}}$, thereby excluding the trivial case where the state is prepared unitarily in $\mathcal{O}'_{\text{det}}$. To ensure this localization, we impose that the support of $f$ is entirely contained within $\mathcal{O}''_{\text{det}}$, i.e., $\text{supp}(f) \subseteq \mathcal{O}''_{\text{det}}$.

The Reeh-Schlieder theorem guarantees the existence of a family of operators $\hat{A}_\zeta(f)$, localized in the region $\mathcal{O}'_{\text{det}}$, such that the coherent state $|f\rangle$ can be approximated by $\hat{A}_\zeta(f)|\Omega\rangle$, with approximation error

$$\mathcal{E}_\zeta(f) = \left\||f\rangle - \hat{A}_\zeta(f)|\Omega\rangle\right\| \quad (10)$$

vanishing as $\zeta \to 0$. In our earlier work [29], we presented an explicit construction of $\hat{A}_\zeta(f)$ for coherent states. A brief outline is given here, with the full derivation available in [29].

We start by selecting a coordinate system $x = (x^0, x^1, x^2, x^3)$ such that the region $\mathcal{O}_{\rm det}$ is entirely contained within the left wedge $\mathcal{W}_{\rm L}$, defined by the condition $x^1 < -|x^0|$. Let $J$ denote the spacetime reflection about the $x^0$ and $x^1$ directions, $J(x^0, x^1, x^2, x^3) = (-x^0, -x^1, x^2, x^3)$, and let $\Lambda_1(\eta)$ represent a Lorentz boost along the $x^1$ direction, $\Lambda_1(\eta)(x^0, x^1, x^2, x^3) = [\cosh(\eta)x^0 + \sinh(\eta)x^1, \cosh(\eta)x^1 + \sinh(\eta)x^0, x^2, x^3]$. Based on these definitions, we construct the family of operators $\hat{A}_\zeta(f)$ as

$$\hat{A}_\zeta(f) = \int_{\mathbb{R}} d\eta G_\zeta(\eta - i\pi)\hat{W}[f \circ J \circ \Lambda_1(-\eta)], \quad (11)$$

where $G_\zeta(\eta) = \exp(-\eta^2/2\zeta)/\sqrt{2\pi\zeta}$ is a Gaussian function with variance $\zeta$. Each operator $\hat{A}_\zeta(f)$ is localized in $\mathcal{O}'_{\rm det}$, and its action on the vacuum $|\Omega\rangle$ approximates the target coherent state $|f\rangle$ with approximation error

$$\mathcal{E}_\zeta(f) = \sqrt{1 - \int_{\mathbb{R}} d\eta [2G_\zeta(\eta) - G_{2\zeta}(\eta)] \exp\{W_2[f, f \circ \Lambda_1(\eta)] - W_2(f, f)\}}, \quad (12)$$

where $W_2(f_1, f_2) = \langle\Omega|\hat{\phi}(f_1)\hat{\phi}(f_2)|\Omega\rangle$ denotes the two-point Wightman function smeared with test functions $f_1$ and $f_2$. The right-hand side of Eq. (12) converges to zero as $\zeta \to 0$, thereby ensuring that $\hat{A}_\zeta(f)|\Omega\rangle$ approximates $|f\rangle$ to arbitrary precision.

An upper bound on the operator norm of $\hat{A}_\zeta(f)$ can be obtained by applying the triangle inequality to Eq. (11):

$$\left\|\hat{A}_\zeta(f)\right\| \le \int_{\mathbb{R}} d\eta |G_\zeta(\eta - i\pi)| \left\|\hat{W}[f \circ J \circ \Lambda_1(-\eta)]\right\|. \quad (13)$$

Since every unitary operator has norm 1, it follows that $\|\hat{W}[f \circ J \circ \Lambda_1(-\eta)]\| = 1$, giving

$$\left\|\hat{A}_\zeta(f)\right\| \le \exp\left(\frac{\pi^2}{2\zeta}\right). \quad (14)$$

This bound diverges as $\zeta \to 0$, which is consistent with expectations: achieving an increasingly accurate approximation of the state $|f\rangle$ via operators localized in the causal complement $\mathcal{O}'_{\rm det}$ requires $\hat{A}_\zeta(f)$ to become unbounded in the limit $\zeta \to 0$.

We denote the probability that the detector clicks in the coherent state $|f\rangle$ as $P_{\rm click}(f) = \langle f|\hat{E}_{\rm click}|f\rangle$. By plugging Eqs. (12) and (14) into the general bound (8), with the natural identifications $P_{\rm click} \mapsto P_{\rm click}(f)$, $\mathcal{E}_\zeta \mapsto \mathcal{E}_\zeta(f)$ and $\hat{A}_\zeta \mapsto \hat{A}_\zeta(f)$, we obtain a concrete expression of the inequality for the specific case where $|\psi\rangle = |f\rangle$. The resulting bound reads as follows:

$$P_{\rm click}(f) \le \left[\mathcal{E}_\zeta(f) + \exp\left(\frac{\pi^2}{2\zeta}\right)\sqrt{P_{\rm dark}}\right]^2. \quad (15)$$

This inequality is valid for any positive value of $\zeta$. To obtain the tightest possible bound, we minimize the right-hand side of Eq. (15) with respect to $\zeta > 0$:

$$P_{\rm click}(f) \le \min_{\zeta>0}\left[\mathcal{E}_\zeta(f) + \exp\left(\frac{\pi^2}{2\zeta}\right)\sqrt{P_{\rm dark}}\right]^2. \quad (16)$$

The coordinate system $x$ can still be chosen freely, provided $\mathcal{O}_{\rm det}$ remains entirely within $\mathcal{W}_{\rm L}$. However, to sharpen the bound in Eq. (16), it is advantageous to select coordinates that optimize the estimate. A general guideline is to choose the coordinates such that $\mathcal{O}_{\rm det}$ lies as close as possible to the origin. This choice minimizes the effect of the boost $\Lambda_1(\eta)$ in Eq. (12), thereby reducing the value of $\mathcal{E}_\zeta(f)$. Physically, this corresponds to the fact that the closer is the support of $\hat{A}_\zeta(f)$ to $\mathcal{O}_{\rm det}$, the more easily $\hat{A}_\zeta(f)$ can approximate $|f\rangle$.

*A simple model.* As an illustrative toy model, we consider a detector with spherical geometry. Its associated spacetime region, $\mathcal{O}_{\rm det}$, is taken to be a four-dimensional hypersphere of radius $R_{\rm det}$. We assume that the coherent-state smearing function $f$ is supported in a concentric hypersphere of radius $R_{\rm coh} \le R_{\rm det}$. The coordinate system is chosen such that the common center of these hyperspheres is located at the spacetime point $C = (0, -\sqrt{2}R_{\rm det}, 0, 0)$, ensuring that $\mathcal{O}_{\rm det}$ lies entirely within the left wedge $\mathcal{W}_{\rm L}$. Also, we take the field $\hat{\phi}$ to be massless, thereby focusing on the strictly relativistic regime.

We model the coherent state using the smearing function

$$f(x) = \frac{\alpha}{R_{\rm coh}^3}\vartheta\left(1 - \frac{\|x - C\|}{R_{\rm coh}}\right), \quad (17)$$

where $\alpha$ is a real dimensionless amplitude, $\|\cdot\|$ is the Euclidean norm, and

$$\vartheta(s) = \frac{\Phi(s)}{\Phi(s) + \Phi(1 - s)}, \quad \Phi(s) = \begin{cases} e^{-1/s}, & \text{if } s > 0, \\ 0, & \text{if } s \le 0 \end{cases} \quad (18)$$

are auxiliary functions that ensures $f$ is smooth and supported within a hypersphere of radius $R_{\rm coh}$ centered at $C$.

We numerically evaluate the upper bound $P_{\rm click}^{(\max)}(f)$, defined as the right-hand side of Eq. (16), for different choices of $\alpha$, $R_{\rm det}/R_{\rm coh}$, $P_{\rm dark}$ [30]. Specifically, we consider $\alpha = 1/2, 1, 2$ and $R_{\rm det}/R_{\rm coh} = 1, 2$, while $P_{\rm dark}$ ranges from $10^{-10}$ and 1. Further details are provided in Supplemental Material [31]. The results, shown in the

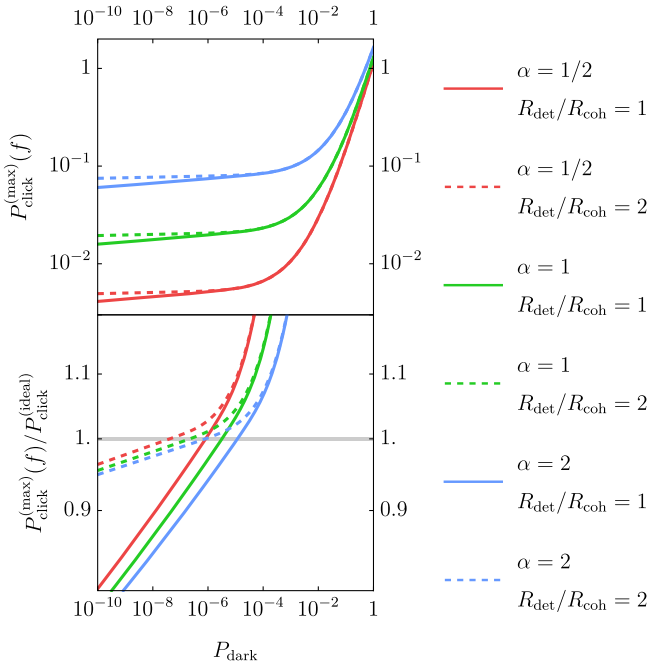



FIG. 1. Upper panel: upper bound $P_{\text{click}}^{(\max)}(f)$ on the click probability $P_{\text{click}}(f)$ as a function of the dark count $P_{\text{dark}}$. Lower panel: ratio between the upper bound $P_{\text{click}}^{(\max)}(f)$ for a finite-size detector and the click probability $P_{\text{click}}^{(\text{ideal})}$ for an idealized infinitely extended detector. Results are shown for different coherent states $|f\rangle$ and detector geometries. The smearing function $f$ is given by Eq. (17), where $\alpha$ denotes the amplitude and $R_{\text{coh}}$ the radius of the sphere supporting the coherent state. The detector is modeled as a concentric sphere of radius $R_{\text{det}}$. The plots are displayed on a log-log scale.

upper panel of Fig. 1, reveal that for all parameter choices $\alpha$ and $R_{\text{det}}/R_{\text{coh}}$, the bound $P_{\text{click}}^{(\max)}(f)$ decreases as $P_{\text{dark}}$ is reduced. This behavior reflects the nonideality of the detector and the intrinsic trade-off between vacuum insensitivity and sensitivity to the coherent state $|f\rangle$: suppressing false positives in the vacuum (i.e., lowering $P_{\text{dark}}$) inevitably tightens the response of the detector to excitations, thereby reducing the maximum possible value of $P_{\text{click}}(f)$.

*Comparison with nonlocal measurements.* We emphasize that Eq. (16) remains quite general, as it rests solely on the assumption that the measurement is local. To better appreciate its implications, it is useful to compare this result with a scenario involving a nonlocal measurement. A notable example is the excitation projector $\hat{P}_{\Omega}^{\perp} = \hat{\mathbb{I}} - |\Omega\rangle\langle\Omega|$, which perfectly distinguishes the vacuum from any excited state: it gives zero when applied to the vacuum and one for any state orthogonal to it. While this operator appears ideal in its ability to detect excitations, it is fundamentally nonlocal, as it does not belong to any local operator algebra. Consequently, it cannot represent a measurement performed by a finite-size detector.

To make this comparison concrete, let us define the probabilities that an idealized infinite-size detector clicks in the coherent state $|f\rangle$ and in the vacuum $|\Omega\rangle$ as $P_{\text{click}}^{(\text{ideal})}(f) = \langle f|\hat{P}_{\Omega}^{\perp}|f\rangle$ and $P_{\text{dark}}^{(\text{ideal})} = \langle\Omega|\hat{P}_{\Omega}^{\perp}|\Omega\rangle$, respectively. By the standard inner product for coherent states, $\langle f_1|f_2\rangle = \exp[W_2(f_1,f_2) - W_2(f_1,f_1)/2 - W_2(f_2,f_2)/2]$, we find that

$$P_{\text{click}}^{(\text{ideal})}(f) = 1 - \exp\left[-W_2(f,f)\right]. \tag{19}$$

The ratio $P_{\text{click}}(f)/P_{\text{click}}^{(\text{ideal})}$ quantifies the deviation of the local measurement from the nonlocal ideal one. An upper bound for this ratio is given by $P_{\text{click}}^{(\max)}(f)/P_{\text{click}}^{(\text{ideal})}$. Whenever this bound falls below unity, it follows that $P_{\text{click}}(f)$ must also lie strictly below $P_{\text{click}}^{(\text{ideal})}$, signaling a definite departure from the ideal case.

As a practical application, we revisit the toy model of a spherically localized coherent state [Eq. (17)] with amplitude $\alpha$ and radius $R_{\text{coh}}$, together with a concentric spherical detector of radius $R_{\text{det}}$ and dark-count probability $P_{\text{dark}}$. Using Eq. (19), we compute the ratio between the upper bound $P_{\text{click}}^{(\max)}(f)$ on the click probability $P_{\text{click}}$ for this finite-size setup and the corresponding click probability $P_{\text{click}}^{(\text{ideal})}$ in the idealized limit of an infinitely extended detector. The results, shown in the lower panel of Fig. 1, reveal that as the detector size $R_{\text{det}}$ decreases relative to $R_{\text{coh}}$, the ratio $P_{\text{click}}^{(\max)}(f)/P_{\text{click}}^{(\text{ideal})}$ diminishes, indicating a departure from ideal detector behavior.

*Concluding remarks.* The main result of this work is the inequality in Eq. (9), which establishes an upper bound on the click probability of a finite-size detector in terms of its dark count rate. This bound depends on the spacetime region $\mathcal{O}_{\text{det}}$ in which the measurement operator $\hat{E}_{\text{click}}$ is localized.

One might raise the objection that actual measurements in laboratory settings are not strictly confined to the detector itself. Rather, the physical process underlying the measurement may extend over a larger region, potentially including parts of the surrounding apparatus. This ambiguity has significant implications for the inequality in Eq. (9) since the bound for $P_{\text{click}}$ depends on $\mathcal{O}_{\text{det}}$. If the effective region of the measurement is larger than the physical detector, then the corresponding bound could differ significantly.

Interestingly, this opens up the possibility of using Eq. (9) as a diagnostic tool: by comparing experimental data against theoretical bounds derived for different choices of $\mathcal{O}_{\text{det}}$, one could, in principle, infer how localized the measurement truly is. Such comparisons may help shed light on the operational scale at which quantum measurements occur.

These conclusions, of course, rest on the premise that local measurements can be faithfully modeled by local POVMs of the form $\{\hat{E}_{\text{click}}, \hat{\mathbb{I}} - \hat{E}_{\text{click}}\}$. If the region $\mathcal{O}_{\text{det}}$

accurately captures the operational extent of the measurement, then any empirical violation of the bound would imply that localized POVMs within the AQFT framework fall short of describing the full measurement process. Such a result would offer new insight into the measurement problem in relativistic quantum physics.

Finally, we remark that the inequality (9) depends only on the choice of spacetime region $\mathcal{O}_{\mathrm{det}}$, and not on the microscopic modeling of the detector. In our illustrative example, we take $\mathcal{O}_{\mathrm{det}}$ to be a four-dimensional ball for simplicity. Still, the construction extends straightforwardly to more realistic geometries representing spatially extended detectors operating during a finite time interval.

*Acknowledgments.* We acknowledge support from HORIZON EIC-2022-PATHFINDERCHALLENGES-01 HEISINGBERG Project No. 101114978.

*Data availability.* The data that support the findings of this article are openly available [30].

[1] H. Reeh and S. Schlieder, Bemerkungen zur unitäräquivalenz von lorentzinvarianten feldern, Il Nuovo Cimento 1955–1965 **22**, 1051 (1961).
[2] S. J. Summers and R. Werner, The vacuum violates Bell's inequalities, Phys. Lett. A **110**, 257 (1985).
[3] S. J. Summers and R. Werner, Bell's inequalities and quantum field theory. II. Bell's inequalities are maximally violated in the vacuum, J. Math. Phys. (N.Y.) **28**, 2448 (1987).
[4] M. Redhead, More ado about nothing, Found. Phys. **25**, 123 (1995).
[5] H. Halvorson and R. Clifton, Generic Bell correlation between arbitrary local algebras in quantum field theory, J. Math. Phys. (N.Y.) **41**, 1711 (2000).
[6] B. Reznik, Entanglement from the vacuum, Found. Phys. **33**, 167 (2003).
[7] S. J. Summers, Yet more ado about nothing: The remarkable relativistic vacuum state, in *Deep beauty: Understanding the Quantum World Through Mathematical Innovation*, edited by H. Halvorson (Cambridge University Press, Cambridge, England, 2011), pp. 317–342.
[8] A. Peres and D. R. Terno, Quantum information and relativity theory, Rev. Mod. Phys. **76**, 93 (2004).
[9] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, 2nd ed., Theoretical and Mathematical Physics (Springer, Berlin, Heidelberg, 1992).
[10] J. Yngvason, The role of type III factors in quantum field theory, Rep. Math. Phys. **55**, 135 (2005).
[11] R. D. Sorkin, Impossible measurements on quantum fields, arXiv:gr-qc/9302018.
[12] D. M. T. Benincasa, L. Borsten, M. Buck, and F. Dowker, Quantum information processing and relativistic quantum fields, Classical Quantum Gravity **31**, 075007 (2014).
[13] L. Borsten, I. Jubb, and G. Kells, Impossible measurements revisited, Phys. Rev. D **104**, 025012 (2021).
[14] C. J. Fewster and R. Verch, Quantum fields and local measurements, Commun. Math. Phys. **378**, 851 (2020).
[15] D. Grimmer, B. S. L. Torres, and E. Martín-Martínez, Measurements in QFT: Weakly coupled local particle detectors and entanglement harvesting, Phys. Rev. D **104**, 085014 (2021).
[16] M. Papageorgiou and D. Fraser, Eliminating the 'impossible': Recent progress on local measurement theory for quantum field theory, Found. Phys. **54**, 26 (2024).
[17] D. R. Terno, Localization of relativistic particles and uncertainty relations, Phys. Rev. A **89**, 042111 (2014).
[18] C. De Rosa and V. Moretti, Quantum particle localization observables on Cauchy surfaces of Minkowski spacetime and their causal properties, Lett. Math. Phys. **114**, 72 (2024).
[19] J. S. Bell, On the Einstein Podolsky Rosen paradox, Phys. Phys. Fiz. **1**, 195 (1964).
[20] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed experiment to test local hidden-variable theories, Phys. Rev. Lett. **23**, 880 (1969).
[21] A. J. Leggett and A. Garg, Quantum mechanics versus macroscopic realism: Is the flux there when nobody looks?, Phys. Rev. Lett. **54**, 857 (1985).
[22] C. Emary, N. Lambert, and F. Nori, Leggett–Garg inequalities, Rep. Prog. Phys. **77**, 016001 (2013).
[23] C. Budroni, A. Cabello, O. Gühne, M. Kleinmann, and J.-Å. Larsson, Kochen-Specker contextuality, Rev. Mod. Phys. **94**, 045007 (2022).
[24] The notion of ideal detectors, as discussed here, should not be confused with that of ideal quantum measurements. In this context, ideal detectors are defined as those that remain inactive in the vacuum state but produce clicks in certain excited states. In contrast, ideal quantum measurements refer to projective measurements onto the eigenbasis of the corresponding observable.
[25] R. Clifton and H. Halvorson, Entanglement and open systems in algebraic quantum field theory, Stud. Hist. Philos. Sci. Part B Stud. Hist. Philos. Mod. Phys. **32**, 1 (2001).
[26] This follows from microcausality, which ensures that $\hat{U}$ and $\hat{E}_{\mathrm{click}}$ commute [25], and from the unitarity of $\hat{U}$, so that $\langle\Omega|\hat{U}^\dagger\hat{E}_{\mathrm{click}}\hat{U}|\Omega\rangle = \langle\Omega|\hat{U}^\dagger\hat{U}\hat{E}_{\mathrm{click}}|\Omega\rangle = \langle\Omega|\hat{E}_{\mathrm{click}}|\Omega\rangle$ [27].
[27] E. Witten, APS medal for exceptional achievement in research: Invited article on entanglement properties of quantum field theory, Rev. Mod. Phys. **90**, 045003 (2018).
[28] This can be shown by contradiction: suppose the family $\hat{A}_\zeta$ converges to a bounded operator $\hat{A}$. Then, $(\hat{U}-\hat{A})|\Omega\rangle = 0$, and by the separating property of the vacuum [4], it would follow that $\hat{U} = \hat{A}$. This contradicts the fact that the two

operators are localized in disjoint regions. Hence, $\hat{A}_\zeta$ must become unbounded as $\mathcal{E}_\zeta \to 0$.

[29] R. Falcone and C. Conti, Reeh-Schlieder approximation for coherent states, arXiv:2509.09021.

[30] R. Falcone and C. Conti, An inequality for relativistic local quantum measurements, Zenodo (2026), computational notebook, 10.5281/zenodo.18863792.

[31] See Supplemental Material at http://link.aps.org/supplemental/10.1103/ydmg-5vjm for a detailed derivation of the upper bound $P_{\text{click}}^{(\max)}(f)$ for the configuration specified in Eq. (17). The Supplemental Material includes Ref. [32].

[32] L. Grafakos, *Classical Fourier Analysis*, 3rd ed., Graduate Texts in Mathematics Vol. 249 (Springer, New York, NY, 2014).