

Gravitational turbulence: The small-scale limit of the cold-dark-matter power spectrum

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The matter power spectrum, $P(k)$, is one of the fundamental quantities in the study of large-scale structure in cosmology. Here, we study its small-scale asymptotic limit, and show that for cold dark matter in d spatial dimensions, $P(k)$ has a universal k^{-d} asymptotic scaling with the wave number k , for $k \gg k_{\text{nl}}$, where k_{nl}^{-1} denotes the length scale at which nonlinearities in gravitational interactions become important. We propose a theoretical explanation for this scaling, based on a nonperturbative analysis of the system's phase-space structure. Gravitational collapse is shown to drive a turbulent phase-space flow of the quadratic Casimir invariant, where the linear and nonlinear time scales are balanced, and this balance dictates the k dependence of the power spectrum. A parallel is drawn to Batchelor turbulence in hydrodynamics, where large scales mix smaller ones via tidal interactions. The k^{-d} scaling is also derived by expressing $P(k)$ as a phase-space integral in the framework of kinetic field theory, which is analyzed by the saddle-point method; the dominant critical points of this integral are precisely those where the time scales are balanced. The coldness of the dark-matter distribution function—its nonvanishing only on a d -dimensional submanifold of phase space—underpins both approaches. The theory is accompanied by 1D Vlasov-Poisson simulations, which confirm it.

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I. INTRODUCTION

One of the basic observable quantities in the study of the large-scale structure of the Universe is the two-point correlation function of the density field, whose Fourier transform is the power spectrum, $P(k, t)$ [1,2]. The two-point correlation function is of fundamental importance, for it allows us to probe theories of the early Universe, dark matter, and inflation, and to study gravity [1,3–5]. In this paper, we explore the small-scale asymptotic behavior of $P(k)$ (we omit the explicit time dependence when it is not confusing to do so), in the limit $k \gg k_{\text{nl}}$, where the (inverse)

nonlinear scale k_{nl} is defined by $k_{\text{nl}}^3 P_{\text{lin}}(k_{\text{nl}}) = 2\pi^2$ [6], with P_{lin} denoting the linearly evolved overdensity power spectrum. The small-scale limit of $P(k)$ is theoretically important for the understanding of the gravitational N -body problem in the large- N limit [7], and the formation of large-scale structure, nonlinear clustering and self-similarity [1,8], but also for the understanding of the nature of dark matter and gravitational backreaction of small scales on large ones—both relativistic [9] and in the context of the effective field theory of large-scale structure [10] (see, e.g., [6,11,12] for reviews and references) and the general bias expansion [3]. Even before modifying gravity, it is important to know what nonlinear phenomena occur in the standard theory.

Data from cosmological dark-matter-only simulations are consistent with $P(k, t)$ developing a k^{-3} tail on small scales (e.g., Fig. 6 of [13]) in 3D, and a k^{-1} tail in 1D, as shown in Fig. 6 of Ref. [14] (cf. [15–18]). The emergence of a power-law tail—and the simplicity of its exponent—hint that a

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fundamental physical reason for it must exist, ultimately stemming from the nature of the gravitational interaction of cold dark matter. Here, we describe the mechanism that produces this asymptotic scaling, by studying the mass distribution in the velocity space as well as in position space.

We restrict ourselves to the strictly collisionless case where the particle mass $m \rightarrow 0$, while the total particle number $N \rightarrow \infty$, so that $M \equiv Nm$ remains fixed (and so does the volume). In this limit, the phase-space distribution of particles is well described by the Vlasov equation [1,14,19–23]¹:

$$\frac{\partial f}{\partial \eta} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{g} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0, \quad (1)$$

where f is the distribution function (the one-point probability density in phase space), η is the conformal time, defined by $dt = a d\eta$, where a is the scale factor of the background (which is taken to be a Friedmann-Lemaître-Robertson-Walker space-time), $\mathbf{x}$ and $\mathbf{v}$ are the comoving position and peculiar velocity, and the gravitational field $\mathbf{g}$ is self-consistently determined by Poisson's equation

$$\nabla \cdot \mathbf{g} \equiv -\nabla^2 \Phi = -4\pi G a^2 \left[\int f(\mathbf{x}, \mathbf{v}, \eta) d^d v - \bar{\rho}_m \right], \quad (2)$$

where $\bar{\rho}_m$ is the mean background matter density and Φ is the gravitational potential. The gravitational field $\mathbf{g}$ thus defined gives $d(\mathbf{a}\mathbf{v})/dt$, when evaluated along a particle's trajectory, and $\mathbf{v}$ is defined so that the 4-velocity in space-time coordinates $x^\mu = (\eta, \mathbf{x})$ is $u^\mu = (1 - \Phi/c^2, \mathbf{v})/a$. This system of equations applies on scales much smaller than the horizon, for particles much slower than the speed of light, c . Henceforth, we work in general spatial dimensions $d \in \{1, 2, 3\}$, to enable comparison between our theory and our $d = 1$ simulations (or future $d = 2$ ones).

Dark matter is assumed to be cold initially, with a thermal velocity $v_{\text{th}} \rightarrow 0$, so that the initial gravitational potential energy is much larger than the initial thermal energy (this assumption is excellent for our Universe [4,24–29]). Despite being cold, dark matter is inherently kinetic, and cannot be described by fluid equations adequately on nonlinear scales, because these equations cease to be valid after streams cross [23,30,31].

Below we will derive the $k \gg k_{\text{nl}}$ limit of $P(k)$ in two complementary ways. First, we study the problem in phase space, and show that the small-scale asymptotics arise when the time scales involved in the Vlasov equation balance with each other in a particular way, to be

explained below. Two ingredients comprise this argument: the balance of time scales in the Vlasov equation during gravitational collapse, and the conservation of the second Casimir invariant [32–36]

$$C_2 \equiv \iint f^2 d^d x d^d v. \quad (3)$$

In plasma physics, this invariant is sometimes referred to as “enstrophy” [32], “phasesstrophy” [33], or “ f -strophy” [37]. The Vlasov equation conserves an infinite number of phase-space invariants—not only C_2 —but we use C_2 here because it is directly related to the power spectrum (*vide infra*). We show that the spectrum characterizes a turbulent cascade (i.e., flow) of C_2 from large scales to small ones. The system's evolution is nonlinear, and involves a mixing in phase space with gravitational collapses on many scales. Gravitational turbulence is different from other types of turbulence in that the flux of C_2 is not constant over the range of scales of interest, but rather turns out to be scale dependent, and yet there is a universal scaling regime of the phase-space power spectrum; this is in contrast to the usual situation in turbulent systems [35,36,38,39], where there exists an inertial range of scales with a constant-flux cascade of an energy-like quadratic invariant.

For the second, alternative approach, we use nonperturbative kinetic field theory for cosmic structure formation (for a review, see [40]). Here, $P(k)$ is expressed as an integral over the initial particle positions and velocities (weighted by the initial-condition probability distribution) of the characteristic function of the displacement field; this integral turns out to have an explicit k dependence, and we utilize this to perform an asymptotic saddle-point analysis.

The two approaches complement each other, both relying on the same assumptions, but highlighting their roles in different ways. We remark that using phase-space expressions ensures that the validity of our results extends beyond the bounds of configuration-space-based approaches, such as Lagrangian [18,41,42] or Eulerian [30] techniques; in particular, it is regular at stream crossing, and accounts for free streaming automatically. Indeed, as we already mentioned above, the inherently kinetic phenomenon of multiscale structure of the distribution of dark matter, developing by virtue of strongly nonlinear interactions, suggests that a type of turbulence in phase space is involved. As early as [42,43], it was realized that the phenomena of gravitation and turbulence might be linked—here we make the analogy precise and characterize this gravitational turbulence.

The rest of this paper is organised as follows. In Sec. II, we formulate the first approach, based on the Vlasov-Poisson system, and use a phase-space analysis of gravitational collapse to derive the small-scale asymptotics of the phase-space power spectrum, from which $P(k)$ may be

¹The collisionless Vlasov equation, of course, ignores dissipation via collisions (equivalently, finite- N effects). However, we find that due to turbulence, such dissipation will inevitably be accessed; this point is discussed further in Appendix A 2.

computed, and then describe the phase-space cascade that it supports. In Sec. III, as promised above, we derive the same asymptotic scaling of $P(k)$ again, but via a saddle-point analysis of an integral expression for $P(k)$. We test our theory by comparing it with numerical Vlasov-Poisson simulations throughout the paper. Our conclusions are discussed in Sec. IV and summarized in Sec. V.

II. PHASE-SPACE TURBULENCE

We start by describing the initial condition in Sec. II A—a cold stream or a collection of streams—and show what its evolution looks like, and then derive the equation that governs the phase-space Fourier transform of f in Sec. II B. We then analyze the time scales involved in the Vlasov-Poisson system in Sec. II C, and show how gravitational collapse via the Jeans instability [44] imposes stringent conditions on these time scales, and on the form of the power spectrum, in Sec. II D. We describe in Sec. II E the phase-space turbulence that characterizes the dynamics. To do that, we derive a transport equation for the integrand of C_2 in the Fourier dual of the phase space. This has a (nonlinear) source term, which is found to receive contributions from all larger scales in Sec. II F. We present simulation results throughout this section, to test the theory.

A. Cold streams

We assume that the system has an initial condition consisting of a superposition of streams, each one of the form

$$f(\eta = 0, \mathbf{x}, \mathbf{v}) = f_{\text{in}}(\mathbf{x}, \mathbf{v}) \equiv \frac{\rho_{\text{in}}(\mathbf{x})}{(2\pi v_{\text{th}}^2)^{d/2}} e^{-\frac{|\mathbf{v} - \mathbf{u}_{\text{in}}(\mathbf{x})|^2}{2v_{\text{th}}^2}}, \quad (4)$$

where $v_{\text{th}}^2 \ll \min\{u_{\text{in}}^2, \int |\Phi_{\text{in}}| f_{\text{in}} d^d x d^d v\}$, where Φ_{in} is derived from ρ_{in} via Eq. (2); that is, the initial condition is very cold—the thermal energy is negligible in comparison with the gravitational potential energy of the system or the kinetic energy of mean flows. In Eq. (4), ρ_{in} and $\mathbf{u}_{\text{in}}$ are typically Gaussian random fields [1,3], and $\mathbf{u}_{\text{in}}$ is a gradient flow [2], i.e., $\nabla \times \mathbf{u}_{\text{in}} = 0$. We also choose the coordinates so that $\int \rho(\mathbf{x}) \mathbf{u}(\mathbf{x}) d^d x = 0$, where $\rho(\mathbf{x})$ is the density and $\rho \mathbf{u} \equiv \int \mathbf{v} f d^d v$. The distribution (4) is essentially a single stream, and, in fact, its Maxwellian shape may be approximated by a Dirac delta function. The thermal speed v_{th} is chosen to be the smallest velocity scale in the problem, so the entire analysis of this paper focuses on the limit where $s v_{\text{th}} \ll 1$, where s is the Fourier conjugate to velocity (see Sec. II B). The distribution function remains a collection of streams as it evolves in time, by Liouville's theorem (e.g. [45]): locally almost everywhere in phase space, i.e., in a sufficiently small neighborhood of almost any point $(\mathbf{x}_0, \mathbf{v}_0)$ where $f \neq 0$, it can be written as $\rho_{\mathbf{x}_0, \mathbf{v}_0}(\mathbf{x}, \eta) \delta^D(\mathbf{v} - \mathbf{u}_{\mathbf{x}_0, \mathbf{v}_0}(\mathbf{x}, \eta)) + O(v_{\text{th}})$, where δ^D is the Dirac delta function; it remains so as long as collisions or

finite- N effects may be ignored. The local functions $\rho_{\mathbf{x}_0, \mathbf{v}_0}(\mathbf{x}, \eta)$ and $\mathbf{u}_{\mathbf{x}_0, \mathbf{v}_0}(\mathbf{x}, \eta)$ might in general differ from the mean density $\rho(\mathbf{x}, \eta)$ and the mean velocity $\mathbf{u}(\mathbf{x}, \eta)$.

To study the evolution of the initial condition (4), we conduct a suite of Vlasov-Poisson simulations in 1D on a static background [$a(\eta) = 1$], using the `Gkeyll` code [46], which is originally a Vlasov solver for kinetic plasmas; by setting the vacuum permittivity to $\epsilon_0 < 0$, we can use the code to study gravity (as opposed to electrostatics). Details on the numerical method and simulation setup may be found in Appendix A. The simulation's units are chosen so that $\tau_0^{-2} \equiv 4\pi G M / L = 1$ (unit of time),² $k_0 \equiv 2\pi / L = 1$ (unit of length) and $v_0 \equiv (k_0 \tau_0)^{-1} = 1$ (unit of speed), where L is the length of the simulation “box” (with periodic boundary conditions) and M is the total mass in the box.

The time evolution of a cold system of three streams—three copies of Eq. (4)—is displayed in Fig. 1. This figure shows that each stream is distorted quickly, by rotating and twisting in phase space. Evidently, this motion generates small-scale structure, which we consider to be a type of turbulence in phase space. As usual, in order to characterize the turbulence and its spectrum of fluctuations, one must identify an invariant quantity that cascades to small scales, what the cascade's time scale is, and also the flux of that invariant quantity [47] (which is not scale independent in our case). We will do so in Secs. II C, II E, and II F, respectively.

That the turbulence is in phase space and not merely in position space is manifest in Fig. 1: already at $t = 6\tau_0$, the system can no longer be described as three streams almost anywhere, so standard cosmological perturbation theory would already be inadequate, and the rms peculiar velocity $v_{\text{rms}} \equiv [\int dv (v - u(x))^2 f(x, v) / \rho(x)]^{1/2}$ is much larger than v_{th} , and is of order v_0 . However, in phase space, we see that the topology of three single lines is preserved.

As explained above, the fact that the system is a collection of streams implies that *locally in phase space*, one can describe each stream with fluid equations: inserting $\rho_{\mathbf{x}_0, \mathbf{v}_0}(\mathbf{x}, \eta) \delta^D(\mathbf{v} - \mathbf{u}_{\mathbf{x}_0, \mathbf{v}_0}(\mathbf{x}, \eta))$ into the Vlasov equation gives the continuity and Euler equations for $\rho_{\mathbf{x}_0, \mathbf{v}_0}$ and $\mathbf{u}_{\mathbf{x}_0, \mathbf{v}_0}$, respectively, and the divergence of the latter yields the Raychaudhuri equation [49], which describes gravitational collapse under the Jeans instability [23,41,44,50–56]. Written in the frame of reference that moves with a given stream, it reads

$$\frac{d\theta}{d\eta} + \mathcal{H}\theta + \frac{\theta^2}{3} + \sigma^{ij}\sigma_{ij} = \nabla \cdot \mathbf{g} + \frac{\Sigma}{\rho_{\mathbf{x}_0, \mathbf{v}_0}} + 2\omega_i \omega^i, \quad (5)$$

where $\mathcal{H}$ is the conformal Hubble constant, $\theta \equiv \nabla \cdot \mathbf{u}_{\mathbf{x}_0, \mathbf{v}_0}$ is the stream's divergence, $\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}_{\mathbf{x}_0, \mathbf{v}_0} / 2$ is its vorticity,

²Note that Poisson's equation (2) implies that G has different dimensions in 1D from 3D.

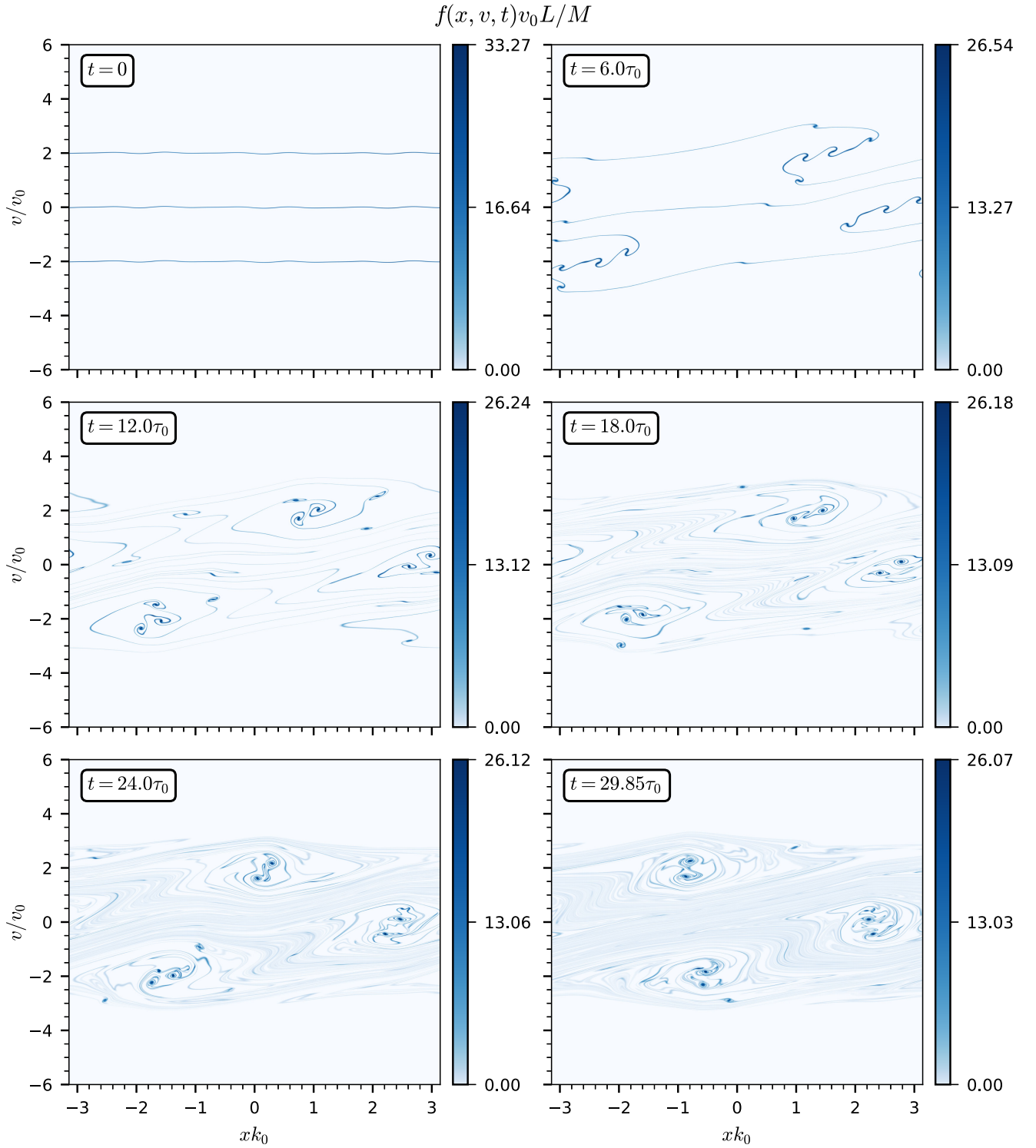


FIG. 1. Color plots of the distribution function showing the time evolution of three cold streams, with $v_{\text{th}}/v_0 = 5 \times 10^{-3}$. The color bar in each panel extends from 0 to max f at the corresponding time in the simulation. See the text and Appendix A for details. A video is available in the Supplementary Material [48].

$\sigma_{ij} \equiv [\partial_i(u_{\mathbf{x}_0, \mathbf{v}_0})_j + \partial_j(u_{\mathbf{x}_0, \mathbf{v}_0})_i - 2\theta\delta_{ij}/3]/2$ is its shear (rate of strain), $d/d\eta$ is the Lagrangian derivative along the stream, $\Sigma/\rho_{\mathbf{x}_0, \mathbf{v}_0}$ is a pressure-related term that is $O(v_{\text{th}}^2)$, and $\mathbf{g}$ is the gravitational field felt by the stream. We will use Eq. (5) in Sec. IID to estimate the time scale τ_j of gravitational collapse, due to the Jeans instability.

B. Spectral transport equations

The evolution of the system is inherently a multiscale process, so it is more convenient to study Eqs. (1) and (2) in Fourier space.

1. Fourier transform

Let us define the Fourier transform (marked by a circumflex) as follows:

$$\hat{f}(\mathbf{k}, \mathbf{s}) \equiv \iint f(\mathbf{x}, \mathbf{v}) e^{i\mathbf{k}\cdot\mathbf{x} - i\mathbf{s}\cdot\mathbf{v}} d^d x d^d v, \quad (6)$$

and similarly for all other functions of $(\mathbf{x}, \mathbf{v})$. For $d > 1$, we denote $k \equiv |\mathbf{k}|$, $s \equiv |\mathbf{s}|$, etc. Under this Fourier transform, the Vlasov-Poisson system (1) and (2) becomes

$$\frac{\partial \hat{f}}{\partial \eta} + \mathbf{k} \cdot \frac{\partial \hat{f}}{\partial \mathbf{s}} + i\mathbf{s} \cdot \int \frac{d^d k'}{(2\pi)^d} \hat{\mathbf{g}}(\mathbf{k}') \hat{f}(\mathbf{k} - \mathbf{k}', \mathbf{s}) = 0, \quad (7)$$

$$k^2 \hat{\Phi} = -4\pi G a^2 \hat{\rho}, \quad (8)$$

where $\hat{\rho}$ is the Fourier-transformed density. The second Casimir invariant, defined by Eq. (3), is given in Fourier space by Parseval's theorem:

$$C_2 = \frac{1}{(2\pi)^{2d}} \iint |\hat{f}|^2 d^d k d^d s. \quad (9)$$

The integrand $|\hat{f}|^2$ is directly related to the density power spectrum. Indeed, let the *phase-space power spectrum* be

$$\hat{F}(\mathbf{k}, \mathbf{s}) \equiv \langle |\hat{f}|^2 \rangle, \quad (10)$$

where $\langle \cdot \rangle$ is an ensemble average over many random realizations of the initial conditions; then the density power spectrum is

$$P(k) \equiv \frac{1}{V} \langle \hat{\rho}(\mathbf{k}) \hat{\rho}^*(\mathbf{k}) \rangle = \frac{\hat{F}(\mathbf{k}, 0)}{V}, \quad (11)$$

where V is the spatial volume.

2. Transport of C_2

Let us now see how the integrand of the second Casimir invariant (9) evolves: multiplying Eq. (7) by $\hat{f}^*$ and taking the real part, we find

$$\frac{\partial |\hat{f}|^2}{\partial \eta} + \mathbf{k} \cdot \frac{\partial |\hat{f}|^2}{\partial \mathbf{s}} + i\mathbf{s} \cdot \int \frac{d^d k'}{(2\pi)^d} [\hat{\mathbf{g}}(\mathbf{k}') \hat{f}^*(\mathbf{k}, \mathbf{s}) \hat{f}(\mathbf{k} - \mathbf{k}', \mathbf{s}) - \hat{\mathbf{g}}^*(\mathbf{k}') \hat{f}(\mathbf{k}, \mathbf{s}) \hat{f}^*(\mathbf{k} - \mathbf{k}', \mathbf{s})] = 0. \quad (12)$$

Integrating Eq. (12) over all $\mathbf{k}$ and $\mathbf{s}$ yields the conservation of C_2 , as it should.

Let δg_r be the amplitude of a typical fluctuation in $\mathbf{g}$ on scales $r = 1/k$ and smaller, viz.,

$$\begin{aligned} \delta g_r^2 &\equiv \frac{2}{V} \int \frac{d^d k'}{(2\pi)^d} \langle |\hat{\mathbf{g}}(\mathbf{k}')|^2 \rangle (1 - e^{i\mathbf{k}'\cdot\mathbf{r}}) \\ &\sim \frac{1}{V} \int_{k' > k} d^d k' \langle |\hat{\mathbf{g}}(\mathbf{k}')|^2 \rangle. \end{aligned} \quad (13)$$

Intuitively, δg_r describes the amplitude of a typical tidal field on a scale r . We take $\mathbf{g}$ to be a sufficiently continuous field, so that for sufficiently small values of r ,

$$\delta g_r \sim \kappa r^\lambda + \text{h.o.t.} \quad (14)$$

with the Hölder exponent $0 < \lambda \leq 1$, and κ is a constant coefficient. A smooth gravitational field has $\lambda = 1$, because one may Taylor expand $\mathbf{g}$, meaning that the fluctuations are dominated by tidal forces (and κ has dimensions of

[time] $^{-2}$).³ We prove in Sec. IIC that $\lambda = 1$ follows from Poisson's equation (2), in conjunction with a balance of time scales, in $d \leq 3$ spatial dimensions, but, for the purposes of the qualitative discussion that follows, we take $\lambda = 1$ now, *a priori*.

Let us now take the large- k limit of Eq. (12). In this limit, for a smooth gravitational field ($\lambda = 1$), the last term on the left-hand side is dominated by two contributions: (i) $k' \ll k$ and (ii) $|\mathbf{k} - \mathbf{k}'| \ll k$. These are sometimes referred to as “squeezed” triangles; case (i) is also known as the Batchelor limit [35,39,57,58]. This approximation is justified by its subsequent success in producing a self-consistent solution (a spectrum that is steep enough for λ to be 1). In Fig. 2, it is validated for the simulation described in Sec. IIA.

³The case $\lambda > 1$ is also smooth, but highly atypical, where the tidal forces vanish; while this could happen in isolated points with exactly zero overdensity, we ignore it here.

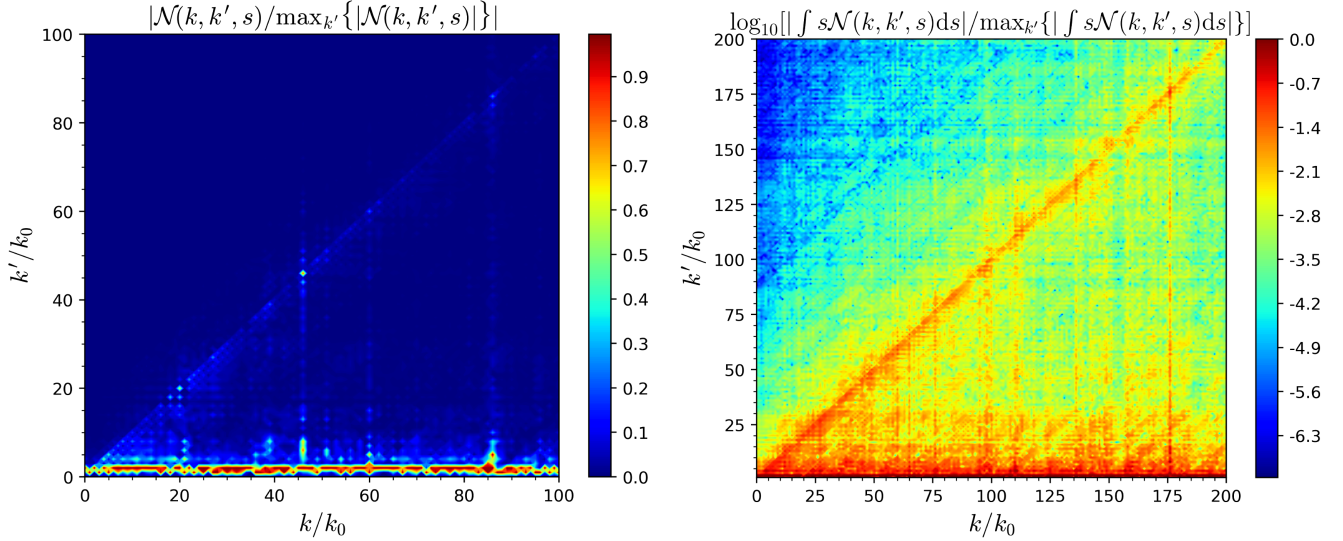


FIG. 2. Plots of the contributions to the nonlinear term in Eq. (12), for the simulation shown in Fig. 1. Defining $\mathcal{N}(\mathbf{k}, \mathbf{k}', \mathbf{s}) \equiv \langle \Im[\hat{\mathbf{g}}(\mathbf{k}')\hat{f}^*(\mathbf{k} - \mathbf{k}', \mathbf{s})\hat{f}^*(\mathbf{k}, \mathbf{s})] \rangle_T$, the left panel displays $|\mathcal{N}|/\max_{k'}\{|\mathcal{N}|\}$, at $sv_0 = 0.52$, as a function of both k and k' (the average is calculated as in Sec. II E). The Batchelor peak is very prominent at low k' , and another peak is visible on the diagonal $k' = k$, corresponding to the limit (ii) of $\mathcal{N}$. In this paper we only require the limit (ii) when integrated over s (in Sec. II F below); to gauge the validity of the approximation in that case, the right panel shows the integral of $s\mathcal{N}$ over $0 \leq s \leq k\tau_0/2$ (in logarithmic scale), normalized in the same way, exhibiting the same two prominent peaks.

Taylor expanding $\hat{f}(\mathbf{k} - \mathbf{k}', \mathbf{s})$ in the Batchelor limit turns the nonlinear term in (12) into

$$\begin{aligned}
 & \text{is} \cdot \int \frac{d^d k'}{(2\pi)^d} \left[\hat{\mathbf{g}}(\mathbf{k}')\hat{f}^*(\mathbf{k}, \mathbf{s}) \left(\hat{f}(\mathbf{k}, \mathbf{s}) - \frac{\partial \hat{f}}{\partial k^i} k'^i \right) - \hat{\mathbf{g}}^*(\mathbf{k}')\hat{f}(\mathbf{k}, \mathbf{s}) \left(\hat{f}^*(\mathbf{k}, \mathbf{s}) - \frac{\partial \hat{f}^*}{\partial k^i} k'^i \right) \right] + \text{h.o.t.} \\
 & \simeq -i \int \frac{d^d k'}{(2\pi)^d} s^j \left[\hat{g}_j(\mathbf{k}')k'^i \hat{f}^*(\mathbf{k}, \mathbf{s}) \frac{\partial \hat{f}}{\partial k^i} - \hat{g}_j^*(\mathbf{k}')k'^i \hat{f}(\mathbf{k}, \mathbf{s}) \frac{\partial \hat{f}^*}{\partial k^i} \right] \\
 & = -i \int \frac{d^d k'}{(2\pi)^d} s^j \hat{g}_j(\mathbf{k}')k'^i \left[\hat{f}^*(\mathbf{k}, \mathbf{s}) \frac{\partial \hat{f}}{\partial k^i} + \hat{f}(\mathbf{k}, \mathbf{s}) \frac{\partial \hat{f}^*}{\partial k^i} \right] \equiv -s^j \Phi_j^i \frac{\partial |\hat{f}|^2}{\partial k^i}, \tag{15}
 \end{aligned}$$

where we have defined $\Phi_j^i \equiv i \int k'^i \hat{g}_j(\mathbf{k}') d^d k' / (2\pi)^d$ —the matrix $\Phi_j^i = \delta^{\text{in}} \partial_n \partial_j \Phi$ is the Hessian matrix of the gravitational potential, i.e., the tidal matrix. The transition from the first line to the second in (15) is valid because, in the center-of-mass frame, the leading-order terms are proportional to $\int d^d k' \hat{\mathbf{g}}(\mathbf{k}') = 0$. Thus, the part of the nonlinear term in the limit (i) reduces to a tidal interaction: on a given (small) scale corresponding to a (large) wave number k , the distribution function f is distorted by the gravitational field at the same (large) energy-containing scale.

We now have, from Eqs. (12) and (15),

$$\frac{\partial |\hat{f}|^2}{\partial \eta} + \mathbf{k} \cdot \frac{\partial |\hat{f}|^2}{\partial \mathbf{s}} - s^j \Phi_j^i \frac{\partial |\hat{f}|^2}{\partial k^i} = -(\text{ii}), \tag{16}$$

where (ii) represents the part of the nonlinear term in Eq. (12) in the limit (ii), $|\mathbf{k} - \mathbf{k}'| \ll k$; upon substituting $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$ in Eq. (12), one has

$$\begin{aligned}
 (\text{ii}) & \simeq \text{is} \cdot \int_{k'' \ll k} \frac{d^d k''}{(2\pi)^d} \left[\hat{\mathbf{g}}(\mathbf{k} - \mathbf{k}'')\hat{f}^*(\mathbf{k}, \mathbf{s})\hat{f}(\mathbf{k}'', \mathbf{s}) - \hat{\mathbf{g}}^*(\mathbf{k} - \mathbf{k}'')\hat{f}(\mathbf{k}, \mathbf{s})\hat{f}^*(\mathbf{k}'', \mathbf{s}) \right] + \text{h.o.t.} \\
 & \simeq \frac{i\mathbf{s}}{V} \cdot [\hat{\mathbf{g}}(\mathbf{k})\hat{f}(k, \mathbf{s})\hat{f}^*(\mathbf{k}, \mathbf{s}) - \hat{\mathbf{g}}^*(\mathbf{k})\hat{f}^*(k, \mathbf{s})\hat{f}(\mathbf{k}, \mathbf{s})], \tag{17}
 \end{aligned}$$

where $\hat{f} \equiv (2\pi)^{-d} V \int_{k'' \ll k} d^d k'' \hat{f}(\mathbf{k}'', \mathbf{s})$ contains the low-wave-number contributions to f .

Equation (16) is a transport equation in Fourier space, with a source $-(\text{ii})$. Taking its average yields an evolution equation for the power spectrum (10):

$$\frac{\partial \hat{F}}{\partial \eta} + \mathbf{k} \cdot \frac{\partial \hat{F}}{\partial \mathbf{s}} - s^j \frac{\partial}{\partial k^i} \langle \Phi_j^i |\hat{f}|^2 \rangle = \hat{S}, \tag{18}$$

with the source

$$\hat{S} = -\frac{\mathbf{i}\mathbf{s}}{V} \cdot \left\langle \hat{\mathbf{g}}(\mathbf{k})\hat{f}(k, \mathbf{s})\hat{f}^*(\mathbf{k}, \mathbf{s}) - \hat{\mathbf{g}}^*(\mathbf{k})\hat{f}^*(k, \mathbf{s})\hat{f}(\mathbf{k}, \mathbf{s}) \right\rangle. \quad (19)$$

We will estimate $\hat{S}$ in Sec. II F below. Note, that $\hat{S}$, as given by Eq. (19), need not vanish when integrated over all $\mathbf{k}$ and $\mathbf{s}$, because Eq. (18) is only valid in the large- k limit: nonsqueezed contributions would matter at small k . Contrast this to Eq. (12), where there is no “source” of C_2 ; effectively, Eq. (18) represents not an injection of total C_2 into the system, but rather a transfer of C_2 from large scales to small ones.

The time scales appearing on the left-hand side of Eq. (18) are the advection (linear) time scale associated with $\mathbf{k} \cdot (\partial/\partial \mathbf{s})$, and the nonlinear (gravitational) time scale associated with $s^j \Phi_j^i \partial/\partial k^i \sim s \delta g_r$. We will discuss these time scales in Sec. II C. Then, we will describe the left-hand side of (18) in Sec. II E, and the right-hand side in Sec. II F.

C. Critical balance

There are two (conformal) time scales in the Vlasov equation (1): linear, $\tau_1 \sim r/\delta v$, and nonlinear, $\tau_{\text{nl}} \sim \delta v/\delta g_r$, where δg_r is defined in Eq. (13) and δv is a velocity-difference scale. In the Fourier-transformed Vlasov equation (7) the linear (or phase-mixing) time scale is

$$\tau_1 \equiv \frac{s}{k}, \quad (20)$$

and the nonlinear time scale is

$$\tau_{\text{nl}} \equiv \frac{1}{s \delta g_r}, \quad (21)$$

where $k = 1/r$ and $s = 1/\delta v$. In the Batchelor approximation (16), the nonlinear time scale is $\tau_{\text{nl}}^{-1} \sim s \|\Phi_j^i\|/k$, which is the same as (21) (up to an order-unity constant), in the case of a smooth gravitational field.

In the highly nonlinear regime, the two time scales must balance each other: if τ_1 were much shorter than τ_{nl} , so that only the first two terms of Eq. (7) dominated, linear phase mixing would drive velocity gradients up until τ_{nl} shrank to the same order of magnitude as τ_1 . Conversely, if τ_{nl} were much shorter than τ_1 , so the third term in Eq. (7) dominated, then nonlinear mixing would drive spatial gradients of f , and, therefore, of $\mathbf{g}$, up until τ_1 and τ_{nl} matched. This so-called *critical balance* may be thought of as a type of dominant-balance asymptotic argument for the Vlasov equation, where one allows the system enough time to establish this balance (see [35,36,59–61] for some examples of critical balance in various areas of physics).

Critical balance amounts to setting $\tau_1 \sim \tau_{\text{nl}}$. These are the only time scales in Eq. (18), so the *cascade time* τ_c , which is the time scale for the transfer of C_2 from scale to scale, must also be

$$\tau_c \sim \tau_1 \sim \tau_{\text{nl}}. \quad (22)$$

The shared value τ_c of the two time scales τ_1 and τ_{nl} is also known as *critical-balance time*. We will use this equality of time scales in Sec. II D to fix the scalings of the phase-space power spectrum. Given Eq. (22), for each length scale $r = 1/k$, there exists a corresponding (inverse) velocity scale

$$s_c(k) \equiv \sqrt{\frac{k}{\delta g_r}}. \quad (23)$$

It is also convenient to define $k_c(s)$ as the inverse function of s_c .

The argument that the two time scales must balance applies only to modes well inside the horizon, such that $k \gg \mathcal{H}/c$ (where $\mathcal{H}$ is the conformal Hubble constant). Indeed, the conformal time in any asymptotically de Sitter cosmology (i.e., one with a positive cosmological constant) is bounded from above by some value η_{max} [62]; so, for modes with k too small, linear phase mixing can only generate velocity gradients up to $s_{\text{max}} \simeq k\eta_{\text{max}}$. If the amplitude of $\hat{f}(k, s_{\text{max}})$ is not large enough for the nonlinear term to become important, then the evolution of such a mode will always be primarily linear. Alternatively, this can be understood as a failure of the requirement that $k \gg \mathcal{H}/c$: at late times $\mathcal{H} \sim 1/(\eta_{\text{max}} - \eta)$, so it will eventually exceed kc ; if s does not grow sufficiently by then, that mode can never become nonlinear. Here we are interested in the $k \rightarrow \infty$ limit, so it is safe to ignore this nuance. Additionally, due to the finite age of the Universe and hierarchical structure formation, the decrease of τ_{nl} until it matches τ_1 might not have happened yet for all values of k , as structures on the largest scales have yet to collapse. Again, this does not affect the $k \rightarrow \infty$ limit, and we may simply take $k \gg \max\{k_{\text{nl}}, \mathcal{H}/c\}$, where k_{nl} was defined in the Introduction.

1. Nonsmooth gravitational field is impossible

Let us now show that the gravitational field must be smooth in a critically balanced Vlasov-Poisson system. To do this, let us parametrize $\hat{F}$: to leading order in the large- (k, s) limit, let

$$\hat{F}(\mathbf{k}, \mathbf{s}) \sim \begin{cases} F_1 k^\gamma s^\xi, & \text{if } s \lesssim s_c(k) \ll v_{\text{th}}^{-1}, k \gg k_{\text{nl}}, \\ F_2 s^\delta k^\sigma, & \text{if } k \lesssim k_c(s), s \gg v_{\text{rms}}^{-1}, \end{cases} \quad (24)$$

for some $\gamma, \delta < 0, \xi, \sigma \in \mathbb{R}$. For $P(k) \propto \hat{F}(k, 0)$ to be finite, $\hat{F}$ must be independent of s in the limit $s \rightarrow 0$, $k_c(s) \ll k \ll k_c(v_{\text{th}}^{-1})$, whence $\xi = 0$. Since $\hat{f}$ must have a finite velocity variance, $\sigma = 0$. Furthermore, matching along the critical-balance line (23) gives us

$$F_1 k^\gamma \sim F_2 [s_c(k)]^\delta, \quad (25)$$

which will shortly lead to a relationship between γ and δ .

The exponent γ may be related directly to the Hölder exponent λ in Eq. (14) by Poisson's equation (2). If the gravitational field were not smooth, i.e., $\lambda < 1$, then by the Paley-Wiener theorem and Eqs. (2) and (13), $F(k, 0) \propto k^{2-d}\delta g_r^2$, whence, from Eq. (24),

$$\gamma = 2 - d - 2\lambda. \quad (26)$$

Furthermore, by Eq. (23), $s_c(k) \propto k^{(1+\lambda)/2}$, whence Eq. (25) yields

$$2\gamma = (1 + \lambda)\delta \quad (27)$$

(this equation also applies when $\lambda = 1$). Eliminating λ from Eqs. (26) and (27) gives

$$\delta = \frac{4\gamma}{4 - d - \gamma}. \quad (28)$$

For $\lambda < 1$, Eq. (26) implies that $\gamma > -d$. Then by Eq. (28) we also have, $\delta > -d$, for $d \in \{1, 2, 3\}$.

However, if γ and δ are both greater than $-d$, the power spectrum $\hat{F}$ does not decay sufficiently fast to be critically balanced: consider the amount of C_2 at a scale k^{-1} (or smaller), which is the variance of f over all scales up to $k = 1/r$, viz., analogously to (13),

$$\delta f_r^2 = \frac{1}{(2\pi)^{2d} V v_{\text{rms}}^d} \int_{k' > k} d^d k' \int d^d s \hat{F}(\mathbf{k}', \mathbf{s}). \quad (29)$$

If $\delta > -d$, then the s integral is dominated by the ultraviolet cutoff, v_{th}^{-1} , so

$$\delta f_r^2 \sim \frac{F_2}{V v_{\text{rms}}^d v_{\text{th}}^{d+\delta}} \int_{k' > k} d^d k', \quad (30)$$

and the k' integral will also be dominated by the ultraviolet cutoff. If this were true, the dynamics of f at the length scale k^{-1} could not be critically balanced as most of C_2 at that scale would be concentrated away from the critical-balance line ($k, s_c(k)$). Even if one introduced such a configuration initially, it would, by the argument of Sec. II C, dynamically evolve into a critically balanced state. Likewise, considering the amount of C_2 stored at velocity scale s^{-1} (or smaller), would imply that the system is not in critical balance at any $s \ll v_{\text{th}}^{-1}$.

Thus, critical balance, in conjunction with Poisson's equation, precludes the possibility of a nonsmooth gravitational field, whence it must be the case that $\lambda = 1$ in Eq. (14), and, therefore, $\gamma, \delta \leq -d$.

2. Smooth gravitational field

If $\delta g_r \sim \kappa r$, as it must be for a smooth field, then

$$\tau_c \sim \frac{1}{\sqrt{\kappa}}, \quad (31)$$

$$s_c(k) = \frac{k}{\sqrt{\kappa}}, \quad (32)$$

$$k_c(s) = \sqrt{\kappa} s, \quad (33)$$

and the critical-balance time is a scale-independent constant.

D. Jeans collapse

We will find below that the phase-space motion induced by gravitational collapse implies, together with critical balance, that $\gamma \geq -d$, so the only possibility compatible with a smooth gravitational field is $\gamma = \delta = -d$.

As stated in Sec. II A, our analysis focuses on the cold-dark-matter limit, where $sv_{\text{th}} \ll 1$, which is equivalent to $k \ll k_c(v_{\text{th}}^{-1})$. On these scales, f is just a collection of streams, so when zooming in on one of them, passing through a phase-space point $(\mathbf{x}_0, \mathbf{v}_0)$, and applying the Raychaudhuri equation (5) in its frame of reference, one finds that the stream is unstable to gravitational collapse. This applies if the stream's density, denoted $\rho_{\mathbf{x}_0, \mathbf{v}_0}$ in Sec. II A, is larger than the background density, ρ_b , due to the rest of the distribution. Indeed, let V_0 be a spatial region around $\mathbf{x}_0$, where the mean density of the stream,

$$\rho_{\mathbf{x}_0, \mathbf{v}_0}(V_0) = \frac{1}{\text{Vol}(V_0)} \int_{V_0} d^d x \rho_{\mathbf{x}_0, \mathbf{v}_0}(\mathbf{x}), \quad (34)$$

is larger than ρ_b , and let $r = 1/k \sim [\text{Vol}(V_0)]^{1/d}$ be the spatial size of V_0 . The gravitational effect of ρ_b may be absorbed into a redefinition of the background (an effective cosmology) felt by the stream, via the “separate universe” principle [63]. If $\rho_{\mathbf{x}_0, \mathbf{v}_0}(V_0) \geq \rho_b$, the stream is overdense (with respect to the redefined effective background) and will therefore collapse. As the entire system is cold, homogeneous and isotropic, such overdensities generically exist on all scales $r \ll k_{\text{nl}}^{-1}$.⁴

All the terms in Eq. (5) are of dimension $[\text{time}]^{-2}$, and the last two terms on the right-hand side are $O(k^2 v_{\text{th}}^2)$,⁵ while the $\mathcal{H}\theta$ term is completely negligible for $k \gg \mathcal{H}/c$. Equation (5) then reduces to

$$\frac{d\theta}{d\eta} = -\frac{\theta^2}{3} + \nabla \cdot \mathbf{g} + O(k^2 v_{\text{th}}^2), \quad (35)$$

⁴Initial conditions f_{in} where this does not occur can be constructed, but these require additional symmetries, and are unstable—to the Jeans instability—so we ignore them here.

⁵Indeed, if initially the system had $\omega^i = 0$, then, in the limit $v_{\text{th}} \rightarrow 0$, this will stay this way locally in phase space [49].

which means that for $s_c(k)v_{\text{th}} \ll 1$, the gravity term $\nabla \cdot \mathbf{g}$ dominates and there is nothing to stop a collapse. This collapse occurs on a time scale τ_g , the *gravitational time*, given by

$$\tau_g^{-2} \sim |\nabla \cdot \mathbf{g}| \sim 4\pi G \delta \rho_r, \quad (36)$$

where again $r = 1/k$, and,⁶ analogously to (13),

$$\delta \rho_r^2 \sim \int_{k' > k} P(k') d^d k'. \quad (37)$$

This collapse is nothing but the Jeans instability occurring in every stream individually, in its own reference frame; this is not dissimilar to the instabilities described in, e.g., [50,53]. Indeed, a linear stability analysis of a spatially homogeneous distribution gives the Jeans-instability growth rate [44]

$$\tau_J^{-2} = \tau_g^{-2} - k^2 v_{\text{th}}^2, \quad (38)$$

which is the same as can be inferred from the Raychaudhuri equation (35). One special case where the collapse time may be deduced explicitly is described in Appendix B.

Thus, the system experiences collapse on all spatial scales smaller than the outer scale and larger than $1/k_c(v_{\text{th}}^{-1})$, at a rate $\tau_J^{-1} \sim \tau_g^{-1}$, by Eq. (38), provided that $\tau_g k v_{\text{th}} \ll 1$. Gravitational collapse is a critically balanced process, so if we recall the other time scales of our system, identified in Sec. II C, we immediately see that the gravitational time is

$$\tau_g \sim \tau_c \sim \frac{1}{\sqrt{k}}, \quad (39)$$

where the last equality follows from Eq. (31), valid for a smooth gravitational field. Then, requiring consistency with Poisson's equation (36), we must immediately conclude that $\delta \rho_r$ is constant in r . Equivalently, $\gamma = -d$ in Eq. (24), which is the main result of the paper:

$$P(k) \sim F_1 k^{-d}. \quad (40)$$

Then, by Eq. (28) and the matching condition (25), $\delta = \gamma$ and $F_1 \sim F_2 \tau_g^d$. We have thus obtained that, for cold dark matter,

$$\hat{F}(k, s) \sim F_1 \begin{cases} k^{-d}, & \text{if } s \ll s_c(k) \ll v_{\text{th}}^{-1}, \\ \tau_g^{-d} s^{-d}, & \text{if } k \ll k_c(s) \ll k_c(v_{\text{th}}^{-1}), \end{cases} \quad (41)$$

provided that $k \gg k_{\text{nl}}$ and $s \gg v_{\text{rms}}^{-1}$. These are the leading-order asymptotics: they hold for large k at $s \rightarrow 0$, and large

s at $k \rightarrow 0$. In general, $\hat{F}$ can also depend on the angle between $\mathbf{s}$ and $\mathbf{k}$, viz., on $\mathbf{k} \cdot \mathbf{s}$. This angular dependence arises only at the next order in $s \ll s_c(k)$ or $k \ll k_c(s)$ (whichever obtains), because it must not exist at $s = 0$, or at $k = 0$.

The analysis of time scales may be recast in terms of length scales as follows. There are only two physically meaningful length scales in the system studied in this paper: k_{nl}^{-1} , the scale below which critical balance has had enough time to establish itself since the big bang, and $v_{\text{th}} \tau_c \sim 1/k_c(v_{\text{th}}^{-1})$, below which f may no longer be treated as a collection of streams, but rather describes “warm” dark matter (which is no longer Jeans unstable). If it were the case that $\gamma < -d$, then necessarily, these two length scales would be equal, for otherwise τ_g would be shorter than τ_c . In this paper, however, we are interested in the case $v_{\text{th}} \tau_c \ll k_{\text{nl}}^{-1}$, and in structures on scales k^{-1} between these two values. Thus, $k \ll k_c(v_{\text{th}}^{-1})$ implies that $\gamma \geq -d$ (otherwise the collapse time would be much longer than τ_c on scales where the system should be unstable), whereas $k \gg k_{\text{nl}}$ necessitates—via critical balance—that $\gamma \leq -d$.

The k^{-d} asymptotic (40) of the density power spectrum may be derived by an altogether different, yet systematic approach—by performing an asymptotic analysis of the integral expression for $P(k)$ and examining its critical points. We shall do this in Sec. III, which is self-contained, but before that, let us see what kind of flow in $(\mathbf{k}, \mathbf{s})$ space is engendered by Eq. (18), and see how the power spectrum (41) is supported by this flow (and vice versa).

E. Phase-space cascade

Consider first the homogeneous part of Eq. (18), ignoring $\hat{S}$ until Sec. II F. The positive eigenvalues of Φ_j^i drive a rotation in $(\mathbf{k}, \mathbf{s})$, where small-scale velocity structure interchanges with small-scale spatial structure, while the negative eigenvalues drive a flow of both to ever smaller scales. In fact, one can analyze Eq. (16) directly—before ensemble averaging—while still ignoring the right-hand side. As Φ_j^i generically depends on time, this analysis holds locally in time (and space, on scales below the outer scale), but by critical balance, the long mode (large-scale) Φ_j^i cannot vary on a time scale shorter than the critical-balance time. We therefore approximate it as constant (in which case there are analytical solutions), but the qualitative features described here—namely, a rotation in $(\mathbf{k}, \mathbf{s})$ space and a flow to larger values of k and s —remain also true for a time-dependent Φ_j^i . Generically, in $d > 1$, Φ_j^i would have both positive and negative eigenvalues.

Consider a positive eigenvalue of Φ_j^i . If $\mathbf{s}_+$ is its corresponding eigenvector, then for $\mathbf{s} \parallel \mathbf{s}_+$, Eq. (16) reduces to a transport equation under the action of a harmonic-oscillator potential, i.e., a rotation in the $(\mathbf{k}, \mathbf{s})$ space. This ensures that the large- s structure in the initial condition

⁶We take $k \gg \mathcal{H}/c$ and assume that $\tau_g \ll \mathcal{H}^{-1}$ so that the scale factor a in Eq. (2) is effectively a constant.

(cold streams) is transported to large k , and vice versa. The negative eigenvalues ensure that there is a flow to ever smaller scales, because then the solution is a linear combination of hyperbolic functions (cf. [36]).

In 1D, it would appear naively that there is only phase-space rotation when Φ_j^i is evaluated in an initially overdense region, because there is only one, necessarily positive, eigenvalue. This, however, is misleading. For any d , as the system evolves, matter moves around, and thus that region generically changes from being overdense to underdense (alternatively, it does so for different realizations of the initial conditions), and therefore the sign of $\text{tr}\Phi_j^i$ also changes. Thus, there is a temporal sequence of phase-space rotations and stretchings (see Fig. 1)—essentially, differential phase-space rotation, leading to the generation of small-scale structure.

Thus, phase-space rotations transfer small-scale structure from s to k and back, and that is supplemented by a flow of $|\hat{f}|^2$ to ever larger k and s (similar to the plasma echo joint with a phase-space cascade in [35,36]).⁷ Hence, Eq. (16) describes a type of phase-space turbulence, where C_2 , being the integral of $|\hat{f}|^2$ over scales, is cascaded to smaller scales by larger-scale tidal fields, in a critically balanced manner.

To test this conclusion numerically, we calculate the time-averaged phase-space power spectrum $\hat{F}$, as a function of $|k|$ and $|s|$ (with the negative values folded on the positive ones), for the simulation shown in Fig. 1. This is plotted in Fig. 3, which shows the contours of the time average (as a proxy for an ensemble average) $\langle |\hat{f}|^2 / C_2(t) \rangle_T$. They are arranged in a rectangular shape: the critical-balance line (32) is the diagonal of this rectangle. This rectangular structure is brought about by the aforementioned rotation in the (k, s) space, conjoined with the effect of the negative eigenvalues of Φ_j^i , which stretch structures along the critical-balance line. Both of these processes establish a symmetry with respect to this line. The line $s = k\tau_0$ is plotted in Fig. 3 and identified as the critical-balance line, yielding $\kappa \sim \tau_0^{-2} = 4\pi G\bar{\rho}_m$ for this simulation.

F. Scale-dependent nonlinear fluxes and sources for a cold system

Having described the homogeneous part of Eq. (16), let us now consider the source term on its right-hand side. Equation (18) is a conservation equation with a source, of the form

$$\frac{\partial \hat{F}}{\partial \eta} + \nabla_{\mathbf{k},s} \cdot \mathbf{\Gamma} = \hat{S}. \quad (42)$$

⁷This behavior is generic: chaos—the exponential separation of nearby trajectories in phase space—combined with Liouville’s theorem, necessitates the formation of structure on smaller and smaller scales.

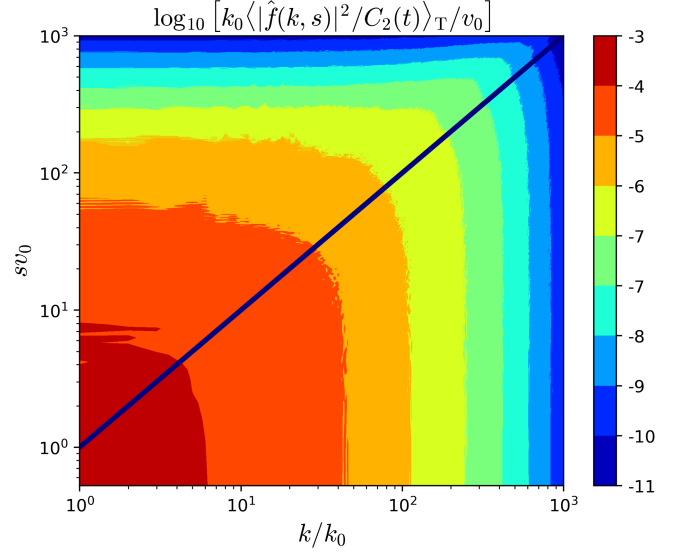


FIG. 3. A contour plot of the time-averaged power spectrum $\langle |\hat{f}|^2(t, k, s) / C_2(t) \rangle_T$, for the same simulation as in Fig. 1 (note that C_2 decays because of collisions and finite-grid effects, so it is sensible to normalize the power spectrum by C_2 at every time). The line $s = k\tau_0$ is also plotted in blue.

Here $\nabla_{\mathbf{k},s}$ is a six-dimensional phase-space gradient, and $\mathbf{\Gamma}$ is a six-dimensional flux, whose components are

$$\Gamma_i^s = k_i \hat{F}, \quad (43)$$

$$\Gamma_i^k = -s^j \langle \Phi_{ij} |\hat{f}|^2 \rangle. \quad (44)$$

The flux $\mathbf{\Gamma}$ of C_2 towards larger k and s is the “cascade” that we described in Sec. II E. Let us calculate the phase-space flux $\mathcal{F}^s$ flowing through a sphere in $\mathbf{s}$ space of radius s , where $s_c(k_{\text{nl}}) \ll s \ll v_{\text{th}}^{-1}$. In steady state, integrating Eq. (42) over $\mathbf{k}$ yields

$$\frac{\partial}{\partial s} \cdot \int \Gamma^s d^d k = \int \hat{S} d^d k. \quad (45)$$

Integrating both sides over the ball in $\mathbf{s}$ space with radius s (at all $\mathbf{k}$) and using Gauss’ theorem yields

$$\mathcal{F}^s \equiv \iint s^{d-2} s^i \Gamma_i^s d^{d-1} \Omega_s d^d k = \iint_{s' < s} \hat{S} d^d k d^d s', \quad (46)$$

where the $d^{d-1} \Omega_s$ integral is over the d -dimensional solid angle and the integral $d^d k$ is over all $\mathbf{k}$. Similarly, the flux in $\mathbf{k}$ is

$$\mathcal{F}^k \equiv \iint k^{d-2} k^i \Gamma_i^k d^{d-1} \Omega_k d^d s = \iint_{k' < k} \hat{S} d^d k' d^d s. \quad (47)$$

Hence, the accumulated nonlinear source $\iint \hat{S}$ plays the role of a mechanism that injects C_2 , which is balanced at

each k by the flux passing this C_2 on to smaller scales. Qualitatively, since the system is Jeans unstable at every scale $k' < k$, and since the resulting collapse time scale τ_g is the same at all scales, the integrated source (47) should receive contributions from the collapses occurring on all of these scales, adding up coherently.

Let us consider how phase-space fluxes depend on k and s in a turbulent state described by the phase-space spectrum (41). Using Eqs. (44) and (47) and the estimate $s^j \Phi_j^i / k \sim \tau_{\text{nl}}^{-1}$, we have

$$\mathcal{F}^k \sim \frac{k^d}{\tau_{\text{nl}}} \int_{s < s_c(k)} d^d s \hat{F} \sim \frac{F_1}{\tau_{\text{nl}}} [s_c(k)]^d \sim F_1 \kappa^{-(d-1)/2} k^d, \quad (48)$$

the last expression having been obtained via critical balance ($\tau_{\text{nl}} \sim \tau_c$) and Eqs. (31) and (32). Similarly, from Eqs. (43) and (46), and $s/k \sim \tau_1$, we have

$$\mathcal{F}^s \sim \frac{s^d}{\tau_1} \int_{k < k_c(s)} d^d k \hat{F} \sim \frac{F_2}{\tau_1} \tau_g^{-d} [k_c(s)]^{-d} \sim F_1 \sqrt{\kappa} s^d, \quad (49)$$

using $\tau_1 \sim \tau_c$ and Eqs. (31) and (32). The balances (46) and (47) imply that the accumulated source terms will have the same scalings, which indeed can be confirmed by direct estimates of the corresponding integrals, using the expression (19) for the nonlinear source, critical balance, and the phase-space power spectrum (41) [$\mathbf{s} \cdot \hat{\mathbf{g}}$ is estimated similarly to $s^j \Phi_j^i / k$ in Eq. (48)].

Thus, due to Jeans collapses occurring at all scales, the C_2 fluxes accumulate to become $\propto k^d$ in k space—and, by critical balance, $\propto s^d$ in s space. This is in contrast to how this works in Kolmogorov or Batchelor-style turbulence [38,39], where the fluxes of the corresponding invariants (energy of the fluid motion or passive-scalar variance, respectively) are constant across the inertial or convective ranges, because injection of these quantities occurs only at the outer scale. In kinetic systems, including our gravitational one, such a situation obtains at $sv_{\text{th}} \gg 1$ and $k \gg k_c(v_{\text{th}}^{-1})$ —for warm dark matter, or an ordinary plasma. The phase-space spectrum, derived for the latter case in Ref. [36], is then steeper in both k and s , with the k^{-d} and s^{-d} scalings replaced by k^{-2d} and s^{-2d} , respectively (see Appendix A 3)—we shall see in the next subsection that this is indeed how our scalings (41) are cut off at the thermal scale.

G. Numerical results

Let us now check whether the theoretical asymptotic scalings (41) are reproduced in our 1D Vlasov-Poisson simulation. The power spectra of the system illustrated in Fig. 1 are shown in Fig. 4: a k^{-1} power law establishes itself quickly, persisting up to $k \sim 100k_0$, which is of the order of $k_c(v_{\text{th}}^{-1})$, as expected. The spectra that we plot are $|\hat{f}|^2 / C_2(t)$, averaged over short time intervals (compared with the simulation's duration) as a proxy for the ensemble average. We rescale $|\hat{f}|^2$ by $C_2(t)$ at every time step because C_2 decays due to collisions and finite-grid effects. In Fig. 5,

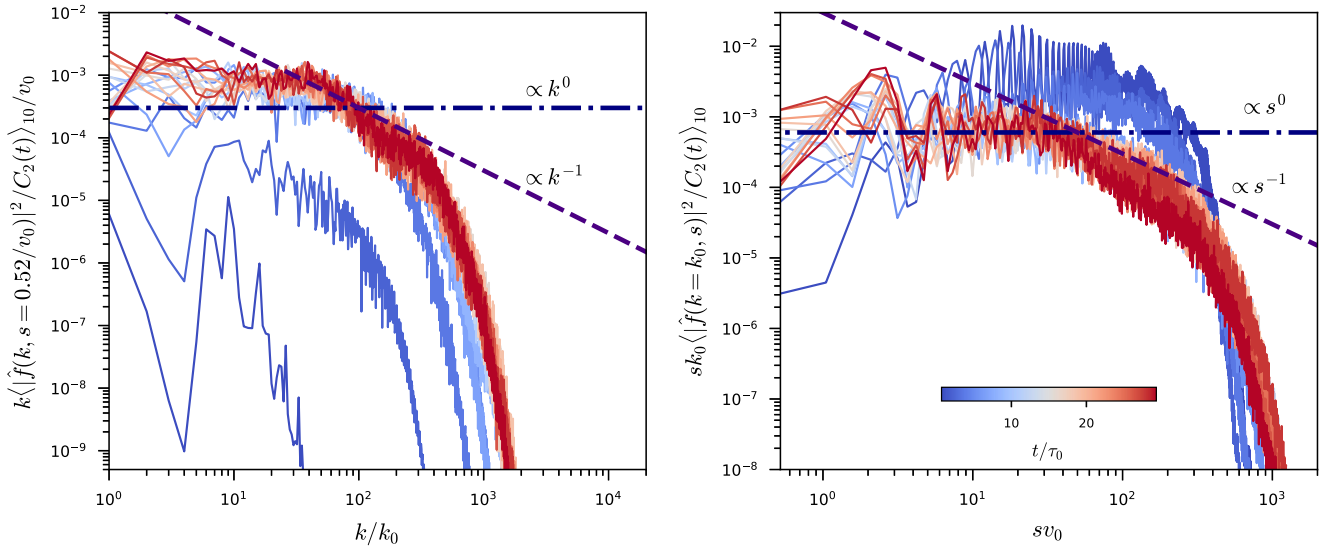


FIG. 4. The time evolution of the power spectra $\langle |\hat{f}|^2 / C_2(t) \rangle_{10}$ for the simulation illustrated in Fig. 1 (early times: blue; late times: red). The time averaging is over 10 simulation outputs (corresponding to time windows of duration $1.36\tau_0$). Left panel: the k spectrum at $s = 0.52/v_0$. Right panel: the s spectrum at $k = k_0$. Both spectra are compensated by the expected asymptotics (multiplied by k^1 and s^1 , respectively). A k^{-1} power law establishes itself quickly, extending to $k \sim 200k_0$, which is of the order of $k_c(v_{\text{th}}^{-1})$, as expected; similarly, the s^{-1} power law extends to $sv_0 \sim 200$. The steeper spectra at smaller scales are the scalings that prevail below the thermal scale (see Appendix A 3).

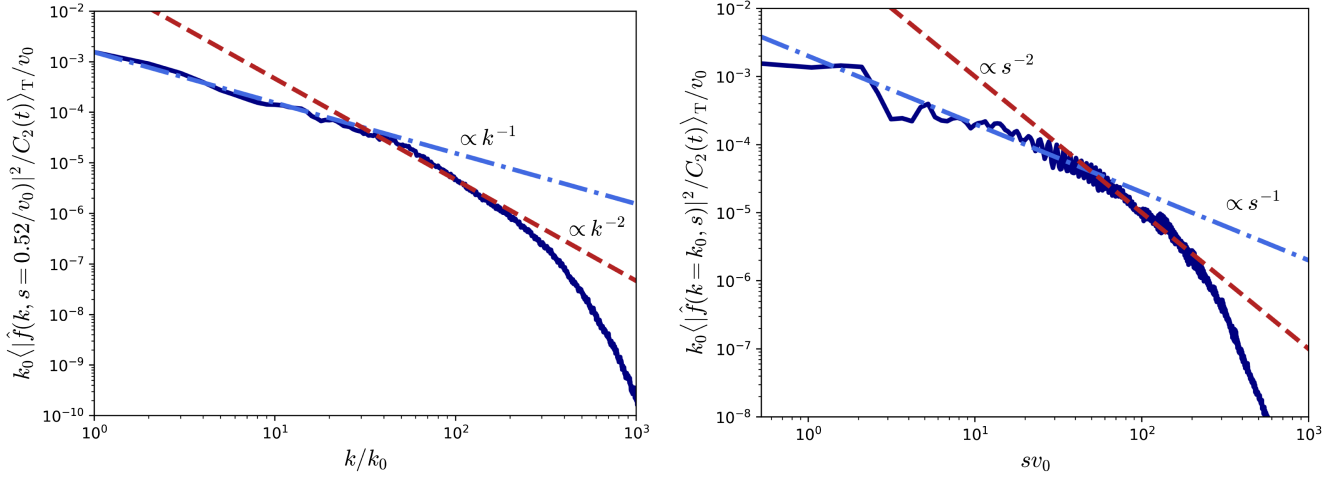


FIG. 5. The power spectra for the same simulation as in Fig. 1, but now averaged over $t \in [0, 30]\tau_0$, as a proxy for the ensemble average. Unlike in Fig. 4, they are uncompensated, and display the theoretical k^{-d} and s^{-d} scalings explicitly ($d = 1$).

we average over the entire simulation, displaying cleaner power laws in the appropriate ranges of k and s . While the system is not stationary, this only affects the overall amplitude—not the overall scaling—and Fig. 4 shows that the rescaling by C_2 corrects for that; hence, we can trust Fig. 5 to give an adequate approximation for $\hat{F}$.

As the system is sourced by the gravitational-collapse instability, which was shown in Sec. II D to continue until $(kv_{\text{th}})^{-1}$ matches the collapse time, the scaling of the power spectrum must be truncated at v_{th} . To test this proposition, it is necessary to see whether it is indeed the case that the asymptotics (41) persist until v_{th} is reached. We ran two identical simulations, differing only by the value of v_{th} , to verify this. The result is presented in Fig. 6: the left column has $v_{\text{th}} = 0.01v_0$ while the right column has $v_{\text{th}} = 0.05v_0$. The s spectra in the bottom row show that indeed, the s^{-1} power law is truncated at a lower value of s , by a factor that matches the ratio of v_{th} for the two runs. The s^{-2} (or k^{-2}) scaling of $\hat{F}$ at $sv_{\text{th}} \gg 1$ (or $k \gg k_c(v_{\text{th}}^{-1})$) is a known result in plasma systems [35,36], discussed in Appendix A 3.

III. SADDLE-POINT APPROACH

In this section, we focus on $d = 3$ spatial dimensions, and derive the k^{-d} scaling of the density power spectrum using a stationary-phase approximation. For pure gravitational Newtonian evolution of N identical particles, the density power spectrum is given exactly by [40]

$$P(k, t) = \frac{M^2}{V} \prod_{n=1}^N \int d^3 q_n d^3 p_n \mathcal{P}(\{q\}, \{p\}) e^{i\mathbf{k} \cdot [\mathbf{x}_1(t) - \mathbf{x}_2(t)]}, \quad (50)$$

where $(\mathbf{q}_n, \mathbf{p}_n)$ is the initial phase-space position of particle n , $(\mathbf{x}_n(t), \mathbf{v}_n(t))$ is the phase-space position of particle

n at time t , and $\mathcal{P}(\{q\}, \{p\})$ is the joint probability distribution of the initial phase-space positions $(\{q\}, \{p\}) \equiv \{(\mathbf{q}_n, \mathbf{p}_n)\}_{n=1}^N$ of all particles. This equation is permutation invariant, and, therefore, the choice of two particles is arbitrary.

For cold dark matter with Gaussian initial conditions, the initial distribution is

$$\mathcal{P}(\{q\}, \{p\}) = \frac{V^{-N} \mathcal{C}(\{q\}, \{p\})}{\sqrt{(2\pi)^{3N} \det C_{pp}^N}} e^{-\{p\}^T (C_{pp}^N)^{-1} \{p\} / 2}, \quad (51)$$

where $C_{pp}^N = C_{pp}^N(\{q\})$ is the $3N \times 3N$ covariance matrix of $\{p\}$, and $\mathcal{C}$ encapsulates initial density-density and density-momentum correlations [64], both of whose functional dependence on particle positions depends on the cosmology (we take a Λ CDM background).

The usual procedure to obtain the power spectrum $P(k, t)$ from Eq. (50) would involve integrating out particles $3, \dots, N$, leaving only a 12-dimensional integral over the phase-sub-space of particles 1 and 2. We, however, go the other way round, and integrate *first* over the relative position and momentum of this pair, and only then over all other particles—we will see that this order of integration is well suited to deriving the asymptotics of $P(k, t)$ as $k \rightarrow \infty$. Let us change the integration variables from $\mathbf{q}_1$ and $\mathbf{q}_2$ to $\mathbf{q} \equiv \mathbf{q}_1 - \mathbf{q}_2$ and $\mathbf{Q} \equiv (\mathbf{q}_1 + \mathbf{q}_2)/2$, and to their conjugate momenta, $\mathbf{p}$ and $\mathbf{P}$, respectively. As $\{p\}^T (C_{pp}^N)^{-1} \{p\}$ is quadratic in $\{p\}$, we can make the $\mathbf{p}$ dependence explicit, viz.,

$$\{p\}^T (C_{pp}^N)^{-1} \{p\} \equiv \mathbf{p}^T \Sigma^{-1}(\mathbf{q}) \mathbf{p} - 2\mathbf{a}(\mathbf{q}, \mathbf{Q}, \mathbf{P}, \{(q, p)\}_3^N) \cdot \mathbf{p} - 2B(\mathbf{q}, \mathbf{Q}, \mathbf{P}, \{(q, p)\}_3^N), \quad (52)$$

where $\mathbf{a}$ is a linear function of momenta and B a quadratic one. Below we will require the following properties of C_{pp}^N ,

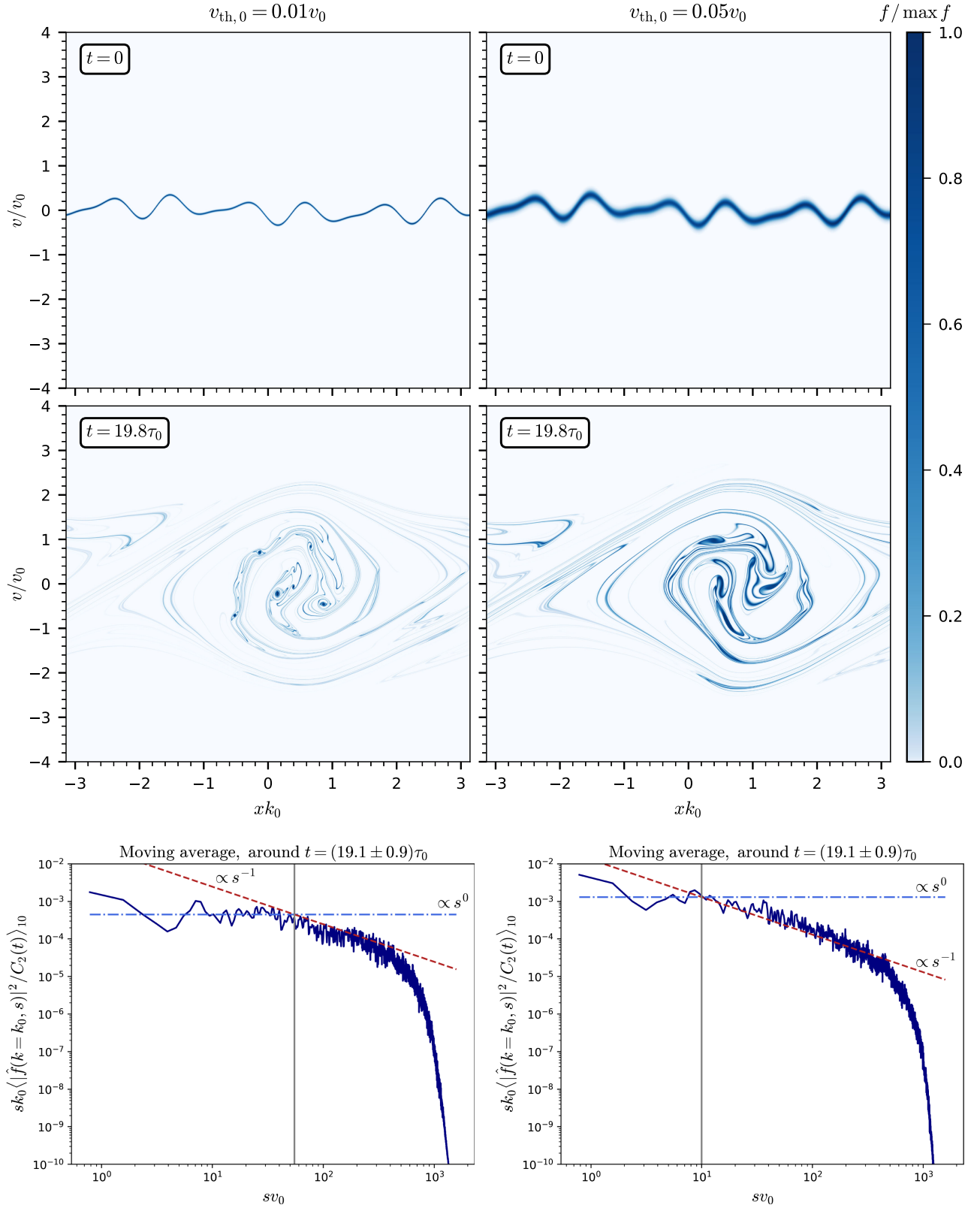


FIG. 6. A comparison between the evolution and the spectra of two systems with identical initial conditions, save for the value of v_{th} . Left panels: a colder case, $v_{\text{th}} = 0.01v_0$; right panels: a warmer case, $v_{\text{th}} = 0.05v_0$. The plots in the bottom row show the s spectra at the end of the simulations, compensated by s . In the warmer case, the spectrum ceases to scale as s^{-1} at around $sv_0 \sim 10$, while in the colder one, it does so at $sv_0 \sim 50$ (marked by gray lines). In both cases the s^{-1} scaling is replaced at $sv_{\text{th}} \gg 1$ by s^{-2} . See the text and Appendix A for details. Videos are available in the Supplementary Material [48].

derived in Appendix C, based on the assumption $v_{\text{th}} \rightarrow 0$ [so the conclusions again will be valid at $k \ll k_c(v_{\text{th}}^{-1})$, as in Sec. II]: (i) $\Sigma \simeq Aq^2$ at small q for some order-unity matrix A , (ii) $|\mathbf{a}| \sim O(q^{-1})$ (at most) in this limit, and (iii) at $q \rightarrow \infty$, the correlation matrix C_{pp}^N tends to a constant, i.e., Σ is order unity in the limit $q \rightarrow \infty$.

Using Duhamel's principle for Hamilton's equations of motion, one has [40]

$$\mathbf{i}\mathbf{k} \cdot [\mathbf{x}_1(t) - \mathbf{x}_2(t)] = \mathbf{i}\mathbf{q} \cdot \mathbf{k} + \mathbf{i}g_{qp}(t, t_{\text{initial}})\mathbf{p} \cdot \mathbf{k} + \mathbf{i}k\psi_I, \quad (53)$$

where g_{qp} is a “propagator” that encapsulates the linear evolution (defined below), and the “interaction term” is

$$k\psi_I(\mathbf{P}, \mathbf{Q}, \{(q, p)\}_3^N) \equiv \mathbf{k} \cdot \int_0^t dt' g_{qp}(t', t_{\text{initial}}) \times (\mathbf{F}_1(t') - \mathbf{F}_2(t')), \quad (54)$$

with $\mathbf{F}_n$ being the additional acceleration of particle n , relative to its motion in the part already included in g_{qp} , caused by all other particles. For example, if $g_{qp} = t - t_{\text{initial}}$, one reobtains regular Newtonian dynamics. Using results of first-order cosmological perturbation theory (or, alternatively, resummed kinetic field theory [65]), we take $g_{qp}(t) \equiv [D_+(t) - D_+(t_{\text{initial}})]/[dD_+/dt]$, where D_+ is the Λ CDM growth factor. Then ignoring the ψ_I term in (53) would just yield the Zel'dovich approximation, whose asymptotics were studied in [66,67].

A. Possible saddle points

Using Eqs. (52) and (53), we define the exponent

$$\varphi \equiv -\frac{1}{2}\mathbf{p}^T \Sigma^{-1} \mathbf{p} + \mathbf{a} \cdot \mathbf{p} + \mathbf{i}\mathbf{q} \cdot \mathbf{k} + \mathbf{i}g_{qp}\mathbf{p} \cdot \mathbf{k} + \mathbf{i}k\psi_I + B, \quad (55)$$

so the integral (50) for the power spectrum turns into

$$P(k, t) \propto \prod_{n=3}^N \int \frac{d^3 q_n d^3 p_n d^3 Q d^3 P d^3 q d^3 p}{V^N \sqrt{(2\pi)^{3N} \det C_{pp}^N}} \mathcal{C}(\{q\}, \{p\}) e^\varphi. \quad (56)$$

We start by integrating over $\mathbf{p}$ and $\mathbf{q}$, applying the saddle-point approximation. This is a movable-saddle problem, so it requires care in handling [68,69].⁸ The function ψ_I is a smooth function of $\mathbf{q}$ and $\mathbf{p}$, because it arises from the Hamiltonian flow in phase space, generated by a smooth

⁸Note also that the saddle point may be complex, but that is innocuous, since the exponent can be continued analytically to the complex plane.

gravitational potential (recall that we neglect collisions and that time is bounded). This assertion is just the statement that $\lambda = 1$ as in Sec. II. Letting $\psi_q \equiv \partial\psi_I/\partial\mathbf{q}$ and $\psi_p \equiv \partial\psi_I/\partial\mathbf{p}$, the exponent (55) is stationary when

$$\begin{aligned} -\Sigma^{-1}\mathbf{p} + \mathbf{a} + \mathbf{i}g_{qp}\mathbf{k} + \mathbf{i}k\psi_p &= 0, \\ -\frac{1}{2}p^i \frac{\partial(\Sigma^{-1})_{ij}}{\partial\mathbf{q}} p^j + p_i \frac{\partial a^i}{\partial\mathbf{q}} + \frac{\partial B}{\partial\mathbf{q}} + \mathbf{i}\mathbf{k} + \mathbf{i}k\psi_q &= 0. \end{aligned} \quad (57)$$

Let us parametrize the solution of these equations as

$$\mathbf{p} = k^\alpha \mathbf{c}, \quad \mathbf{q} = k^\beta \mathbf{d}, \quad (58)$$

where $\mathbf{c}$ and $\mathbf{d}$ are complex vectors whose magnitude remains finite as $k/k_{\text{nl}} \rightarrow \infty$. Together with α, β , they are to be determined by Eq. (57), by seeking a dominant balance, i.e., such a balance where the exponent (55) has the weakest k dependence around the stationary point that solves Eq. (57).

A priori, Eq. (57) could permit a balance that is independent of the initial-condition distribution, i.e., that same balance would exist for uniform initial conditions. But in that case, Liouville's theorem—after changing variables from the initial phase-space positions $\{(\mathbf{q}_n, \mathbf{p}_n)\}^N$ to the current ones $\{(\mathbf{x}_n, \mathbf{v}_n)\}^N$ —ensures that this yields a contribution to $P(k)$ proportional to $\delta^D(\mathbf{k})$, whence this balance is not dominant at large k . Conversely, the dominant balance must involve the first terms on the left-hand sides of Eq. (57). If $\beta > 0$, then, as $\Sigma \rightarrow_{q \rightarrow \infty} \text{const}$, the first term in the first equation in Eq. (57) is proportional to p . As this must play a part in a dominant balance, this implies that $\alpha = 1$. Substituting such a saddle point into Eq. (55) gives an exponentially suppressed contribution to $P(k)$, of order $\sim O[\exp(\alpha - k^2)]$ at most. Thus, stationary points with $\beta < 0$ are dominant, as they could contribute a power-law tail to $P(k)$. For $\beta < 0$, Eqs. (57) imply $1 = \alpha - 2\beta$ and $1 = 2\alpha - 3\beta$, whence $\alpha = \beta = -1$. This is consistent as long as

$$\lim_{\substack{(q,p) \sim k^{-1} \\ k \rightarrow \infty}} (|\psi_q|, |\psi_p|) = O(1), \quad (59)$$

which is generically valid, and so the interaction term is potentially as important as the linear term; that both are equally important is essentially a statement of critical balance (cf. Sec. II C).

B. Evaluation of the asymptotics

Having proven that the asymptotic expansion of the power spectrum is given, up to exponentially small (in k) contributions, by its dominant saddle point at $q, p \sim k^{-1}$, one is allowed to replace φ with its expansion at small q and p . This yields

$$\varphi \simeq -\frac{1}{2} \mathbf{p}^T \Sigma^{-1} \mathbf{p} + \mathbf{a} \cdot \mathbf{p} + i\mathbf{q} \cdot \mathbf{k} + ig_{qp} \mathbf{p} \cdot \mathbf{k} + ik[\mathbf{p} \cdot \boldsymbol{\psi}_p(0) + \mathbf{q} \cdot \boldsymbol{\psi}_q(0)] + B(q=0). \quad (60)$$

With this asymptotic approximation for φ , we need to integrate Eq. (56) over $\mathbf{p}$ and $\mathbf{q}$ to obtain its asymptotic

scaling with k . Observe that $\det C_{pp}^N$ factorizes into $\det \Sigma$ multiplied by normalization factors for the other momenta variables, and the only other pieces that still depend on $\mathbf{Q}$, $\mathbf{P}$, and the initial positions and momenta of the other particles are $\mathcal{C}$, $\mathbf{a}$, B , and $\boldsymbol{\psi}_{q,p}$. Thus,

$$P(k, t) \simeq \frac{M^2}{V^2} \left\langle \int \frac{d^3 q d^3 p \mathcal{C}}{[(2\pi)^3 \det \Sigma]^{1/2}} \exp \left\{ -\frac{1}{2} \mathbf{p}^T \Sigma^{-1} \mathbf{p} + \mathbf{a} \cdot \mathbf{p} + i\mathbf{q} \cdot \mathbf{k} + ig_{qp} \mathbf{p} \cdot \mathbf{k} + ik[\mathbf{p} \cdot \boldsymbol{\psi}_p(0) + \mathbf{q} \cdot \boldsymbol{\psi}_q(0)] \right\} \right\rangle, \quad (61)$$

where the average is over the position and momentum of the center of mass of particles 1 and 2, as well as the positions and momenta of all the other particles (e^B is absorbed into the average). Integrating over $\mathbf{p}$, we get

$$P(k, t) \simeq \frac{M^2}{V^2} \left\langle \int d^3 q \mathcal{C} \exp \left\{ -\frac{k^2}{2} \left[g_{qp} \hat{\mathbf{k}} + \boldsymbol{\psi}_p(0) - \frac{i\mathbf{a}}{k} \right]^T \Sigma(\mathbf{q}) \left[g_{qp} \hat{\mathbf{k}} + \boldsymbol{\psi}_p(0) - \frac{i\mathbf{a}}{k} \right] + i\mathbf{q} \cdot \mathbf{k} + ik\mathbf{q} \cdot \boldsymbol{\psi}_q(0) \right\} \right\rangle. \quad (62)$$

This integration does not introduce any powers of k because both $d^3 p$ and $\sqrt{\det \Sigma}$ are proportional to k^{-3} in the vicinity of the stationary point, so the two cancel. Recalling that the saddle point is at $\mathbf{q} = \mathbf{d}/k$, where $\mathbf{d}$ is a finite constant, we change variables to $\mathbf{y} = k\mathbf{q}$, whence

$$P(k, t) \simeq \frac{M^2}{V^2 k^3} \left\langle \int d^3 y \mathcal{C} e^{-k^2 [g_{qp} \hat{\mathbf{k}} + \boldsymbol{\psi}_p(0) - \frac{i\mathbf{a}}{k}(\mathbf{y}/k)]^T \Sigma(\mathbf{y}/k) [g_{qp} \hat{\mathbf{k}} + \boldsymbol{\psi}_p(0) - \frac{i\mathbf{a}}{k}(\mathbf{y}/k)]/2 + i\mathbf{y} \cdot [\hat{\mathbf{k}} + \boldsymbol{\psi}_q(0)]} \right\rangle \equiv \frac{F_1}{V} k^{-3}. \quad (63)$$

Therefore, the integral $d^3 q$, as opposed to the momentum integral in (61), is not compensated by any function that scales like k^{-3} , so it is this integration that yields the k^{-3} scaling, which emerges from $q \sim k^{-1}$ at the stationary point. The exponential in Eq. (63) is order unity in the limit $k \rightarrow \infty$, because $\Sigma \sim y^2/k^2$, and also because $\mathbf{a}/k$ is at most order unity,⁹ whence the $\mathbf{y}$ integral is a Gaussian integral, and the entire factor multiplying k^{-3} in Eq. (63), denoted by F_1/V , is of order unity.

Thus, we have established that for collisionless, cold dark matter in the nonrelativistic limit, the asymptotic behavior of the power spectrum as $k \rightarrow \infty$ is dominated by the contribution of the saddle point at $q, p \sim k^{-1}$. This yields $P(k) \propto k^{-3}$, with an order-unity coefficient; that this coefficient is nonzero follows from the more qualitative argument of Sec. II.

IV. DISCUSSION

A. The importance of being cold

The k^{-d} power spectrum arose from a turbulent state, where the gravitational field was treated as a smooth field—this is an analogue of Batchelor turbulence in hydrodynamics [39], where a tracer is advected by a large-scale, smooth velocity field. A crucial step in the derivation of the form of the power spectrum was the

occurrence of gravitational collapse on all k such that $\tau_g k v_{\text{th}} \ll 1$ —that the initial configuration was cold, and Jeans unstable. This was also necessary in Sec. III, where the initial conditions were special to cold dark matter, with the particular functional form of the C_{pp}^N matrix, $\Sigma \sim Aq^2$, reflecting the fact that if two particles started out at exactly the same spatial position, they would remain together forever.

The coldness of the system means that locally in phase space, the distribution function is always a stream, and therefore, locally, everywhere and on every scale larger than the thermal scale, the Raychaudhuri equation (5) applies. This, in turn, implies that the stream is unstable to gravitational collapse, with the collapse time scale (36), matching the critical-balance time (22). As the collapse occurs on all scales, one deduces from Poisson's equation that the only way in which critical balance could be satisfied is if the density power spectrum is $P(k) \propto k^{-d}$ —otherwise, the two physical length scales k_{nl} and $v_{\text{th}} \tau_c$ would have to coincide, thereby violating the coldness assumption.

When the system is not cold, these assumptions fail. Indeed, the source term would not accumulate when integrated over k and s up to small, nonlinear scales, because at $sv_{\text{th}} \gg 1$ and $s_c(k)v_{\text{th}} \gg 1$, there is no instability to drive the collapse, so one would expect the flux to be $\mathcal{F}^{\mathbf{k}} = \text{const}$, rather than $\propto k^d$ in Eq. (48). In other words, for a warm system, the two length scales k_{nl} and $v_{\text{th}} \tau_c$ are indeed the same.

⁹The exponential might depend on $\hat{\mathbf{k}}$, the direction of $\mathbf{k}$, before being ensemble averaged.

While in a collisionless setting, v_{th} would not change with time,¹⁰ in reality finite- N effects do increase v_{th} . Additionally, baryons would influence the matter power spectrum significantly on small scales, probably leading to a deviation from the power-law scalings found here. The approach used in this paper does not apply once interactions with baryons are included, and this is its main limitation. As mentioned, it only applies insofar as dark matter behaves as a *cold*, classical *phase-space* fluid—other dark-matter candidates might follow different scalings, and we defer this to future work.

Were it not for baryons, a full theory of the constituent particle of dark matter should predict both the initial value of v_{th} and the functional form of the effective collision operator; this should allow one to find how C_2 changes with time, and thence how v_{th} evolves. Measuring the break in the dark-matter power spectrum—the transition from k^{-3} to a different, steeper power law of k^{-2d} [35,36] (in $d = 1$, from k^{-1} to a k^{-2} scaling, as in Fig. 5)—would hypothetically allow one to find the present-day value of v_{th} , which is directly connected to the nature of dark matter (if indeed it held that $u_\nu \ll v_{\text{th}}$, where the collisional velocity scale u_ν is defined in Appendix A). For example, for weakly interacting massive particles, which decouple from the photons when nonrelativistic, one would have

$$v_{\text{th}}(z) = c \frac{T(z)}{T_{\text{kd}}} \sqrt{\frac{2k_B T_{\text{kd}}}{mc^2}} \approx 3.3 \times 10^{-12} c(1+z) \left[\frac{10 \text{ MeV}}{T_{\text{kd}}} \right]^{1/2} \left[\frac{\text{GeV}}{m} \right]^{1/2}, \quad (64)$$

where $T(z)$ is the photon temperature at redshift z and T_{kd} is the kinetic-decoupling temperature [24,25,70]. This is far too small to be observed practically, and moreover, it corresponds to (cf. Appendix B)

$$k_c(v_{\text{th}}^{-1}) \sim \delta_{\text{typ}}^{1/2} \frac{H}{v_{\text{th}}} \approx \frac{10^8 h(z)}{1+z} \delta_{\text{typ}}^{1/2} \text{ Mpc}^{-1} \left[\frac{m}{\text{GeV}} \right]^{1/2} \left[\frac{T_{\text{kd}}}{10 \text{ MeV}} \right]^{1/2}, \quad (65)$$

where δ_{typ} is a typical overdensity and the Hubble constant is $H(z) = 100h(z) \text{ km s}^{-1} \text{ Mpc}^{-1}$ (this value is of the same order as the free-streaming scale [24] for $\delta_{\text{typ}} \sim 1$).

¹⁰Because, if we denote $\|f\|_\infty = \max f$, $C_2 \sim \|f\|_\infty^2 v_{\text{th}}^d V$, and both C_2 and $\|f\|_\infty$ are conserved (in fact, one can define $v_{\text{th}} \equiv C_2^{1/d} \|f\|_\infty^{-2/d} V^{-1/d} / \sqrt{\pi}$, as a measure of the thinness of the distribution). There is of course a change in $\|f\|_\infty$ because of the Universe's expansion, but this is very slow for the large- k limit, where $k \gg \mathcal{H}/c$.

B. Some like it hot: Similarities with plasma physics

A phase-space Batchelor-type cascade was recently proposed to be the universal regime of (plasma) Vlasov-Poisson turbulence at Debye and sub-Debye scales [35,36]. This has been verified numerically in 1D simulations of turbulence driven by an external forcing [36] and by the two-stream instability [71]. Like in the cold-dark-matter turbulence presented here, the (electric) field fluctuations are spatially smooth, so the phase-space mixing of the distribution function is dominated by the outer-scale fields. However, unlike cold-dark-matter turbulence, which is sourced at every Jeans-unstable scale, the cascade in the plasma case has a constant flux of C_2 , because C_2 is only sourced at the outer scale; this changes the scalings of the power spectrum and of the field spectrum. This situation is analogous to the gravitational phase-space turbulence at $s \gg v_{\text{th}}^{-1}$ that appears in the simulations presented in Fig. 6. We will investigate this regime further in future work.

C. Implications for dark-matter haloes

The $P(k) \propto k^{-3}$ scaling (in 3D) derived here for the nonlinear power spectrum sheds some light on universal properties of dark matter haloes. Within the hierarchical-clustering paradigm [72], the small-scale limit of the power spectrum is dominated by the one-halo term,

$$P_{1h}(k) = \int dM \frac{dn}{dM} [R^3 \hat{\delta}(kR)]^2, \quad (66)$$

where dn/dM is the halo mass function, $R(M) \equiv R_{200}(M)/c(M)$, $c(M)$ is the halo's concentration, R_{200} is the radius where the density is 200 times the critical density of the Universe, and $\hat{\delta}$ is the Fourier transform of the normalized halo density profile $\delta(x/R)$ [73]. If we define $\Lambda_c \equiv -d \ln c / d \ln M$ and $\Lambda_n \equiv d \ln (dn/dM) / d \ln M$, both at $M \rightarrow 0$, and also

$$\Lambda_\delta \equiv - \lim_{kR \rightarrow \infty} \frac{d \ln \hat{\delta}(kR)}{d \ln(kR)}, \quad (67)$$

then, as $k \rightarrow \infty$,

$$P_{1h}(k) \sim k^\gamma, \quad \gamma = - \min \left\{ \frac{3(3 + \Lambda_n)}{1 + 3\Lambda_c}, 2\Lambda_\delta \right\}. \quad (68)$$

Setting $\gamma = -3$, as would be required by our result, yields a relation between the universal concentration-to-mass relation and the halo mass function for cold dark matter: either $\Lambda_\delta = 3/2$ (cf. [74–77]) and $\Lambda_n > 3\Lambda_c - 2$, or $\Lambda_\delta > 3/2$ and

$$\Lambda_n = 3\Lambda_c - 2. \quad (69)$$

This constrains semianalytic prescriptions for halo mass functions [78]. We remark that $\Lambda_\delta > 3/2$ for both the Navarro-Frenk-White [79] and the Einasto profiles [80].

V. SUMMARY

In this paper, we have proposed a physical mechanism that produces the k^{-d} asymptotic scaling of the dark-matter power spectrum naturally. This was done in two ways: by expressing $P(k)$ as a phase-space integral and analyzing it with a stationary-phase method (Sec. III), and via a phenomenological study of a critically balanced phase-space turbulent state, akin to Batchelor turbulence, for a cold, collisionless, self-gravitating system (Sec. II). Both methods are phase-space based, and so remain valid even after streams cross. The fact that the phase-space distribution function is cold (i.e., that it is only nonzero in a d -dimensional submanifold of phase space) was crucial to both approaches. Gravitational collapse sources a cascade of the quadratic Casimir invariant, which is sustained by the joint action of phase mixing and tidal forces (by smooth fields). The balance between linear free streaming and tidal forces, which ultimately dictates the form of the power spectrum, also appears in the saddle-point argument of Sec. III. In contrast with many turbulent systems, gravitational turbulence is different in that the flux of the Casimir invariant is not constant over the range of scales of interest, but there still exists a universal scaling regime of the phase-space power spectrum.

The validity of our approach is supported by 1D Vlasov-Poisson simulations, which confirm our theoretical predictions.

Our determination of the small-scale asymptotics of the dark-matter power spectrum may allow for a nontrivial test of effective field theories of the large scales, either by imposing these asymptotics on them, or by using the phase-space power spectrum found here as a closure for these theories. We intend to investigate these possibilities in future work.

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DATA AVAILABILITY

The simulations presented in this paper were conducted with the publicly available code Gkeyll [46]. Input files for the simulations are openly available [81].

APPENDIX A: SIMULATION METHODS

Here we describe the details of the Vlasov-Poisson simulations whose results are shown in Sec. II and Figs. 1–6. The simulations are conducted using the Gkeyll code,¹¹ which is an Eulerian solver in phase-space, originally designed for the Vlasov-Maxwell system. Gkeyll uses a discontinuous Galerkin algorithm to compute distribution functions [46]. We set the vacuum permittivity $\epsilon_0 = -1$, which turns electrostatic interactions into gravitational ones—and, for unit particle charges, it is equivalent to choosing units so that $4\pi G = 1$. Our simulations are 1D in position space (so the phase space is 2D), with periodic spatial boundary conditions; the units of length are such that the box size is $L = 2\pi$. We always take the particle mass to be unity, which implies that the system's total mass is normalized to $M = 2\pi$. These three choices specify the units for the simulations, and are equivalent to choosing $\tau_0 = k_0 = v_0 = 1$ (these are defined in Sec. II A). Thus, the outer scale $k_{nl} \sim k_0$ is of order unity.

1. Initial conditions

We simulate a single-species distribution function, with the initial condition (4), where we chose $v_{th} = 0.005v_0$ (Fig. 1), $0.01v_0$, or $0.05v_0$ (the left and right columns of Fig. 6, respectively), and

$$\rho_{in}(x) \frac{L}{M} = 1 - \sum_{n=1}^5 \frac{a_n}{k_n} \sin(k_n x + \phi_n), \quad (A1)$$

$$u_{in}(x) = v_0 \sum_{n=1}^5 b_n \cos(k_n x + \phi_n), \quad (A2)$$

where $\phi_n \in [0, 2\pi]$ are random phases, the amplitudes a_n/k_0 are uniformly sampled from the interval $[0, 0.2]$, and $k_n = k_0 U_n$, with $U_n \in \{1, 2, \dots, 10\}$ —uniformly distributed random integers. For runs with one initial stream (Fig. 6), $b_n = a_n/k_0$, while for runs with three initial streams (Fig. 1), we set $b_n = 0.05a_n/k_0$ and triple the initial conditions (4) by shifting $u_{in}(x)$ by $\pm 2v_0$ and duplicating for two additional streams. We ensure that the resolution is sufficiently fine for a Maxwellian of width

¹¹<https://gkeyll.readthedocs.io/>

v_{th} still to be resolved: we use an $N_x \times N_v$ phase-space grid with $N_x = N_v = 4032$ for the single-stream initial condition and $N_x = N_v = 7680$ for the multistream one. The simulation box contains v_0/v_{th} Jeans lengths, corresponding to 200, 100 and 20 Jeans lengths for the three choices of v_{th} above.

Because of numerical errors, f can become slightly negative in very small and isolated areas. This is because the discontinuous Galerkin algorithm that Gkeyll uses is not positivity preserving, and errors arise from overshooting due to large derivatives. These problems are somewhat alleviated by the small collision operator (see Appendix A 2), which smooths large velocity derivatives. In any case, these regions do not cause the simulation to become unstable because they are isolated and $|f|$ is still very small there; the total mass occupied by negative f is $\iint f \Theta(-f) dx dv < 10^{-5} M$ for the simulation in Fig. 1 and $< 0.02 M$ for Fig. 6. Therefore, this does not invalidate any of our conclusions in this paper, and we set $f \mapsto f \Theta(f)$ for the purpose of plotting Figs. 1 and 6.

2. Collisions

One would like to run collisionless simulations, but the finite resolution of any numerical representation of phase space induces an unavoidable effective collisionality due to the grid. Many dark-matter models imply that dark matter in the Universe consists of a finite number of particles [82]—and finite- N effects also induce a effective collisionality. To control the cutoff scale in velocity space and the dissipation of C_2 , we add a collision operator $(\partial f / \partial t)_c \propto \nu$ with a small collision frequency ν . The collision operator used is the Dougherty operator—a type of Fokker-Planck operator in velocity space [83,84]:

$$\left(\frac{\partial f}{\partial t} \right)_c = \nu \frac{\partial}{\partial v} \left[(v - u) f + v_t^2 \frac{\partial f}{\partial v} \right], \quad (\text{A3})$$

where $u(x)$ is first velocity moment of f and $v_t^2 \equiv \int dv (v - u)^2 f / \rho$. Details can be found in [84], which described the implementation of Eq. (A3) in Gkeyll. We use a Dorland number [85] of $\text{Do} \equiv (\nu \tau_0)^{-1} = 10^6$ for all runs.

The inevitability of some collisions—whether due to the grid, finite- N effects, or a collision operator—implies the existence of another scale in the problem. The collision time scale is [35,84]

$$\tau_\nu \sim \frac{1}{\nu s^2 v_{\text{rms}}^2}, \quad (\text{A4})$$

where v_{rms}^2 is the second velocity cumulant. The velocity scale where collisions become competitive with the cascade rate ($\tau_\nu \sim \tau_g$) is, therefore,

$$s_\nu = \frac{1}{v_{\text{rms}} \sqrt{\nu \tau_g}}. \quad (\text{A5})$$

By critical balance (see Sec. II C), this also gives a length scale

$$k_\nu = \frac{1}{L \sqrt{\nu \tau_g}} \sim \frac{k_{\text{nl}}}{\sqrt{\nu \tau_g}}, \quad (\text{A6})$$

which is the collisional cutoff in position space. For a sufficiently small ν , one can have¹²

$$k_{\text{nl}} \ll k \ll k_c(v_{\text{th}}^{-1}) \ll k_\nu, \quad (\text{A7})$$

$$v_{\text{rms}}^{-1} \ll s \ll v_{\text{th}}^{-1} \ll s_\nu, \quad (\text{A8})$$

where $k_c(s)$ is the critical-balance scale defined by Eq. (33). While this hierarchy justifies neglecting collisions in this paper, as phase-space structure cascades to ever smaller scales by (18), the collisional scales, where it is erased, are eventually reached [35,36]. Likewise, in reality, a particle-noise floor $\hat{F} \approx M^2/N$ (which is not present in our simulations) is eventually reached, too. Determining whether this occurs on scales smaller or larger than l_ν is deferred to future work (cf. [36]).

3. Phase-space power spectra

In the 1D setup described above, our prediction (41) for the phase-space power spectrum becomes

$$\hat{F}(k, s) \sim \begin{cases} F_1 k^{-1}, & \text{if } \max\{k_{\text{nl}}, k_c(s)\} \ll k \ll \min\{k_c(v_{\text{th}}^{-1}), k_\nu\}, \\ F_2 s^{-1}, & \text{if } \max\{v_{\text{rms}}^{-1}, s_c(k)\} \ll s \ll \min\{v_{\text{th}}^{-1}, s_\nu\}. \end{cases} \quad (\text{A9})$$

We are not concerned here with the asymptotics of the power spectrum $\hat{F}$, defined in Eq. (10), at values of s and k above v_{th}^{-1} and $k_c(v_{\text{th}}^{-1})$, respectively, even if they lie below the collisional cutoffs s_ν and k_ν [see Eqs. (A5) and (A6)]. In this region, gravitational collapse halts by Eq. (35), but if one extrapolates the findings of Ref. [36]—who

¹²In reality, it might be that $s_\nu v_{\text{th}} \lesssim 1$, for both parameters depend sensitively on the nature of the constituent particle(s) of dark matter. But as long as s is smaller than both s_ν and v_{th}^{-1} , none of the conclusions of this paper are invalidated by this possibility.

studied turbulence in electrostatic plasmas with v_{th} of order unity—from electrostatics to gravity, then one should have, assuming $s_\nu v_{\text{th}} \gg 1$,

$$\hat{F}(k, s) \sim \begin{cases} F_1 k^{-1}, & \text{if } \max\{k_{\text{nl}}, k_c(s)\} \ll k \ll k_c(v_{\text{th}}^{-1}), \\ F_2 s^{-1}, & \text{if } \max\{v_{\text{rms}}^{-1}, s_c(k)\} \ll s \ll v_{\text{th}}^{-1}, \\ F_3 k^{-2}, & \text{if } \max\{k_{\text{nl}}, k_c(s), k_c(v_{\text{th}}^{-1})\} \ll k \ll k_\nu, \\ F_4 s^{-2}, & \text{if } \max\{v_{\text{rms}}^{-1}, s_c(k), v_{\text{th}}^{-1}\} \ll s \ll s_\nu. \end{cases} \quad (\text{A10})$$

The spectrum is truncated exponentially at $k > k_\nu$ or $s > s_\nu$ (or submerged into particle noise as discussed in [36]).

Figure 1 shows an example of the time evolution of a system that starts with three streams with $v_{\text{th}} = 0.005 v_0$ and has $\text{Do} = 10^6$. This corresponds to $u_\nu \sim 10^{-3} v_0$, so scales are separated in the way that we have assumed. The grid size is $N_x = N_y = 7680$ and the velocity box size is $12 v_0$, so the Nyquist velocity scale is $s_{\text{Ny}} \approx 2011 v_0^{-1}$. Figure 1 shows that the streams collapse quickly, by rotating and twisting in phase space. In Fig. 4, we show the evolution of the power spectra for this simulation.

APPENDIX B: EINSTEIN-DE SITTER UNIVERSE

Although the analysis of Sec. II, resulting in the prediction (41) for the phase-space power spectrum, is valid for ΛCDM , and in general for any cosmology where there is a separation of scales between $\mathcal{H}/c$ and the nonlinear scale k_{nl} , in the special case of an Einstein-de Sitter (EdS) cosmology, there exists a scale invariance (see, e.g., [1,86]) that provides additional insight, which we sketch in this appendix. This allows us to propose a global analogue of the local analysis of the main text. We focus on the case of $d = 3$ for simplicity.

In an EdS space-time, the Vlasov-Poisson system is invariant under the transformation

$$(t, \mathbf{x}, \mathbf{v}, f) \mapsto (\zeta t, \zeta^\varpi \mathbf{x}, \zeta^{\varpi-1/3} \mathbf{v}, \zeta^{3\varpi+1} f), \quad (\text{B1})$$

where ϖ is fixed by assuming the scaling of the density power spectrum. This means that for every (inverse) length scale k , there exists a (cosmic) time scale $t \propto k^{1/\varpi}$. Hence, one obtains a (conformal) time scale, which is just the collapse conformal time in the standard spherical-collapse model,

$$\tau_g(k) \equiv \frac{2}{H_0} \left(\frac{\pi}{2H_0 t_0} \right)^{1/3} \left(\frac{3}{5\delta_{\text{rms}}(k)} \right)^{1/2}, \quad (\text{B2})$$

where H_0 and t_0 are the present-day Hubble constant and cosmic time, and $\delta_{\text{rms}}(k)$ is the rms amplitude of the dark-matter density fluctuation at scale k , normalized by the EdS growth factor $D_+(t_{\text{initial}}) = a(t_{\text{initial}})$. The time scale (B2) turns out to be nothing but the standard gravitational time scale $\tau_g \sim 1/\sqrt{G\delta\rho}$ [cf. Eq. (36)], where $\delta\rho \sim \delta_{\text{rms}}\rho_{\text{crit}}$, with the critical density defined by $\rho_{\text{crit}} = 3H_0^2/(8\pi G)$. Relating $\delta_{\text{rms}}(k)$ to the overdensity power spectrum $P_\delta(k) = P_0(k/k_p)^n$ for some constant k_p , via $\delta_{\text{rms}}(k) = \sqrt{k^3 P_\delta(k)}$, yields

$$\tau_g \sim \frac{2\sqrt{3}}{\sqrt{5}H_0} \left(\frac{\pi}{2H_0 t_0} \right)^{1/3} \left(\frac{k_p^n}{P_0 k^{3+n}} \right)^{1/4}. \quad (\text{B3})$$

By critical balance (Sec. II C), the collapse time scale τ_g and the linear phase-mixing time τ_1 must be the same, whence

$$s_c(k) \propto k^{(1-n)/4}. \quad (\text{B4})$$

Consequently, a C_2 flux $\propto k^d$ (cf. Sec. II F) would imply

$$k^{4+\gamma} \propto s_c(k). \quad (\text{B5})$$

Using Eq. (B3), one concludes from Eq. (B5) that the exponent of the density power spectrum in (24) is $\gamma = -(15+n)/4$. In a steady state, the power spectrum must be invariant under the collapse process, and so in a self-sustaining scenario, $\gamma = n$, which is solved by $n = -3$. This is unsurprising, because an EdS space-time is just a matter-dominated Friedmann-Lemaître-Robertson-Walker space-time; as this paper is focused on small scales, locally, everything is matter dominated in any cosmology.

APPENDIX C: INITIAL MOMENTUM CORRELATIONS

The purpose of this appendix is to prove, for Sec. III, that (i) $\Sigma \sim Aq^2$ and (ii) $\Sigma \mathbf{a} = O(q)$ as $q \rightarrow 0$, and that (iii) in the opposite limit, $q \rightarrow \infty$, Σ is order unity. These quantities are defined in Eq. (52). What follows holds for $v_{\text{th}} = 0$.

Consider Eq. (52). Since $2\mathbf{p}_1 = \mathbf{p} + 2\mathbf{P}$ and $2\mathbf{p}_2 = 2\mathbf{P} - \mathbf{p}$, the part involving $\mathbf{a}$ comes solely from the cross-correlations between $\mathbf{p}_1$, $\mathbf{p}_2$ and $(\mathbf{p}_3, \dots, \mathbf{p}_N)$, viz.,

$$\begin{aligned} \mathbf{p}_1^T (C_{pp}^N)^{-1}_{1n} \mathbf{p}_n + \mathbf{p}_2^T (C_{pp}^N)^{-1}_{2n} \mathbf{p}_n &= \mathbf{P}[(C_{pp}^N)^{-1}_{1n} + (C_{pp}^N)^{-1}_{2n}] \mathbf{p}_n + \frac{1}{2} \mathbf{P}[(C_{pp}^N)^{-1}_{1n} - (C_{pp}^N)^{-1}_{2n}] \mathbf{p}_n \\ &+ \mathbf{P}[(C_{pp}^N)^{-1}_{11} - (C_{pp}^N)^{-1}_{22}] \mathbf{P} + 2\mathbf{P}[(C_{pp}^N)^{-1}_{11} + (C_{pp}^N)^{-1}_{21} + (C_{pp}^N)^{-1}_{12} + (C_{pp}^N)^{-1}_{22}] \mathbf{P}, \end{aligned} \quad (\text{C1})$$

where $(C_{pp}^N)^{-1}_{mn}$ specifies the submatrix of $(C_{pp}^N)^{-1}$ that pertains to particles m and n , and the Einstein summation convention is used. Now, $(C_{pp}^N)^{-1}_{mn}$ is a function of the positions of the particles only. When $q \equiv |\mathbf{q}_1 - \mathbf{q}_2| \rightarrow 0$, $\mathbf{q}_1$ and $\mathbf{q}_2$ both tend to $\mathbf{Q} = (\mathbf{q}_1 + \mathbf{q}_2)/2$, and, by the indistinguishability of the constituent particles,

$$(C_{pp}^N)^{-1}_{1n}(\mathbf{q}_1 = \mathbf{Q}, \mathbf{q}_2 = \mathbf{Q}, \mathbf{q}_3, \dots, \mathbf{q}_N) = (C_{pp}^N)^{-1}_{2n}(\mathbf{q}_1 = \mathbf{Q}, \mathbf{q}_2 = \mathbf{Q}, \mathbf{q}_3, \dots, \mathbf{q}_N), \quad (\text{C2})$$

$$(C_{pp}^N)^{-1}_{11}(\mathbf{q}_1 = \mathbf{Q}, \mathbf{q}_2 = \mathbf{Q}, \mathbf{q}_3, \dots, \mathbf{q}_N) = (C_{pp}^N)^{-1}_{22}(\mathbf{q}_1 = \mathbf{Q}, \mathbf{q}_2 = \mathbf{Q}, \mathbf{q}_3, \dots, \mathbf{q}_N). \quad (\text{C3})$$

Consequently, for $q \rightarrow 0$ and finite $\mathbf{Q}, \mathbf{q}_3, \dots, \mathbf{q}_N, \mathbf{P}, \mathbf{p}_3, \dots, \mathbf{p}_N$, we have

$$[(C_{pp}^N)^{-1}_{1n} - (C_{pp}^N)^{-1}_{2n}] = O(q), \quad (\text{C4})$$

$$[(C_{pp}^N)^{-1}_{11} - (C_{pp}^N)^{-1}_{22}] = O(q). \quad (\text{C5})$$

Therefore, in the limit $q \rightarrow 0$, the matrix $(C_{pp}^N)^{-1}$, *qua* a linear operator in momentum space, splits into a block matrix in the basis $(\mathbf{p}, \mathbf{P}, \mathbf{p}_3, \dots, \mathbf{p}_N)$, with one block for $\mathbf{p}$ and another for $(\mathbf{P}, \mathbf{p}_3, \dots, \mathbf{p}_N)$. Hence its inverse C_{pp}^N is also a block matrix in this basis. Namely,

$$C_{pp}^N = \left(\begin{array}{c|cc} (C_{pp}^N)_{11} + (C_{pp}^N)_{22} - (C_{pp}^N)_{12} - (C_{pp}^N)_{21} & 0 & (C_{pp}^N)_{1n} - (C_{pp}^N)_{2n} \\ \hline 0 & (C_{pp}^N)_{11} + (C_{pp}^N)_{22} + (C_{pp}^N)_{12} + (C_{pp}^N)_{21} & (C_{pp}^N)_{1n} + (C_{pp}^N)_{2n} \\ (C_{pp}^N)_{1n} - (C_{pp}^N)_{2n} & (C_{pp}^N)_{1n} + (C_{pp}^N)_{2n} & (C_{pp}^N)_{nm} \end{array} \right). \quad (\text{C6})$$

The matrix Σ^{-1} is the top-left block of $(C_{pp}^N)^{-1}$:

$$\Sigma = [(C_{pp}^N)_{11} + (C_{pp}^N)_{22} - (C_{pp}^N)_{12} - (C_{pp}^N)_{21}] - [(C_{pp}^N)_{1n} - (C_{pp}^N)_{2n}](C_{pp}^N)^{-1}_{nm}[(C_{pp}^N)_{1m} - (C_{pp}^N)_{2m}]. \quad (\text{C7})$$

This, in conjunction with results on the initial correlation submatrix of C_{pp}^N pertaining to particles 1 and 2 [40,66,67] to the effect that, when $v_{\text{th}} = 0$, $(C_{pp}^N)_{11} + (C_{pp}^N)_{22} - (C_{pp}^N)_{12} - (C_{pp}^N)_{21} = O(q^2)$ (with the coefficient determined by the cosmology, which we take to be standard Λ CDM), implies that indeed $\Sigma \sim Aq^2$ as $q \rightarrow 0$. Besides, the components a_k of $\mathbf{a}$ satisfy

$$a_k \propto -(\Sigma^{-1})_{kn}[(C_{pp}^N)_{1n} - (C_{pp}^N)_{2n}](C_{pp}^N)^{-1}_{nm}\mathbf{p}_m, \quad (\text{C8})$$

so, as promised,

$$\Sigma \mathbf{a} \sim [(C_{pp}^N)_{1n} - (C_{pp}^N)_{2n}] = O(q). \quad (\text{C9})$$

Additionally, the results of Sec. 3.1.2 of [40], in conjunction with Eq. (C6), imply that the entire C_{pp}^N matrix tends to a constant as $q \rightarrow \infty$, which means that Σ , $\mathbf{a}$ and B become $O(1)$ in that limit, too.

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