

More on 8D nonsupersymmetric branes and heterotic strings

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We determine the maximal gauge groups arising in E_8 heterotic string theory on T^2 . Our analysis proceeds along two approaches. First, we start the moduli space of the supersymmetric heterotic string theory on T^2 focusing on points of maximal gauge enhancement. At these special points, the charge lattice can exhibit a $\mathbb{Z}_2$ outer automorphism corresponding to the bulk gauge symmetry. By orbifolding the worldsheet theory by it with the fermion parity, we obtain the maximal gauge group of the E_8 theory. Second, we directly study the toroidal compactification of 10D E_8 heterotic string. Both approaches agree, yielding a classification of 22 maximal gauge groups. For each case, we present the corresponding massless spectrum. In light of the no global symmetry/cobordism conjecture in quantum gravity, our result also offers a classification of nonsupersymmetric branes in 8D supersymmetric heterotic string theories.

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I. INTRODUCTION

Nonsupersymmetric vacua in string theory form a wide landscape that has only begun to be explored. Recently, new objects called nonsupersymmetric branes have been focused on and found in various types of superstring theories [1–10]. These objects can be understood within nonsupersymmetric string constructions, so studying such theories is important for learning their properties and possibly finding new kinds of extended objects.

It is also known that many nonsupersymmetric string theories contain tachyons and are therefore unstable. However, progress in the study of tachyon condensation has shown that such instabilities can sometimes lead to new vacua that are stable and free of tachyons [11–17]. This indicates that even nonsupersymmetric string theories with tachyons, which have not been studied in detail before, may still be valuable for understanding consistent string vacua.

In the context of heterotic strings [18], it is known that there are two supersymmetric theories [$E_8 \times E_8$ and $SO(32)$] and seven nonsupersymmetric theories [$SO(16) \times SO(16)$, $SO(16) \times E_8$, $SO(32)$, $(E_7 \times SU(2))^2$,

$U(1) \times SU(16)$, $SO(8) \times SO(24)$, E_8] in ten dimensions [19–21]. The complete classification of ten-dimensional string theories was already established in [21]. More recently, this classification has been revisited and further clarified, confirming the absence of additional ten-dimensional heterotic string theories beyond the known ones [22–24]. Among nonsupersymmetric theories, only the $SO(16) \times SO(16)$ theory is nontachyonic, and only the E_8 theory has the reduced-rank gauge symmetry.

The compactification has been well investigated in light of the swampland conjectures [25] as well as the string lampposts/string universality [26–33]. For instance, the toroidal compactifications of the supersymmetric heterotic string and Chaudhuri-Hockney-Lykken (CHL) string theories [34] are studied in [35–38].

Regarding the nonsupersymmetric strings, the S^1 compactification of nonsupersymmetric heterotic string theory is studied in [39].¹ Moreover, two of the authors constructed nine-dimensional nonsupersymmetric heterotic strings with reduced-rank gauge symmetry [8]. There, we obtain all of the possible nine-dimensional theories but only a few eight-dimensional examples. In this paper, building on the previous study, we construct all possible maximal gauge enhancements arising in E_8 heterotic string theory on T^2 . This is achieved by two approaches (see Fig. 1). The first approach is similar to the one in [8], where we start from the moduli space of the supersymmetric heterotic string theory on T^2 and focus on points of

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¹See also e.g., for [40–55] for recent studies on nonsupersymmetric heterotic strings.

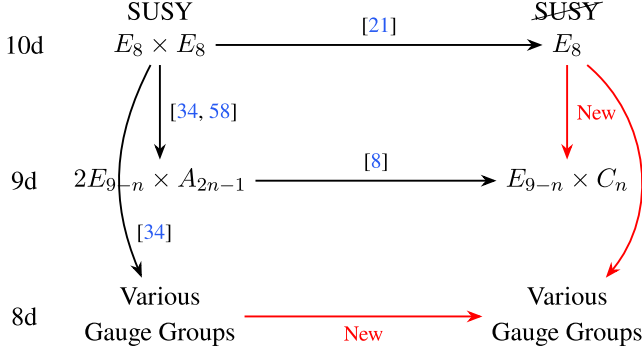


FIG. 1. Schematic of the relation between the 10D, 9D, and 8D heterotic theories [8,21,35,59].

maximal gauge enhancement [36]. Then, we find that the charge lattice sometimes exhibits a $\mathbb{Z}_2$ outer automorphism corresponding to the bulk gauge symmetry. When this happens, we can obtain the maximal gauge group of the E_8 theory by orbifolding the worldsheet theory by this outer automorphism and the spacetime fermion parity. We also investigate toroidal compactifications of the E_8 heterotic string theory directly and determine the moduli that lead to the nine- and eight-dimensional theories with reduced-rank gauge symmetries. Furthermore, we propose the generalized Dynkin diagrams for the E_8 string theory that can be used to identify the maximally enhanced theories. Using the no global symmetry conjecture or the cobordism conjecture [56] of the quantum gravity, our classification can be interpreted as a classification of the nonsupersymmetric codimension-two branes in supersymmetric heterotic string theory in 8D.

Finally, we briefly comment on the possible physical relevance of our results. The nonsupersymmetric vacua classified in this work may provide useful starting points for phenomenological string constructions, in light of recent discussions on compactifications of tachyonic ten-dimensional string vacua (see e.g., [57,58]).

The paper is organized as follows. In Sec. II, we review the asymmetric orbifolding of the heterotic string theory by the folding of Dynkin diagrams. In Sec. III, we review our previous results in 9D [8]. In Sec. IV, we perform the analysis of the eight-dimensional theories obtained by the asymmetric orbifolding of the supersymmetric theories. In Sec. V, we discuss the direct construction of the eight-dimensional theories from the ten-dimensional E_8 heterotic string theory on T^2 .

II. ASYMMETRIC ORBIFOLDING OF HETEROTIC STRING THEORY

In this section, we review the asymmetric orbifolding of heterotic string theory and the folding of simply-laced (ADE-type) Dynkin diagrams, namely those of A_n , D_n , and $E_{6,7,8}$.

A. Construction

The starting point is the torus partition function of $(10-d)$ -dimensional supersymmetric heterotic string theory given by

$$Z_{\text{SUSY}}^{(10-d)}(\tau, \bar{\tau}) = Z_B^{(8-d)}(\bar{V}_8 - \bar{S}_8) \frac{1}{\eta^{16+d} \bar{\eta}^d} \sum_{p \in \Gamma_{16+d,d}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2}, \quad (2.1)$$

where the $(16+d, d)$ lattice $\Gamma_{16+d,d}$ is even and self-dual, so-called Narain lattice, $\tau \in \mathbb{C}$ where $\text{Im } \tau > 0$ is a complex modulus of a torus, and $q = e^{2\pi i \tau}$. We used

$$Z_B^{(8-d)} := \frac{1}{(\text{Im } \tau)^{\frac{8-d}{2}}} \frac{1}{(\eta \bar{\eta})^{8-d}}, \quad (2.2)$$

where η is the Dedekind eta function, and V_8, S_8 are the D_4 character vector and spinor conjugacy classes defined in Appendix A.

If the lattice $\Gamma_{16+d,d}$ admits a $\mathbb{Z}_2$ symmetry g , then a $\mathbb{Z}_2$ orbifolding can be discussed using its symmetry. One can construct the $(8+d, d)$ lattice $I_{8+d,d}$ by the action of g on $\Gamma_{16+d,d}$ as follows:

$$I_{8+d,d} := \{p \in \Gamma_{16+d,d} | g(p) = p\}, \quad (2.3)$$

which is called invariant lattice. A new theory obtained by this orbifolding is based on $I_{8+d,d}$, and this process is called asymmetric orbifolding [60] because a symmetry g acts only on the left part of the lattice.

To obtain the nonsupersymmetric string theories, we consider the asymmetric orbifolding by $g(-1)^F$, where F is the spacetime fermion number. The partition function of the untwisted sector is given by

$$Z^{(\text{untwisted})} = \frac{1}{2} Z_B^{(8-d)} ((\bar{V}_8 - \bar{S}_8) Z(1, 1) + (\bar{V}_8 + \bar{S}_8) Z(1, g)), \quad (2.4)$$

where

$$Z(1, 1) := \frac{1}{\eta^{16+d} \bar{\eta}^d} \sum_{p \in \Gamma_{16+d,d}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2},$$

$$Z(1, g) := \frac{1}{\eta^{16+d} \bar{\eta}^d} \left(\frac{\eta^3}{\theta_2} \right)^4 \sum_{p \in I_{8+d,d}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2}. \quad (2.5)$$

The modular invariance of the partition function requires the twisted sectors should be added. Using the Poisson summation formula and the modular transformation property of theta functions (see Appendix A), we obtain

$$Z^{(\text{twisted})} = \frac{1}{2} Z_B^{(8-d)} ((\bar{O}_8 - \bar{C}_8)Z(g, 1) - (\bar{O}_8 + \bar{C}_8)Z(g, g)), \quad (2.6)$$

where

$$Z(g, 1) = \frac{1}{\eta^{16+d} \bar{\eta}^d} \left(\frac{\eta^3}{\theta_4} \right)^4 \sum_{p \in I_{8+d,d}^*} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2},$$

$$Z(g, g) = -\frac{1}{\eta^{16+d} \bar{\eta}^d} \left(\frac{\eta^3}{\theta_3} \right)^4 \sum_{p \in I_{8+d,d}^*} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} e^{\pi i p^2}. \quad (2.7)$$

Note that $p^2 = p_L^2 - p_R^2$. Here O_8, C_8 are the D_4 character scalar and cospinor conjugacy classes defined in Appendix A and $I_{8+d,d}^*$ is the dual lattice of $I_{8+d,d}$. For the modular invariance, $Z(g, 1)$ should be invariant under $\tau \rightarrow \tau + 2$, so every element $(p_L, p_R) \in I_{8+d,d}^*$ satisfies $p^2 \in \mathbb{Z}$. Then it holds that $I_{8+d,d}^* \subset I_{8+d,d}/2$, since $I_{8+d,d} \subset \Gamma_{18,2}$ is an even lattice.

Adding all of these together, the partition function of the nonsupersymmetric theory constructed by this orbifolding is given as follows:

$$Z_{\text{SUSY}}^{(10-d)} = \frac{1}{2} Z_B^{(8-d)} ((\bar{V}_8 - \bar{S}_8)Z(1, 1) + (\bar{V}_8 + \bar{S}_8)Z(1, g) + (\bar{O}_8 - \bar{C}_8)Z(g, 1) - (\bar{O}_8 + \bar{C}_8)Z(g, g)). \quad (2.8)$$

B. Folding of Dynkin diagram

As an action g , we consider the outer automorphism of the Lie algebra, which corresponds to the folding of Dynkin diagrams. Here, we introduce the foldings relevant to our analysis.

1. $A_{2n-1} \rightarrow C_n$

As an example, we describe the folding A_{2n-1} into C_n . Let us consider the root lattice of A_{2n-1} :

$$\Lambda_R(A_{2n-1}) = \bigoplus_{i=1}^{2n-1} \mathbb{Z} \alpha_i^{(A_{2n-1})}. \quad (2.9)$$

The specific form of $\alpha_i^{(A_{2n-1})}$ is summarized in Appendix B. This lattice admits a $\mathbb{Z}_2$ symmetry g which flips $\alpha_i^{(A_{2n-1})} \leftrightarrow \alpha_{2n-i}^{(A_{2n-1})}$. A new lattice obtained by taking the g -invariant part of $\Lambda_R(A_{2n-1})$ is close to the root lattice of C_n :

$$\begin{aligned} \Lambda_R^g(A_{2n-1}) &:= \{p \in \Lambda_R(A_{2n-1}) | g(p) = p\} \\ &= \mathbb{Z}(\alpha_1^{(A_{2n-1})} + \alpha_{2n-1}^{(A_{2n-1})}) \oplus \dots \\ &\quad \oplus \mathbb{Z}(\alpha_{n-1}^{(A_{2n-1})} + \alpha_{n+1}^{(A_{2n-1})}) \oplus \mathbb{Z}\alpha_n^{(A_{2n-1})} \\ &\cong \sqrt{2} \left(\Lambda_R(C_n) + \frac{1}{2} \mathbb{Z} \alpha_n^{(C_n)} \right), \end{aligned} \quad (2.10)$$

where $\cong$ means an isomorphism of the Abelian group, and here we identified C_n roots from A_{2n-1} roots as follows:

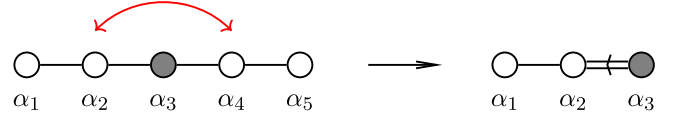


FIG. 2. Folding A_{2n-1} into C_n .

$$\begin{aligned} \alpha_i^{(C_n)} &= \frac{1}{\sqrt{2}} (\alpha_i^{(A_{2n-1})} + \alpha_{2n-i}^{(A_{2n-1})}), \quad \text{for } 1 \leq i \leq n-1, \\ \alpha_n^{(C_n)} &= \sqrt{2} \alpha_n^{(A_{2n-1})}. \end{aligned} \quad (2.11)$$

This process is called *folding*, since it corresponds to the folding of the Dynkin diagram (see Fig. 2).

2. $D_4 \rightarrow B_3$

Another example is folding D_4 into B_3 (see Fig. 3).

The root lattice of D_4 admits a $\mathbb{Z}_2$ symmetry g which flips $\alpha_1^{(D_4)} \leftrightarrow \alpha_3^{(D_4)}$. Then we identified B_3 roots from D_4 roots as follows:

$$\begin{aligned} \alpha_1^{(B_3)} &= \alpha_4^{(D_4)}, \\ \alpha_2^{(B_3)} &= \alpha_2^{(D_4)}, \\ \alpha_3^{(B_3)} &= \frac{1}{2} (\alpha_1^{(D_4)} + \alpha_3^{(D_4)}). \end{aligned} \quad (2.12)$$

The invariant sublattice of the D_4 root lattice under g is

$$\begin{aligned} \mathbb{Z} \alpha_4^{(D_4)} \oplus \mathbb{Z} \alpha_2^{(D_4)} \oplus \mathbb{Z} (\alpha_1^{(D_4)} + \alpha_3^{(D_4)}) \\ = \mathbb{Z} \alpha_1^{(B_3)} \oplus \mathbb{Z} \alpha_2^{(B_3)} \oplus \mathbb{Z} 2\alpha_3^{(B_3)}. \end{aligned} \quad (2.13)$$

3. $E_6 \rightarrow F_4$

The last one used in this paper is folding E_6 into F_4 (see Fig. 4).

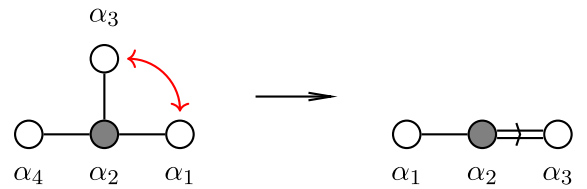


FIG. 3. Folding D_4 into B_3 .

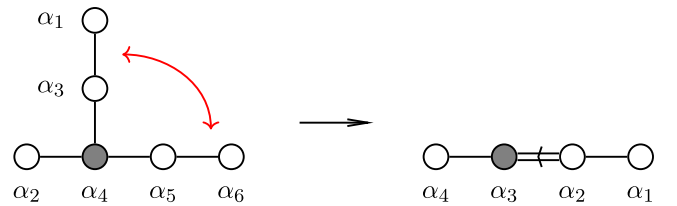


FIG. 4. Folding E_6 into F_4 .

TABLE I. List of 9D nonsupersymmetric heterotic strings obtained from 9D supersymmetric heterotic strings by orbifolding the outer automorphism and the spacetime fermion parity.

SUSY theory	SUSY theory	Twisted sector	
		$\bar{O}_8$	$\bar{C}_8$
A_{17}	C_9	1	152
$A_{15} + 2A_1$	$C_8 + A_1$	1	$(119, 1) \oplus (1, 3)$
$A_{11} + 2A_2 + 2A_1$	$C_6 + A_2 + A_1$	1	$(65, 1, 1) \oplus (1, 8, 1) \oplus (1, 1, 3)$
$A_9 + 2A_4$	$C_5 + A_4$	1	$(44, 1) \oplus (1, 24)$
$A_7 + 2D_5$	$C_4 + D_5$	1	$(27, 1) \oplus (1, 45)$
$A_5 + 2E_6$	$C_3 + E_6$	1	$(14, 1) \oplus (1, 78)$
$A_3 + 2E_7$	$C_2 + E_7$	1	$(5, 1) \oplus (1, 133)$
$A_1 + 2E_8$	$C_1 + E_8$	1	$(1, 248)$

The root lattice of E_6 admits a $\mathbb{Z}_2$ symmetry g which flips $\alpha_1^{(E_6)} \leftrightarrow \alpha_6^{(E_6)}$ and $\alpha_3^{(E_6)} \leftrightarrow \alpha_5^{(E_6)}$ simultaneously. Then we identified F_4 roots from E_6 roots as follows:

$$\begin{aligned}
\alpha_1^{(F_4)} &= \frac{1}{\sqrt{2}}(\alpha_1^{(E_6)} + \alpha_6^{(E_6)}), \\
\alpha_2^{(F_4)} &= \frac{1}{\sqrt{2}}(\alpha_3^{(E_6)} + \alpha_5^{(E_6)}), \\
\alpha_3^{(F_4)} &= \frac{1}{\sqrt{2}}\alpha_4^{(E_6)}, \\
\alpha_4^{(F_4)} &= \frac{1}{\sqrt{2}}\alpha_2^{(E_6)}.
\end{aligned} \tag{2.14}$$

The invariant sublattice of the E_6 root lattice under g is

$$\begin{aligned}
&\mathbb{Z}(\alpha_1^{(E_6)} + \alpha_6^{(E_6)}) \oplus \mathbb{Z}(\alpha_3^{(E_6)} + \alpha_5^{(E_6)}) \oplus \mathbb{Z}\alpha_4^{(E_6)} \oplus \mathbb{Z}\alpha_2^{(E_6)} \\
&= \sqrt{2}\alpha_1^{(F_4)} \oplus \sqrt{2}\alpha_2^{(F_4)} \oplus \frac{1}{\sqrt{2}}\alpha_3^{(F_4)} \oplus \frac{1}{\sqrt{2}}\alpha_4^{(F_4)}.
\end{aligned} \tag{2.15}$$

III. REVIEW OF 9D RESULTS

In this section, we review the 9D theories obtained in [8]. The results are summarized in Table I. The massless states in the untwisted sectors consist of the graviton, B -field, dilaton, gauge fields, gauginos, and the adjoint scalars. The spectrum of the twisted sectors can be read off from (2.6) by $q, \bar{q}$ expansions. Due to the level matching condition, we

only have to study terms in which the powers of q and $\bar{q}$ are matched.

We note that as discussed in [35], the gauge groups for the 9D SUSY theories can be obtained by removing two appropriate nodes from the generalized Dynkin diagram shown in Fig. 5. Similarly, the gauge groups for the 9D SUSY theories can be obtained by removing one appropriate node from the generalized Dynkin diagram shown in Fig. 6. After investigating the maximal enhancements of the E_8 string theory on S^1 in Sec. VB, we propose the generalized Dynkin diagram for the 9D SUSY theories which can be used for obtaining the allowed gauge groups. The rules for the 9D generalized Dynkin diagram is explained in Sec. VC.

IV. GAUGING LATTICE CFTS BY FOLDING DYNKIN DIAGRAMS IN 8D

In this section, we perform the orbifolding from the 8D SUSY theories whose charge lattices have a $\mathbb{Z}_2$ symmetry, and study the massless spectrum. We obtain 22 gauge symmetries, as is summarized in Table II. Here the number # indicates the one in Table 12 of [35].

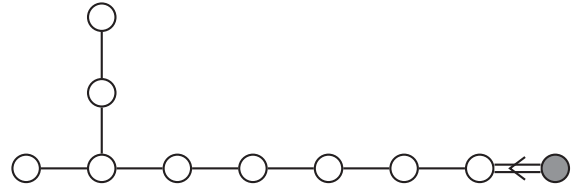


FIG. 6. Generalized Dynkin diagram for 9D SUSY theories.

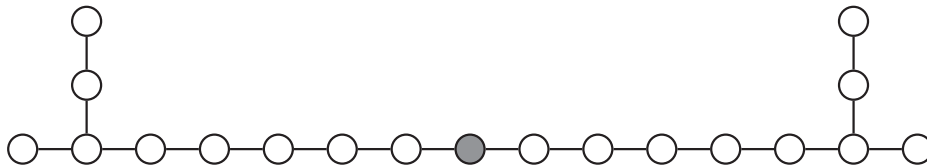


FIG. 5. Generalized Dynkin diagram for 9D SUSY theories.

TABLE II. The list of 8D (non)supersymmetric heterotic strings and lighter particles in the twisted sector. $\bar{O}_8$ is the singlet representation of spacetime $SO(8)$, and $\bar{C}_8$ is the conjugacy spinor representation. **1** in $\bar{O}_8$ is a tachyon, and representations in $\bar{C}_8$ are fermions.

#	SUSY theory	SUSY theory	Twisted sector	
			$\bar{O}_8$	$\bar{C}_8$
1	$6A_3$	$2C_2 + 2A_3$	1	$((\mathbf{5}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{5}), \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, (\mathbf{15}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{15}))$
2	$2A_1 + 4A_4$	$2C_1 + 2A_4$	1	$(\mathbf{1}, \mathbf{1}, (\mathbf{24}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{24}))$
5	$4A_2 + 2A_5$	$2A_2 + 2C_3$	1	$((\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}), \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, (\mathbf{14}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{14}))$
6	$A_3 + 3A_5$	$C_2 + C_3 + A_5$	1	$(\mathbf{5}, \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{14}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{35})$
25	$4A_1 + 2A_7$	$2A_1 + 2C_4$	1	$((\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}), \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, (\mathbf{27}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{27}))$
28	$2A_1 + 3A_3 + A_7$	$A_1 + C_2 + A_3 + C_4$	1	$(\mathbf{1}, \mathbf{5}, \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{15}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{27})$
54	$2A_9$	$2C_5$	1	$(\mathbf{44}, \mathbf{1}), (\mathbf{1}, \mathbf{44})$
57	$A_1 + 2A_4 + A_9$	$C_1 + A_4 + C_5$	1	$(\mathbf{1}, \mathbf{24}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{44})$
81	$3A_1 + 2A_2 + A_{11}$	$A_1 + C_1 + A_2 + C_6$	1	$(\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{8}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{65})$
83	$2A_2 + A_3 + A_{11}$	$A_2 + C_2 + C_6$	1	$(\mathbf{8}, \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{5}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{65})$
87	$2A_1 + A_5 + A_{11}$	$A_1 + C_3 + C_6$	1	$(\mathbf{3}, \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{14}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{65})$
106	$3A_1 + A_{15}$	$A_1 + C_1 + C_8$	1	$(\mathbf{3}, \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{119})$
108	$A_3 + A_{15}$	$C_2 + C_8$	1	$(\mathbf{5}, \mathbf{1}), (\mathbf{1}, \mathbf{119})$
111	$A_1 + A_{17}$	$C_1 + C_9$	1	$(\mathbf{1}, \mathbf{152})$
121	$A_1 + A_7 + 2D_5$	$C_1 + C_4 + D_5$	1	$(\mathbf{1}, \mathbf{27}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{44})$
137	$2A_3 + 2D_6$	$2C_2 + D_6$	1	$((\mathbf{5}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{5}), \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{66})$
219	$3E_6$	$E_6 + F_4$	1	$(\mathbf{78}, \mathbf{1}), (\mathbf{1}, \mathbf{26})$
222	$A_1 + A_5 + 2E_6$	$C_1 + C_3 + E_6$	1	$(\mathbf{1}, \mathbf{14}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{78})$
257	$A_1 + A_3 + 2E_7$	$C_1 + C_2 + E_7$	1	$(\mathbf{1}, \mathbf{5}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{133})$
279	$D_4 + 2E_7$	$B_3 + E_7$	1	$(\mathbf{7}, \mathbf{1}), (\mathbf{1}, \mathbf{133})$
296	$2A_1 + 2E_8$	$2C_1 + E_8$	1	$(\mathbf{1}, \mathbf{248})$
297	$A_2 + 2E_8$	$A_2 + E_8$	1	$(\mathbf{1}, \mathbf{248})$

In the previous work [8], the eight non-SUSY theories with the 9D gauge groups plus a C_1 factor were obtained from the 8D SUSY theories whose gauge groups are expressed as the 9D possible gauge groups and an extra A_1 . We skip the calculations for these cases in this paper,

but we include them into Table II. These eight cases correspond to #57, 81, 106, 111, 121, 222, 257, 296.

A. $6A_3 \rightarrow 2C_2 + 2A_3$ (#1)

We start from the even self-dual lattice:

$$\Gamma_{18,2}^{(6A_3)} = \Lambda_R(6A_3) + \mathbb{Z}(0, 2, 1, 1, 1, 1)\omega_1^{(A_3)} + \mathbb{Z}(1, 1, 0, 2, 1, -1)\omega_1^{(A_3)} + \mathbb{Z}(0; 2, 0) + \mathbb{Z}(0; 0, 2) + \mathbb{Z}\left((0, 1, 1, 0, 0, 1)\omega_1^{(A_3)}; \frac{1}{2}, 0\right) + \mathbb{Z}\left((1, 0, 0, -1, 1, 0)\omega_1^{(A_3)}; 0, \frac{1}{2}\right). \quad (4.1)$$

This lattice admits the following symmetry:

$$g: (x_1^{(A_3)}, x_2^{(A_3)}, x_3^{(A_3)}, x_4^{(A_3)}, x_5^{(A_3)}, x_6^{(A_3)}) \mapsto (\tilde{x}_1^{(A_3)}, \tilde{x}_2^{(A_3)}, -x_4^{(A_3)}, -x_3^{(A_3)}, -x_6^{(A_3)}, -x_5^{(A_3)}), \quad (4.2)$$

where $\tilde{x}^{(A_3)}$ is obtained from exchanging α_i and α_{2n-i} in $x^{(A_3)}$ for all $i = 1, \dots, 2n - 1$. The invariant lattice and its dual lattice are

$$\begin{aligned} I &= \sqrt{2} \left(\Lambda_R(2C_2 + 2A_3) + \frac{1}{2} \mathbb{Z} \alpha_2^{(C_2)} + \frac{1}{2} \mathbb{Z} \alpha_2^{(C_2)} + \mathbb{Z}(0; \sqrt{2}, 0) + \mathbb{Z}(0; 0, \sqrt{2}) \right) \\ &\quad + \sqrt{2} \mathbb{Z}((0, 0), (2, -2)\omega_1^{(A_3)}) + \sqrt{2} \mathbb{Z}\left(\left(\frac{1}{2}, \frac{1}{2}\right)\omega_2^{(C_2)}, (0, 2)\omega_1^{(A_3)}\right) \\ &\quad + \sqrt{2} \mathbb{Z}\left((0, 0), (1, -1)\omega_1^{(A_3)}; \frac{1}{\sqrt{2}}, 0\right) + \sqrt{2} \mathbb{Z}\left((0, 0), (1, 1)\omega_1^{(A_3)}; 0, \frac{1}{\sqrt{2}}\right), \\ I^* &= I + \frac{1}{\sqrt{2}} \Lambda_R(2C_2 + 2A_3). \end{aligned} \quad (4.3)$$

By gathering the eight neutral elements and elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $2C_2 + 2A_3$ as massless fermions:

$$((\mathbf{5}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{5}), \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, (\mathbf{15}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{15})). \quad (4.4)$$

The gauge symmetry is given by

$$\frac{\mathrm{Sp}(2) \times \mathrm{Sp}(2) \times \mathrm{SU}(4) \times \mathrm{SU}(4) \times \mathrm{U}(1) \times \mathrm{U}(1)}{\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2}, \quad (4.5)$$

where the action of $\mathbb{Z}_n$ in the denominator on the fundamental representation is

$$\begin{aligned} \mathbb{Z}_4 &: (\mathbf{1}, \mathbf{1}, e^{2\pi i \frac{1}{4}} \mathbf{1}, e^{2\pi i \frac{1}{4}} \mathbf{1}, 1, -1), \\ \mathbb{Z}_4 &: (-\mathbf{1}, \mathbf{1}, \mathbf{1}, e^{2\pi i \frac{1}{4}} \mathbf{1}, e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{4}}), \\ \mathbb{Z}_2 &: (-\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, 1, 1). \end{aligned} \quad (4.6)$$

B. $2A_1 + 4A_4 \rightarrow 2C_1 + 2A_4$ (#2)

The even self-dual lattice is

$$\begin{aligned} \Gamma^{(2A_1+4A_4)} &= \Lambda_R(2A_1 + 4A_4) + \mathbb{Z}(0, 0, (1, 1, 2, 2)\omega_1^{(A_4)}) + \mathbb{Z}(0; \sqrt{10}, 0) + \mathbb{Z}(0; \sqrt{10}, 0) \\ &+ \mathbb{Z}\left(\omega_1^{(A_1)}, 0, (1, -1, 0, 0)\omega_1^{(A_4)}; \frac{1}{\sqrt{10}}, 0\right) + \mathbb{Z}\left(0, \omega_1^{(A_1)}, (0, 0, 1, -1)\omega_1^{(A_4)}; 0, \frac{1}{\sqrt{10}}\right). \end{aligned} \quad (4.7)$$

This lattice admits the following symmetry:

$$g: (x_1^{(A_1)}, x_2^{(A_1)}, x_1^{(A_4)}, x_2^{(A_4)}, x_3^{(A_4)}, x_4^{(A_4)}) \mapsto (x_1^{(A_1)}, x_2^{(A_1)}, -x_2^{(A_4)}, -x_1^{(A_4)}, -x_4^{(A_4)}, -x_3^{(A_4)}). \quad (4.8)$$

The invariant lattice and its dual lattice are

$$\begin{aligned} I &= \sqrt{2} \left(\Lambda_R(2C_1 + 2A_4) + \mathbb{Z} \frac{1}{2} \alpha_1^{(C_1)} + \mathbb{Z} \frac{1}{2} \alpha_1^{(C_1)} + \mathbb{Z}(0; \sqrt{5}, 0) + \mathbb{Z}(0; 0, \sqrt{5}) \right) \\ &+ \sqrt{2} \mathbb{Z} \left(\left(\frac{1}{2}, 0 \right) \omega_1^{(C_1)}, (1, 0) \omega_1^{(A_4)}; \frac{1}{\sqrt{20}}, 0 \right) + \sqrt{2} \mathbb{Z} \left(\left(0, \frac{1}{2} \right) \omega_1^{(C_1)}, (0, 1) \omega_1^{(A_4)}; 0, \frac{1}{\sqrt{20}} \right), \\ I^* &= I + \frac{1}{\sqrt{2}} \Lambda_R(2C_1 + 2A_4). \end{aligned} \quad (4.9)$$

By gathering the eight neutral elements and the elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $2C_1 + 2A_4$ as massless fermions:

$$(\mathbf{1}, \mathbf{1}, (\mathbf{24}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{24})). \quad (4.10)$$

The gauge symmetry is

$$\frac{\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{SU}(5) \times \mathrm{SU}(5) \times \mathrm{U}(1) \times \mathrm{U}(1)}{\mathbb{Z}_5 \times \mathbb{Z}_{10} \times \mathbb{Z}_{10}}, \quad (4.11)$$

where the action of $\mathbb{Z}_n$ in the denominator on the fundamental representation is

$$\begin{aligned} \mathbb{Z}_5 &: (\mathbf{1}, \mathbf{1}, e^{2\pi i \frac{1}{5}} \mathbf{1}, \mathbf{1}, e^{2\pi i \frac{3}{5}}, 1), \\ \mathbb{Z}_{10} &: (-\mathbf{1}, -\mathbf{1}, \mathbf{1}, e^{2\pi i \frac{1}{10}} \mathbf{1}, -1, e^{2\pi i \frac{1}{10}}), \\ \mathbb{Z}_{10} &: (-\mathbf{1}, \mathbf{1}, \mathbf{1}, e^{2\pi i \frac{1}{10}} \mathbf{1}, -1, e^{2\pi i \frac{3}{10}}). \end{aligned} \quad (4.12)$$

C. $4A_2 + 2A_5 \rightarrow 2A_2 + 2C_3$ (#5)

The even self-dual lattice is

$$\begin{aligned} \Gamma_{18,2}^{(4A_2+2A_5)} &= \Lambda_R(4A_2 + 2A_5) + \mathbb{Z}((0, 0, 1, 1)\omega_1^{(A_2)}, (2, 2)\omega_1^{(A_5)}) + \mathbb{Z}((1, 1, 0, 0)\omega_1^{(A_2)}, (2, -2)\omega_1^{(A_5)}) \\ &+ \mathbb{Z}\left((1, 0, 0, 2)\omega_1^{(A_2)}, (0, 1)\omega_1^{(A_5)}; \frac{1}{\sqrt{6}}, 0\right) + \mathbb{Z}\left((0, 1, 2, 0)\omega_1^{(A_2)}, (1, 0)\omega_1^{(A_5)}; 0, \frac{1}{\sqrt{6}}\right) \\ &+ \mathbb{Z}(0; \sqrt{6}, 0) + \mathbb{Z}(0; 0, \sqrt{6}). \end{aligned} \quad (4.13)$$

This lattice admits the following symmetry:

$$g: \left(x_1^{(A_2)}, x_2^{(A_2)}, x_3^{(A_2)}, x_4^{(A_2)}, x_1^{(A_5)}, x_2^{(A_5)} \right) \mapsto \left(-x_2^{(A_2)}, -x_1^{(A_2)}, -x_4^{(A_2)}, -x_3^{(A_2)}, \tilde{x}_1^{(A_5)}, \tilde{x}_2^{(A_5)} \right). \quad (4.14)$$

The invariant lattice and its dual lattice are

$$\begin{aligned} I = & \sqrt{2} \left(\Lambda_R(2A_2 + 2C_3) + \frac{1}{2} \mathbb{Z} \alpha_3^{(C_3)} + \frac{1}{2} \mathbb{Z} \alpha_3^{(C_3)} + \mathbb{Z}(0; \sqrt{3}, 0) + \mathbb{Z}(0; 0, \sqrt{3}) \right) \\ & + \sqrt{2} \mathbb{Z} \left((1, 1) \omega_1^{(A_2)}, (0, 0); \frac{1}{\sqrt{3}}, 0 \right) + \sqrt{2} \mathbb{Z} \left((-1, -1) \omega_1^{(A_2)}, (0, 0); 0, \frac{1}{\sqrt{3}} \right) \\ & + \sqrt{2} \mathbb{Z} \left((0, 0), \left(0, \frac{1}{2} \omega_3^{(C_3)} \right); \frac{\sqrt{3}}{2}, 0 \right) + \sqrt{2} \mathbb{Z} \left((0, 0), \left(\frac{1}{2} \omega_3^{(C_3)}, 0 \right); 0, \frac{\sqrt{3}}{2} \right), \\ I^* = & I + \frac{1}{\sqrt{2}} \Lambda_R(2A_2 + 2C_3). \end{aligned} \quad (4.15)$$

By gathering the eight neutral elements and the elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $2A_2 + 2C_3$ as massless fermions:

$$((\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}), \mathbf{1}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{1}, (\mathbf{14}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{14})). \quad (4.16)$$

The gauge group is

$$\frac{\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{Sp}(3) \times \mathrm{Sp}(3) \times \mathrm{U}(1) \times \mathrm{U}(1)}{\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2}, \quad (4.17)$$

where the action of $\mathbb{Z}_n$ in the denominator on the fundamental representation is

$$\begin{aligned} \mathbb{Z}_3: & (e^{2\pi i \frac{1}{3}} \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, e^{2\pi i \frac{2}{3}}, e^{2\pi i \frac{1}{3}}), \\ \mathbb{Z}_3: & (e^{2\pi i \frac{1}{3}} \mathbf{1}, e^{2\pi i \frac{2}{3}} \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}), \\ \mathbb{Z}_2: & (\mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}), \\ \mathbb{Z}_2: & (\mathbf{1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{1}). \end{aligned} \quad (4.18)$$

D. $A_3 + 3A_5 \rightarrow C_2 + C_3 + A_5$ (#6)

We start from the even self-dual lattice:

$$\begin{aligned} \Gamma_{18,2}^{(A_3+3A_5)} = & \Lambda_R(A_3 + 3A_5) + \mathbb{Z}(0; 2, 0) + \mathbb{Z}(0; 0, \sqrt{6}) + \mathbb{Z}(2\omega_1^{(A_3)}, (2, 5, 5)\omega_1^{(A_5)}; 0, 0) \\ & + \mathbb{Z}\left(\omega_1^{(A_3)}, (0, 3, 0)\omega_1^{(A_5)}; \frac{1}{2}, 0\right) + \mathbb{Z}\left(0, (1, 2, 0)\omega_1^{(A_5)}; 0, \frac{1}{\sqrt{6}}\right). \end{aligned} \quad (4.19)$$

This lattice has the following symmetry:

$$g: \left(x^{(A_3)}, x^{(A_5)}, x_1^{(A_5)}, x_2^{(A_5)}; x^{(R)} \right) \mapsto \left(\tilde{x}^{(A_3)}, \tilde{x}^{(A_5)}, -x_2^{(A_5)}, -x_1^{(A_5)}; x^{(R)} \right). \quad (4.20)$$

The invariant lattice I and its dual lattice I^* are

$$\begin{aligned} I = & \sqrt{2} \left(\Lambda_R(C_2 + C_3 + A_5) + \mathbb{Z}(0; \sqrt{2}, 0) + \mathbb{Z}(0; 0, \sqrt{3}) + \frac{1}{2} \mathbb{Z} \alpha_2^{(C_2)} + \frac{1}{2} \mathbb{Z} \alpha_3^{(C_3)} \right) \\ & + \sqrt{2} \mathbb{Z} \left(\frac{1}{2} \omega_2^{(C_2)}, 0, 3\omega_1^{(A_5)}; 0, 0 \right) + \sqrt{2} \mathbb{Z} \left(\frac{1}{2} \omega_2^{(C_2)}, 0, 0; \frac{1}{\sqrt{2}}, 0 \right) + \sqrt{2} \mathbb{Z} \left(0, 0, 2\omega_1^{(A_5)}; 0, \frac{1}{\sqrt{3}} \right) \\ & + \sqrt{2} \mathbb{Z} \left(0, \frac{1}{2} \omega_3^{(C_3)}, \omega_1^{(A_5)}; \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{3}} \right), \\ I^* = & I + \frac{1}{\sqrt{2}} \Lambda_R(C_2 + C_3 + A_5). \end{aligned} \quad (4.21)$$

By gathering the eight neutral elements and the elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $C_2 + C_3 + A_5$ as massless fermions:

$$(\mathbf{5}, \mathbf{1}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{14}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{1}, \mathbf{35}). \quad (4.22)$$

The gauge group is

$$\frac{\mathrm{Sp}(2) \times \mathrm{Sp}(3) \times \mathrm{SU}(6) \times \mathrm{U}(1) \times \mathrm{U}(1)}{\mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_2}, \quad (4.23)$$

where the action of $\mathbb{Z}_n$ in the denominator on the fundamental representation is

$$\begin{aligned} \mathbb{Z}_6: & (\mathbf{1}, \mathbf{1}, e^{2\pi i \frac{1}{6}} \mathbf{1}, -1, e^{2\pi i \frac{2}{3}}), \\ \mathbb{Z}_2: & (\mathbf{1}, -\mathbf{1}, \mathbf{1}, 1, -1), \\ \mathbb{Z}_2: & (-\mathbf{1}, \mathbf{1}, \mathbf{1}, -1, 1). \end{aligned} \quad (4.24)$$

E. $2A_7 + 4A_1 \rightarrow 2C_4 + 2A_1$ (#25)

The even self-dual lattice is

$$\begin{aligned} \Gamma_{18,2}^{(2A_7+4A_1)} = & \Lambda_R(2A_7 + 4A_1) + \mathbb{Z}(0; 2, 0) + \mathbb{Z}(0; 0, 2) \\ & + \mathbb{Z}((2, 2)\omega_1^{(A_7)}, (1, 1)\omega_1^{(A_1)}, (0, 0)) + \mathbb{Z}((4, 0)\omega_1^{(A_7)}, (1, 1)\omega_1^{(A_1)}, (1, 1)\omega_1^{(A_1)}) \\ & + \mathbb{Z}\left((1, 1)\omega_1^{(A_7)}, (0, 1)\omega_1^{(A_1)}, (0, 0); \frac{1}{2}, 0\right) + \mathbb{Z}\left((1, -1)\omega_1^{(A_7)}, (0, 0), (0, 1)\omega_1^{(A_1)}; 0, \frac{1}{2}\right). \end{aligned} \quad (4.25)$$

This lattice admits the following symmetry:

$$g: (x_1^{(A_7)}, x_2^{(A_7)}, x_1^{(A_1)}, x_2^{(A_1)}, x_3^{(A_1)}, x_4^{(A_1)}; x^{(R)}) \mapsto (\tilde{x}_1^{(A_7)}, \tilde{x}_2^{(A_7)}, -x_2^{(A_1)}, -x_1^{(A_1)}, -x_4^{(A_1)}, -x_3^{(A_1)}; x^{(R)}). \quad (4.26)$$

The invariant lattice and its dual lattice are

$$\begin{aligned} I = & \sqrt{2} \left(\Lambda_R(2C_4 + 2A_1) + \mathbb{Z}(0; \sqrt{2}, 0) + \mathbb{Z}(0; 0, \sqrt{2}) + \frac{1}{2} \mathbb{Z} \alpha_4^{(C_4)} + \frac{1}{2} \mathbb{Z} \alpha_4^{(C_4)} \right) \\ & + \sqrt{2} \mathbb{Z} \left(\frac{1}{2} (1, 1)\omega_1^{(C_4)}, (0, 0) \right) + \sqrt{2} \mathbb{Z} \left(\frac{1}{2} (1, 0)\omega_1^{(C_4)}, (1, 1)\omega_1^{(A_1)} \right) \\ & + \sqrt{2} \mathbb{Z} \left((0, 0), (1, 0)\omega_1^{(A_1)}; \frac{1}{\sqrt{2}}, 0 \right) + \sqrt{2} \mathbb{Z} \left(\frac{1}{2} (1, 0)\omega_1^{(C_4)}, (0, 0); 0, \frac{1}{\sqrt{2}} \right), \\ I^* = & I + \frac{1}{\sqrt{2}} \Lambda_R(2C_4 + 2A_1). \end{aligned} \quad (4.27)$$

By gathering the eight neutral elements and the elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $2C_4 + 2A_1$ as massless fermions:

$$((\mathbf{27}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{27}), \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})). \quad (4.28)$$

The gauge group is

$$\frac{\mathrm{Sp}(4) \times \mathrm{Sp}(4) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)}{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2}, \quad (4.29)$$

where the action of $\mathbb{Z}_n$ in the denominator on the fundamental representation is

$$\begin{aligned} \mathbb{Z}_2: & (\mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{1}, 1, -1), \\ \mathbb{Z}_2: & (\mathbf{1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}, -1, 1), \\ \mathbb{Z}_2: & (-\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, -1, -1), \\ \mathbb{Z}_2: & (\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, -1, -1). \end{aligned} \quad (4.30)$$

F. $A_7 + 3A_3 + 2A_1 \rightarrow C_4 + C_2 + A_3 + A_1$ (#28)

The even self-dual lattice is

$$\begin{aligned}\Gamma_{18,2}^{(A_7+3A_3+2A_1)} &= \Lambda_R(A_7 + 3A_3 + 2A_1) + \mathbb{Z}(0; 2, 0) + \mathbb{Z}(0; 0, 2\sqrt{2}) \\ &+ \mathbb{Z}(2\omega_1^{(A_7)}, (2, 1, 1)\omega_1^{(A_3)}, (0, 0)) + \mathbb{Z}(4\omega_1^{(A_7)}, (2, 0, 0)\omega_1^{(A_3)}, (1, 1)\omega_1^{(A_1)}) \\ &+ \mathbb{Z}\left(0, (1, 2, 0)\omega_1^{(A_3)}, (0, 1)\omega_1^{(A_1)}; \frac{1}{2}, 0\right) + \mathbb{Z}\left(-\omega_1^{(A_7)}, (0, 0, 1)\omega_1^{(A_3)}, (1, 0)\omega_1^{(A_1)}; 0, \frac{1}{2\sqrt{2}}\right).\end{aligned}\quad (4.31)$$

This lattice admits the following symmetry:

$$g: (x^{(A_7)}, x^{(A_3)}, x_1^{(A_3)}, x_2^{(A_3)}, x_1^{(A_1)}, x_2^{(A_1)}; x^{(R)}) \mapsto (\tilde{x}^{(A_7)}, \tilde{x}^{(A_3)}, -x_2^{(A_3)}, -x_1^{(A_3)}, -x_2^{(A_1)}, -x_1^{(A_1)}; x^{(R)}). \quad (4.32)$$

The invariant lattice and its dual lattice are

$$\begin{aligned}I &= \sqrt{2}\left(\Lambda_R(C_4 + C_2 + A_3 + A_1) + \mathbb{Z}(0; \sqrt{2}, 0) + \mathbb{Z}(0; 0, 2) + \frac{1}{2}\mathbb{Z}\alpha_4^{(C_4)} + \frac{1}{2}\mathbb{Z}\alpha_2^{(C_2)}\right) \\ &+ \sqrt{2}\mathbb{Z}\left(\frac{1}{2}\omega_4^{(C_4)}, 0, 2\omega_1^{(A_3)}, 0\right) + \sqrt{2}\mathbb{Z}\left(\frac{1}{2}\omega_4^{(C_4)}, \frac{1}{2}\omega_2^{(C_2)}, 0, \omega_1^{(A_1)}\right) \\ &+ \sqrt{2}\mathbb{Z}\left(0, \frac{1}{2}\omega_2^{(C_2)}, 0, 0; \frac{1}{\sqrt{2}}, 0\right) + \sqrt{2}\mathbb{Z}\left(\frac{1}{2}\omega_4^{(C_4)}, \frac{1}{2}\omega_2^{(C_2)}, -\omega_1^{(A_3)}, 0; 0, \frac{1}{2}\right), \\ I^* &= I + \frac{1}{\sqrt{2}}\Lambda_R(C_4 + C_2 + A_3 + A_1).\end{aligned}\quad (4.33)$$

By gathering the eight neutral elements and the elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $C_4 + C_2 + A_3 + A_1$ as massless fermions:

$$(\mathbf{27}, \mathbf{1}, \mathbf{1}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{5}, \mathbf{1}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{1}, \mathbf{15}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}). \quad (4.34)$$

The gauge group is

$$\frac{\mathrm{Sp}(4) \times \mathrm{Sp}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(4) \times \mathrm{U}(1)}{\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2}, \quad (4.35)$$

where the action of $\mathbb{Z}_n$ in the denominator on the fundamental representation is

$$\begin{aligned}\mathbb{Z}_4: & (\mathbf{1}, -\mathbf{1}, e^{2\pi i \frac{1}{4}}\mathbf{1}, \mathbf{1}, \mathbf{1}, e^{2\pi i \frac{1}{4}}), \\ \mathbb{Z}_4: & (\mathbf{1}, \mathbf{1}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}), \\ \mathbb{Z}_2: & (\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}), \\ \mathbb{Z}_2: & (-\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{1}).\end{aligned}\quad (4.36)$$

G. $2A_9 \rightarrow 2C_5$ (#54)

The even self-dual lattice is

$$\begin{aligned}\Gamma_{18,2}^{(2A_9)} &= \Lambda_R(2A_9) + \mathbb{Z}(2\omega_1^{(A_9)}, 4\omega_1^{(A_9)}) \\ &+ \mathbb{Z}(0; \sqrt{2}, 0) + \mathbb{Z}(0; 0, \sqrt{2}) \\ &+ \mathbb{Z}\left(5\omega_1^{(A_9)}, 0; \frac{1}{\sqrt{2}}, 0\right) + \mathbb{Z}\left(0, 5\omega_1^{(A_9)}; 0, \frac{1}{\sqrt{2}}\right).\end{aligned}\quad (4.37)$$

This lattice admits the following symmetry:

$$g: (x_1^{(A_9)}, x_2^{(A_9)}; x^{(R)}) \mapsto (\tilde{x}_1^{(A_9)}, \tilde{x}_2^{(A_9)}; x^{(R)}). \quad (4.38)$$

The invariant lattice and its dual lattice are

$$\begin{aligned}I &= \sqrt{2}\Lambda_R(2C_5) + \mathbb{Z}(0; \sqrt{2}, 0) + \mathbb{Z}(0; 0, \sqrt{2}) \\ &+ \sqrt{2}\left(\frac{1}{2}\mathbb{Z}\alpha_5^{(C_5)} + \frac{1}{2}\mathbb{Z}\alpha_5^{(C_5)} + \mathbb{Z}\left(\frac{1}{2}\omega_5^{(C_5)}, 0; \frac{1}{2}; 0\right)\right) \\ &+ \mathbb{Z}\left(0, \frac{1}{2}\omega_5^{(C_5)}; 0, \frac{1}{2}\right), \\ I^* &= I + \frac{1}{\sqrt{2}}\Lambda_R(2C_5).\end{aligned}\quad (4.39)$$

By gathering the eight neutral elements and the elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $2C_5$ as massless fermions:

$$(\mathbf{44}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{44}). \quad (4.40)$$

The gauge symmetry is

$$\frac{\mathrm{Sp}(5) \times \mathrm{U}(1)}{\mathbb{Z}_2} \times \frac{\mathrm{Sp}(5) \times \mathrm{U}(1)}{\mathbb{Z}_2}. \quad (4.41)$$

H. $A_{11} + 2A_2 + A_3 \rightarrow C_6 + A_2 + C_2$ (#83)

The even self-dual lattice is

$$\begin{aligned} \Gamma_{18,2}^{(A_{11}+2A_2+A_3)} &= \Lambda_R(A_{11} + 2A_2 + A_3) + \mathbb{Z}(10\omega_1^{(A_{11})}, (1, 1)\omega_1^{(A_2)}, 2\omega_1^{(A_3)}) + \Lambda\left(\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right) \\ &\quad + \mathbb{Z}(\omega_1^{(A_{11})}, (0, 2)\omega_1^{(A_2)}, 3\omega_1^{(A_3)}; e_1^*) + \mathbb{Z}(0, (2, 1)\omega_1^{(A_2)}, 2\omega_1^{(A_3)}; e_2^*), \end{aligned} \quad (4.42)$$

where e_1^*, e_2^* are duals of generators of the lattice $\Lambda(\frac{4}{2}, \frac{2}{4})$, which is a lattice with gram matrix $(\frac{4}{2}, \frac{2}{4})^2$.

This lattice admits the following symmetry:

$$g: (x^{(A_{11})}, x_1^{(A_2)}, x_2^{(A_2)}, x^{(A_3)}; x^{(R)}) \mapsto (\tilde{x}^{(A_{11})}, -x_2^{(A_2)}, -x_1^{(A_2)}, \tilde{x}^{(A_3)}; x^{(R)}). \quad (4.43)$$

The invariant lattice and its dual lattice are

$$\begin{aligned} I &= \sqrt{2} \left(\Lambda_R(C_6 + A_2 + C_2) + \frac{1}{2} \mathbb{Z}\alpha_6^{(C_6)} + \frac{1}{2} \mathbb{Z}\alpha_2^{(C_2)} + \frac{1}{2} \mathbb{Z}(\omega_6^{(C_6)}, 0, \omega_2^{(C_2)}) \right) \\ &\quad + \Lambda\left(\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right) + \mathbb{Z}(0, \sqrt{2}\omega_1^{(A_2)}, 0; 2e_1^*) + \mathbb{Z}\left(0, 2\sqrt{2}\omega_1^{(A_2)}, \frac{1}{\sqrt{2}}\omega_2^{(C_2)}; e_2^*\right), \\ I^* &= I + \frac{1}{\sqrt{2}} \Lambda_R(C_6 + A_2 + C_2). \end{aligned} \quad (4.44)$$

By gathering the eight neutral elements and the elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $C_6 + A_2 + C_2$ as massless fermions:

$$(\mathbf{65}, \mathbf{1}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{8}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{1}, \mathbf{5}). \quad (4.45)$$

The gauge group is

$$\frac{\mathrm{Sp}(6) \times \mathrm{SU}(3) \times \mathrm{Sp}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)}{\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2}, \quad (4.46)$$

where the action of $\mathbb{Z}_n$ in the denominator on the fundamental representation is

$$\begin{aligned} \mathbb{Z}_3: & (\mathbf{1}, e^{2\pi i \frac{2}{3}} \mathbf{1}, \mathbf{1}, e^{2\pi i \frac{1}{3}} \mathbf{1}, e^{2\pi i \frac{1}{3}} \mathbf{1}), \\ \mathbb{Z}_2: & (-\mathbf{1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}), \\ \mathbb{Z}_2: & (-\mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}). \end{aligned} \quad (4.47)$$

$$\mathbf{I. A_{11} + A_5 + 2A_1 \rightarrow C_6 + C_3 + A_1 (\#87)}$$

$$\begin{aligned} \Gamma_{18,2}^{(A_{11}+A_5+2A_1)} &= \Lambda_R(A_{11} + A_5 + 2A_1) + \mathbb{Z}(10\omega_1^{(A_{11})}, 2\omega_1^{(A_5)}, (1, 1)\omega_1^{(A_1)}) + \mathbb{Z}(0; \sqrt{2}, 0) + \mathbb{Z}(0; 0, 2) \\ &\quad + \mathbb{Z}\left(8\omega_1^{(A_{11})}, \omega_1^{(A_5)}, (1, 1)\omega_1^{(A_1)}; \frac{1}{\sqrt{2}}, 0\right) + \mathbb{Z}\left(3\omega_1^{(A_{11})}, 3\omega_1^{(A_5)}, (0, 1)\omega_1^{(A_1)}; 0, \frac{1}{2}\right). \end{aligned} \quad (4.48)$$

This lattice admits the following symmetry:

$$g: (x^{(A_{11})}, x^{(A_5)}, x_1^{(A_1)}, x_2^{(A_1)}; x^{(R)}) \mapsto (\tilde{x}^{(A_{11})}, \tilde{x}^{(A_5)}, -x_2^{(A_1)}, -x_1^{(A_1)}; x^{(R)}). \quad (4.49)$$

The invariant lattice and its dual lattice are

$$\begin{aligned} I &= \sqrt{2} \left(\Lambda_R(C_6 + C_3 + A_1) + \mathbb{Z}(0; 1, 0) + \mathbb{Z}(0; 0, \sqrt{2}) + \frac{1}{2} \mathbb{Z}\alpha_6^{(C_6)} + \frac{1}{2} \mathbb{Z}\alpha_3^{(C_3)} \right) \\ &\quad + \sqrt{2} \mathbb{Z}\left(\frac{1}{2}\omega_6^{(C_6)}, 0, \omega_1^{(A_1)}\right) + \sqrt{2} \mathbb{Z}\left(0, \frac{1}{2}\omega_3^{(C_3)}, \omega_1^{(A_1)}; \frac{1}{2}, 0\right) + \sqrt{2} \mathbb{Z}\left(\frac{1}{2}\omega_6^{(C_6)}, 0, 0; 0, \frac{1}{\sqrt{2}}\right), \\ I^* &= I + \frac{1}{\sqrt{2}} \Lambda_R(C_6 + C_3 + A_1). \end{aligned} \quad (4.50)$$

²More explicitly, $e_1^* = e_1/3 - e_2/6$ and $e_2^* = -e_1/6 + e_2/3$.

By gathering the eight neutral elements and the elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $C_6 + C_3 + A_1$ as massless fermions:

$$(\mathbf{65}, \mathbf{1}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{14}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{1}, \mathbf{3}). \quad (4.51)$$

The gauge group is

$$\frac{\mathrm{Sp}(6) \times \mathrm{Sp}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)}{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2}, \quad (4.52)$$

where the action of $\mathbb{Z}_n$ in the denominator on the fundamental representation is

$$\begin{aligned} \mathbb{Z}_2: & (\mathbf{1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}, -\mathbf{1}), \\ \mathbb{Z}_2: & (\mathbf{1}, -\mathbf{1}, \mathbf{1}, -\mathbf{1}, -\mathbf{1}), \\ \mathbb{Z}_2: & (-\mathbf{1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}). \end{aligned} \quad (4.53)$$

$$\mathbf{J. A_{15} + A_3 \rightarrow C_8 + C_2 (\#108)}$$

The even self-dual lattice is

$$\begin{aligned} \Gamma_{18,2}^{(A_{15}+A_3)} = & \Lambda_R(A_{15} + A_3) + \mathbb{Z}(0; \sqrt{2}, 0) + \mathbb{Z}(0; 0, \sqrt{2}) + \mathbb{Z}(4\omega_1^{(A_{15})}, 2\omega_1^{(A_3)}; 0, 0) \\ & + \mathbb{Z}\left(2\omega_1^{(A_{15})}, \omega_1^{(A_3)}; \frac{1}{\sqrt{2}}, 0\right) + \mathbb{Z}\left(2\omega_1^{(A_{15})}, -\omega_1^{(A_3)}; 0, \frac{1}{\sqrt{2}}\right). \end{aligned} \quad (4.54)$$

This lattice admits the following symmetry:

$$g: (x^{(A_{15})}, x^{(A_3)}; x^{(R)}) \mapsto (\tilde{x}^{(A_{15})}, \tilde{x}^{(A_3)}; x^{(R)}). \quad (4.55)$$

By gathering the eight neutral elements and the elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $C_8 + C_2$ as massless fermions:

The invariant lattice I and its dual lattice I^* are

$$(\mathbf{119}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{5}). \quad (4.57)$$

$$\begin{aligned} I = & \sqrt{2} \left(\Lambda_R(C_8 + C_2) + \mathbb{Z}(0; 1, 0) \right. \\ & + \mathbb{Z}(0; 0, 1) + \frac{1}{2} \mathbb{Z} \alpha_8^{(C_8)} + \mathbb{Z} \frac{1}{2} \alpha_2^{(C_2)} \\ & \left. + \mathbb{Z} \frac{1}{2} \omega_8^{(C_8)} + \mathbb{Z} \left(0, \frac{1}{2} \omega_2^{(C_2)}; \frac{1}{2}, \frac{1}{2} \right) \right), \end{aligned}$$

$$I^* = I + \frac{1}{\sqrt{2}} \Lambda(C_8 + C_2). \quad (4.56)$$

The gauge group is

$$\frac{\mathrm{Sp}(8)}{\mathbb{Z}_2} \times \frac{\mathrm{Sp}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)}{\mathbb{Z}_2}. \quad (4.58)$$

$$\mathbf{K. 2A_3 + 2D_6 \rightarrow 2C_2 + D_6 (\#137)}$$

The even self-dual lattice is

$$\begin{aligned} \Gamma_{18,2}^{(2A_3+2D_6)} = & \Lambda_R(2A_3 + 2D_6) + \mathbb{Z}(0; 2, 0) + \mathbb{Z}(0; 0, 2) + \mathbb{Z}((0, 2)\omega_1^{(A_3)}, (1, 1)\omega_6^{(D_6)}) + \mathbb{Z}((2, 0)\omega_1^{(A_3)}, (1, 1)\omega_5^{(D_6)}) \\ & + \mathbb{Z}\left((1, 0)\omega_1^{(A_3)}, (1, 0)\omega_5^{(D_5)}; \frac{1}{2}, 0\right) + \mathbb{Z}\left((0, 1)\omega_1^{(A_3)}, (0, 1)\omega_6^{(D_5)}; 0, \frac{1}{2}\right). \end{aligned} \quad (4.59)$$

This lattice admits the following symmetry:

$$g: (x_1^{(A_3)}, x_2^{(A_3)}, x_1^{(D_6)}, x_2^{(D_6)}; x^{(R)}) \mapsto (\tilde{x}_1^{(A_3)}, \tilde{x}_2^{(A_3)}, -x_2^{(D_6)}, -x_1^{(D_6)}; x^{(R)}). \quad (4.60)$$

The invariant lattice and its dual lattice are

$$\begin{aligned} I = & \sqrt{2} \left(\Lambda_R(2C_2 + D_6) + \mathbb{Z}(0; \sqrt{2}, 0) + \mathbb{Z}(0; 0, \sqrt{2}) + \mathbb{Z} \frac{1}{2} \alpha_2^{(C_2)} + \mathbb{Z} \frac{1}{2} \alpha_2^{(C_2)} \right) \\ & + \sqrt{2} \mathbb{Z} \left(\left(0, \frac{1}{2} \right) \omega_2^{(C_2)}, \omega_6^{(D_6)} \right) + \sqrt{2} \mathbb{Z} \left(\left(\frac{1}{2}, 0 \right) \omega_2^{(C_2)}, \omega_5^{(D_6)} \right) \\ & + \sqrt{2} \mathbb{Z} \left(\left(\frac{1}{2}, 0 \right) \omega_2^{(C_2)}, 0; \frac{1}{\sqrt{2}}, 0 \right) + \sqrt{2} \mathbb{Z} \left(\left(0, \frac{1}{2} \right) \omega_2^{(C_2)}, 0; 0, \frac{1}{\sqrt{2}} \right), \\ I^* = & I + \frac{1}{\sqrt{2}} \Lambda_R(2C_2 + D_6). \end{aligned} \quad (4.61)$$

By gathering the eight neutral elements and the elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $2C_2 + D_6$ as massless fermions:

$$((\mathbf{5}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{5}), \mathbf{1}), \quad (\mathbf{1}, \mathbf{1}, \mathbf{66}). \quad (4.62)$$

The gauge group is

$$\frac{\text{Sp}(2) \times \text{Sp}(2) \times \text{Spin}(12) \times \text{U}(1) \times \text{U}(1)}{\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2}, \quad (4.63)$$

where

$$\begin{aligned} \mathbb{Z}_4 &: (\mathbf{1}, \mathbf{1}, e^{2\pi i \frac{1}{4}} \mathbf{1}, e^{2\pi i \frac{3}{4}} \mathbf{1}, e^{2\pi i \frac{1}{4}}), \\ \mathbb{Z}_2 &: (-\mathbf{1}, \mathbf{1}, \mathbf{1}, -\mathbf{1}, \mathbf{1}), \\ \mathbb{Z}_2 &: (-\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}, \mathbf{1}). \end{aligned} \quad (4.64)$$

L. $3E_6 \rightarrow E_6 + F_4$ (#219)

The even self-dual lattice is

$$\begin{aligned} \Gamma_{18,2}^{(3E_6)} &= \Lambda_R(3E_6) + \mathbb{Z}(1, 1, 1) \omega_1^{(E_6)} + \Lambda \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\ &+ \mathbb{Z}((1, -1, 0) \omega_1^{(E_6)}; e_1^*) + \mathbb{Z}((-1, 1, 0) \omega_1^{(E_6)}; e_2^*), \end{aligned} \quad (4.65)$$

where e_1, e_2 are the generators of $\Lambda(\frac{2}{1} \frac{1}{2})$, which is a lattice with gram matrix $(\frac{2}{1} \frac{1}{2})$.³

This lattice admits the following symmetry:

$$g: (x_1^{(E_6)}, x_2^{(E_6)}, x^{(E_6)}; x^{(R)}) \mapsto (-x_2^{(E_6)}, -x_1^{(E_6)}, \tilde{x}^{(E_6)}; x^{(R)}), \quad (4.66)$$

where $\tilde{x}^{(E_6)}$ is obtained by exchanging $\alpha_1^{(E_6)} \leftrightarrow \alpha_6^{(E_6)}$ and $\alpha_3^{(E_6)} \leftrightarrow \alpha_5^{(E_6)}$ in $x^{(E_6)}$.

The invariant lattice and its dual lattice are

$$\begin{aligned} I &= \sqrt{2} \Lambda_R(E_6 + F_4) + \Lambda \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\ &+ \mathbb{Z}(\sqrt{2} \omega_1^{(E_6)}, e_1^*) + \mathbb{Z}(-\sqrt{2} \omega_1^{(E_6)}, e_2^*), \\ I^* &= I + \frac{1}{\sqrt{2}} (\Lambda_R(E_6) + \mathbb{Z} \alpha_1^{(F_4)} + \mathbb{Z} \alpha_2^{(F_4)}). \end{aligned} \quad (4.67)$$

By gathering the eight neutral elements and the elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $E_6 + F_4$ as massless fermions:

$$(\mathbf{78}, \mathbf{1}), \quad (\mathbf{1}, \mathbf{26}). \quad (4.68)$$

The gauge group is

$$F_4 \times \frac{E_6 \times \text{U}(1) \times \text{U}(1)}{\mathbb{Z}_3 \times \mathbb{Z}_3}. \quad (4.69)$$

M. $D_4 + 2E_7 \rightarrow B_3 + E_7$ (#279)

The even self-dual lattice is

$$\begin{aligned} \Gamma_{18,2}^{(D_4+2E_7)} &= \Lambda_R(D_4 + 2E_7) + \mathbb{Z}(\omega_4^{(D_4)}, \omega_7^{(E_7)}, \omega_7^{(E_7)}) + \mathbb{Z}(0; \sqrt{2}, 0) + \mathbb{Z}(0; 0, \sqrt{2}) \\ &+ \mathbb{Z}\left(\omega_3^{(D_4)}, \omega_7^{(E_7)}, 0; \frac{1}{\sqrt{2}}, 0\right) + \mathbb{Z}\left(\omega_3^{(D_4)}, 0, \omega_7^{(E_7)}; 0, \frac{1}{\sqrt{2}}\right). \end{aligned} \quad (4.70)$$

This lattice admits the following symmetry:

$$g: (x^{(D_4)}, x_1^{(E_7)}, x_2^{(E_7)}; x^{(R)}) \mapsto (\tilde{x}^{(D_4)}, -x_2^{(E_7)}, -x_1^{(E_7)}; x^{(R)}), \quad (4.71)$$

where $\tilde{x}^{(D_4)}$ is obtained by exchanging $\alpha_1^{(D_4)} \leftrightarrow \alpha_3^{(D_4)}$ in $x^{(D_4)}$.

The invariant lattice and its dual lattice are

$$\begin{aligned} I &= \mathbb{Z} \alpha_1^{(B_3)} + \mathbb{Z} \alpha_2^{(B_3)} + \mathbb{Z} 2 \alpha_3^{(B_3)} + \sqrt{2} \Lambda_R(E_7) + \mathbb{Z}(0; \sqrt{2}, 0) + \mathbb{Z}(0; 0, \sqrt{2}) \\ &+ \mathbb{Z}(\alpha_3^{(B_3)}, \sqrt{2} \omega_7^{(E_7)}) + \mathbb{Z}\left(0, \sqrt{2} \omega_7^{(E_7)}; \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \\ I^* &= I + \frac{1}{\sqrt{2}} \Lambda_R(E_7) + \mathbb{Z}\left(\frac{1}{2} (\alpha_1^{(B_3)} + \alpha_3^{(B_3)}) + \frac{1}{\sqrt{2}} \omega_7^{(E_7)}; \frac{1}{\sqrt{2}}, 0\right). \end{aligned} \quad (4.72)$$

By gathering the eight neutral elements and the elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $B_3 + E_7$ as massless fermions:

³More explicitly, $e_1^* = 2e_1/3 - e_2/3$ and $e_2^* = -e_1/3 + 2e_2/3$.

$$(77, 1), \quad (1, 133). \quad (4.73)$$

The gauge group is

$$\text{Spin}(7) \times \frac{E_7 \times \text{U}(1) \times \text{U}(1)}{\mathbb{Z}_2 \times \mathbb{Z}_2}. \quad (4.74)$$

$$\mathbf{N. } 2E_8 + A_2 \rightarrow E_8 + A_2 (\#297)$$

The even self-dual lattice is

$$\begin{aligned} \Gamma_{18,2}^{(2E_8+A_2)} &= \Lambda_R(2E_8 + A_2) + \Lambda \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\ &+ \mathbb{Z}(\omega_1^{(A_2)}, e_1^*) + \mathbb{Z}(-\omega_1^{(A_2)}, e_2^*), \end{aligned} \quad (4.75)$$

where e_1, e_2 are generators of $\Lambda \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, which is a lattice with gram matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. This lattice admits the following symmetry:

$$g: (x_1^{(E_8)}, x_2^{(E_8)}, x^{(A_2)}; x^{(R)}) \mapsto (-x_2^{(E_8)}, -x_1^{(E_8)}, x^{(A_2)}; x^{(R)}). \quad (4.76)$$

The invariant lattice and its dual lattice are

$$\begin{aligned} I &= \sqrt{2}\Lambda_R(E_8) + \Lambda \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \Lambda_R(A_2) + \mathbb{Z}(\omega_1^{(A_2)}, e_1^*) \\ &+ \mathbb{Z}(-\omega_1^{(A_2)}, e_2^*), \\ I^* &= I + \frac{1}{\sqrt{2}}\Lambda_R(E_8). \end{aligned} \quad (4.77)$$

By gathering the eight neutral elements and the elements $(p_L, p_R) \in I^*$ that satisfy $p_L^2 = 1, p_R^2 = 0$, we obtain the following representation of $E_8 + A_2$ as massless fermions:

$$(248, 1). \quad (4.78)$$

The gauge group is

$$E_8 \times \frac{\text{SU}(3) \times \text{U}(1) \times \text{U}(1)}{\mathbb{Z}_3 \times \mathbb{Z}_3}. \quad (4.79)$$

O. Nonsupersymmetric branes

In this section, we have seen the various points of gauge enhancement with the $\mathbb{Z}_2$ outer automorphism symmetry g ,

as summarized in Table II. This indicates the disconnected gauge groups of the supersymmetric heterotic strings. Applying the no global symmetry/cobordism conjecture, the existence of the codimension-two nonsupersymmetric brane is predicted in supersymmetric heterotic theory. It is known that, in 10D heterotic theory, such a brane is characterized by the monodromy exchanging two E_8 s [5]. Similarly, in [8], we find that the monodromy involves the exchange as well as the charge conjugation symmetry of A_n . In this paper, we have found the new features that the monodromy corresponds to the folding of D_4 and E_6 Dynkin diagrams.

V. E_8 STRING THEORY ON T^d

In this section, we consider S^1 and T^2 compactifications of the ten-dimensional nonsupersymmetric E_8 string theory, and investigate the moduli space of maximal gauge enhancement. We confirm that the gauge enhancement is precisely the same as the result in Sec. IV. Note that the E_8 string theory on T^d always has a moduli-independent tachyon that cannot be removed by turning on moduli [61].

A. Construction

The ten-dimensional nonsupersymmetric E_8 string theory can be constructed by orbifolding the ten-dimensional supersymmetric $E_8 \times E'_8$ string theory (2.1) by $g = R(-1)^F$, where R is the outer automorphism of the $E_8 \times E'_8$ lattice $\Gamma_{E_8} \oplus \Gamma_{E'_8}$ and F is the spacetime fermion number. Its partition function can be expressed as the sum of these blocks (see e.g., [61] and Appendix B of [49]):

$$\begin{aligned} Z^{(10)}(1, 1) &= \frac{1}{2} Z_B^{(8)}(\bar{V}_8 - \bar{S}_8) \times \frac{1}{\eta^{16}} \sum_{\pi \in \Gamma_{E_8}} \sum_{\pi' \in \Gamma_{E'_8}} q^{\frac{1}{2}(\pi^2 + \pi'^2)}, \\ Z^{(10)}(1, g) &= \frac{1}{2} Z_B^{(8)}(\bar{V}_8 + \bar{S}_8) \times \frac{1}{\eta^{16}} \left(\frac{2\eta^3}{\theta_2} \right)^4 \sum_{\pi \in \sqrt{2}\Gamma_{E_8}} q^{\frac{1}{2}\pi^2}, \\ Z^{(10)}(g, 1) &= \frac{1}{2} Z_B^{(8)}(\bar{O}_8 - \bar{C}_8) \times \frac{1}{\eta^{16}} \left(\frac{\eta^3}{\theta_4} \right)^4 \sum_{\pi \in \frac{1}{\sqrt{2}}\Gamma_{E_8}} q^{\frac{1}{2}\pi^2}, \\ Z^{(10)}(g, g) &= \frac{1}{2} Z_B^{(8)}(\bar{O}_8 + \bar{C}_8) \times \frac{1}{\eta^{16}} \left(\frac{\eta^3}{\theta_3} \right)^4 \sum_{\pi \in \frac{1}{\sqrt{2}}\Gamma_{E_8}} q^{\frac{1}{2}\pi^2} e^{\pi i \pi^2}. \end{aligned} \quad (5.1)$$

Compactifying this theory on T^d , we obtain

$$\begin{aligned} Z^{(10-d)}(1, 1) &= \frac{1}{2} Z_B^{(8-d)}(\bar{V}_8 - \bar{S}_8) \times \frac{1}{\eta^{16+d} \bar{\eta}^d} \sum_{(p_-, P) \in \Gamma_{16+d,d}} q^{\frac{1}{2}(p_-^2 + P^2)} \bar{q}^{\frac{1}{2}P^2}, \\ Z^{(10-d)}(1, g) &= \frac{1}{2} Z_B^{(8-d)}(\bar{V}_8 + \bar{S}_8) \times \frac{1}{\eta^{16+d} \bar{\eta}^d} \left(\frac{2\eta^3}{\theta_2} \right)^4 \sum_{P \in I_{8+d,d}} q^{\frac{1}{2}P^2} \bar{q}^{\frac{1}{2}P^2}, \end{aligned}$$

$$\begin{aligned}
Z^{(10-d)}(g, 1) &= \frac{1}{2} Z_B^{(8-d)}(\bar{O}_8 - \bar{C}_8) \times \frac{1}{\eta^{16+d} \bar{\eta}^d} \left(\frac{\eta^3}{\theta_4} \right)^4 \sum_{P \in I_{8+d,d}^*} q^{\frac{1}{2} P_L^2} \bar{q}^{\frac{1}{2} P_R^2}, \\
Z^{(10-d)}(g, g) &= \frac{1}{2} Z_B^{(8-d)}(\bar{O}_8 + \bar{C}_8) \times \frac{1}{\eta^{16+d} \bar{\eta}^d} \left(\frac{\eta^3}{\theta_3} \right)^4 \sum_{P \in I_{8+d,d}^*} q^{\frac{1}{2} P_L^2} \bar{q}^{\frac{1}{2} P_R^2} e^{\pi i P^2}.
\end{aligned} \tag{5.2}$$

The elements of $\Gamma_{16+d,d}$ are expressed in terms of moduli such as the metric $G = ee^t$,⁴ the antisymmetric tensor B , and the Wilson lines $A = (a, a)$,⁵

$$\begin{aligned}
p_- &= \frac{1}{\sqrt{2}} \pi_-, \\
p_+ &= \frac{1}{\sqrt{2}} (\pi_+ + 2wa), \\
p_L &= \frac{1}{\sqrt{2}} (n + w(2G - E) - \pi_+ \cdot a) e^{-t} = p_R + \sqrt{2} we, \\
p_R &= \frac{1}{\sqrt{2}} (n - wE - \pi_+ \cdot a) e^{-t},
\end{aligned} \tag{5.3}$$

where we define $E = G + B + a \cdot a$. We have adopted the notations of row vectors

$$\pi_{\pm} := \pi \pm \pi' \in \Gamma_{E_8}, \quad w \in \mathbb{Z}^d, \quad n \in \mathbb{Z}^d. \tag{5.4}$$

The elements of $I_{8+d,d}$ and $I_{8+d,d}^*$ are written by $P = (P_L; p_R) = (p_+, p_L; p_R)$ which satisfies

$$P^2 = P_L^2 - p_R^2 = \frac{1}{2} \pi_+^2 + 2wn^t. \tag{5.5}$$

Note that $\pi_+ \in 2\Gamma_{E_8}$ for $P \in I_{8+d,d}$.

B. Massless states and maximal enhancements of 9D theories

1. Massless vectors

To identify the maximal enhancements, we first focus on the massless gauge bosons which come from the untwisted sectors combined with the D_4 character vector conjugacy class in 9D theories. The massless conditions for the gauge bosons which depend on the moduli are the same as those for the untwisted sectors of CHL string (for details, see e.g., [36]):

$$P_L^2 = \frac{1}{2} \pi_+^2 + 2wn = \begin{cases} 1 & \text{with } \pi_-^2 = 1, \\ 2 & \text{with } \pi_- = 0, \end{cases} \tag{5.6}$$

$$p_R = n - wE - \pi_+ \cdot a = 0, \tag{5.7}$$

⁴Here e is the vierbein, and t is the transpose.

⁵By the consistency with orbifolding, Wilson lines are restricted to take the form as $A = (a, a)$.

where $E = R^2 + a^2$. The solutions for $P_L^2 = 1$ correspond to short roots while those for $P_L^2 = 2$ correspond to long roots. Note that for the $P_L^2 = 2$ case, π_+ is more restricted to $\pi_+ \in 2\Gamma_8$. In the 9D CHL string, n must be even and there are no solutions for $P_L^2 = 2$ in the untwisted sectors with $d = 1$, which does not allow nonsimply laced gauge symmetries in the 9D CHL string. The E_8 string on S_1 , however, has the solutions which give nonsimply laced gauge enhancements even in 9D theories with specific moduli.

Let us consider the simplest case where $a = 0$. The condition $p_R = 0$ gives $n = wR^2$, thus we get the solutions for any R :

$$\pi_+^2 = 2, \quad w = n = 0, \tag{5.8}$$

which satisfy $P_L^2 = 1$ and correspond to the nonzero roots of E_8 . One can also find the solutions for $P_L^2 = 2$ when $R = 1$, which give long roots:

$$\pi_+ = 0, \quad w = n = \pm 1. \tag{5.9}$$

Note that these states cannot be found in the 9D CHL string. From these solutions, the gauge group is $E_8 \times C_1$, rather than $E_8 \times A_1$, because of the long roots existing in the case with $R = 1$, $a = 0$.

Specifying the massless vectors in the case of nonzero a is also straightforward. Here we briefly comment on the outline. For short roots, the same solutions as in the 9D CHL can be found if the Wilson line takes the form as $a = \omega_i / \kappa_i$ where ω_i and κ_i are the fundamental weight and the Kac mark of E_8 shown in Table III.

TABLE III. Kac marks, simple roots, and fundamental weights of E_8 .

i	κ_i	α_i	ω_i
1	3	$(1, -1, 0, 0, 0, 0, 0, 0)$	$-(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{5}{2})$
2	6	$(0, 1, -1, 0, 0, 0, 0, 0)$	$-(0, 0, 1, 1, 1, 1, 1, -5)$
3	5	$(0, 0, 1, -1, 0, 0, 0, 0)$	$-(0, 0, 0, 1, 1, 1, 1, -4)$
4	4	$(0, 0, 0, 1, -1, 0, 0, 0)$	$-(0, 0, 0, 0, 1, 1, 1, -3)$
5	3	$(0, 0, 0, 0, 1, -1, 0, 0)$	$-(0, 0, 0, 0, 0, 1, 1, -2)$
6	2	$(0, 0, 0, 0, 0, 1, -1, 0)$	$(0, 0, 0, 0, 0, 0, -1, 1)$
7	4	$-(1, 1, 0, 0, 0, 0, 0, 0)$	$-(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{7}{2})$
8	2	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(0, 0, 0, 0, 0, 0, 0, 2)$
0	1	$(0, 0, 0, 0, 0, 0, 1, -1)$	$(0, 0, 0, 0, 0, 0, 0, 0)$

TABLE IV. Maximal enhancements and corresponding moduli in the $E_8 \times E'_8$ and E_8 string on S^1 , where Wilson line a is expressed by data in Table III. The latter ones can be obtained by deleting the i th node of the 9D EDD shown in Fig. 7.

i	$E_8 \times E'_8$	E_8	E	a
1	A_{17}	C_9	1	$\frac{\omega_1}{3}$
2	$2A_1 + 2A_2 + A_{11}$	$C_6 + A_2 + A_1$	1	$\frac{\omega_2}{6}$
3	$2A_4 + A_9$	$C_5 + A_4$	1	$\frac{\omega_3}{5}$
4	$A_7 + 2D_5$	$C_4 + D_5$	1	$\frac{\omega_4}{4}$
5	$A_5 + 2E_6$	$C_3 + E_6$	1	$\frac{\omega_5}{3}$
6	$A_3 + 2E_7$	$C_2 + E_7$	1	$\frac{\omega_6}{2}$
7	$2A_1 + A_{15}$	$C_8 + A_1$	1	$\frac{\omega_7}{4}$
0	$A_1 + 2E_8$	$C_1 + E_8$	1	ω_0

For example, we find

$$\pi_+^2 = 2, \quad w = n = 0, \quad \pi_+ \cdot \omega_i = 0. \quad (5.10)$$

From the first and third conditions, $\pi_+ = \alpha_j, j \neq i$ can survive, but $\pi_+ = \alpha_i$ is projected out, where α_i is the simple roots of E_8 . For long roots, we always obtain (5.9) when $E = 1$.

For solving (5.6) with the moduli $E = 1, a = \omega_i/\kappa_i$, we finally obtain the eight maximal enhanced gauge groups shown in Table IV. Note that we cannot obtain a maximal enhancement with $a = \omega_8/2$ which is not consistent with $E = 1$, since $a^2 = 1$ gives $R = 0$. This leads to the technique of the generalized Dynkin diagram for the 9D theories that can identify the maximal enhancements, discussed in Sec. VC.

2. Massless matters

Next we consider the matter spectra. In particular, to compare the results in Sec. III, we focus on the massless cospinors which come from the untwisted sectors combined with the D_4 character co-spinor conjugacy class in 9D theories. From 96, cospinor states are paired with the dual lattice. The massless conditions for the cospinors are then given by

$$P_L^2 = \frac{1}{2}\pi_+^2 + 2wn = 1, \quad (5.11)$$

$$p_R = n - wE - \pi_+ \cdot a = 0. \quad (5.12)$$

These conditions are nothing but (5.6) with $P_L^2 = 1$, which correspond to the short roots. Then the massless matters are expressed by the quasi-minuscule representation of C_n . We finally obtain the same results as in Table II by solving (5.10) with the moduli $E = 1, a = \omega_i/\kappa_i, i \neq 8$.

C. Generalized Dynkin diagram for S^1

We propose the generalized Dynkin diagram (EDD) for the E_8 string on S^1 , which can be used for the gauge group with the maximal rank. The difference from the EDD for the 9D CHL string [36] is the C node that gives the condition of E .

We define the charge vector

$$Z = |2w, n; \pi_+\rangle, \quad (5.13)$$

and the inner product of charge vectors is defined through the metric

$$\eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_8 \end{pmatrix}. \quad (5.14)$$

Note that $Z^2 = \pi_+^2 + 4wn = 2P^2$. The nodes 1 through 8 are the simple roots of E_8

$$Z_i = |0, 0; \alpha_i\rangle, \quad i = 1, \dots, 8, \quad (5.15)$$

and the 0 node is the lowest root of E_8 with nonzero momentum

$$Z_0 = |0, -1; \alpha_0\rangle. \quad (5.16)$$

The lowest root α_0 is expressed as

$$\alpha_0 = -\sum_{i=1}^8 \kappa_i \alpha_i. \quad (5.17)$$

Finally, the C node is

$$Z_C = |2, 1; 0^8\rangle, \quad (5.18)$$

which corresponds to the long root and differs from the C node of the 9D CHL string $|1, 1; 0\rangle$. The difference in the first component comes from the fact that the winding number can take a half-integer value in the 9D CHL string while that is impossible in the E_8 string on S^1 . That also leads to the condition $E = 1$ for the maximal enhancements, unlike $E = 2$ in the 9D CHL string.

We show the generalized Dynkin diagram for the E_8 string theory on S^1 in Fig. 7. We colored the nodes that have

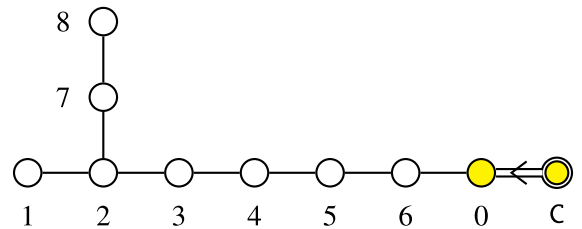


FIG. 7. Generalized Dynkin diagram for 9D theories.

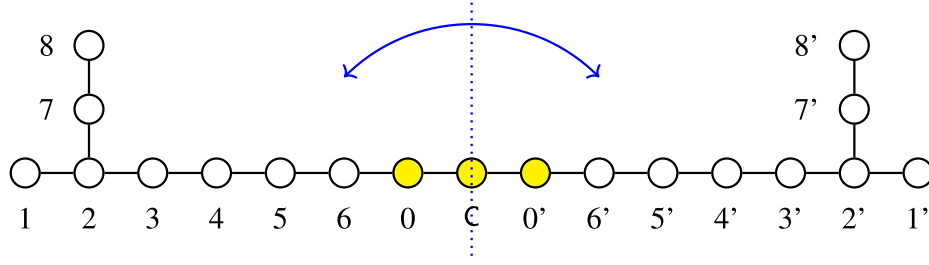


FIG. 8. EDD for the 9D theories can be obtained by folding the SUSY one.

momentum and/or winding charge and the double border of the node implies that it corresponds to a vector with $P^2 = 2$. From this diagram, we can determine the maximal enhancements of this theory by deleting one node and we can also read off the corresponding moduli as shown in Table IV. Note that the case of deleting the node 8 is excluded since it gives the unphysical condition such that $a^2 = (\frac{\omega_8}{2})^2 = 1$ and $R^2 = E - a^2 = 0$. These results are completely the same as in [8].

This diagram can also be obtained by folding the generalized Dynkin diagram for the heterotic superstrings on S^1 [35], as shown in Fig. 8. That is consistent with the statement that this theory can be constructed from the 9D heterotic $E_8 \times E_8$ superstring by orbifolding of the outer automorphism as well as the fermion parity. Note that the nodes 8 and 8' cannot be removed at the same time in the SUSY EDD since it does not give a possible Dynkin diagram. That is also consistent with the fact that node 8 cannot be removed in the EDD in Fig. 7. From this operation, we can see that removing the i th node in Fig. 7 corresponds to removing the i th and i' th node in the SUSY Dynkin diagram, which implies that the maximal enhancements of the E_8 string can be obtained by folding the only eight maximal enhancements of SUSY theories in Table 11 of [35], where the Wilson line takes the form $A = (a, a)$.

D. Massless states and maximal enhancements of 8D theories

1. Massless vectors

Let us consider the massless vectors in 8D theories. The massless conditions for 8D are the generalized ones of (5.6):

$$P_L^2 = \frac{1}{2}\pi_+^2 + 2w^1 n_1 + 2w^2 n_2 = \begin{cases} 1 & \text{with } \pi_-^2 = 1, \\ 2 & \text{with } \pi_- = 0, \end{cases} \quad (5.19)$$

$$p_{R,i} = n_i - w^j E_{ij} - \pi_+ \cdot a_i = 0, \quad i = 1, 2. \quad (5.20)$$

In order to obtain the 8D maximal enhancements, we use the data for the gauge groups with maximal rank of the 8D SUSY heterotic string theory and the corresponding moduli, as shown in Table 12 of [35]. The criteria for choosing the

theories is that both of the two Wilson lines of SUSY theories are expressed as $A_1 = (a_1, a_1)$, $A_2 = (a_2, a_2)$, and use the same moduli a_1 , a_2 , and E . Solving (5.17) with these moduli, we can then obtain the 22 maximal enhanced theories, as shown in Table V. The maximally enhanced gauge groups in Table V are consistent with the results obtained by folding the lattice in Sec. IV. In the 8D SUSY theories, #1 and #2 were derived using the fixed Wilson line algorithm, while the others were obtained using the method of generalized Dynkin diagrams. In Sec. V E, we show that those 20 theories can be obtained with the method of EDDs.

2. Massless matters

As in the case with 9D, the massless conditions for the cospinor states are given by (5.17) with $P_L^2 = 1$. Solving these conditions with the corresponding moduli, we can obtain the same results in Table II.

E. Generalized Dynkin diagram for T^2

We propose the procedure for identifying the maximal enhanced gauge groups in the E_8 string on T^2 , which is similar to the method with generalized Dynkin diagrams proposed in the SUSY theories. For the T^2 case, the charge vector is generalized as

$$Z = |2w_1, 2w_2, n_1, n_2; \pi_+\rangle, \quad (5.21)$$

and the metric is expressed as

$$\eta = \begin{pmatrix} 0 & \mathbf{1}_2 & 0 \\ \mathbf{1}_2 & 0 & 0 \\ 0 & 0 & \mathbf{1}_8 \end{pmatrix}. \quad (5.22)$$

Note again $Z^2 = \pi_+^2 + 4w^1 n_1 + 4w^2 n_2 = 2P^2$. As discussed in [35], to obtain the maximal enhancements, we take the Wilson line as $a_1 = \omega_k / \kappa_k$, where k is the number of the deleted node. The subgroup left invariant by a_1 is denoted by H_k , which can be obtained from Fig. 7 by deleting the node corresponding to the root α_k . For a_2 , which breaks H_k further, we use the shift algorithm and consider the following procedures: we append the one affine root $\hat{\alpha}_k$ of the subgroup H_k , and then delete one node corresponding to the root α_p with $p \neq k$ of the subgroup.

TABLE V. Maximal enhancements and corresponding moduli in the $E_8 \times E'_8$ and E_8 strings on T^2 , where Wilson lines a_1, a_2 are expressed by data in Table III. All of them except for #1, 2 can be obtained with the method of generalized Dynkin diagrams.

#	$E_8 \times E'_8$	E_8	E_{11}	E_{12}	E_{21}	E_{22}	a_1	a_2
1	$6A_3$	$2C_2 + 2A_3$	1	$\frac{1}{2}$	0	1	$\frac{\omega_6}{2}$	$\frac{\omega_2}{4} - \frac{3\omega_6}{4}$
2	$2A_1 + 4A_4$	$2C_1 + 2A_4$	1	$-\frac{1}{5}$	$\frac{1}{5}$	1	$\frac{\omega_3}{5}$	$w_1 - \frac{3\omega_3}{5}$
5	$4A_2 + 2A_5$	$2A_2 + 2C_3$	1	0	0	1	$\frac{\omega_5}{3}$	$\frac{\omega_2}{3} - \frac{2\omega_5}{3}$
6	$A_3 + 3A_5$	$C_2 + C_3 + A_5$	1	0	0	1	$\frac{\omega_5}{3}$	$\frac{\omega_1}{2} - \frac{\omega_5}{2}$
25	$4A_1 + 2A_7$	$2A_1 + 2C_4$	1	0	0	1	$\frac{\omega_4}{4}$	$\frac{\omega_2}{2} - \frac{3\omega_4}{4}$
28	$2A_1 + 3A_3 + A_7$	$A_1 + C_2 + A_3 + C_4$	1	0	0	1	$\frac{\omega_4}{4}$	$\frac{\omega_4}{2} - \frac{\omega_7}{2}$
54	$2A_9$	$2C_5$	1	0	0	1	$\frac{\omega_3}{5}$	$\omega_1 - \frac{3\omega_3}{5}$
57	$A_1 + 2A_4 + A_9$	$C_1 + A_4 + C_5$	1	0	0	1	$\frac{\omega_3}{5}$	0
81	$3A_1 + 2A_2 + A_{11}$	$A_1 + C_1 + A_2 + C_6$	1	0	0	1	$\frac{\omega_2}{6}$	0
83	$2A_2 + A_3 + A_{11}$	$A_2 + C_2 + C_6$	1	0	0	1	$\frac{\omega_6}{3}$	$\frac{\omega_3}{3} - \frac{5\omega_6}{6}$
87	$2A_1 + A_5 + A_{11}$	$A_1 + C_3 + C_6$	1	0	0	1	$\frac{\omega_2}{6}$	$w_7 - \frac{2\omega_2}{3}$
106	$3A_1 + A_{15}$	$A_1 + C_1 + C_8$	1	0	0	1	$\frac{\omega_7}{4}$	0
108	$A_3 + A_{15}$	$C_2 + C_8$	1	0	0	1	$\frac{\omega_6}{2}$	$\frac{\omega_1}{2} - \frac{3\omega_6}{4}$
111	$A_1 + A_{17}$	$C_1 + C_9$	1	0	0	1	$\frac{\omega_1}{3}$	0
121	$A_1 + A_7 + 2D_5$	$C_1 + C_4 + D_5$	1	0	0	1	$\frac{\omega_4}{4}$	0
137	$2A_3 + 2D_6$	$2C_2 + D_6$	1	0	0	1	$\frac{\omega_6}{2}$	$\frac{\omega_6}{2} - \frac{\omega_8}{2}$
219	$3E_6$	$E_6 + F_4$	1	-1	0	1	$\frac{\omega_5}{3}$	0
222	$A_1 + A_5 + 2E_6$	$C_1 + C_3 + E_6$	1	0	0	1	$\frac{\omega_5}{3}$	0
257	$A_1 + A_3 + 2E_7$	$C_1 + C_2 + E_7$	1	0	0	1	$\frac{\omega_6}{2}$	0
279	$D_4 + 2E_7$	$B_3 + E_7$	1	-1	0	1	$\frac{\omega_6}{2}$	0
296	$2A_1 + 2E_8$	$2C_1 + E_8$	1	0	0	1	0	0
297	$A_2 + 2E_8$	$A_2 + E_8$	1	-1	0	1	0	0

The two Wilson lines a_1, a_2 finally leave a subgroup of E_8 unbroken and the surviving simple roots are α_i , $i \neq k, p$, and $\hat{\alpha}_k$. By the shift algorithm, we can obtain a_2 by the following conditions:

$$a_2 \cdot \alpha_i = 0, \quad i \neq k, p, \quad a_2 \cdot \hat{\alpha}_k = -1. \quad (5.23)$$

The affine root $\hat{\alpha}_k$ can be expressed as

$$\hat{\alpha}_k = -\sum_{i \neq k} \hat{\kappa}_i \alpha_i, \quad (5.24)$$

where $\hat{\kappa}_i$ is the Kac mark of the i th node for the Dynkin diagram of H_k . Then from (5.23), the Wilson line a_2 is given by

$$a_2 = \frac{\kappa_p}{\hat{\kappa}_p} \left(\frac{\omega_p}{\kappa_p} - \frac{\omega_k}{\kappa_k} \right). \quad (5.25)$$

The nodes 0 through 8 are simply generalized to

$$Z_i = |0, 0, 0, 0; \alpha_i\rangle, \quad i = 1, \dots, 8, \quad (5.26)$$

$$Z_0 = |0, 0, -1, 0; \alpha_0\rangle. \quad (5.27)$$

The new affine node associated with $\hat{\alpha}_k$ is

$$Z_{-1} = |0, 0, 0, -1; \hat{\alpha}_k\rangle. \quad (5.28)$$

We need to choose the tensor E_{ij} in the T^2 case to fix the C node. We focus on the two choices as in [35]:

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \quad (5.29)$$

For $E = E_1$, we obtain the two charged vectors

$$Z_{C_1} = |2, 0, 1, 0; 0^8\rangle, \quad Z_{C_2} = |0, 2, 0, 1; 0^8\rangle, \quad (5.30)$$

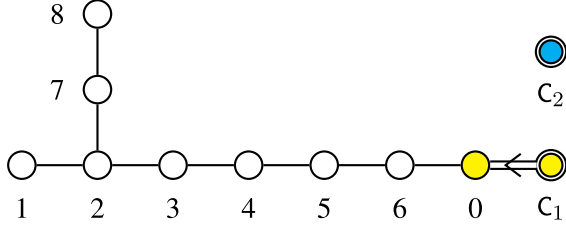
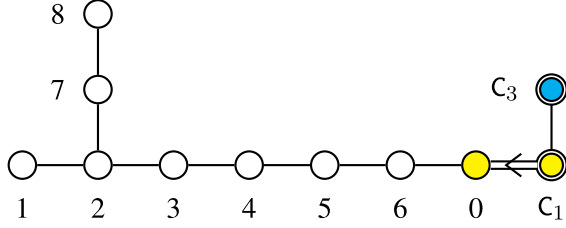
and for $E = E_2$,

$$Z_{C_1} = |2, 0, 1, 0; 0^8\rangle, \quad Z_{C_3} = |0, 2, -1, 1; 0^8\rangle. \quad (5.31)$$

Note that these nodes correspond to long roots since $Z^2 = 2P^2 = 4$. From their inner products, one can see Z_{C_1} and Z_{C_2} are disconnected while Z_{C_1} and Z_{C_3} are connected.

The procedures to obtain the maximal enhancements using generalized Dynkin diagrams are summarized as follows:

- (i) We start from the generalized Dynkin diagram for the E_8 string theory on S^1 shown in Fig. 7 and omit the k th node using the Wilson line $a_1 = \omega_k / \kappa_k$.

FIG. 9. Generalized Dynkin diagram for 8D theories with $E = E_1$ and $a_2 = 0$.FIG. 10. Generalized Dynkin diagram for 8D theories with $E = E_2$ and $a_2 = 0$.

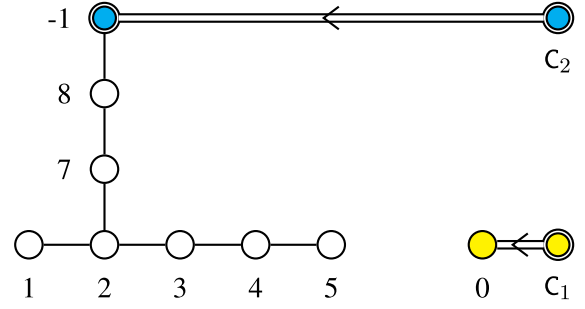
- (ii) Next we append the affine node Z_{-1} to the ADE types of the disconnected diagram, and furthermore connect the node Z_{C_2} or Z_{C_3} to the affine one.
- (iii) Finally we delete the p th node using the Wilson line a_2 given by (5.25) so that the remaining diagrams are Dynkin diagrams.

The diagrams appeared in this procedures can be obtained by folding the ones in the procedures for the SUSY theories.

Let us see some examples. We take the simplest case with $E = E_1, a_2 = 0$ as a first example. Since a_2 vanishes, the node Z_{-1} is deleted. The generalized Dynkin diagram in this case is shown in Fig. 9. Using $a_1 = \omega_i/\kappa_i, i = 0, \dots, 7$, we obtain eight allowed maximal enhancements by deleting the i th node, which correspond to #57, 81, 106, 111, 121, 222, 257, 296. Note that the 8 node cannot be omitted because it does not lead to Dynkin diagrams.

Next we take the case with $E = E_2, a_2 = 0$. The difference from the first example is that the node Z_{C_3} appears instead of Z_{C_2} and it is connected to the node Z_{C_1} . The generalized Dynkin diagram in this case is shown in Fig. 10. As in the case with $E = E_1, a_2 = 0$, we obtain the allowed maximal enhancements by specific a_1 . However, in this case, the nodes that can be deleted are more restricted. To get possible Dynkin diagrams, we have only three choices: the 0th, 5th, or 6th node that we can omit. These correspond to #219, 279, 297. Note that the moduli $E = E_2, a_1 \neq 0, a_2 \neq 0$ cannot lead to any maximal enhancements because deleting any nodes cannot lead to Dynkin diagrams, as in the case of the SUSY theories.

As the last example, we take the case with $E = E_1, a_1 = \omega_6/2$. The generalized Dynkin diagram in

FIG. 11. Generalized Dynkin diagram corresponding to the choice $E = E_1, a_1 = \omega_6/2$.

this case is shown in Fig. 11. Using a_2 given by (5.25) with $k = 6, p = 1, 3, 6, 8$, we obtain four allowed maximal enhancements by deleting the i th node, which correspond to #83, 108, 137, and again #257.

Similarly, the other six maximal enhancements #5, 6, 25, 28, 54, 87 can also be obtained by the above procedures.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: THETA FUNCTIONS AND D_4 CHARACTERS

In this appendix, we summarize the formula used in the paper. Let τ be a complex number with positive imaginary part, and $q = \exp(2\pi i\tau)$. Theta functions are

$$\begin{aligned}
 \theta_1(\tau) &:= i \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} = 0, \\
 \theta_2(\tau) &:= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n-\frac{1}{2})^2}, \\
 \theta_3(\tau) &:= \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2}, \\
 \theta_4(\tau) &:= \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2}.
 \end{aligned} \tag{A1}$$

They can also be expressed as infinite products:

$$\begin{aligned}
 \theta_2(\tau) &= 2q^{\frac{1}{8}} \prod_{m=1}^{\infty} (1 - q^m)(1 + q^m)(1 + q^{2m}), \\
 \theta_3(\tau) &= \prod_{m=1}^{\infty} (1 - q^m)(1 + q^{m-\frac{1}{2}})(1 + q^{m-\frac{3}{2}}), \\
 \theta_4(\tau) &= \prod_{n=0}^{\infty} (1 - q^n)(1 - q^{n-\frac{1}{2}})(1 - q^{n-\frac{3}{2}}). \quad (\text{A2})
 \end{aligned}$$

Modular transformation properties of theta functions are given as follows:

$$(\theta_2, \theta_3, \theta_4)(\tau + 1) = (e^{\frac{\pi i}{4}} \theta_2, \theta_4, \theta_3)(\tau),$$

$$\begin{pmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} \begin{pmatrix} -1 \\ \tau \end{pmatrix} = e^{-\frac{i\pi}{4}\tau^{\frac{1}{2}}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix}(\tau). \quad (\text{A3})$$

The conjugacy classes are

$$\Gamma_g^{(4)} = \left\{ (n_1, \dots, n_4) \mid n_i \in \mathbb{Z}, \sum_{i=1}^4 n_i \in 2\mathbb{Z} \right\}, \quad (\text{A4})$$

$$\Gamma_v^{(4)} = \left\{ (n_1, \dots, n_4) \mid n_i \in \mathbb{Z}, \sum_{i=1}^4 n_i \in 2\mathbb{Z} + 1 \right\}, \quad (\text{A5})$$

$$\Gamma_s^{(4)} = \left\{ \left(n_1 + \frac{1}{2}, \dots, n_4 + \frac{1}{2} \right) \mid n_i \in \mathbb{Z}, \sum_{i=1}^4 n_i \in 2\mathbb{Z} \right\}, \quad (\text{A6})$$

$$\Gamma_c^{(4)} = \left\{ \left(n_1 + \frac{1}{2}, \dots, n_4 + \frac{1}{2} \right) \mid n_i \in \mathbb{Z}, \sum_{i=1}^4 n_i \in 2\mathbb{Z} + 1 \right\}. \quad (\text{A7})$$

These lattices give the D_4 characters:

$$\begin{aligned}
 O_8 &= \frac{1}{\eta^4} \sum_{p \in \Gamma_g^{(4)}} q^{\frac{1}{2}|p|^2} = \frac{1}{2\eta^4} (\theta_3^4(\tau) + \theta_4^4(\tau)), \\
 V_8 &= \frac{1}{\eta^4} \sum_{p \in \Gamma_v^{(4)}} q^{\frac{1}{2}|p|^2} = \frac{1}{2\eta^4} (\theta_3^4(\tau) - \theta_4^4(\tau)), \\
 S_8 &= \frac{1}{\eta^4} \sum_{p \in \Gamma_s^{(4)}} q^{\frac{1}{2}|p|^2} = \frac{1}{2\eta^4} (\theta_2^4(\tau) + \theta_1^4(\tau)), \\
 C_8 &= \frac{1}{\eta^4} \sum_{p \in \Gamma_c^{(4)}} q^{\frac{1}{2}|p|^2} = \frac{1}{2\eta^4} (\theta_2^4(\tau) - \theta_1^4(\tau)). \quad (\text{A8})
 \end{aligned}$$

It follows from Eq. (A3) that they transform as

$$\begin{aligned}
 &(O_8, V_8, S_8, C_8)(\tau + 1) \\
 &= (e^{-\frac{1}{3}\pi i} O_8, -e^{-\frac{1}{3}\pi i} V_8, e^{\frac{2}{3}\pi i} S_8, e^{\frac{2}{3}\pi i} C_8)(\tau), \\
 &\begin{pmatrix} O_8 \\ V_8 \\ S_8 \\ C_8 \end{pmatrix} \begin{pmatrix} -1 \\ \tau \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} O_8 \\ V_8 \\ S_8 \\ C_8 \end{pmatrix}(\tau). \quad (\text{A9})
 \end{aligned}$$

The small q expansions are

$$\begin{aligned}
 O_8 &= \frac{1}{\eta^4} (1 + 24q + \dots), & V_8 &= \frac{1}{\eta^4} (8q^{\frac{1}{2}} + \dots), \\
 S_8 &= \frac{1}{\eta^4} (8q^{\frac{1}{2}} + \dots), & C_8 &= \frac{1}{\eta^4} (8q^{\frac{1}{2}} + \dots). \quad (\text{A10})
 \end{aligned}$$

The definition of the Dedekind eta function is

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \quad (\text{A11})$$

The modular transformation properties are

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau). \quad (\text{A12})$$

APPENDIX B: ROOT AND WEIGHT LATTICES

In this appendix, we summarize the details of the root lattices and weight lattices used in Secs. II, IV, and V. For notation and other details see [62]. Note that for E_8 , we use the same notation as in [35]. We denote the i th standard orthonormal basis of $\mathbb{R}^n$ by ε_i .

1. A_n type

The Dynkin diagram of the A_n algebra is shown in Fig. 12. The root system of A_n is

$$\varepsilon_j - \varepsilon_k, \quad \text{for } 1 \leq j < k \leq n + 1, \quad j \neq k. \quad (\text{B1})$$

The basis is

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad \text{for } 1 \leq i \leq n. \quad (\text{B2})$$

The fundamental weights of A_n are

$$\begin{aligned}
 \omega_i &= \varepsilon_1 + \dots + \varepsilon_i - \frac{i}{n+1} (\varepsilon_1 + \dots + \varepsilon_{n+1}) \\
 &= \frac{1}{n+1} [(n-i+1)(\alpha_1 + 2\alpha_2 + \dots + (i-1)\alpha_{i-1}) \\
 &\quad + i((n-i+1)\alpha_i + (n-i)\alpha_{i+1} + \dots + \alpha_n)]. \quad (\text{B3})
 \end{aligned}$$

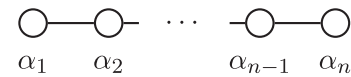


FIG. 12. Dynkin diagram of type A_n .

FIG. 13. Dynkin diagram of type B_n .

2. B_n type

The Dynkin diagram of the B_n algebra is shown in Fig. 13. The root system of B_n is

$$\begin{aligned} \pm \varepsilon_i, & \quad \text{for } 1 \leq i \leq n, \\ \pm \varepsilon_i \pm \varepsilon_j, & \quad \text{for } 1 \leq i < j \leq n. \end{aligned} \quad (\text{B4})$$

The fundamental weights of B_n are

$$\begin{aligned} \omega_{i \neq n} &= \varepsilon_1 + \cdots + \varepsilon_i \\ &= \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_i + \cdots + \alpha_n), \\ \omega_n &= \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n) \\ &= \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n). \end{aligned} \quad (\text{B5})$$

3. C_n type

The Dynkin diagram of the C_n algebra is shown in Fig. 14. The root system of C_n is

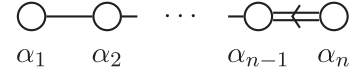
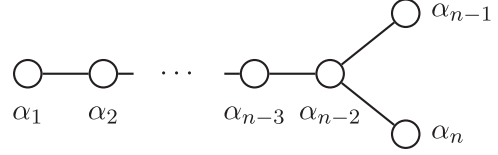
$$\begin{aligned} \pm 2\varepsilon_j, & \quad \text{for } 1 \leq j \leq n, \\ \pm \varepsilon_j \pm \varepsilon_k, & \quad \text{for } 1 \leq j < k \leq n. \end{aligned} \quad (\text{B6})$$

The basis of the C_n root is

$$\begin{aligned} \alpha_i &= \varepsilon_i - \varepsilon_{i+1}, \quad \text{for } 1 \leq i \leq n-1, \\ \alpha_n &= 2\varepsilon_n. \end{aligned} \quad (\text{B7})$$

The fundamental weights of C_n are

$$\begin{aligned} \omega_i &= \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i \\ &= \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} \\ &\quad + i\left(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-1} + \frac{1}{2}\alpha_n\right). \end{aligned} \quad (\text{B8})$$

FIG. 14. Dynkin diagram of type C_n .FIG. 15. Dynkin diagram of type D_n .

The root lattice and weight lattice have the following relationships:

$$\begin{aligned} (\Lambda_R(C_n))^* &= \Lambda_W(C_n) + \mathbb{Z}\frac{1}{2}\omega_n \\ &= \Lambda_R(C_n) + \mathbb{Z}\frac{1}{2}\alpha_n + \mathbb{Z}\frac{1}{2}\omega_n^{(C_n)}, \\ (\Lambda_R(C_n) + \frac{1}{2}\mathbb{Z}\alpha_n^{(C_n)})^* &= (\mathbb{Z}^n)^* \\ &= \Lambda_R(C_n) + \frac{1}{2}\mathbb{Z}\alpha_n^{(C_n)}, \\ \Lambda_W(C_n)/\Lambda_R(C_n) &= \mathbb{Z}_2. \end{aligned} \quad (\text{B9})$$

4. D_n type

The Dynkin diagram of the D_n algebra is shown in Fig. 15. The root system of D_n is

$$\pm \varepsilon_j \pm \varepsilon_k, \quad \pm \varepsilon_j \mp \varepsilon_k, \quad \text{for } 1 \leq j < k \leq n. \quad (\text{B10})$$

The bases are

$$\begin{aligned} \alpha_i &= \varepsilon_i - \varepsilon_{i+1}, \quad \text{for } 1 \leq i \leq n-1, \\ \alpha_n &= \varepsilon_{n-1} + \varepsilon_n. \end{aligned} \quad (\text{B11})$$

The fundamental weights of D_n are

$$\begin{aligned} \omega_i &= \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i \\ &= \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i(\alpha_n + \alpha_{i+1} + \cdots + \alpha_{n-2}) + \frac{1}{2}i(\alpha_{n-1} + \alpha_n), \quad \text{for } 1 \leq i \leq n-2, \\ \omega_{n-1} &= \frac{1}{2}(\varepsilon_n + \varepsilon_2 + \cdots + \varepsilon_{n-2} + \varepsilon_{n-1} - \varepsilon_n) \\ &= \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}n\alpha_{n-1} + \frac{1}{2}(n-2)\alpha_n\right), \end{aligned}$$

$$\begin{aligned}\omega_n &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{n-2} + \varepsilon_{n-1} + \varepsilon_n) \\ &= \frac{1}{2}\left(\alpha_1 + 2\alpha_2 + \cdots + (n-2)\alpha_{n-2} + \frac{1}{2}(n-2)\alpha_{n-1} + \frac{1}{2}n\alpha_n\right).\end{aligned}\quad (\text{B12})$$

The root lattice and weight lattice have the following relationships:

$$\begin{aligned}(\Lambda_R(D_{2n}))^* &= \Lambda_W(D_{2n}) \\ &= \Lambda_R(D_{2n}) + \mathbb{Z}\omega_{2n-1} + \mathbb{Z}\omega_{2n}, \\ (\Lambda_R(D_{2n+1}))^* &= \Lambda_W(D_{2n+1}) \\ &= \Lambda_R(D_{2n}) + \mathbb{Z}\omega_{2n+1}, \\ \Lambda_W(D_{2n})/\Lambda_R(D_{2n}) &= \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \Lambda_W(D_{2n+1})/\Lambda_R(D_{2n+1}) = \mathbb{Z}_4.\end{aligned}\quad (\text{B13})$$

5. E_6

The Dynkin diagram of the E_6 algebra is shown in Fig. 16. The root system of E_6 is

$$\begin{aligned}\pm \varepsilon_i \pm \varepsilon_j, \quad \text{for } 1 \leq i, j \leq 5, \\ \pm \frac{1}{2}\left(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{i=1}^5 (-1)^{\nu_i} \varepsilon_i\right), \quad \text{for } \sum_{i=1}^5 \nu_i \in 2\mathbb{Z}.\end{aligned}\quad (\text{B14})$$

The basis of the E_6 root system is

$$\begin{aligned}\alpha_1 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7), \quad \alpha_2 = \varepsilon_1 + \varepsilon_2, \\ \alpha_3 &= \varepsilon_2 - \varepsilon_1, \quad \alpha_4 = \varepsilon_3 - \varepsilon_2, \\ \alpha_5 &= \varepsilon_4 - \varepsilon_3, \quad \alpha_6 = \varepsilon_5 - \varepsilon_4.\end{aligned}\quad (\text{B15})$$

The fundamental weights of E_6 are

$$\begin{aligned}\omega_1 &= \frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) \\ &= \frac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6), \\ \omega_2 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8) \\ &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \\ \omega_3 &= \frac{5}{6}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5) \\ &= \frac{1}{3}(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6), \\ \omega_4 &= \varepsilon_3 + \varepsilon_4 + \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8 \\ &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6, \\ \omega_5 &= \frac{2}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_4 + \varepsilon_5 \\ &= \frac{1}{3}(4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 5\alpha_6), \\ \omega_6 &= \frac{1}{3}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6) + \varepsilon_5 \\ &= \frac{1}{3}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6).\end{aligned}\quad (\text{B16})$$

The relation between the root and weight lattice is

$$\begin{aligned}(\Lambda_R(E_6))^* &= \Lambda_W(E_6) \\ &= \Lambda_R(E_6) + \mathbb{Z}\omega_1, \\ \Lambda_W(E_6)/\Lambda_R(E_6) &= \mathbb{Z}_3.\end{aligned}\quad (\text{B17})$$

6. E_7

The Dynkin diagram of the E_7 algebra is shown in Fig. 17. The root system of E_7 is

$$\begin{aligned}\pm \varepsilon_i \pm \varepsilon_j, \quad \text{for } 1 \leq i, j \leq 6, \\ \pm (\varepsilon_7 - \varepsilon_8), \\ \pm \frac{1}{2}\left(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{i=1}^6 (-1)^{\nu_i} \varepsilon_i\right), \quad \text{for } \sum_{i=1}^6 \nu_i \in 2\mathbb{Z}.\end{aligned}\quad (\text{B18})$$

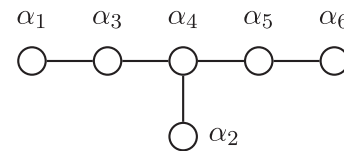
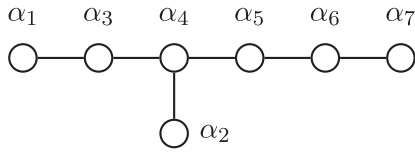


FIG. 16. Dynkin diagram of type E_6 .

FIG. 17. Dynkin diagram of type E_7 .

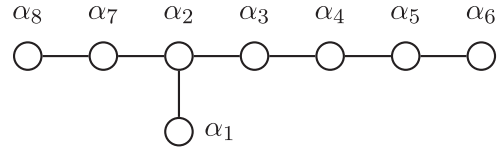
The bases are

$$\begin{aligned}\alpha_1 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_8) - \frac{1}{2}(\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7), \\ \alpha_2 &= \varepsilon_1 + \varepsilon_2, & \alpha_3 &= \varepsilon_2 - \varepsilon_1, \\ \alpha_4 &= \varepsilon_3 - \varepsilon_2, & \alpha_5 &= \varepsilon_4 - \varepsilon_3, \\ \alpha_6 &= \varepsilon_5 - \varepsilon_4, & \alpha_7 &= \varepsilon_6 - \varepsilon_5.\end{aligned}\quad (\text{B19})$$

The fundamental weights of E_7 are

$$\begin{aligned}\omega_1 &= \varepsilon_8 - \varepsilon_7 \\ &= 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \\ \omega_2 &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - 2\varepsilon_7 + 2\varepsilon_8) \\ &= \frac{1}{2}(4\alpha_1 + 7\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 8\alpha_6 + 3\alpha_7), \\ \omega_3 &= \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 - 3\varepsilon_7 + 3\varepsilon_8) \\ &= 3\alpha_1 + 4\alpha_2 + 6\alpha_3 + 8\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7, \\ \omega_4 &= \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + 2(\varepsilon_8 - \varepsilon_7) \\ &= 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7, \\ \omega_5 &= \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \frac{3}{2}(\varepsilon_8 - \varepsilon_7) \\ &= \frac{1}{2}(6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 10\alpha_6 + 5\alpha_7), \\ \omega_6 &= \varepsilon_5 + \varepsilon_6 - \varepsilon_7 + \varepsilon_8 \\ &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7, \\ \omega_7 &= \varepsilon_6 + \frac{1}{2}(\varepsilon_8 - \varepsilon_7) \\ &= \frac{1}{2}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7).\end{aligned}\quad (\text{B20})$$

The relation between the root and weight lattice is

FIG. 18. Dynkin diagram of type E_8 .

$$\begin{aligned}(\Lambda_R(E_7))^* &= \Lambda_W(E_7) \\ &= \Lambda_R(E_7) + \mathbb{Z}\omega_7,\end{aligned}$$

$$\Lambda_W(E_7)/\Lambda_R(E_7) = \mathbb{Z}_2. \quad (\text{B21})$$

7. E_8

The Dynkin diagram of the E_8 algebra is shown in Fig. 18. The bases are

$$\begin{aligned}\alpha_1 &= \varepsilon_1 - \varepsilon_2, & \alpha_2 &= \varepsilon_2 - \varepsilon_3, \\ \alpha_3 &= \varepsilon_3 - \varepsilon_4, & \alpha_4 &= \varepsilon_4 - \varepsilon_5, \\ \alpha_5 &= \varepsilon_5 - \varepsilon_6, & \alpha_6 &= \varepsilon_6 - \varepsilon_7, \\ \alpha_7 &= -(\varepsilon_1 + \varepsilon_2), & \alpha_8 &= \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_8).\end{aligned}\quad (\text{B22})$$

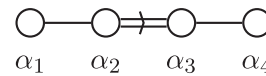
8. F_4

The Dynkin diagram of the F_4 algebra is shown in Fig. 19. The root system of F_4 is

$$\begin{aligned}&\pm \varepsilon_i, \quad \text{for } 1 \leq i \leq 4, \\ &\pm \varepsilon_i \pm \varepsilon_j, \quad \text{for } 1 \leq i, j \leq 4, \\ &\frac{1}{2}(\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4).\end{aligned}\quad (\text{B23})$$

The basis of the root system of F_4 is

$$\begin{aligned}\alpha_1 &= \varepsilon_2 - \varepsilon_3, & \alpha_2 &= \varepsilon_3 - \varepsilon_4, \\ \alpha_3 &= \varepsilon_4, & \alpha_4 &= \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4).\end{aligned}\quad (\text{B24})$$

FIG. 19. Dynkin diagram of type F_4 .

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