

Toward celestial chiral algebras of self-dual black holes

Giuseppe Bogna^{*}*The Mathematical Institute, University of Oxford, Oxford, United Kingdom*Simon Heuveline[†]*Department of Applied Maths and Theoretical Physics, University of Cambridge, Cambridge, United Kingdom*

(Received 16 May 2025; accepted 5 August 2025; published 18 February 2026)

In this article, we show that several self-dual spacetimes previously studied in the context of celestial and twisted holography arise as limits of a certain Taub-NUT AdS_4 metric, the Pedersen metric, in which their mass, NUT charge, and cosmological constant obey a self-duality relation. In particular, self-dual Taub-NUT, a singular double cover of Eguchi-Hanson space, Euclidean AdS_4 , and noncompact $\mathbb{CP}^2$, which is conformally equivalent to Burns space, arise as special limits of the Pedersen metric. The Pedersen metric can be derived from a curved twistor space which we conjecture to arise from a backreaction of self-dual gravity in the presence of a cosmological constant when coupled to a defect operator wrapping a certain $\mathbb{CP}^1$ at infinity. The curved twistor space gives rise to a 2-parameter deformation of the celestial symmetry algebra Lw_Λ which reduces to previously studied algebras in various limits.

DOI: [10.1103/y7ql-z2r1](https://doi.org/10.1103/y7ql-z2r1)

I. INTRODUCTION

Celestial holography suggests, among other things, that collinear singularities of graviton scattering amplitudes are described by the operator product expansion (OPE) of a putative dual conformal field theory (CFT) [1–3]. One of the great successes has been the insight that this duality is true at tree level which led to the discovery of new infinite dimensional symmetry algebras of tree-level amplitudes in flat space closely related to $w_{1+\infty}$ [4–6]. Nontrivial backgrounds have been studied in the celestial context for several reasons [7–30], and it is known that considering (self-dual) gravity on such backgrounds can lead to deformations of these celestial symmetries [7–14] or the absence thereof [29]. Deformed algebras were previously identified, among other backgrounds, for Eguchi-Hanson space [7] and AdS_4 [8,9,15]. In this work, we will derive a 2-parameter deformation of Lw_Λ , which interpolates between these two deformations, and provide its bulk interpretation.

We consider a backreaction from coupling self-dual gravity in the presence of a cosmological constant to a defect operator on twistor space. This defect wraps $\mathbb{CP}^1$

over a spacetime point at infinity, which backreacts to the twistor space of self-dual Taub-NUT when $\Lambda = 0$. When $\Lambda < 0$, we give strong evidence that the resulting backreaction leads to a self-dual limit of Wick-rotated Taub-NUT AdS_4 black holes, which we refer to as the Pedersen metric. In particular, the Pedersen metric can be viewed as a $\Lambda \neq 0$ generalization of the self-dual black holes recently considered in [30–34].

The Pedersen metric depends on two parameters, and further relations can be imposed between these in which it reduces to previously studied self-dual geometries such as self-dual Taub-NUT, a singular double cover of Eguchi-Hanson space, Euclidean AdS_4 , and noncompact $\mathbb{CP}^2$, which is conformally equivalent to Burns space. From its 2-parameter twistor space we derive a conjectured 2-parameter deformation of $Lw_\Lambda = L\mathfrak{ham}(\mathbb{C}^2/\mathbb{Z}_2)$ (following the conventions of [7,35]) which reduces to the expected deformations in various limits.

While our construction is bottom-up and at tree-level, we hope that the algebra we find might be extended beyond tree-level by means of some top-down construction. In the case of Burns space, a fully-fledged holographically dual chiral algebra was identified [19,20] but the gravitational part of the algebra is much larger than the previously discussed symmetries [4] since the four-dimensional bulk gravitational theory is a self-dual subsector of conformal gravity, Mabuchi gravity, rather than Einstein gravity. The Pedersen metric is conformally equivalent to a 2-parameter family of scalar flat Kähler manifolds, i.e., solutions to the classical field equations of Mabuchi gravity [19,36] which

^{*}Contact author: giuseppe.bogna@maths.ox.ac.uk

[†]Contact author: sph48@cam.ac.uk

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](https://creativecommons.org/licenses/by/4.0/). Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

interpolates between Burns space and a singular double cover of Eguchi-Hanson space. So, we expect to obtain the Pedersen twistor space in the context of the recent top-down construction [37] or [19,20]. From this, we hope to obtain amplitudes on a Pedersen background from some chiral algebra computation which could be matched with results in the literature [38]. Since a nontrivial thermodynamic phase structure is known to exist for the Pedersen metric [39–41] such considerations might lead to new insights into the non-perturbative structure of twisted holography.

Throughout this paper we make free use of spinor conventions $\langle \lambda \kappa \rangle = \lambda^\alpha \kappa_\alpha = \epsilon^{\alpha\beta} \lambda_\beta \kappa_\alpha$ and similarly $[\tilde{\lambda} \tilde{\kappa}] = \tilde{\lambda}^{\dot{\alpha}} \tilde{\kappa}_{\dot{\alpha}}$. The twistor space $\mathbb{PT}$ of flat Minkowski space is the total space of $\mathcal{O}(1) \oplus \mathcal{O}(1) \rightarrow \mathbb{CP}^1 \cong \mathbb{CP}^3 \setminus \mathbb{CP}^1$. We will use homogeneous coordinates λ_α on the $\mathbb{CP}^1$ base, and $\mu^{\dot{\alpha}}$ on the fibers, collectively denoting these by $Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha) \in \mathbb{CP}^3$ such that $\lambda_\alpha \neq 0$. Euclidean spinor conjugation is given by $\omega^\alpha = (a, b) \mapsto \hat{\omega}^\alpha = (-\bar{b}, \bar{a})$ and $\kappa^{\dot{\alpha}} = (c, d) \mapsto \hat{\kappa}^{\dot{\alpha}} = (-\bar{d}, \bar{c})$.

II. THE PEDERSEN METRIC AND ITS LIMITS

A. Self-dual Plebański-Demiański spacetimes

A well-known 7-parameter generalization of the Kerr metric was constructed by Plebański-Demiański in [42]. We will not be interested in acceleration or electromagnetic charges, so we set the corresponding parameters to 0. The resulting 4-parameter family of rotating Taub-NUT-AdS black holes reads [43,44]

$$\begin{aligned} ds^2 = & -\frac{\Delta}{\Sigma} \left(dt + (2n \cos \theta - a \sin^2 \theta) \frac{d\phi}{\Xi} \right)^2 \\ & + \frac{\Delta_\theta}{\Sigma} \left(a dt - (r^2 + a^2 + n^2) \frac{d\phi}{\Xi} \right)^2 \\ & + \frac{\Sigma}{\Delta} dr^2 + \frac{\Sigma}{\Delta_\theta} \sin^2 \theta d\theta^2, \end{aligned} \quad (1)$$

where

$$\begin{aligned} \Sigma &= r^2 + (n + a \cos \theta)^2, \\ \frac{\Delta_\theta}{\sin^2 \theta} &= 1 - \frac{4an \cos \theta}{l^2} - \frac{a^2 \cos^2 \theta}{l^2}, \\ \Delta &= r^2 + a^2 - 2mr - n^2 \\ &\quad + \frac{3(a^2 - n^2)n^2 + (a^2 + 6n^2)r^2 + r^4}{l^2}, \\ \Xi &= 1 - \frac{a^2}{l^2}. \end{aligned} \quad (2)$$

The metric (1) is known to solve Einstein's equations in the presence of a cosmological constant $\Lambda = -3/l^2$. The remaining three parameters (m, n, a) are related to the

mass, the NUT charge, and the angular momentum of the black hole.

We will consider the case $a = 0$. In the absence of Λ , i.e., in the limit $l \rightarrow \infty$, the metric (1) has a self-dual limit $n = -iM$, $m = M$, in which it can be analytically continued to give a Euclidean self-dual metric commonly referred to as self-dual Taub-NUT [45]. Alternatively, it can be continued to the Kleinian signature [31,33], where the metric has a genuine horizon beyond which it can be extended with the maximal extension having a singularity at $r = -M$. This justifies the previously used terminology *self-dual black hole*. It was also argued in [31,34] that the parameter a can be reintroduced with a diffeomorphism closely related to the Newman-Janis shift, and it is a natural question if this can be extended to $\Lambda \neq 0$.

In the presence of a nonvanishing cosmological constant, $\Lambda \neq 0$, there exists a similar self-dual limit, which can be obtained from computing the components of the Weyl tensor in the Newman-Penrose formalism using a certain null-tetrad [43]. The only nonvanishing component of the anti-self-dual Weyl tensor is

$$\psi_2 = \frac{4n^3 + l^2(-im + n)}{l^2(n - ir)^3}, \quad (3)$$

together with its complex conjugate $\bar{\psi}_2$. We can set either of them to zero by imposing

$$n = \pm iM, \quad m = M \left(1 - \frac{4M^2}{l^2} \right), \quad (4)$$

which gives an (anti-)self-dual metric that has previously been studied in various contexts [38,39,46,47]. The resulting metric can be brought into triaxial form with a suitable diffeomorphism that we discuss in Sec. IV. This form of the metric was first constructed by Pedersen [48,49], so we will refer to it as the *Pedersen metric*.

We also note that when $a \neq 0$ the (anti-)self-duality condition is further deformed to

$$n = \pm iM, \quad m = M \left(1 - \frac{a^2 + 4M^2}{l^2} \right), \quad (5)$$

which will be further discussed in the near future.

B. The Pedersen metric

The triaxial form of the Euclidean Pedersen metric on the four-dimensional ball of radius l is

$$ds^2 = f^2(r)(h_r(r)dr^2 + h_{12}(r)(\sigma_1^2 + \sigma_2^2) + h_3(r)\sigma_3^2), \quad (6)$$

where

$$\begin{aligned} \nu^2 &= \frac{1}{4M^2} - \frac{1}{l^2}, & f(r) &= \frac{2}{1 - r^2/l^2}, \\ h_r(r) &= \frac{1 + \nu^2 r^2}{1 + \nu^2 r^4/l^2}, & h_{12}(r) &= r^2(1 + \nu^2 r^2), \\ h_3(r) &= \frac{r^2}{h_r(r)}, \end{aligned} \quad (7)$$

and σ_i denote the standard $SU(2)$ left-invariant 1-forms. We can see that the conformal boundary of the spacetime is a biaxially squashed 3-sphere known as Berger's sphere: setting $r = l$ in (6), we find the boundary metric

$$ds_3^2 = \sigma_1^2 + \sigma_2^2 + \frac{1}{1 + \nu^2 l^2} \sigma_3^2, \quad (8)$$

up to an overall constant. Since $(1 + \nu^2 l^2)^{-1} \leq 1$, the metric (8) is an oblate squashing of the 3-sphere. For this reason, we will also refer to the metric in (6) as the *oblate Pedersen metric*. The spacetime metric represents an explicit realization of a theorem by LeBrun [50], where it is shown that any 3-manifold with a Riemannian conformal structure is the conformal infinity of a self-dual Riemannian 4-manifold satisfying Einstein's equations with a negative cosmological constant. The theorem ensures the existence of the bulk 4-manifold only in a collared neighborhood of its conformal boundary, but the Pedersen metric is a complete metric inside the entire 4-ball with radius l . In the language of LeBrun [51], this means that the biaxially squashed 3-sphere is of positive frequency (with the appropriate orientation). More generally, Hitchin showed that every left-invariant conformal structure on S^3 has positive frequency leading to a generalization of the Pedersen metric [52].

C. Relation to Mabuchi gravity

As shown in Sec. IV, the Pedersen metric is diffeomorphic to

$$ds^2 = \frac{\nu l^3}{(\sin \chi - \nu l \cos \chi)^2} (V^{-1}(d\tau + A)^2 + V d\Omega_3^2), \quad (9)$$

where we introduced the Higgs field and gauge potential on the 3-sphere as

$$V = \nu l + \cot \chi, \quad A = -\cos \theta d\phi. \quad (10)$$

$d\Omega_3^2 = d\chi^2 + \sin^2 \chi d\Omega_2^2$ denotes the canonical metric on the 3-sphere. It is straightforward to show that the pair (V, A) satisfies the Bogomolny equation $\star_3 dV = dA$, where $\star_3$ denotes the Hodge star operator on S^3 with respect to $d\Omega_3^2$.

It is also possible to analytically continue the metric (6) by $\nu \mapsto i\nu$. Provided that $\nu l < 1$, the metric is still complete on the open ball $r < l$. We will refer to this as the *prolate*

Pedersen metric, because its boundary conformal structure is a prolate squashing of S^3 in contrast to the oblate Pedersen metric (6). The prolate Pedersen metric can also be written as

$$ds^2 = \frac{\nu l^3}{(\sinh \chi - \nu l \cosh \chi)^2} (V_H^{-1}(d\tau + A_H)^2 + V_H ds_{\mathbb{H}^3}^2), \quad (11)$$

where now the pair (V_H, A_H) is defined on $\mathbb{H}^3$

$$V_H = -\nu l + \coth \chi, \quad A_H = -\cos \theta d\phi, \quad (12)$$

and $ds_{\mathbb{H}^3}^2 = d\chi^2 + \sinh^2 \chi d\Omega_2^2$ denotes the standard metric on $\mathbb{H}^3$. The Pedersen metric can thus be understood as arising from a generalized Gibbons-Hawking ansatz [53] where one replaces the $\mathbb{R}^3$ with S^3 or $\mathbb{H}^3$, respectively, in the oblate and prolate cases [48,49].

A conformal factor can be included to turn (11) into

$$ds^2 = \nu l^3 q(\chi, \theta, \phi)^2 (V_H^{-1}(d\tau + A_H)^2 + V_H ds_{\mathbb{H}^3}^2), \quad (13)$$

which has been shown to be scalar flat Kähler by LeBrun [54]. $q(\chi, \theta, \phi)$ is a horospherical height function [51], which is explicitly given by the coordinate q after changing to half-plane coordinates $ds_{\mathbb{H}^3}^2 = q^{-2}(dx^2 + dy^2 + dq^2)$ on the $\mathbb{H}^3$ factor [55]. Since twistor spaces are *a priori* agnostic about conformal factors, we see that the Pedersen twistor space also leads to a two-parameter family of solutions to the equations of motion of Mabuchi gravity [20,36]. In the next section and IV, we will discuss that this class of metrics particularly interpolates between Burns space and a double cover of Eguchi-Hanson space leading to a conjectured relation to [37].

D. Various limits

The oblate Pedersen metric (6) and its continuation to the prolate Pedersen metric (11) depend on two parameters ν^2 and $\Lambda = -3/l^2$. Note that ν in (6) is either real with any $l > 0$ (oblate case) or purely imaginary with $-i\nu l < 1$ (prolate case) to ensure completeness. The resulting region is displayed in yellow in Fig. 1. There are various limiting cases in which the Pedersen metric turns into spacetimes that were previously studied in the context of celestial and twisted holography [7,8,19,20,30,31,37]. These limits are displayed in Fig. 1 and discussed in Sec. IV.

III. SELF-DUAL GRAVITY ON TWISTOR SPACE

We will consider the twistor uplift of self-dual gravity in the presence of a cosmological constant described by the Mason-Wolf action [56]

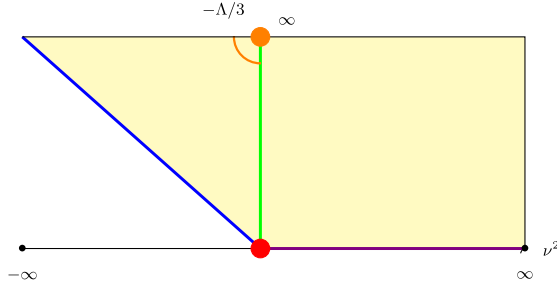


FIG. 1. The Pedersen metric (yellow) has various limits: a singular double cover of Eguchi-Hanson space (orange dot), self-dual Taub-NUT (violet line), $\mathbb{CP}^2$ (blue line), Euclidean AdS_4 (green line), and $\mathbb{R}^4$ (red dot). $\nu^2 < 0$ makes up the prolate case while $\nu^2 > 0$ makes up the oblate case.

$$S_\Lambda[g, h] = \int_{\mathbb{PT}_\Lambda} D^3Z \wedge g \wedge \left(\bar{\partial}h + \frac{1}{2} \{h, h\}_\Lambda \right), \quad (14)$$

where $g \in \Omega^{0,1}(\mathbb{PT}_\Lambda, \mathcal{O}(-6))$ and $h \in \Omega^{0,1}(\mathbb{PT}_\Lambda, \mathcal{O}(-2))$. The holomorphic $\mathcal{O}(-2)$ -valued Jacobi bracket $\{, \}_\Lambda$ is defined through an infinity twistor [57,58], and we choose to work in the following homogeneous coordinates:

$$\{f, g\}_\Lambda = \varepsilon^{\dot{\alpha}\dot{\beta}} \frac{\partial f}{\partial \mu^{\dot{\alpha}}} \frac{\partial g}{\partial \mu^{\dot{\beta}}} + \Lambda \varepsilon_{\alpha\beta} \frac{\partial f}{\partial \lambda_\alpha} \frac{\partial g}{\partial \lambda_\beta}. \quad (15)$$

For a recent discussion of this action, we refer to [8]. Note that working with homogeneous coordinates means that we are formally working on a nonprojective twistor, where (15) is a (twisted) Poisson bracket whereas it becomes a (twisted) Jacobi bracket on projective twistor space. Below, we will formally work on a generalized nonprojective twistor space of a curved spacetime which is formalized by the total space of the Swann-bundle over the curved twistor space [59,60].

A. Deforming twistor space with a defect operator

In [7], it was shown that the twistor space of Eguchi-Hanson space arises as a backreaction from including a defect operator wrapping $\mathbb{CP}^1_{\mu^{\dot{\alpha}}=0}$ in the flat twistor space of $\mathbb{C}^2/\mathbb{Z}_2$. We will now formally include a defect operator wrapping the twistor line $\mathbb{CP}^1_{\lambda_\alpha=0}$ in $\mathbb{CP}^3 \supset \mathbb{PT}$ that couples electrically to g . Doing this can equivalently be viewed as introducing a boundary condition for h , a perspective which relates our work to [30] and will be elaborated on in the future.

1. Self-dual Taub-NUT

As a warm-up, let us consider the special case $\Lambda = 0$ in which the backreacted twistor space is that of self-dual Taub-NUT. The deformed action reads

$$S[g, h] = S_0[g, h] - \frac{\pi^2}{4M} \int_{\mathbb{CP}^1_{\lambda_\alpha=0}} \langle \lambda d\lambda \rangle \wedge \eta^2 g, \quad (16)$$

where $\eta = \mu^{\dot{0}} \mu^{\dot{1}} = [\iota\mu][\mu\hat{\iota}]$ singles out a choice of dotted reference spinors $\iota, \hat{\iota}$ which breaks Lorentz invariance. Below, we will also use $\mu^+ = \mu^{\dot{0}}$ and $\mu^- = \mu^{\dot{1}}$ for notational convenience. The latter term in (16) describes the coupling to the defect. $1/M$ is to be viewed as a coupling constant controlling the strength of the coupling. For $1/M \neq 0$, we can vary g to obtain the deformed equation of motion for h

$$\bar{\partial}h + \frac{1}{2} \{h, h\}_0 = \frac{\pi^2 \eta^2}{2M} \delta^{(2)}(\lambda). \quad (17)$$

The equation for g remains unchanged. Similar to [7], we can solve the sourced equation by

$$h = \frac{\eta^2}{8M} \bar{e}^0, \quad \bar{e}^0 = \frac{\langle \hat{\lambda} d\hat{\lambda} \rangle}{\langle \lambda \hat{\lambda} \rangle^2}, \quad (18)$$

since we have $\{h, h\}_0 = 0$, $\bar{\partial}h = 0$ for $\langle \lambda \hat{\lambda} \rangle \neq 0$ and the correct normalization. Note that up until this point, the backreaction is directly related to the one performed in [7] by exchanging λ_α and $\mu^{\dot{\alpha}}$, which corresponds to a conformal inversion

$$x^{\alpha\dot{\alpha}} \mapsto \frac{2x^{\alpha\dot{\alpha}}}{x^2} \quad (19)$$

on spacetime, since

$$\mu^{\dot{\alpha}} = x^{\alpha\dot{\alpha}} \lambda_\alpha \Leftrightarrow \frac{2x^{\alpha\dot{\alpha}}}{x^2} \mu^{\dot{\alpha}} = \lambda_\alpha. \quad (20)$$

However, the Beltrami differential will not simply arise by exchanging λ_α and $\mu^{\dot{\alpha}}$ since the Poisson structure, or Jacobi structure when $\Lambda \neq 0$, of Eq. (15) is not conformally invariant. We can see this explicitly by considering the deformed Dolbeaut operator which now reads

$$\begin{aligned} \bar{\nabla}_0 &= \bar{\partial} + \{h, \}_0 \\ &= \bar{\partial} + \frac{\eta}{4M} \bar{e}^0 \wedge (\mu^{\dot{1}} \mathcal{L}_{\dot{1}} - \mu^{\dot{0}} \mathcal{L}_{\dot{0}}), \end{aligned} \quad (21)$$

where $\mathcal{L}_{\dot{\alpha}} = \mathcal{L}_{\partial/\partial \mu^{\dot{\alpha}}}$ and $\mathcal{L}^\alpha = \mathcal{L}_{\partial/\partial \lambda_\alpha}$ denote the Lie derivatives [61]. The coordinates λ_α and η are still holomorphic in the deformed complex structure, but $\mu^{\dot{\alpha}}$ are not. We can construct patchwise (over $\mathbb{CP}^1$) holomorphic coordinates

$$\rho^\pm = \mu^\pm \exp\left(\pm \frac{\eta f(\lambda)}{4M}\right), \quad (22)$$

where we introduced the patchwise-defined function

$$f(\lambda) = \begin{cases} \frac{1}{\langle \lambda \hat{\lambda} \rangle} \frac{\hat{\lambda}_0}{\lambda_0} & \lambda_0 \neq 0 \\ \frac{1}{\langle \lambda \hat{\lambda} \rangle} \frac{\hat{\lambda}_1}{\lambda_1} & \lambda_1 \neq 0 \end{cases}. \quad (23)$$

$\rho^\pm$ are holomorphic as a consequence of $\bar{\partial}f = \bar{e}^0$.

On the overlap, the $\rho^\pm$ patch according to the transition function

$$\begin{aligned} \rho^\pm &\mapsto \rho^\pm \exp\left(\mp \frac{\eta \hat{\lambda}_0}{4M \langle \lambda \hat{\lambda} \rangle \lambda_0}\right) \exp\left(\pm \frac{\eta \hat{\lambda}_1}{4M \langle \lambda \hat{\lambda} \rangle \lambda_1}\right) \\ &= \rho^\pm \exp\left(\pm \frac{\eta}{4M \lambda_0 \lambda_1}\right). \end{aligned} \quad (24)$$

Moreover, they also satisfy $\rho^+ \rho^- = \eta$ on each patch. This backreacted geometry precisely matches Hitchin's description of the twistor space of the self-dual Taub-NUT [62–64]. Namely, it is defined as $\rho^+ \rho^- = \eta$ inside the total space of a sum of two line bundles $L^{1/4M}(1)$ and $L^{-1/4M}(1)$ over the minitwistor space [65] of $\mathbb{R}^3$, $\mathbb{MT} = \mathcal{O}(2) \rightarrow \mathbb{CP}^1$:

$$\begin{aligned} \{(\rho^+, \rho^-) \in L^{1/4M}(1) \oplus L^{-1/4M}(1) : \rho^+ \rho^- = \eta\} \\ \subset \text{Tot}(L^{1/4M}(1) \oplus L^{-1/4M}(1) \rightarrow \mathbb{MT}). \end{aligned} \quad (25)$$

The spacetime metric of self-dual Taub-NUT can be explicitly derived from this twistor space [64]. Note that the holomorphic coordinates (22) are formally equivalent to coordinates which previously appeared in the twisted holography context in [66].

2. The Pedersen twistor space

Let us now consider the same backreaction in the presence of a cosmological constant. Consider the action

$$S[g, h] = S_\Lambda[g, h] - \frac{\pi^2 \nu^2}{2} \int_{\mathbb{CP}^1_{\lambda_\alpha=0}} \langle \lambda d\lambda \rangle \wedge \eta^2 g. \quad (26)$$

The solution to the equations of motion is still given by

$$h = \frac{\nu^2 \eta^2}{4} \bar{e}^0, \quad (27)$$

since $\{h, h\}_\Lambda = 0$. However, the presence of the new term in $\{, \}_\Lambda$ changes the Beltrami differential to

$$\begin{aligned} \bar{\nabla}_\Lambda = \bar{\partial} + \{h, -\}_\Lambda \\ = \bar{\partial} + \frac{\nu^2 \eta}{2} \bar{e}^0 \wedge (\mu^i \mathcal{L}_i - \mu^0 \mathcal{L}_0) + \frac{\Lambda \nu^2 \eta^2}{2} \frac{\hat{\lambda}_\alpha}{\langle \lambda \hat{\lambda} \rangle} \bar{e}^0 \wedge \mathcal{L}^\alpha, \end{aligned} \quad (28)$$

where we used the identity $\mathcal{L}^\alpha \bar{e}^0 = 2\hat{\lambda}^\alpha \bar{e}^0 / \langle \lambda \hat{\lambda} \rangle$. The coordinate $\eta = \mu^0 \mu^i$ is still holomorphic but now neither μ^α

nor λ_α are. Similar to [67], the λ_α coordinates get deformed to

$$Y^{\alpha\beta} = \lambda^\alpha \lambda^\beta - \frac{\Lambda \nu^2 \eta^2}{2} \frac{\hat{\lambda}^\alpha \hat{\lambda}^\beta}{\langle \lambda \hat{\lambda} \rangle^2}, \quad (29)$$

which will be shown to obey $\bar{\nabla}_\Lambda Y^{\alpha\beta} = 0$ in Sec. VA. Since the three holomorphic coordinates

$$X = Y^{11}, \quad Y = Y^{00}, \quad Z = -Y^{01}, \quad (30)$$

and η all scale with weight 2 and obey the relation $Y^{\alpha\beta} Y_{\alpha\beta} = -\Lambda \nu^2 \eta^2$, they form a quadric inside $\mathbb{CP}^3$:

$$\mathcal{Q}_{\Lambda \nu^2} = \left\{ (X, Y, Z, \eta) \in \mathbb{CP}^3 : XY - Z^2 + \frac{\Lambda \nu^2}{2} \eta^2 = 0 \right\}, \quad (31)$$

which can be identified with $\mathbb{CP}^1 \times \mathbb{CP}^1$. In analogy to MT in the self-dual Taub-NUT case, this $\mathbb{CP}^1 \times \mathbb{CP}^1$ again plays the role of a minitwistor space. It is to be viewed as the minitwistor space of S^3 [65] or the minitwistor space of $\mathbb{H}^3$ after removing the antiholomorphic diagonal.

We saw the oblate (or prolate) Pedersen metric fibers over S^3 (or $\mathbb{H}^3$) through its description as a generalized Gibbons-Hawking metric in Eqs. (9) and (11). We saw the full twistor space of the Pedersen metric fibers over the corresponding minitwistor space $\mathbb{CP}^1 \times \mathbb{CP}^1$ [48,49]. We already saw that our Beltrami differential (28) leads to this correct base space. Let us now analyze the remaining coordinates $\mu^\pm$.

It will be shown in section VA that the coordinates

$$\phi^\pm = \mu^\pm \left(\frac{\sqrt{\frac{2}{\Lambda \nu^2}} + \eta f(\lambda)}{\sqrt{\frac{2}{\Lambda \nu^2}} - \eta f(\lambda)} \right)^{\pm \sqrt{\frac{\nu^2}{8\Lambda}}} \quad (32)$$

are holomorphic. Equation (32) is defined on patches $\lambda_0 \neq 0$ and $\lambda_1 \neq 1$, which are somewhat unnatural in the given context since λ_α are not holomorphic. However, naively working with these patches gives rise to the following transition functions:

$$\begin{aligned} \left(\frac{\sqrt{\frac{2}{\Lambda \nu^2}} + \eta \frac{\hat{\lambda}_0}{\lambda_0 \langle \lambda \hat{\lambda} \rangle}}{\sqrt{\frac{2}{\Lambda \nu^2}} - \eta \frac{\hat{\lambda}_0}{\lambda_0 \langle \lambda \hat{\lambda} \rangle}} \right)^{\mp \sqrt{\frac{\nu^2}{8\Lambda}}} \left(\frac{\sqrt{\frac{2}{\Lambda \nu^2}} + \eta \frac{\hat{\lambda}_1}{\lambda_1 \langle \lambda \hat{\lambda} \rangle}}{\sqrt{\frac{2}{\Lambda \nu^2}} - \eta \frac{\hat{\lambda}_1}{\lambda_1 \langle \lambda \hat{\lambda} \rangle}} \right)^{\pm \sqrt{\frac{\nu^2}{8\Lambda}}} \\ = \left(\frac{Z + \sqrt{\frac{\Lambda \nu^2}{2}} \eta}{Z - \sqrt{\frac{\Lambda \nu^2}{2}} \eta} \right)^{\pm \sqrt{\frac{\nu^2}{8\Lambda}}}. \end{aligned} \quad (33)$$

The Pedersen twistor space [48,49] is given by a line bundle over $\mathcal{Q}_{\Lambda \nu^2}$ with a transition function that is closely

related to (33), in the sense that one factor of Pedersen's equation (8.15) in [49] matches our (33) but there is a further factor that we do not find. We conjecture that our deformed twistor space, which is also a line bundle over $\mathcal{Q}_{\Lambda\nu^2}$, actually agrees with the full Pedersen twistor space and that the reason for this mismatch is the fact that we consider patches $\lambda_\alpha \neq 0$ in (32), whereas Pedersen considers different patches that are better adapted to the mini-twistor space of S^3 or $\mathbb{H}^3$. We hope to verify this conjecture more directly in the future by overcoming technical difficulties in working with the patches used by Pedersen.

Further evidence for this conjecture is given by the fact that our Beltrami differential has the correct limits according to Fig. 1. For $\Lambda \rightarrow 0$, it is immediate that we obtain the twistor space of self-dual Taub-NUT. For $\Lambda \rightarrow \infty$ and $\nu^2 \rightarrow 0$ with $\Lambda\nu^2$ fixed, we obtain the holomorphic coordinates of Eguchi-Hanson space [7].

B. Celestial symmetries from twistor space

Beyond viewing the twistor space of Eguchi-Hanson space as a backreaction, Ref. [67] obtained the celestial chiral algebra of self-dual gravity from considering the action of $\{\cdot, \cdot\}_0$ on generators built out of the deformed holomorphic coordinates. We will follow the same strategy, but since $\Lambda \neq 0$ we will have to use $\{\cdot, \cdot\}_\Lambda$. Acting with $\{\cdot, \cdot\}_\Lambda$ on generators in the undeformed holomorphic coordinates led to the recent cosmological constant deformation of $L\mathfrak{ham}(\mathbb{C}^2)$ [8,9,68]. In Sec. VB, we derive the action of $\{\cdot, \cdot\}_\Lambda$ on any pair of holomorphic coordinates

$$\begin{aligned} & \{w[p, q, 2i, 2j], w[r, s, 2k, 2l]\}_\Lambda \\ &= (ps - qr)w[p + r - 1, q + s - 1, 2(i + k), 2(j + l)] \\ &+ 4\Lambda(il - jk)w[p + r, q + s, 2(i + k - 1) + 1, 2(j + l - 1) + 1] \\ &- \frac{\Lambda\nu^2}{2}((p - q)(k + l) - (r - s)(i + j))w[p + r + 1, q + s + 1, 2(i + k - 1), 2(j + l - 1)]. \end{aligned} \quad (36)$$

Although (36) and more generally (64) look fairly involved, it is the unique algebra that respects the expected symmetries, obeys the Jacobi identity, and reduces to the known algebras in the limits of Sec. III B. Indeed, when transformed to the slightly different coordinates of [8] it can be seen to be a further deformation of the Λ -deformed algebra considered in [8,9]. In fact, we obtain a $\mathbb{Z}_2$ -quotient of the AdS_4 algebra in the limit $M \rightarrow \infty$, which is an expected feature due to the coordinates (29) in analogy to [8]. After rescaling the generators by Λ , related to the prefactor of (48), and sending $\Lambda \rightarrow \infty$ and $M \rightarrow \infty$, keeping $c^2 = \frac{\Lambda\nu^2}{2}$ constant, the algebra becomes the expected algebra of Eguchi-Hanson space considered in Eq. (4.8) of [67]. The role of μ^α and λ_α is exchanged as expected from the

$\phi^\pm, \eta, X, Y, Z$ leading to the main result in Eq. (60). This can be used to compute the bracket of two generators $(\phi^+)^a(\phi^-)^b\eta^cX^dY^eZ^f$, where $a, b, c \in \mathbb{N}$ and $d, e, f \in \mathbb{Z}$, which is the natural generalization of [6–8]. We still need to impose the right scaling by demanding that the generators have weight 2, i.e., $a + b + \frac{1}{2}(c + d + e + f) = 2$, and impose relations between the different coordinates. For the latter, we note that the subspaces generated by

$$\langle\{\phi^+\phi^- - \eta\}\rangle, \quad \left\langle\left\{XY - Z^2 + \frac{\Lambda\nu^2}{2}\eta^2\right\}\right\rangle, \quad (34)$$

are Poisson ideals, which means we can use these relations to consider only generators of the form

$$\begin{aligned} w[p, q, 2i, 2j] &= (\phi^+)^p(\phi^-)^qX^iY^j, \\ w[p, q, 2i + 1, 2j + 1] &= (\phi^+)^p(\phi^-)^qX^iY^jZ. \end{aligned} \quad (35)$$

For such $w[p, q, a, b]$ we have the condition $p + q + a + b = 2$. Solving for $b = 2 - p - q - a$ would lead to the standard presentation of generators with three labels but we will not do so to keep certain symmetries between μ^α and λ_α manifest. The algebra can then be found by explicitly applying the bracket (60) to the generators (35) and using the constraints (34) which has been spelled out in Sec. VC. For instance, considering two generators of the first kind leads to

conformal inversion in Eq. (47). In the self-dual Taub-NUT limit, $\Lambda \rightarrow 0$, we find an undeformed Lw_Λ .

IV. MORE ON THE PEDERSEN METRIC

In the remaining part of this work, we provide some of the technical material relating the various spacetime metrics to each other, deriving holomorphic coordinates and the form of the deformed Jacobi bracket.

A. From the Plebański-Demiański to the Pedersen metric

Let us first derive the triaxial Pedersen metric (6) from the Plebański-Demiański metric (1). At the self-dual

point (4) with $a = 0$ and after the Wick rotation $t \mapsto it$, the Plebański-Demiański metric is

$$ds^2 = \frac{1}{U(r)}(dt - 2M \cos \theta d\phi)^2 + U(r)dr^2 + (r^2 - M^2)(d\theta^2 + \sin^2 \theta d\phi^2), \quad (37)$$

where $r \geq M$, (θ, ϕ) are coordinates on S^2 and $t \sim t + 8\pi M$ is a coordinate on S^1 . We have also introduced

$$U(r) = \frac{r+M}{r-M} \left(1 + \frac{(r-M)(r+3M)}{l^2} \right)^{-1}. \quad (38)$$

Performing the diffeomorphism

$$\rho = \sqrt{2Ml^2 \frac{r-M}{l^2 + 2M(r-M)}}, \quad \psi = \frac{t}{2M}, \quad (39)$$

and introducing

$$\nu^2 = \frac{1}{4M^2} - \frac{1}{l^2} \quad (40)$$

bring the metric into the form

$$ds^2 = \frac{1}{(1 - \rho^2/l^2)^2} \left[4 \frac{1 + \nu^2 \rho^2}{1 + \nu^2 \rho^4/l^2} d\rho^2 + \rho^2(1 + \nu^2 \rho^2)(d\theta^2 + \sin^2 \theta d\phi^2) + \rho^2 \frac{1 + \nu^2 \rho^4/l^2}{1 + \nu^2 \rho^2} (d\psi - \cos \theta d\phi)^2 \right], \quad (41)$$

where $0 \leq \rho < l$ and $0 \leq \psi < 4\pi$. This is precisely the metric (6), after relabeling $\rho \mapsto r$ and introducing the 1-forms

$$\begin{aligned} \sigma_1 &= \frac{1}{2}(\cos \psi d\theta + \sin \psi \sin \theta d\phi), \\ \sigma_2 &= \frac{1}{2}(-\sin \psi d\theta + \cos \psi \sin \theta d\phi), \\ \sigma_3 &= \frac{1}{2}(d\psi - \cos \theta d\phi). \end{aligned} \quad (42)$$

To obtain the generalized Gibbons-Hawking form of the metric, consider the further diffeomorphism

$$\rho = \sqrt{\frac{l}{\nu}} \tan \chi. \quad (43)$$

This brings the metric into the metric (9). Similarly, if we start from the oblate metric (41) and consider the analytic continuation $\nu \mapsto i\nu$, we find the prolate metric (11) using the diffeomorphism

$$\rho = \sqrt{\frac{l}{\nu}} \tanh \chi. \quad (44)$$

B. Various limits of the Pedersen metric

Let us now discuss the various limits of the Pedersen metric that were previously studied in the context of celestial holography [7,9,20,30,31,37] and explain Fig. 1.

1. Self-dual Taub-NUT

We first consider the limit in which the cosmological constant is vanishing, $l \rightarrow \infty$. It is easier to understand this limit from the form (37) of the Pedersen metric, which reduces to

$$ds^2 = \left(1 + \frac{2M}{r} \right)^{-1} (dt + 2M \cos \theta d\phi)^2 + \left(1 + \frac{2M}{r} \right) (dr^2 + r^2 d\Omega_2^2), \quad (45)$$

after a further shifting $r \mapsto r + M$. This is precisely self-dual Taub-NUT.

2. AdS₄

The second natural limit is the limit $\nu \rightarrow 0$. In this case, the Pedersen metric in the form (6) is

$$ds^2 = \frac{4}{(1 - r^2/l^2)^2} (dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)). \quad (46)$$

This is Euclidean AdS₄. As expected, the boundary metric reduces to the canonical metric on the 3-sphere parametrized by (ψ, θ, ϕ) [69].

3. A singular double cover of Eguchi-Hanson space

In this and the following limit, we focus on the prolate Pedersen metric (11). Consider the limit $\nu, l \rightarrow 0$ with $c^2 \equiv l/\nu$ held constant and set

$$r = \frac{c^2}{\varrho}. \quad (47)$$

The metric in triaxial form then becomes

$$ds^2 = \frac{d\varrho^2}{1 - c^4/\varrho^4} + \varrho^2(\sigma_1^2 + \sigma_2^2) + \varrho^2 \left(1 - \frac{c^4}{\varrho^4} \right) \sigma_3^2, \quad (48)$$

up to an overall factor of $l^4/4c^2$. The spacetime metric can be extended to the region $\varrho > c$ and is seen to be the Eguchi-Hanson metric. More precisely, the singularity at $\varrho = c$ is not removable without taking a $\mathbb{Z}_2$ quotient, so the spacetime is actually a singular double cover of Eguchi-Hanson space [70].

4. Burns space and $\widetilde{\mathbb{CP}}^2$

Setting $\nu l = 1$ in the prolate Pedersen metric (11), we recover the Burns space metric [20] up to a conformal prefactor. In fact, performing the change of coordinates

$$r^2 = \frac{R^2}{2 - R^2/l^2}, \quad (49)$$

in the triaxial form of the metric, we find

$$\begin{aligned} ds^2 &= \frac{2}{(1 - R^2/l^2)^2} (dR^2 + \rho^2 \sigma_3^2) \\ &+ \frac{2}{1 - R^2/l^2} (\sigma_1^2 + \sigma_2^2), \end{aligned} \quad (50)$$

which is the Fubini-Study metric on a noncompact version of $\mathbb{CP}^2$ commonly referred to as Bergmann space and denoted by $\widetilde{\mathbb{CP}}^2$. As a homogeneous space it is described by $\widetilde{\mathbb{CP}}^2 = SU(2, 1)/U(2)$, and it is the noncompact dual of $\mathbb{CP}^2 = SU(3)/U(2)$ in the sense of [71]. Both $\widetilde{\mathbb{CP}}^2$ and $\mathbb{CP}^2$ with a point removed are conformally equivalent to Burns space [20] and (13) gives the corresponding conformal factor. Note that $\widetilde{\mathbb{CP}}^2$ has a degenerate boundary conformal structure since the prolate Pedersen metric generically has the boundary conformal structure

$$ds_3^2 = \sigma_1^2 + \sigma_2^2 + \frac{1}{1 - \nu^2 l^2} \sigma_3^2, \quad (51)$$

and we set $\nu l = 1$. The similarity to the boundary of Minkowski space has been previously commented on in [72].

V. THE CURVED TWISTOR SPACE AND CELESTIAL SYMMETRIES

A. Holomorphicity of the deformed coordinates

Let us check explicitly that the coordinates discussed in Sec. III are holomorphic for $\bar{\nabla}_\Lambda$ of Eq. (28). First, we have

$$Y^{\alpha\beta} = \lambda^\alpha \lambda^\beta - \frac{\Lambda \nu^2 \eta^2}{2} \frac{\hat{\lambda}^\alpha \hat{\lambda}^\beta}{\langle \lambda \hat{\lambda} \rangle^2}, \quad (52)$$

which obeys

$$\begin{aligned} \bar{\nabla}_\Lambda Y^{\alpha\beta} &= 2\lambda^{(\alpha} \bar{\nabla}_\Lambda \lambda^{\beta)} - 2 \frac{\Lambda \nu^2 \eta^2}{2} \frac{\hat{\lambda}^{(\alpha}}{\langle \lambda \hat{\lambda} \rangle} \bar{\nabla}_\Lambda \frac{\hat{\lambda}^{\beta)}}{\langle \lambda \hat{\lambda} \rangle} \\ &= 2 \frac{\Lambda \nu^2 \eta^2}{2} \bar{e}^0 \frac{\lambda^{(\alpha} \hat{\lambda}^{\beta)}}{\langle \lambda \hat{\lambda} \rangle} - 2 \frac{\Lambda \nu^2 \eta^2}{2} \bar{e}^0 \frac{\hat{\lambda}^{(\alpha} \lambda^{\beta)}}{\langle \lambda \hat{\lambda} \rangle} \\ &= 0. \end{aligned} \quad (53)$$

$\eta = \mu^+ \mu^-$ is manifestly holomorphic since h depends only on μ^α through η and $\{\eta, \eta\}_\Lambda = 0$. For $\phi^\pm$, we can easily see that

$$\bar{\nabla}_\Lambda \phi^+ = \frac{\nu^2 \eta \phi^+}{2} \left(-\bar{e}^0 + \frac{1}{1 - \frac{\Lambda \nu^2 \eta^2 f(\lambda)^2}{2}} \bar{\nabla}_\Lambda f(\lambda) \right) = 0, \quad (54)$$

where we used that $\frac{\hat{\lambda}_\alpha}{\langle \lambda \hat{\lambda} \rangle} \frac{\partial f}{\partial \lambda_\alpha} = -f^2$ holds in both patches. Similarly $\bar{\nabla}_\Lambda \phi^- = 0$.

B. Jacobi bracket in deformed coordinates

Let us now derive the action of $\{\cdot, \cdot\}_\Lambda$ on any pair of holomorphic coordinates $\phi^\pm, \eta, X, Y, Z$. We derive it in part by acting on holomorphic coordinates with

$$\{f, g\}_\Lambda = \epsilon^{\dot{\alpha}\dot{\beta}} \frac{\partial f}{\partial \mu^{\dot{\alpha}}} \frac{\partial g}{\partial \mu^{\dot{\beta}}} + \Lambda \epsilon_{\alpha\beta} \frac{\partial f}{\partial \lambda_\alpha} \frac{\partial g}{\partial \lambda_\beta}, \quad (55)$$

and in part from the two consistency conditions that the Jacobi identity needs to hold and that the subspaces generated by (34) need to be Poisson ideals.

First, we act with (15) on a pair of $Y_{\alpha\beta}$ giving

$$\{Y_{\alpha\beta}, Y_{\gamma\delta}\}_\Lambda = 2\Lambda(\epsilon_{\beta\gamma} Y_{\alpha\delta} + \epsilon_{\alpha\delta} Y_{\beta\gamma}). \quad (56)$$

This is equivalent to X, Y, Z obeying the defining relations of $\mathfrak{sl}_2$ similar to [7]

$$\{X, Y\}_\Lambda = 4\Lambda Z, \quad \{X, Z\}_\Lambda = 2\Lambda X, \quad \{Y, Z\}_\Lambda = -2\Lambda Y. \quad (57)$$

Moreover, using (55) we can immediately see

$$\begin{aligned} \{\phi^\pm, \eta\}_\Lambda &= \pm \phi^\pm, & \{Y_{\alpha\beta}, \eta\}_\Lambda &= 0, \\ \{\phi^+, \phi^-\}_\Lambda &= \{\mu^+, \mu^-\}_\Lambda = 1. \end{aligned} \quad (58)$$

Demanding now that the subspace generated by $\{XY - Z^2 + \frac{\Lambda \nu^2}{2} \eta^2\}$ is a Poisson ideal and the resulting bracket has to obey the Jacobi identity uniquely fixes the remaining brackets (under the additional assumption that X and Y should be treated symmetrically) to be

$$\begin{aligned} \{\phi^\pm, Y\}_\Lambda &= \mp \frac{\Lambda \nu^2}{2} \phi^\pm \frac{\eta}{X}, & \{\phi^\pm, X\}_\Lambda &= \mp \frac{\Lambda \nu^2}{2} \phi^\pm \frac{\eta}{Y}, \\ \{\phi^\pm, Z\}_\Lambda &= 0. \end{aligned} \quad (59)$$

We hope to give a more direct derivation of Eq. (59) in the future.

Although a direct derivation is absent, the above discussion leads to strong evidence for the conjectured formula

$$\begin{aligned} \{\cdot, \cdot\}_\Lambda &= 2\Lambda X \frac{\partial}{\partial X} \wedge \frac{\partial}{\partial Z} - 2\Lambda Y \frac{\partial}{\partial Y} \wedge \frac{\partial}{\partial Z} + 4\Lambda Z \frac{\partial}{\partial X} \wedge \frac{\partial}{\partial Y} \\ &+ \frac{\partial}{\partial \phi^+} \wedge \frac{\partial}{\partial \phi^-} + \phi^+ \frac{\partial}{\partial \phi^+} \wedge \frac{\partial}{\partial \eta} - \phi^- \frac{\partial}{\partial \phi^-} \wedge \frac{\partial}{\partial \eta} \\ &+ \frac{\Lambda \nu^2}{2} \eta \left(\frac{\phi^-}{X} \frac{\partial}{\partial \phi^-} \wedge \frac{\partial}{\partial Y} + \frac{\phi^+}{Y} \frac{\partial}{\partial \phi^+} \wedge \frac{\partial}{\partial X} \right. \\ &\left. - \frac{\phi^+}{X} \frac{\partial}{\partial \phi^+} \wedge \frac{\partial}{\partial Y} - \frac{\phi^-}{Y} \frac{\partial}{\partial \phi^-} \wedge \frac{\partial}{\partial X} \right). \end{aligned} \quad (60)$$

In Sec. III B, we discussed that the correct limits are obtained from Eq. (60) according to Fig. 1.

C. Deriving the 2-parameter deformation of Lw_Λ

Equation (60) can be used to compute the bracket of two generators

$$(\phi^+)^a(\phi^-)^b\eta^cX^dY^eZ^f, \quad (61)$$

where $a, b, c \in \mathbb{N}$ and $d, e, f \in \mathbb{Z}$. Since $\phi^\pm, \eta$ are deformations of the μ^α coordinates and X, Y, Z are deformations of the λ_α coordinates, these generators and their index ranges are

the natural generalization of the treatment in the literature [6–8]. We still need to impose the right scaling by demanding that the generators have weight 2, i.e., $a + b + \frac{1}{2}(c + d + e + f) = 2$, and impose the relations (34) between the different coordinates. Let us initially only work on the support of $\phi^+\phi^- = \eta$ with no relation between X, Y, Z, η imposed. A generator of weight 2 is then given by

$$w[a, b, c, d, e] = (\phi^+)^a(\phi^-)^bX^cY^dZ^e, \quad (62)$$

where $a, b \in \mathbb{N}_0; c, d, e \in \mathbb{Z}$; and $a + b + \frac{1}{2}(c + d + e) = 2$. Using (60) leads to the Lie algebra

$$\begin{aligned} \{w[p, q, i, j, k], w[r, s, l, m, n]\}_\Lambda &= (ps - qr)w[p + r - 1, q + s - 1, i + l, j + m, k + n] \\ &+ 2\Lambda((i - j)n - (l - m)k)w[p + r, q + s, i + l, j + m, k + n - 1] \\ &+ 2(im - jl)w[p + r, q + s, i + l - 1, j + m - 1, k + n + 1] \\ &- \frac{\Lambda\nu^2}{2}((p - q)(l + m) - (r - s)(i + j))w[p + r + 1, q + s + 1, i + l - 1, j + m - 1, k + n], \end{aligned} \quad (63)$$

which has been explicitly checked to obey the Jacobi identity. Imposing the further constraint $Z^2 = XY + \frac{\Lambda\nu^2}{2}\eta^2$ means that we can always solve for X, Y, η whenever we have 2 or more powers of Z . This leaves us with the generators (35).

Using the relation $XY - Z^2 = -\frac{\Lambda\nu^2}{2}\eta^2$ in (63) then leads us to the final algebra

$$\begin{aligned} \{w[p, q, 2i, 2j], w[r, s, 2k, 2l]\}_\Lambda &= (ps - qr)w[p + r - 1, q + s - 1, 2(i + k), 2(j + l)] \\ &+ 4\Lambda(il - jk)w[p + r, q + s, 2(i + k - 1) + 1, 2(j + l - 1) + 1] \\ &- \frac{\Lambda\nu^2}{2}((p - q)(k + l) - (r - s)(i + j))w[p + r + 1, q + s + 1, 2(i + k - 1), 2(j + l - 1)], \\ \{w[p, q, 2i, 2j], w[r, s, 2k + 1, 2l + 1]\}_\Lambda &= (ps - qr)w[p + r - 1, q + s - 1, 2(i + k) + 1, 2(j + l) + 1] \\ &+ 2\Lambda(i(2l + 1) - j(2k + 1))w[p + r, q + s, 2(i + k), 2(j + l)] \\ &+ 2\Lambda^2\nu^2(il - jk)w[p + r + 2, q + s + 2, 2(i + k - 1), 2(j + l - 1)] \\ &- \frac{\Lambda\nu^2}{2}((p - q)(k + l) - (r - s)(i + j)) \\ &\times w[p + r + 1, q + s + 1, 2(i + k - 1) + 1, 2(j + l - 1) + 1], \\ &\times \{w[p, q, 2i + 1, 2j + 1], w[r, s, 2k + 1, 2l + 1]\}_\Lambda \\ &= (ps - qr)w[p + r - 1, q + s - 1, 2(i + k + 1), 2(j + l + 1)] \\ &+ \frac{\Lambda\nu^2}{2}(ps - qr)w[p + r + 2, q + s + 2, 2(i + k), 2(j + l)] \\ &+ \Lambda((2i + 1)(2l + 1) - (2j + 1)(2k + 1))w[p + r, q + s, 2(i + k) + 1, 2(j + l) + 1] \\ &+ 2\Lambda^2\nu^2(il - jk)w[p + r + 2, q + s + 2, 2(i + k - 1) + 1, 2(j + l - 1) + 1] \\ &- \frac{\Lambda\nu^2}{2}((p - q)(k + l) - (r - s)(i + j))w[p + r + 1, q + s + 1, 2(i + k), 2(j + l)] \\ &- \left(\frac{\Lambda\nu^2}{2}\right)^2((p - q)(k + l) - (r - s)(i + j)) \\ &\times w[p + r + 3, q + s + 3, 2(i + k - 1), 2(j + l - 1)]. \end{aligned} \quad (64)$$

This is the unique Lie-algebra that respects the expected symmetries, obeys the Jacobi identity, and reduces to the known algebras in certain limits as discussed in Sec. III B.

ACKNOWLEDGMENTS

We thank Tim Adamo, William Biggs, Roland Bittleston, Kasia Budzik, Kevin Costello, Adam Kmec, Lionel Mason, Chris Pope, Maria Rodriguez, Sean Seet, Atul Sharma, David Skinner, and Andrew Strominger for helpful discussions. Moreover, we thank Atul Sharma for encouragement and comments on the draft. S.H. also thanks the organizers of the “2024 summer school on celestial holography” and the hospitality of Perimeter

Institute and Roland Bittleston. G.B. is supported by a joint Clarendon Fund and Merton College Mathematics Scholarship. S.H. is supported by the STFC HEP Theory Consolidated Grant No. ST/T000694/1 and St. John’s College, Cambridge.

DATA AVAILABILITY

No data were created or analyzed in this study.

-
- [1] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory* (Princeton University Press, Princeton, New Jersey, 2017); [arXiv:1703.05448](#).
 - [2] A.-M. Raclariu, [arXiv:2107.02075](#).
 - [3] S. Pasterski, M. Pate, and A.-M. Raclariu, in *Snowmass 2021* (2021), [arXiv:2111.11392](#).
 - [4] A. Strominger, *Phys. Rev. Lett.* **127**, 221601 (2021).
 - [5] A. Guevara, E. Himwich, M. Pate, and A. Strominger, *J. High Energy Phys.* **11** (2021) 152.
 - [6] T. Adamo, L. Mason, and A. Sharma, *SIGMA* **18**, 016 (2022).
 - [7] R. Bittleston, S. Heuveline, and D. Skinner, *J. High Energy Phys.* **09** (2023) 008.
 - [8] R. Bittleston, G. Bogna, S. Heuveline, A. Kmec, L. Mason, and D. Skinner, *J. High Energy Phys.* **07** (2024) 010.
 - [9] T. R. Taylor and B. Zhu, *Phys. Rev. Lett.* **132**, 221602 (2024).
 - [10] W. Bu, S. Heuveline, and D. Skinner, *J. High Energy Phys.* **12** (2022) 011.
 - [11] W. Melton, S. A. Narayanan, and A. Strominger, *J. High Energy Phys.* **03** (2023) 233.
 - [12] K. Costello and N. M. Paquette, *J. High Energy Phys.* **10** (2022) 193.
 - [13] J. Mago, L. Ren, A. Y. Srikant, and A. Volovich, *SIGMA* **19**, 044 (2023).
 - [14] M. Carrillo González, A. Lipstein, and S. Nagy, *J. High Energy Phys.* **10** (2024) 183.
 - [15] A. Lipstein and S. Nagy, *Phys. Rev. Lett.* **131**, 081501 (2023).
 - [16] T. McLoughlin, N. Moynihan, and A. Puhm, [arXiv:2408.13234](#).
 - [17] T. Adamo, W. Bu, and B. Zhu, *J. High Energy Phys.* **06** (2024) 076.
 - [18] T. Adamo, W. Bu, P. Tourkine, and B. Zhu, *J. High Energy Phys.* **10** (2024) 192.
 - [19] K. Costello, N. M. Paquette, and A. Sharma, *Phys. Rev. Lett.* **130**, 061602 (2023).
 - [20] K. Costello, N. M. Paquette, and A. Sharma, *J. High Energy Phys.* **10** (2023) 174.
 - [21] N. Garner and N. M. Paquette, *J. High Energy Phys.* **08** (2023) 088.
 - [22] N. Garner and N. M. Paquette, *J. High Energy Phys.* **05** (2025) 228.
 - [23] R. Gonzo, T. McLoughlin, and A. Puhm, *J. High Energy Phys.* **10** (2022) 073.
 - [24] W. Melton, A. Sharma, and A. Strominger, *J. High Energy Phys.* **07** (2024) 132.
 - [25] W. Melton, F. Niewinski, A. Strominger, and T. Wang, *J. High Energy Phys.* **08** (2024) 046.
 - [26] E. Casali, W. Melton, and A. Strominger, *J. High Energy Phys.* **11** (2022) 140.
 - [27] W. Fan, A. Fotopoulos, S. Stieberger, T. R. Taylor, and B. Zhu, *J. High Energy Phys.* **08** (2022) 213.
 - [28] A. Kmec, L. Mason, R. Ruzziconi, and A. Yellespur Srikant, *J. High Energy Phys.* **10** (2024) 250.
 - [29] T. Adamo, G. Bogna, L. Mason, and A. Sharma, *Classical Quantum Gravity* **41**, 015030 (2024).
 - [30] E. Crawley, A. Guevara, E. Himwich, and A. Strominger, *J. High Energy Phys.* **09** (2023) 109.
 - [31] E. Crawley, A. Guevara, N. Miller, and A. Strominger, *J. High Energy Phys.* **10** (2022) 135.
 - [32] A. Guevara and U. Kol, [arXiv:2311.07933](#).
 - [33] D. A. Easson and M. W. Pezzelle, *Phys. Rev. D* **109**, 044007 (2024).
 - [34] J. Desai, G. Herczeg, D. McNutt, and M. Pezzelle, *J. High Energy Phys.* **12** (2024) 044.
 - [35] R. Bittleston, *J. High Energy Phys.* **01** (2023) 018.
 - [36] K. J. Costello, [arXiv:2111.08879](#).
 - [37] R. Bittleston, K. Costello, and K. Zeng, [arXiv:2412.02680](#).
 - [38] K. Zoubos, *J. High Energy Phys.* **12** (2002) 037.
 - [39] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers, *Phys. Rev. D* **59**, 064010 (1999).
 - [40] S. W. Hawking, C. J. Hunter, and D. N. Page, *Phys. Rev. D* **59**, 044033 (1999).
 - [41] D. Martelli, A. Passias, and J. Sparks, *Nucl. Phys.* **B876**, 810 (2013).
 - [42] J. F. Plebanski and M. Demianski, *Ann. Phys. (N.Y.)* **98**, 98 (1976).
 - [43] J. B. Griffiths and J. Podolsky, *Int. J. Mod. Phys. D* **15**, 335 (2006).
 - [44] N. H. Rodríguez and M. J. Rodríguez, *J. High Energy Phys.* **10** (2022) 044.
 - [45] S. W. Hawking, *Phys. Lett.* **60A**, 81 (1977).
 - [46] G. W. Gibbons and C. N. Pope, *Commun. Math. Phys.* **61**, 239 (1978).

- [47] E. Ivanov and G. Valent, *Classical Quantum Gravity* **16**, 1039 (1999).
- [48] H. Pedersen, *Math. Ann.* **274**, 35 (1986).
- [49] H. Pedersen, Geometry and magnetic monopoles, Ph.D. thesis, Citeseer, 1985.
- [50] C. R. LeBrun, *Proc. R. Soc. A* **380**, 171 (1982).
- [51] C. LeBrun, *Duke Math. J.* **63**, 723 (1991).
- [52] N. J. Hitchin, *J. Diff. Geom.* **42**, 30 (1995).
- [53] G. W. Gibbons and S. W. Hawking, *Phys. Lett.* **78B**, 430 (1978).
- [54] C. LeBrun, *J. Diff. Geom.* **34**, 223 (1991).
- [55] S. S. Costa, [arXiv:math-ph/0112039](#).
- [56] L. J. Mason and M. Wolf, *Commun. Math. Phys.* **288**, 97 (2009).
- [57] R. S. Ward, *Commun. Math. Phys.* **78**, 1 (1980).
- [58] R. Penrose, *Gen. Relativ. Gravit.* **7**, 171 (1976).
- [59] T. Adamo, L. Mason, and A. Sharma, *Adv. Theor. Math. Phys.* **27**, 623 (2023).
- [60] A. Swann, HyperKähler and quaternionic Kähler geometry, Ph.D. thesis, University of Oxford, 1990.
- [61] We thank David Skinner for pointing this backreaction out to us.
- [62] N. J. Hitchin, *Math. Proc. Cambridge Philos. Soc.* **85**, 465 (1979).
- [63] N. Hitchin, Séminaire Bourbaki **1991**, 137 (1992).
- [64] A. L. Besse, *Einstein Manifolds* (Springer-Verlag, Berlin, Heidelberg, New York, 1987).
- [65] D. Calderbank and H. Pedersen, *Surv. Diff. Geom.* **6**, 387 (2001).
- [66] K. Budzik, in *Proceedings of the 21st String Phenomenology 2022* (American Mathematical Society, Providence, USA, 2023); [arXiv:2310.16721](#).
- [67] R. Bittleston, A. Sharma, and D. Skinner, *Commun. Math. Phys.* **403**, 1543 (2023).
- [68] L. F. Alday, T. Hansen, and J. A. Silva, *Phys. Rev. Lett.* **131**, 161603 (2023).
- [69] Be careful not to confuse this 3-sphere with the 3-sphere parametrized by (χ, θ, ϕ) .
- [70] T. Eguchi and A. J. Hanson, *Ann. Phys. (N.Y.)* **120**, 82 (1979).
- [71] S. Helgason, in *Battelle Rencontres* (W. A. Benjamin, New York, NY, 1968), pp. 1–71.
- [72] R. Britto-Pacumio, A. Strominger, and A. Volovich, *J. High Energy Phys.* **11** (1999) 013.