

Higher order electroweak radiative corrections in lepton-proton scattering using covariant approach

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We perform detailed calculations of electroweak radiative corrections to parity violating lepton scattering with a proton target (ep and μp) up to quadratic and reducible two-loop level using a covariant approach. Our numerical results are presented at energies relevant for a variety of existing and proposed experimental programs such as Q_{weak} , P2, Measurement of a Lepton-Lepton Electroweak Reaction (MOLLER), Muon Scattering Experiment, and experiments at the Electron-Ion Collider. Analysis shows that such corrections at the next-to-next-to-leading order are quite significant and have to be included in searches of physics beyond the standard model, matching the increasing precision of the future experimental programs at low-energy scales.

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I. INTRODUCTION

The Standard Model (SM) of particle physics has been tested in countless experiments with extraordinary precision. Despite this success, it has limitations in explaining the concept of gravity, dark matter and dark energy, matter-antimatter asymmetry in the Universe, and hierarchies of scale related to the Higgs boson. This opens the door to search for beyond the Standard Model (BSM) physics via direct production of additional particles at high-energy accelerators. However, to date, there is no direct evidence of BSM physics even at the TeV energy scales achievable at the Large Hadron Collider (LHC), at CERN. In this scenario, low-energy precision physics plays an essential role in reaching the mass-energy scales not directly accessible at the existing high-energy colliders. These indirect searches are achieved through the precise measurements of well-predicted SM observables, allowing the highly targeted alternative tests for BSM physics.

There are many proposed and working experimental programs that aim to find BSM physics by precisely measuring the physical quantities at low-energy scales. One such observable, widely measured, is the weak charge of the proton Q_W^p , which defines the strength of proton interaction with other particles via the well-known neutral

electroweak force. The electroweak interaction violates parity symmetry first postulated in 1956 [1] and experimentally proven in 1957 [2]. In this work, we calculate lepton-proton scattering by considering a longitudinally polarized lepton beam having positive and negative helicities. Parity-violating asymmetry provides a tool to isolate the weak interaction and is given by

$$A_{\text{PV}} = \frac{\sigma^+ - \sigma^-}{\sigma^+ + \sigma^-}, \quad (1)$$

where $\sigma^{\pm}$ corresponds to the scattering cross sections in case of a polarized fermion with a negative or positive helicity, respectively.

Recently, many parity-violating electron scattering experiments have been performed and have been proposed to be designed following the latest improvements to precisely measure the SM parameters. One such example is the Q_{weak} experiment [3,4] at the Thomas Jefferson National Accelerator Facility which used a longitudinally polarized beam of electrons accelerated to 1.16 GeV and scattered from a liquid-hydrogen target at a small 4-momentum transfer squared $-q^2 = 0.0248 \text{ GeV}^2$. The most up-to-date value of A_{PV} measured by the Q_{weak} experiment is -226.5 ± 7.3 (statistical) ± 5.8 (systematic) parts per billion (ppb). [5]. This asymmetry was then used to determine the Q_W^p which was reported to be 0.0719 ± 0.0045 , where the uncertainty is 1 standard deviation.

Several proposed experiments are performed using the same technique of polarized electron beam scattering on a liquid-hydrogen target to measure the precise value of A_{PV} . One such example is the P2 experiment on the MESA accelerator [6] that will operate at the small beam energy of 155 MeV. The first measurement of the P2 experiment

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comprises a sensitive test of the Standard Model which requires measuring $A_{PV} = -39.94$ ppb with a precision of $\Delta A_{PV} = 0.56$ ppb (1.4%) with a small $-q^2 = 4.5 \times 10^{-3} \text{ GeV}^2$ [6]. Another one is the MOLLER experiment that will be performed in the Jefferson laboratory at an upgraded beam energy of 12 GeV and aims to measure the parity-violating asymmetry in the scattering of longitudinally polarized electrons off unpolarized electrons with a precision of 0.73 ppb. That would allow a measurement of the weak charge of the electron at a fractional accuracy of 2.3% and a determination of the weak mixing angle with uncertainty of ± 0.00026 (statistical) ± 0.00013 (systematic) [7].

The Electron-Ion Collider (EIC) [8] at Brookhaven National Laboratory is another highly anticipated facility that aims to make precision measurements of the constituent quarks and gluons of the proton. It will be the first polarized electron-proton collider where the spins of both electron and proton beams are aligned in a controllable way. The polarized beams will then collide in center-of-mass energies ranging from ~ 20 to ~ 100 GeV and upgradable to ~ 140 GeV.

In case of muon beam scattering, two important experiments have been proposed, one corresponding to the $\mu p \rightarrow \mu p$ scattering process is a Paul Scherrer Institute MUon Scattering Experiment (MUSE) [9] that aims to measure and directly compare ep to μp elastic scattering at the subpercentage level and low momentum transfer. This scattering experiment will be the best test of lepton universality to date and has the potential to demonstrate whether the interactions μp and ep are consistent or different. The unexplained large discrepancy of the proton charge radius measurements with muonic hydrogen Lamb shift and determinations from elastic electron scattering and Lamb shift in regular hydrogen is known as the proton radius puzzle [10]. The MUSE experimental uncertainties are such that if the discrepancy is real, it should be confirmed with significance of $\approx 5\sigma$.

The above-mentioned high-precision parity-violating experiments require a new level of accuracy of electroweak radiative corrections which include higher-order effects. These are the processes that are quite more complicated than the actual process but are indistinguishable from it experimentally. A complete set of one-loop-level electroweak radiative corrections in A_{PV} has already been obtained in the case of lepton-nucleon scattering [11–16] as well as electron-electron scattering [17–19], while some calculations have been performed in order to study quadratic [20] and two-loop effects in low-energy electroweak measurements [21,22]. In 1976, the covariant approach was first introduced by Bardin and Shumeiko [23] to extract the infrared divergence from the lowest-order bremsstrahlung cross section. This approach has also been used to get explicit expressions in the case of QED radiative corrections up to the one-loop level for elastic electron-nucleon scattering [24]. In this work, we used the covariant approach and calculated the most precise

electroweak leptonic tensor up to next-to-next-to leading order (NNLO) (quadratic and reducible two loop). This leptonic tensor once obtained can be used to calculate the scattering cross section for any distinguishable hadronic target. In previous calculations of A_{PV} , the mass of the electron (m_e) was treated as a small parameter, but in our case we kept it to account for better precision. We calculate electroweak one-loop (vertex corrections and self-energies), quadratic (squaring of vertex correction and self-energy graphs), and reducible two-loop level A_{PV} using FeynArts [25], FormCalc [26], FeynCalc [27], and LoopTools [26] as the base languages. Our calculated results are in good agreement with the measured and proposed values of asymmetries of the Q_{weak} and P2 experiments, respectively. The theoretical predictions in this work will be important for the above-mentioned MOLLER, EIC, and MUSE experimental programs (either directly or as background studies) searching for physics beyond the Standard Model at the precision frontier. This work is done considering elastic lepton-proton scattering and by using an unpolarized proton. In the future, we would like to consider a hadronic target in an inelastic regime and proton being polarized. Some recent work related to inclusive deep inelastic scattering with an unpolarized proton target can be found in [28]. In this work, we did not consider the box diagrams; however, one can calculate them using the approaches [29,30].

The paper is organized as follows. The basic notation and introduction to the covariant approach with the electroweak tree-level and one-loop level unpolarized hadronic tensor calculations are presented in Sec. II. The tree-level polarized electroweak leptonic tensor and contributions from higher order (next-to-leading (NLO) and next-to-next-to-next-leading (NNLO)) corrections in A_{PV} are described in Sec. III. In order to treat infrared (IR) divergence, we use the bremsstrahlung process in soft photon approximation, and details are given in Sec. IV. The numerical analysis is presented in Sec. V and conclusions in Sec. VI.

II. COVARIANT APPROACH AND PARITY-VIOLATING ASYMMETRY

The covariant approach involves a covariant formulation by applying the “cutting rules.” In the covariant approach, when a propagator “cut” is applied as shown in Fig. 1, it divides the differential cross section of the Feynman diagram into two parts, leptonic and hadronic. The differential cross section is given by

$$d\sigma = \frac{1}{4(2\pi)^2} \frac{|\mathcal{M}|^2}{\Phi\sqrt{s}} d\Omega, \quad (2)$$

where Φ is the flux factor given by $\Phi = 4\sqrt{s}$, $d\Omega$ is the solid angle, and $\sqrt{s}$ is the center of mass energy. Considering tree-level QED case as an example, the amplitude of the lepton-proton scattering using Feynman gauge is given by

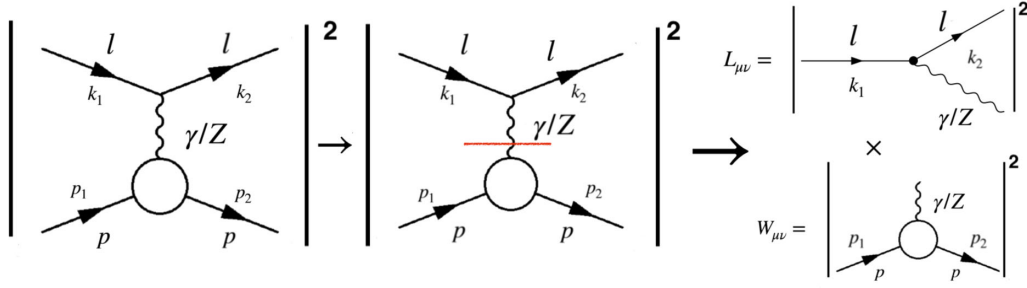


FIG. 1. Tree-level Feynman diagrams for lepton-proton ($l-p$) elastic scattering using a covariant approach. The cross sign represents the contraction between the leptonic $L_{\mu\nu}$ and hadronic $W^{\mu\nu}$ tensors.

$$i\mathcal{M} = [\bar{u}(k_2)(-ie\gamma^\mu)u^{s_1}(k_1)] \times [\bar{u}(p_2)(-ie\Gamma^\nu(q^2))u(p_1)] \left(\frac{-ig_{\mu\nu}}{q^2} \right). \quad (3)$$

Using the covariant approach, one can separate the leptonic and hadronic currents and write the amplitude squared in terms of the leptonic and hadronic tensors as

$$L^{\mu\nu} = [\bar{u}(k_2)(-ie\gamma^\mu)u^{s_1}(k_1)][\bar{u}^{s_1}(k_1)(ie\gamma^\nu)u(k_2)],$$

$$W_{\mu\nu} = [\bar{u}(p_2)(-ie\Gamma_\mu(q^2))u(p_1)][\bar{u}(p_1)(ie\Gamma_\nu(q^2))u(p_2)], \quad (4)$$

where $L^{\mu\nu}$ and $W_{\mu\nu}$ are the leptonic and hadronic currents, respectively. Here, $q^2 = (k_1 - k_2)^2 = (p_1 - p_2)^2$, with momenta for leptonic and hadronic currents are shown on Fig. 1. The amplitude squared $|\mathcal{M}|^2$ can be written as

$$|\mathcal{M}|^2 = \frac{L^{\mu\nu}W_{\mu\nu}}{q^4}. \quad (5)$$

The term $\Gamma_\nu(q^2)$ is the coupling between proton and photon. Momenta k_1 and k_2 corresponds to the momenta of incident and scattered leptons, whereas p_1 and p_2 are the momenta of the incident and scattered protons. The term s_1 refers to the helicity reference vector of the incident lepton. Using the relationship in the case of an incoming polarized lepton, i.e.,

$$u_\beta^{s_1}(k_i)\bar{u}_\beta^{s_1}(k_i) = \frac{1}{2}(1 + \beta\gamma_5\not{s}_1)(\not{k}_i + m), \quad (6)$$

and applying Casimir's trick, one can write

$$L^{\mu\nu} = 2\pi\alpha\text{Tr}[(\not{k}_2 + m_l)\gamma^\mu(1 + \beta\gamma_5\not{s}_1)(\not{k}_1 + m_l)\gamma^\nu],$$

$$W_{\mu\nu} = 4\pi\alpha\text{Tr}[(\not{p}_2 + m_p)\Gamma_\mu(q^2)(\not{p}_1 + m_p)\Gamma_\nu(q^2)], \quad (7)$$

where $\beta = \pm 1$. We follow the same process of covariant approach in the calculations of one-loop, reducible two-loop, and quadratic level lepton-proton scattering amplitude

squared. Owing to the complexity of the expressions, we do not present the full analytical results here; the details of the tree-level tensors $L^{\mu\nu}$ and $W_{\mu\nu}$ are given in Appendix A.

In this way, one gets two parts of the scattering diagram, which are calculated separately and contracted in the end to get the total scattering cross sections. The formula for calculating the tree-level parity-violating asymmetry is given in Eq. (1). Since $\sigma^+ \propto |\mathcal{M}^+|^2$ and $\sigma^- \propto |\mathcal{M}^-|^2$, one can rewrite A_{PV} in terms of amplitude squared as

$$A_{\text{PV}} = \frac{|\mathcal{M}_Z^+|^2 - |\mathcal{M}_Z^-|^2 + 2\Re(\mathcal{M}_\gamma^+\mathcal{M}_Z^{+\dagger}) - 2\Re(\mathcal{M}_\gamma^-\mathcal{M}_Z^{-\dagger})}{|\mathcal{M}_{\gamma,Z}^+|^2 + |\mathcal{M}_{\gamma,Z}^-|^2}, \quad (8)$$

where $\mathcal{M}_{\gamma,Z}$ are the amplitudes in the case of electromagnetic and weak neutral interactions, respectively. The term $\mathcal{M}_\gamma\mathcal{M}_Z^\dagger$ represents the cross term between weak and electromagnetic interactions. The denominator of Eq. (8) corresponds to an unpolarized amplitude by absolute value squared, i.e., $|\mathcal{M}_{\gamma,Z}^{+,-}|^2 = |\mathcal{M}_\gamma^{+,-}|^2 + |\mathcal{M}_Z^{+,-}|^2 + 2\Re(\mathcal{M}_\gamma^{(+,-)}\mathcal{M}_Z^{(+,-)\dagger})$. Owing to the parity-conserving nature of electromagnetic interactions, $|\mathcal{M}_\gamma^+|^2 = |\mathcal{M}_\gamma^-|^2$, the numerator of Eq. (8) contains only terms that violate parity.

In the Standard Model, considering the case of lepton (l) scattering on a proton (p) target, one can write

$$l(k_1, s_1) + p(p_1) \rightarrow l(k_2) + p(p_2), \quad (9)$$

where k_1 and k_2 are the momenta of the incident and scattered lepton, whereas p_1 and p_2 are the momenta of the incident and recoiled proton. Considering $l-p$ scattering, the tree-level electromagnetic and weak neutral amplitudes are given as

$$\mathcal{M}_\gamma = [\bar{u}(k_2)(-ie\gamma_\mu)u^{s(1)}(k_1)] \left(\frac{-i}{q^2} \right) \times [\bar{u}(p_2)(-ie\Gamma_\mu^\mu(q^2))u(p_1)],$$

$$\mathcal{M}_Z = [\bar{u}(k_2)(-ie(a_V\gamma_\mu + a_{AV}\gamma_\mu\gamma_5))u^{s(1)}(k_1)] \times \left(\frac{-i}{q^2 - m_Z^2} \right) [\bar{u}(p_2)(-ie\Gamma_{Zp}^\mu(q^2))u(p_1)], \quad (10)$$

where the term s_1 refers to the helicity reference vector of the incident lepton. The coupling of the proton with the photon as a function of momentum transfer squared q^2 is written as

$$\Gamma_{\gamma p}^\mu(q^2) = \gamma^\mu F_{1p}^\gamma(q^2) + \frac{i}{2m_p} \sigma^{\mu\nu} q_\nu F_{2p}^\gamma(q^2), \quad (11)$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. The terms $F_{1p}^\gamma(q^2)$ and $F_{2p}^\gamma(q^2)$ are the Dirac and Pauli form factors which depend on the momentum transfer squared $q^2 = -Q^2$. We can also write them in terms of Sachs electric (G_E) and Sachs magnetic (G_M) form factors as

$$\begin{aligned} F_{1p}^\gamma(q^2) &= \frac{\tau G_M(q^2) + G_E(q^2)}{1 + \tau}, \\ F_{2p}^\gamma(q^2) &= \frac{G_M(q^2) - G_E(q^2)}{1 + \tau}, \end{aligned} \quad (12)$$

where $\tau = -\frac{q^2}{4m^2}$. The Sachs form factors are the fourier transform of the electric charge density and magnetization distributions in the nonrelativistic limit.

This paper presents the formulation of the electroweak leptonic tensor up to NNLO, explicitly retaining the electron mass. At this stage, we adopt a simple dipole form factor parametrization based on fitting procedures commonly used in the Rosenbluth separation method [31] and the polarization transfer technique [32]. This approach allows us to obtain analytical results by contracting the NNLO (quadratic and two-loop reducible) leptonic tensor with a simplified model of the hadronic tensor.

We have tested the validity of our predictions by comparing them with the Q_{weak} measured asymmetry and find good agreement [Fig. (19)], although box diagrams and hard photon bremsstrahlung processes are not yet included. In the next phase, we plan to refine this analysis by incorporating more sophisticated models of hadronic couplings to vector bosons. This will include updated parametrizations of nucleon form factors, accounting for deviations from the dipole form reported in recent studies [33–35]. Such parametrizations have already been successfully applied in studies of various scattering processes, including neutrino-nucleon charged-current differential cross sections [36].

The vector (a_V) and axial-vector (a_{AV}) coupling strengths in Eq. (10) are defined as

$$a_V = \frac{I_3 - 2s_W^2 Q_f}{2s_W c_W}. \quad (13)$$

$$a_{AV} = \frac{I_3}{2s_W c_W}. \quad (14)$$

Here $s_W \equiv \sin \theta_W$ and $c_W \equiv \cos \theta_W$, with θ_W being the Weinberg mixing angle. The term Q_f in Eq. (13) is the

electric charge, which in the case of e/μ is -1 , while $I_3 = -\frac{1}{2}$ is the lepton's weak isospin.

The coupling of the proton with the Z -boson is given as

$$\Gamma_{Zp}^\mu(q^2) = \gamma^\mu F_{1p}^Z(q^2) + \frac{i}{2m_p} \sigma^{\mu\nu} q_\nu F_{2p}^Z(q^2) + \gamma^\mu \gamma_5 G_A^Z(q^2), \quad (15)$$

where $F_{(1,2)p}^Z(q^2)$ [37] are the form factors of the proton neutral weak current and m_p is the mass of the proton. $G_A^Z(q^2)$ stands for the isovector axial form factor which is given by

$$G_A^Z(q^2) = -\tau G_A(q^2) + \Delta s,$$

where $\tau = +1(-1)$ for protons (neutrons) and Δs stands for the strange-quark contribution which we ignored in this work. The isovector axial form factor (G_A) is normalized at $q^2 = 0 \text{ GeV}^2$ using information from neutron β -decay. Although neutron β -decay is a charged-current process, it involves the same type of axial-vector form factor as neutral weak interactions. Under the assumption of perfect isospin symmetry, results from charged β -decay can therefore be used to fix (normalize) the axial-vector coupling in neutral weak processes. This implies that any strong interaction effects that modify the charged β -decay form factors will also modify the corresponding part of the Z -boson coupling in a consistent and predictable way [38]. In this work, we used $G_A(0) = +1.2670 \pm 0.0035$ [37], whereas the recent Particle Data Group (PDG) value is $G_A(0) = +1.2753 \pm 0.0013$ [39]. The difference leads to only a subpercentage-level deviation (about 0.0003%) in the asymmetry.

The weak form factors are related to the proton and neutron electromagnetic form factors by the expression

$$F_{(1,2)p}^Z(q^2) = \frac{1}{4s_W c_W} ((1 - 4s_W^2) F_{(1,2)p}^\gamma(q^2) - F_{(1,2)n}^\gamma(q^2)). \quad (16)$$

A. Electroweak hadronic tensor

In this subsection we calculate a tree-level and one-loop level electroweak hadronic tensor for an unpolarized proton target, which is necessary to include in our tree and NLO calculations.

1. Tree-level electroweak hadronic tensor

The tree-level electroweak hadronic tensor in the case of unpolarized incoming and outgoing protons consists of two diagrams with γ and Z -boson propagators as shown in Fig. 2. The Higgs boson (H) interaction is suppressed by its induced propagator and is ignored.

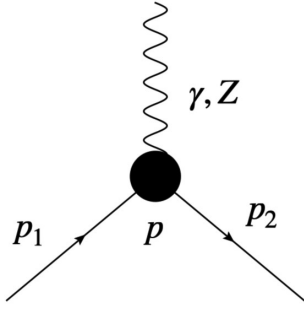


FIG. 2. Tree-level electroweak hadronic diagrams with off shell γ and Z -bosons.

Using the couplings given in Eqs. (11) and (15), we can calculate the Feynman amplitudes of the diagrams shown in Fig. 2. The tree-level unpolarized hadronic tensor is calculated as

$$W_0^{\mu\nu, VV} = H_1^{VV} g^{\mu\nu} + H_2^{VV} p_2^\mu p_1^\nu + H_3^{VV} p_1^\mu p_2^\nu + H_4^{VV} p_1^\mu p_1^\nu + H_5^{VV} p_2^\mu p_2^\nu + H_6^{VV} \epsilon^{\mu, \nu, p_1, p_2}. \quad (17)$$

Here, $\epsilon^{\mu, \nu, p_1, p_2} = \epsilon^{\mu, \nu, \alpha\beta} p_{1,\alpha} p_{2,\beta}$ corresponds to the contraction of Levi-Civita tensor with incoming and outgoing hadron momenta. In Eq. (17), VV represents $\gamma\gamma$, ZZ , or γZ mixing. The terms $W_0^{\mu\nu, \gamma\gamma}$, $W_0^{\mu\nu, ZZ}$, and $W_0^{\mu\nu, \gamma Z}$ are obtained from the products of photon couples to proton $\times$ photon couples to proton, Z -boson couples to proton $\times$ Z -boson couples to proton, and photon couples to proton $\times$ Z -boson couples to proton, respectively. The terms H_{1-6}^{VV} represent the hadronic structure functions corresponding to the

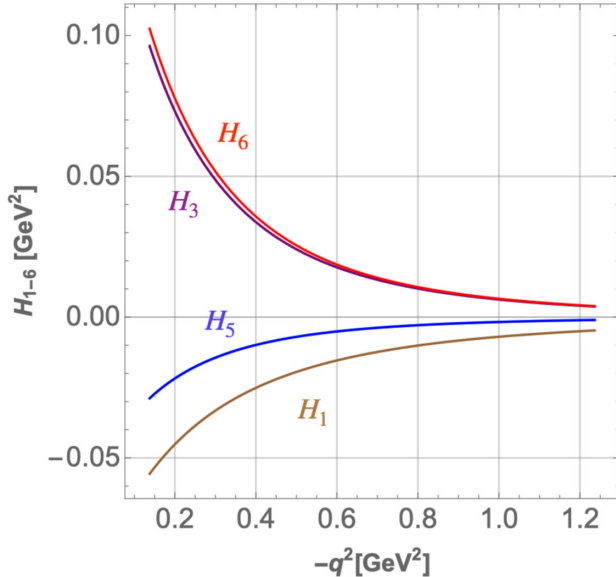


FIG. 3. Tree level electroweak hadronic tensor structure functions plotted versus momentum transfer squared ($-q^2$). Here the graphs with colors brown, purple, blue, and red are for H_1 , H_3 , H_5 , and H_6 , respectively. The structure functions $H_3 = H_2$ and $H_4 = H_5$. The graphs are plotted at $E_{\text{CMS}} = 20$ GeV.

Lorentz structures $g^{\mu\nu}$, $p_2^\mu p_1^\nu$, $p_1^\mu p_2^\nu$, $p_1^\mu p_1^\nu$, $p_2^\mu p_2^\nu$, and $\epsilon^{\mu, \nu, p_1, p_2}$, respectively. These structure functions are written in terms of QED and weak proton form factors which are functions of momentum transfer. These structure functions are shown in Fig. 3. Here we apply the same approach as in lepton structure functions to keep units consistent in terms of GeV^2 . The analytical details of these structure functions are given in Appendix B.

2. One-loop level electroweak hadronic tensor

As discussed in Sec. III A 4, in order to calculate the reducible two-loop level graphs, we split one of the self-energy to the lower (hadronic) part of the diagram as shown in Fig. 13. After the splitting of reducible two-loop graph into one-loop self-energy, we calculate the electroweak hadronic self-energy part in the same way as the electroweak leptonic self-energy, except now we consider the unpolarized protons. The vertex correction to the hadronic current is suppressed by the dipole structure of the form factors and ignored here. We replace the $\gamma - e$ and $Z - e$ couplings with $\gamma - p$ and $Z - p$ ones as given in Eq. (15).

Following the syntax of the FeynArts package [25] in our code, we write the tree level and NLO matrix elements of the hadronic tensor with an incoming and outgoing proton having couplings given in Eqs. (11) and (15). After that we use the FormCalc package [26] to calculate amplitude squared which automatically implements tensor decomposition and tensor reduction techniques and expresses the final result in the Passarino-Veltman basis [40].

The structure of a one-loop level unpolarized hadronic tensor obtained in this way is given by

$$W_{\text{NLO}}^{\mu\nu, VV} = h_1^{VV} g^{\mu\nu} + h_2^{VV} p_2^\mu p_1^\nu + h_3^{VV} p_1^\mu p_2^\nu + h_4^{VV} p_1^\mu p_1^\nu + h_5^{VV} p_2^\mu p_2^\nu + h_6^{VV} \epsilon^{\mu, \nu, p_1, p_2}. \quad (18)$$

There are in total six one-loop level unpolarized hadronic tensor structure functions for each photon (γ), Z or $\gamma - Z$ interference case. In Eq. (18), h_{1-6}^{VV} are the hadronic structure functions at the one-loop level. These structure functions depend on the momentum transfer squared $q^2 = (p_2 - p_1)^2$ between the incoming and outgoing protons and are given in terms of the Passarino-Veltman integral functions. We plot these structure functions as shown in Fig. 4. The units of all these structure functions are made consistent by using a scaling parameter $\delta^2 = 1 \text{ GeV}^2$.

The relation between tree and one-loop level hadronic tensor structure functions is given in terms of the truncated transverse self-energy ($\hat{\Sigma}_{VV}^T$):

$$\hat{\Pi}_{\mu\nu}^{VV} = i \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \hat{\Sigma}_{VV}^T + \frac{k_\mu k_\nu}{k^2} \hat{\Sigma}_{VV}^L, \quad (19)$$

$$h_i = \frac{H_i}{q^2} (\hat{\Sigma}_{\gamma\gamma}^T + 2\hat{\Sigma}_{\gamma Z}^T + \hat{\Sigma}_{ZZ}^T),$$

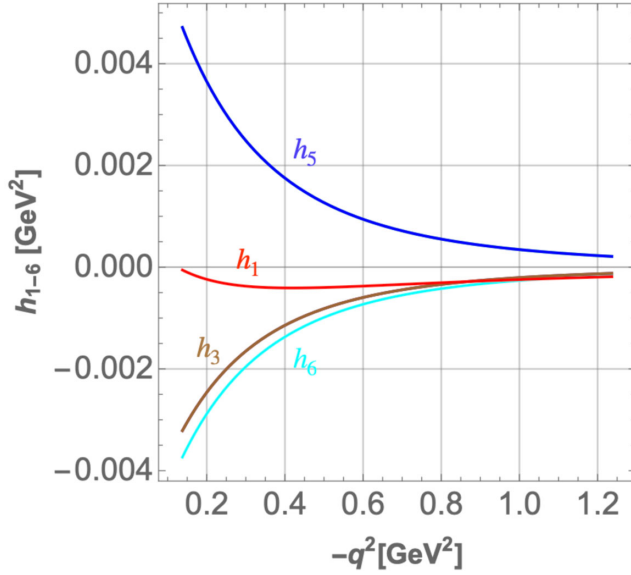


FIG. 4. NLO level electroweak hadronic tensor structure functions plotted versus momentum transfer squared ($-q^2$). Here the graphs with colors red, purple, brown, green, blue, and cyan are for h_1 , h_2 , h_3 , h_4 , h_5 , and h_6 , respectively. The structure functions $h_4 = h_5$ and $h_2 = h_3$. The graphs are plotted at $E_{\text{CMS}} = 20$ GeV.

where $\hat{\Pi}_{\mu\nu}^{VV}$ is the $V - V$ mixing tensor and the terms $\hat{\Sigma}_{VV}^T$ and $\hat{\Sigma}_{VV}^L$ are the transverse and longitudinal parts of the truncated self-energy. The longitudinal part does not contribute to the cross section.

Once the electroweak (tree and NLO) hadronic tensors are obtained, we contract them with the electroweak leptonic tensors as calculated in Sec. III. This contraction gives the amplitude squared $|\mathcal{M}|^2$ which is used to calculate the parity-violating asymmetry given by Eq. (8). Numerical results are obtained in the end after contraction using the LoopTools package [26].

III. LEADING ORDER (LO) ELECTROWEAK LEPTONIC TENSOR

The LO (tree-level) electroweak polarized lepton (e, μ) scattering with an unpolarized proton target involves two leptonic tensor diagrams as shown in Fig. 5, having photon (γ) and Z -boson propagators. Higgs boson interaction is suppressed by its induced propagator and coupling proportional to the mass of lepton, so we ignore it.

After adding and taking the amplitude squared of the two diagrams with γ and Z -boson propagators as shown in Fig. 5, one gets the tree-level electroweak leptonic tensor as follows:

$$\begin{aligned} L_{\mu\nu}^0 = & 4\pi\alpha[l_1 g_{\mu\nu} + l_2 k_{2\mu} k_{1\nu} + l_3 k_{1\mu} k_{2\nu} \\ & + l_4 \epsilon_{s_1, \mu, \nu, k_1} + l_5 \epsilon_{s_1, \mu, \nu, k_2} + l_6 \epsilon_{\mu, \nu, k_1, k_2} \\ & + l_7 k_{2\mu} s_{1\nu} + l_8 k_{2\nu} s_{1\mu}]. \end{aligned} \quad (20)$$

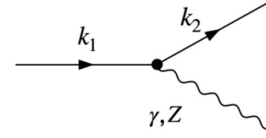


FIG. 5. Tree-level electroweak leptonic diagram in the case of incoming and outgoing leptons (l) with off shell γ , and Z bosons.

FIG. 6. The amplitude squared in case of tree level electroweak $l - p$ scattering.

The term $\epsilon_{\alpha, \mu, \nu, \beta}$ is the Levi-Civita tensor. One can write the tensor, e.g., $\epsilon_{s_1, \mu, \nu, k_2}$ in Eq. (20) in terms of the Levi-Civita tensor as $s_1^\alpha k_2^\beta \epsilon_{\alpha, \mu, \nu, \beta}$. Here s_1 is the helicity reference vector of the incoming lepton given by

$$s_1 = \frac{1}{m_l}(p, 0, 0, E_1).$$

The 3-momentum (p) and energy (E_1) of the incoming lepton are given by

$$\begin{aligned} p &= \sqrt{\frac{E_{\text{CMS}}^2 + m_p^2 - m_l^2}{4E_{\text{CM}}^2 - m_p^2}}, \\ E_1 &= \sqrt{p^2 + m_l^2}, \end{aligned}$$

where m_l and m_p are the masses of the lepton and proton, respectively. The term E_{CMS} is the center-of-mass energy, and the relation we used in the case of polarized leptons is given in Eq. (6). The leptonic structure functions are represented by the terms $l_1 - l_8$. In the case of QED, there are only five tensor structures expressed by $g_{\mu\nu}$, $k_{2\mu} k_{1\nu}$, $k_{1\mu} k_{2\nu}$, $\epsilon_{s_1, \mu, \nu, k_1}$, and $\epsilon_{s_1, \mu, \nu, k_2}$. The analytical details for LO electroweak leptonic tensor structure functions are given in Appendix A.

Finally, the amplitude squared in the case of tree-level electroweak lepton-proton scattering is obtained by contracting the upper leptonic tensor part with the lower hadronic tensor part of the diagram, as shown in Fig. 6.

A. Higher order corrections: NLO and NNLO level electroweak leptonic tensor

The higher-order radiative corrections in a scattering process involve the contribution of virtual particles in the form of self-energies, vertex corrections, real photon emissions (bremsstrahlung process), and two-boson exchange (boxes). In this paper, we use the covariant approach to calculate the NLO and NNLO level-boson self-energy and vertex correction graphs, as well as the bremsstrahlung

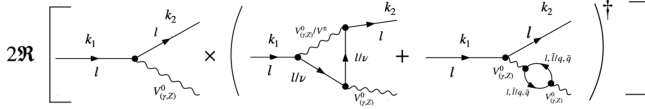


FIG. 7. One-loop level electroweak leptonic tensor diagram with an incoming and outgoing lepton. The symbols V^0 and $V^\pm$ are used for γ , Z -boson, and $W^\pm$ vector boson propagators, respectively.

process. Since box diagrams [29,30] form a gauge-invariant set, and their calculation with a leptonic tensor approach requires a different treatment, this topic will be discussed in a separate paper.

In this work we use the 't Hooft-Feynman gauge in the case of NLO and NNLO corrections which simplifies our calculations considerably. The electroweak vertices and self-energies are gauge dependent; separately, however the contributions coming from both are gauge independent. Since we did not consider the vertex correction on hadronic current (due to the small contributions), we assume a small degree of violation of the gauge invariance arising in the longitudinal part of VV mixing graphs. However, in the case of fermion-fermion scattering this part goes to zero, and hence only the transverse part contributes in this process.

The boson self-energy graphs do not contain IR divergences, and we use an on shell renormalization scheme for the treatment of ultraviolet (UV) divergences. The IR divergence in the vertex correction graphs is treated by using a small parameter for the photon mass. We shall see in Sec. IV how to get IR finite results analytically by adding bremsstrahlung contributions.

1. NLO level electroweak leptonic tensor

The electroweak corrected cross section at the NLO (one-loop) is of the order of α^3 as compared to the LO in the

perturbation expansion. It is obtained by multiplying the tree-level diagrams with the sum of the vertex correction and self-energy graphs at the one-loop level, as shown in Fig. 7. This contribution to the amplitude squared can be expressed as

$$2\Re[\mathcal{M}_{LO} \times \mathcal{M}_{NLO}^\dagger], \quad (21)$$

where

$$\mathcal{M}_{NLO} = \hat{\mathcal{M}}_{SE} + \hat{\mathcal{M}}_{TR}. \quad (22)$$

In Eq. (22), the terms $\hat{\mathcal{M}}_{(SE,TR)}$ correspond to the renormalized photon and Z -boson amplitudes in the on shell renormalization scheme [41,42] whereas the subscripts SE and TR represent the self-energy and vertex correction terms, respectively. The vertex correction graphs are infrared divergent that arises due to the massless photon in the vertex loop integral. We will treat this divergence in Sec. IV.

Using Eq. (21), we calculate the one-loop electroweak leptonic tensor given by

$$\begin{aligned} L_{\mu\nu}^{NLO} = & r_1 g_{\mu\nu} + r_2 k_{2\mu} k_{1\nu} + r_3 k_{1\mu} k_{2\nu} + r_4 \epsilon_{s_1, \mu, \nu, k_1} \\ & + r_5 \epsilon_{s_1, \mu, \nu, k_2} + r_6 \epsilon_{\mu, \nu, k_1, k_2} + r_7 k_{2\mu} s_{1\nu} + r_8 k_{2\nu} s_{1\mu} \\ & + r_9 k_{2\mu} k_{2\nu} + r_{10} \epsilon_{s_1, \mu, k_1, k_2} k_{2\nu} + r_{11} \epsilon_{s_1, \nu, k_1, k_2} k_{2\mu} \\ & + r_{12} k_{1\nu} k_{1\mu} + r_{13} \epsilon_{\mu, \nu, k_2, k_1} + r_{14} s_{1\nu} k_{1\mu} + r_{15} s_{1\mu} k_{1\nu} \\ & + r_{16} \epsilon_{s_1, \mu, k_1, k_2} k_{1\nu} + r_{17} \epsilon_{s_1, \mu, k_2, k_1} k_{2\nu} \\ & + r_{18} \epsilon_{s_1, \nu, k_2, k_1} k_{1\mu} + r_{19} \epsilon_{s_1, \nu, k_2, k_1} k_{2\mu}. \end{aligned} \quad (23)$$

There are in total 19 leptonic tensor structure functions obtained by the products of photon couples to lepton $\times$ photon couples to lepton, Z -boson couples to lepton $\times$ Z -boson couples to lepton, and photon couples to lepton $\times$ Z -boson couples to lepton, respectively.

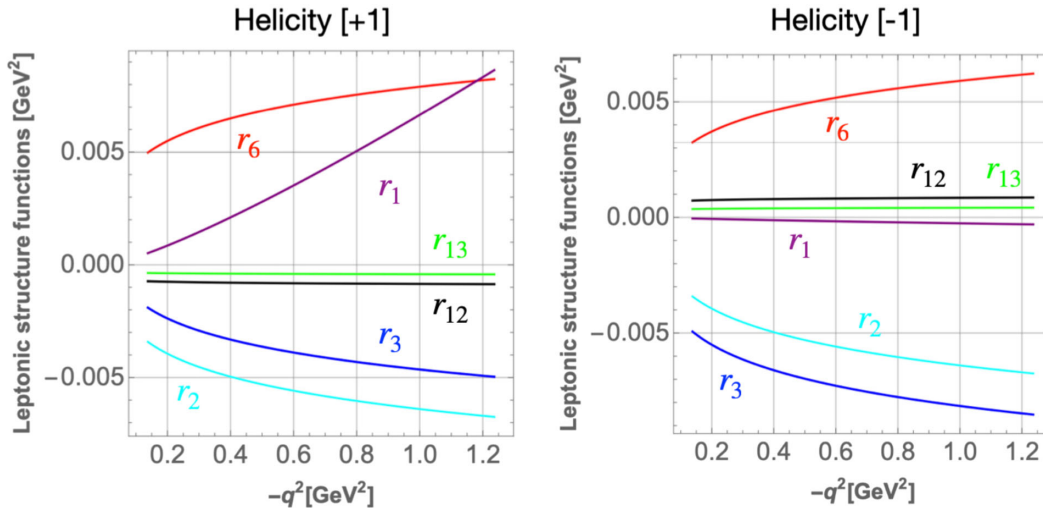


FIG. 8. NLO level leptonic tensor structure functions plotted versus $-q^2$ with an incoming polarized lepton having helicity ± 1 . Here the graphs with colors purple, cyan, blue, red, black, and green are for r_1 , r_2 , r_3 , r_6 , r_{12} , and r_{13} , respectively. The graphs are plotted at $E_{CMS} = 20$ GeV.

In Eq. (23), r_{1-19} are the one-loop level leptonic structure functions. These structure functions depend on momentum transfer squared ($-q^2 = Q^2$) and are written in terms of the Passarino-Veltman functions. We use the `LoopTools` package to calculate these functions numerically. We plot these structure functions as a function of $-q^2$ as shown in Fig. 8. By applying the sensitivity study, it was noticed that the significant contribution to parity-violating asymmetry comes from only six structure functions ($r_1, r_2, r_3, r_6, r_{12}$, and r_{13}) which are plotted in Fig. 8. However, the A_{PV} is calculated by considering all of the above mentioned 19 structure functions. The graphs are plotted at center-of-mass energy $E_{\text{CMS}} = 20$ GeV. The structure function r_1 has the units of GeV^2 . Hence, in order to make units consistent for all the structure functions, we multiply and divide r_2, r_3, r_6, r_{12} , and r_{13} by a scaling parameter $\delta^2 = 1 \text{ GeV}^2$.

There are in total 307 self-energy and vertex correction graphs in the case of electroweak leptonic tensor at the one-loop containing all SM particles inside the loop. A snapshot of such graphs generated in `FeynArts` is shown in Fig. 9.

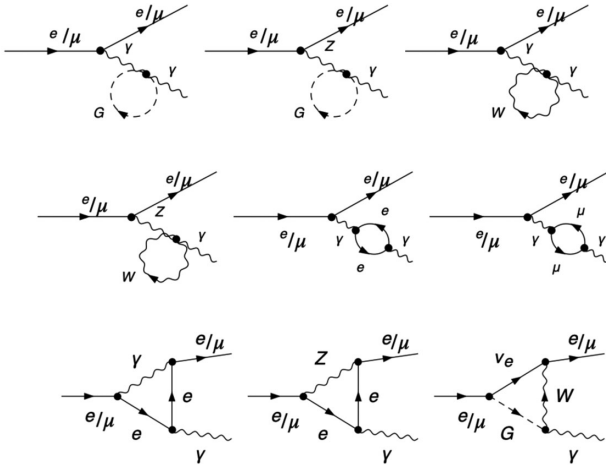


FIG. 9. Examples of self-energy and vertex correction diagrams in the case of electroweak leptonic tensor at one-loop level. In total there are 307 graphs.

2. NNLO electroweak leptonic tensor

The squared amplitude related to the calculations of NNLO electroweak leptonic tensor can be written as

$$|\mathcal{M}_{\text{NNLO}}|^2 = |\mathcal{M}_{1l}|^2 + 2\Re \mathcal{M}_{\text{LO}}(\mathcal{M}_{2lr}^\dagger + \mathcal{M}_{2li}^\dagger), \quad (24)$$

where $\mathcal{M}_{1l}$, $\mathcal{M}_{2lr}$, and $\mathcal{M}_{2li}$ are the one-loop and two-loop reducible as well as two-loop irreducible amplitudes, respectively. At NNLO, the electroweak corrections to the cross section are of the order of α^4 . The quadratic electroweak leptonic tensor is obtained by squaring the sum of self-energy and vertex correction graphs at the one-loop level, shown in Fig. 10.

3. Quadratic NNLO level electroweak leptonic tensor

The electroweak quadratic leptonic tensor ($L_{\mu\nu}^{\text{QD}}$) obtained in this way has the following form:

$$\begin{aligned} L_{\mu\nu}^{\text{QD}} = & n_1 g_{\mu\nu} + n_2 k_{2\mu} k_{1\nu} + n_3 k_{1\mu} k_{2\nu} + n_4 \epsilon_{s_1, \mu, \nu, k_1} \\ & + n_5 \epsilon_{s_1, \mu, \nu, k_2} + n_6 \epsilon_{\mu, \nu, k_1, k_2} + n_7 k_{2\mu} s_{1\nu} + n_8 k_{2\nu} s_{1\mu} \\ & + n_9 k_{2\mu} k_{2\nu} + n_{10} \epsilon_{s_1, \mu, k_1, k_2} k_{2, \nu} + n_{11} \epsilon_{s_1, \nu, k_1, k_2} k_{2\mu} \\ & + n_{12} k_{1\nu} k_{1\mu} + n_{13} \epsilon_{\mu, \nu, k_2, k_1} + n_{14} s_{1\nu} k_{1\mu} + n_{15} s_{1\mu} k_{1\nu} \\ & + n_{16} \epsilon_{s_1, \mu, k_1, k_2} k_{1\nu} + n_{17} \epsilon_{s_1, \mu, k_2, k_1} k_{2\nu} \\ & + n_{18} \epsilon_{s_1, \nu, k_2, k_1} k_{1\mu} + n_{19} \epsilon_{s_1, \nu, k_2, k_1} k_{2\mu} \\ & + n_{20} \epsilon_{s_1, \mu, k_2, k_1} k_{1\nu} + n_{21} \epsilon_{s_1, \nu, k_1, k_2} k_{1\mu}, \end{aligned} \quad (25)$$

where n_{1-21} are the quadratic leptonic structure functions and are calculated using the `FormCalc` and `LoopTools` packages. Among these structure functions, only eight ($n_1, n_2, n_3, n_6, n_{12}, n_{13}, n_{14}, n_{15}$) have a significant contribution to A_{PV} which are plotted in Fig. 11 as a function of momentum transfer squared. All these structure functions have different units. In order to keep units consistent (GeV^2), we multiply n_{14}, n_{15} by a scaling parameter $\delta = 1 \text{ GeV}$ and n_2, n_3, n_6, n_{12} , and n_{13} by $\delta^2 = 1 \text{ GeV}^2$.

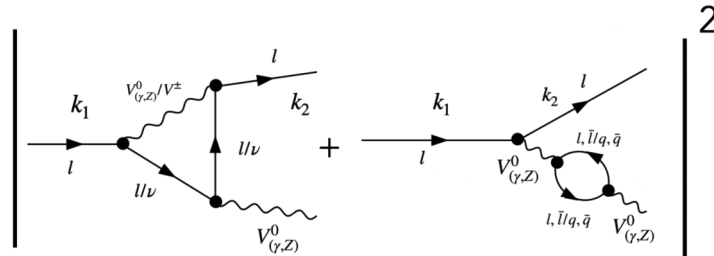


FIG. 10. Quadratic level electroweak graphs obtained by squaring the sum of one-loop level self-energy and vertex correction. Here $V_{(\gamma, Z)}^0$ could be either photon or Z boson, and $V^\pm$ is for $W^\pm$ boson.

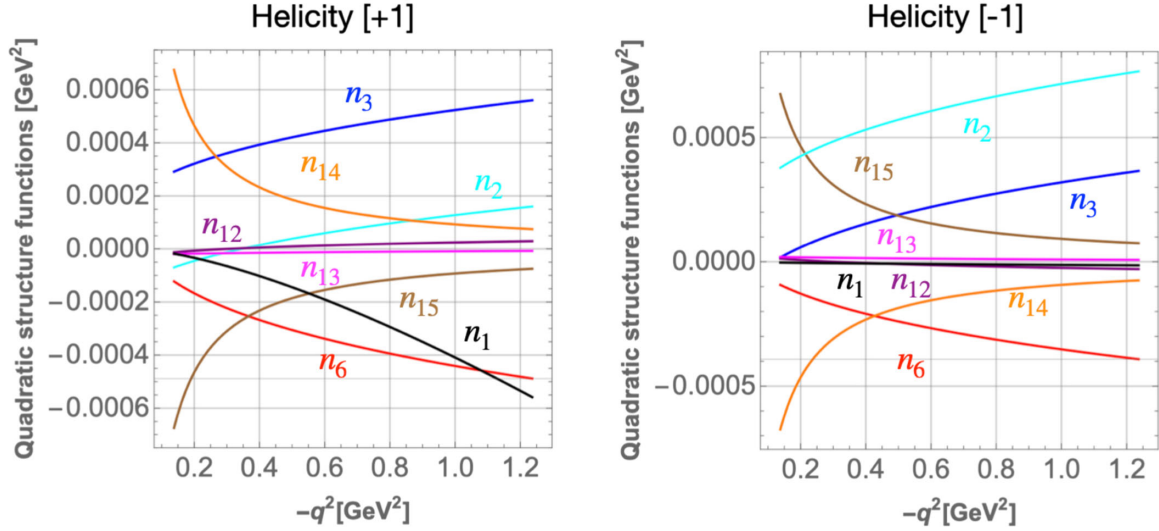


FIG. 11. Quadratic level leptonic tensor structure functions plotted versus $-q^2$ with helicities ± 1 . Here the graphs with colors black, cyan, blue, red, purple, magenta, orange, and brown are for $n_1, n_2, n_3, n_6, n_{12}, n_{13}, n_{14}$, and n_{15} , respectively. The graphs are plotted at $E_{\text{CMS}} = 20$ GeV.

The LO leptonic tensor given in Eq. (20) is calculated using Feynman rules built-in *FeynArts* package [25]. The NLO and NNLO (quadratic) leptonic tensors given in Eqs. (23) and (25) are calculated using the tensor decomposition and tensor reduction techniques which are implemented in our code using the *FormCalc* package [26] followed by G. Passarino and M. Veltman's paper [40]. These loop level decompositions are complete and free from ultraviolet and infrared singularities by using an on shell renormalization scheme and soft photon bremsstrahlung processes.

4. Reducible two-loop level electroweak leptonic tensor

Another way of obtaining the electroweak radiative corrections which are of the order of α^4 is by multiplying the tree-level diagram with the reducible two-loop graphs. Such graphs include a double self-energy and a vertex correction attached with self-energy diagrams, as shown in Fig. 12.

The leptonic tensor obtained in this case has the same form as given in Eq. (23), having a total of 19 lepton structure functions for each γ, Z , and $\gamma - Z$ interaction case. There are two ways of calculating such reducible two-loop level diagrams in *Mathematica*; one by using an effective

propagator approach and the other by considering one of the loops at the upper part (leptonic side) and the other one at the lower part (hadronic side) of the diagram. The final squared amplitude is then obtained by contracting both the upper and lower parts of the diagram.

In this work, we use the second approach to calculate the squared amplitude of the reducible two-loop level diagram by splitting the two loops at each end of the leptonic-hadronic sides, as shown in Fig. 13. The one-loop level self-energy and vertex correction parts can then be calculated using the *FeynArts*, *FormCalc*, and *LoopTools* packages.

IV. SOFT PHOTON BREMSSTRAHLUNG (SPB)

The IR divergence appears in the vertex correction graphs and fermion self-energies due to a lower integration limit of the photon loop momentum $k_\gamma \rightarrow 0$ GeV. This IR divergence is regularized in our calculations by giving a small mass (λ) to the photon. Photon mass parameter λ is unphysical and exactly cancels out with the lower integration limit of the soft photon bremsstrahlung contribution. In this section we are using the covariant approach previously

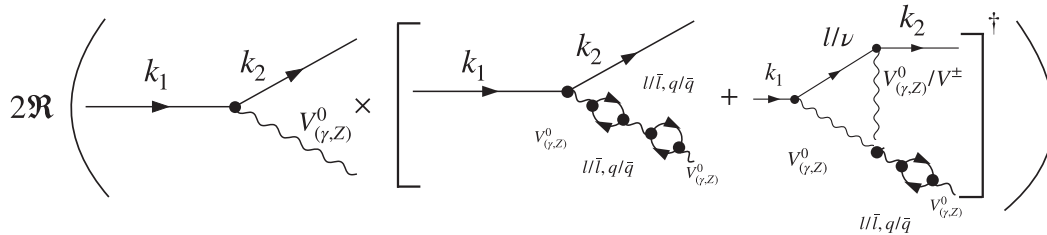


FIG. 12. Example of contributions coming in reducible two-loop level electroweak leptonic tensor.

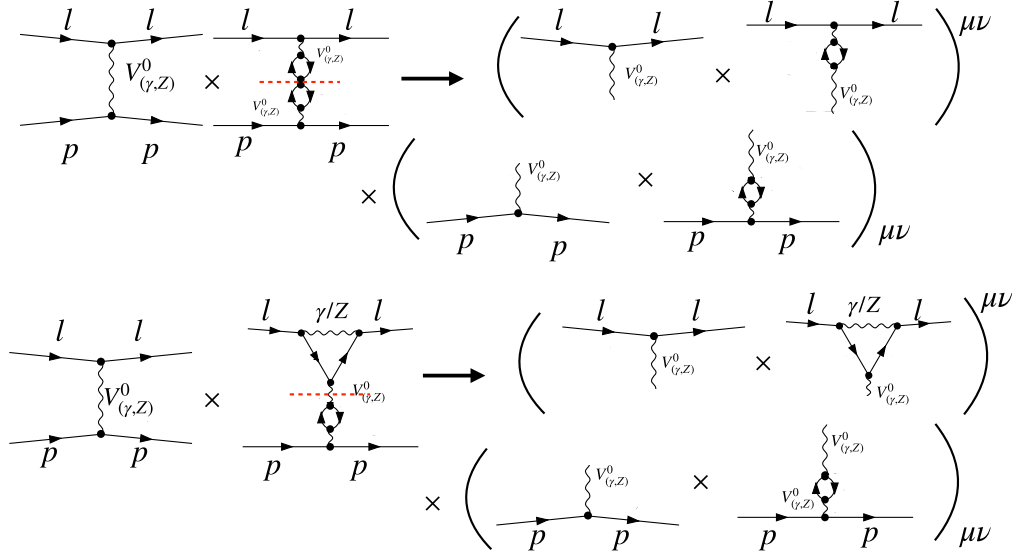


FIG. 13. Splitting of the reducible two-loop level diagram into a single loop one at each end of the leptonic and hadronic sides.

introduced by Bardin and Shumeiko [23] for the separation of the IR divergent part of the lowest order bremsstrahlung cross section. Bardin and Shumeiko used it for the calculation of LO QED bremsstrahlung processes, whereas we are using it to calculate up to NNLO electroweak soft photon bremsstrahlung processes.

A. Infrared divergence treatment at NLO level

We calculate the electroweak bremsstrahlung contribution as shown in Fig. [14]. The amplitude for the initial state bremsstrahlung $(\mathcal{M}_{in,SPB})^\beta$ can be written as

$$\begin{aligned}
 (\mathcal{M}_{in,SPB}^\gamma)^\beta &= \left[\bar{u}(k_2)(ie\gamma^\beta) \right. \\
 &\quad \times \left(\frac{m_l + \not{k}_1 - \not{k}_\gamma}{(k_1 - k_\gamma)^2 - m_l^2} \right) (ie\gamma^\alpha) u(k_1) \left. \right] \epsilon_\alpha^*(k_\gamma), \\
 (\mathcal{M}_{in,SPB}^Z)^\beta &= \left[\bar{u}(k_2)(ie(a_v\gamma^\beta + a_p\gamma^\beta\gamma_5)) \right. \\
 &\quad \times \left(\frac{m_l + \not{k}_1 - \not{k}_\gamma}{(k_1 - k_\gamma)^2 - m_l^2} \right) (ie\gamma^\alpha) u(k_1) \left. \right] \epsilon_\alpha^*(k_\gamma).
 \end{aligned} \tag{26}$$

The 4-momentum of the radiated soft photon is given by k_γ and polarization vector of the emitted soft photon is represented by $\epsilon_\alpha^*(k_\gamma)$.

Since the energy of the real photon emitted in the SPB process is so small, we can ignore it in the numerator of Eq. (26). Similarly, the denominator of Eq. (26) can also be simplified as $-2(k_1 \cdot k_\gamma)$, since $k_1^2 = m_l^2$ and $k_\gamma^2 = 0$ GeV² for the real photon. Using these simplifications, Eq. (26) can be modified as

$$\begin{aligned}
 (\mathcal{M}_{in,SPB}^\gamma)^\beta &= [\bar{u}(k_2)ie\gamma^\beta u(k_1)] \left(\frac{ie(k_1 \cdot \epsilon^*(k_\gamma))}{-(k_1 \cdot k_\gamma)} \right), \\
 (\mathcal{M}_{in,SPB}^Z)^\beta &= [\bar{u}(k_2)ie(a_v\gamma^\beta + a_p\gamma^\beta\gamma_5)u(k_1)] \\
 &\quad \times \left(\frac{ie(k_1 \cdot \epsilon^*(k_\gamma))}{-(k_1 \cdot k_\gamma)} \right).
 \end{aligned} \tag{27}$$

In a similar way, the amplitude for the final state bremsstrahlung $(\mathcal{M}_{fin,SPB})^\beta$ can be written as

$$\begin{aligned}
 (\mathcal{M}_{fin,SPB}^\gamma)^\beta &= [\bar{u}(k_2)ie\gamma^\beta u(k_1)] \left(\frac{ie(k_2 \cdot \epsilon^*(k_\gamma))}{(k_2 \cdot k_\gamma)} \right), \\
 (\mathcal{M}_{fin,SPB}^Z)^\beta &= [\bar{u}(k_2)ie(a_v\gamma^\beta + a_p\gamma^\beta\gamma_5)u(k_1)] \\
 &\quad \times \left(\frac{ie(k_2 \cdot \epsilon^*(k_\gamma))}{(k_2 \cdot k_\gamma)} \right).
 \end{aligned} \tag{28}$$

After taking the amplitude squared and applying the polarization sum, we get for $L_{\alpha\beta}^{SPB}$

$$L_{\alpha\beta}^{SPB} = L_{\alpha\beta}^0 \delta_{SPB}, \tag{29}$$

where $L_{\alpha\beta}^{SPB}$ is the SPB leptonic tensor and δ_{SPB} is the SPB factor. The term $L_{\alpha\beta}^0$ is the tree-level electroweak leptonic tensor.

Integrating on the emitted soft photon phase space, we can write the soft photon factor:

$$\begin{aligned}
 \delta_{SPB} &= -\frac{4\pi\alpha}{2(2\pi)^3} \sum_{i,j=1}^2 \int_{|\vec{k}_\gamma| < \Delta E} \frac{d^3\vec{k}_\gamma}{\omega} \frac{(k_i \cdot k_j)}{(k_i \cdot k_\gamma)(k_j \cdot k_\gamma)} \\
 &= -\frac{\alpha}{4\pi^2} \sum_{i,j=1}^2 (k_i \cdot k_j) I(k_i, k_j).
 \end{aligned} \tag{30}$$

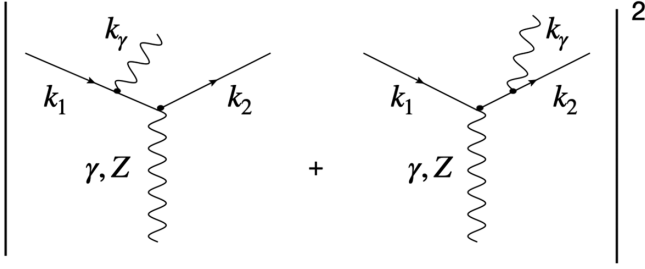


FIG. 14. One-loop level electroweak bremsstrahlung process with off shell γ and Z bosons.

In Eq. (30), ω is the energy of the emitted photon, whereas ΔE is the cut on the soft photon's energy, which is usually defined by the detector threshold. This SPB contribution in Eq. (30) cancels the unphysical photon mass λ , as confirmed numerically by the λ independence shown in Fig. 15. As different detectors have different threshold energies, we choose different ΔE values for each detector. This ΔE value is arbitrary, and we choose it to be 5% of the center-of-mass energy during a collision, i.e., $\Delta E = 0.05 E_{\text{CMS}}$ GeV. This energy cut is treated as a breakdown between the soft and hard photon bremsstrahlung (HPB) processes. Above this energy threshold, photons become hard, and the HPB process needs to be accounted for. In this scenario the scattering is treated as a $2 \rightarrow 3$ process [43,44] with an emitted hard photon. We are considering the calculations of the HPB process, and our plan is to include it in our next project.

The soft photon integral $I(k_i, k_j)$ has already been calculated in [45], and details are given in Appendix C.

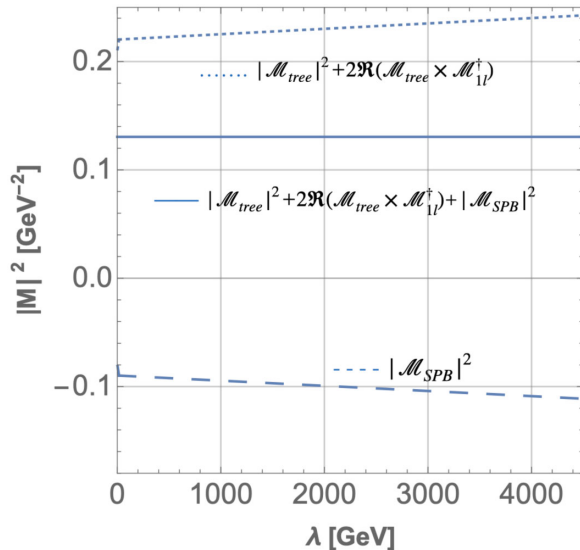


FIG. 15. Test for IR-divergence cancellation using MUSE ($\mu - p$) kinematics at $\theta_{\text{lab}} = 35^\circ$ and beam energy $E_{\text{lab}} = 0.235$ GeV. Dashed line is for the SPB amplitude squared, dotted line is the up to NLO level corrections, and solid line is the sum of the two mentioned contributions.

Using the example of muon-proton scattering, we demonstrated the cancellation of IR regularization parameter λ by adding one-loop corrected and one photon emission $|\mathcal{M}_{\text{SPB}}|^2$. The result is shown in Fig. 15. We applied the same SPB technique to our NNLO results (quadratic and two-loop reducible contributions).

B. Infrared divergence treatment at NNLO

The IR divergence at the reducible two-loop level can be removed analytically by adding a product of tree-level and one-loop level graphs that both have initial- and final-state soft-photon emission as shown in Fig. 16. In this way, one gets an IR finite two-loop reducible interference term:

$$\begin{aligned} & 2\Re(\mathcal{M}_{\text{tree}} \times \mathcal{M}_{2lr}^\dagger)_{\text{IR-fin.}} \\ &= 2\Re(\mathcal{M}_{\text{tree}} \times \mathcal{M}_{2lr}^\dagger)_{\text{IR-div.}} \\ &+ \left(\frac{1}{2}\right) \times 2\Re(\mathcal{M}_{\text{tree}} \times \mathcal{M}_{\text{SE}}^\dagger)\delta_{\text{SP}}, \end{aligned} \quad (31)$$

where $\mathcal{M}_{\text{tree}}$ and $\mathcal{M}_{\text{SE}}$ are the electroweak tree-level and one-loop self-energy amplitudes respectively. Here a factor of $\frac{1}{2}$ appears due to the fact that one half portion of $2\Re(\mathcal{M}_{\text{tree}} \times \mathcal{M}_{\text{SE}}^\dagger)\delta_{\text{SP}}$ goes to the IR-divergence treatment of reducible two-loop graphs, whereas the other half goes into the treatment of quadratic graphs.

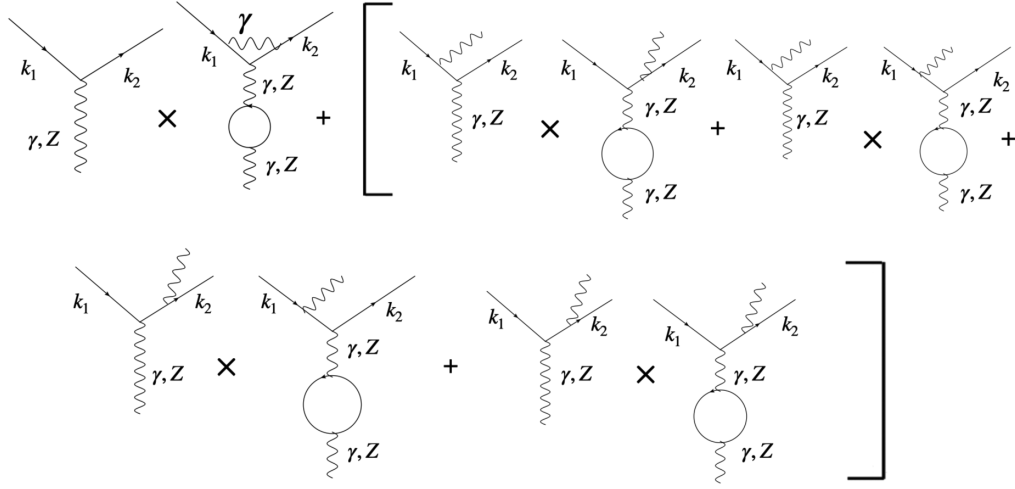
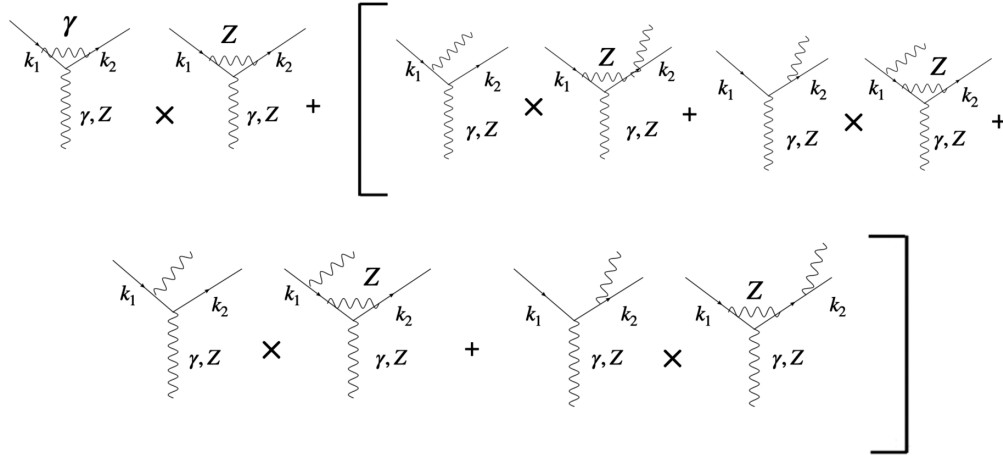
The NNLO quadratic electroweak lepton scattering involves a product of vertex and self-energy as well as vertex-squared diagrams that contain IR divergence. The IR divergence due to the product of the vertex and self-energy graph is removed by adding the product of tree and vertex correction graphs that both have initial- and final-state soft-photon emission as shown in Fig. 17. Solving these SPB graphs one gets a one-loop interference term multiplied by a soft-photon factor δ_{SP} , i.e., $2\Re(\mathcal{M}_{\text{tree}} \times \mathcal{M}_{\text{vertex}}^\dagger)\delta_{\text{SP}}$.

However, in order to remove photon vertex-squared IR divergence, one needs to include six two-photon emission diagrams with initial- and final-state radiated photons along with $\Re(\mathcal{M}_{\text{tree}} \times \mathcal{M}_{\gamma\text{-vertex}}^\dagger)$ SPB diagrams, as shown in Fig. 18.

Hence, the final quadratic level IR finite squared amplitude can be written as

$$\begin{aligned} |\mathcal{M}_{1l}|_{\text{IR-fin.}}^2 &= |\mathcal{M}_{1l}|_{\text{IR-div.}}^2 + \frac{1}{2} \\ &\times 2\Re(\mathcal{M}_{\text{tree}} \times \mathcal{M}_{\text{vertex}}^\dagger + \mathcal{M}_{\text{tree}} \times \mathcal{M}_{\text{SE}}^\dagger)\delta_{\text{SP}} \\ &+ 2\Re(\mathcal{M}_{\text{tree}} \times \mathcal{M}_{\gamma\text{-vertex}}^\dagger)\delta_{\text{SP}} + |\mathcal{M}_{\text{tree}}|^2\delta_{\text{SP}}^2, \end{aligned} \quad (32)$$

where δ_{SP}^2 is the soft-photon bremsstrahlung factor squared that we already calculated and $\mathcal{M}_{\text{tree}}$, $\mathcal{M}_{\text{SE}}$, and $\mathcal{M}_{\text{vertex}}$ are the electroweak tree-level, one-loop self-energy, and vertex correction amplitudes, respectively.

FIG. 16. Treatment of IR divergence at reducible two-loop level by including $\Re(\mathcal{M}_{\text{tree}}\mathcal{M}_{\text{SE}}^\dagger)$ SPB diagrams.FIG. 17. Treatment of IR divergence due to the product of vertex and self-energy graphs at quadratic level by including $\Re(\mathcal{M}_{\text{tree}} \times \mathcal{M}_{Z\text{-vertex}}^\dagger)$ SPB diagrams.

V. NUMERICAL ANALYSIS

We have calculated the tree-level, NLO, and NNLO (tree + one – loop + quadratic + reducible two-loop) electroweak parity-violating asymmetry (A_{PV}) for the lepton-proton elastic scattering by using the covariant approach. The Feynman diagrams are generated using the *FeynArts Mathematica* package. The leptonic and hadronic tensors are calculated using *FormCalc*, which are then contracted to obtain the squared amplitudes with the *FeynCalc* package.

We compared our NLO and NNLO (A_{PV}) results with the measured Q_{weak} experimental value. Our results are in good agreement; however, box diagrams and hard photon bremsstrahlung contributions to the cross section still need to be included. One can expect these box contributions to change the asymmetry by only a few percentages. The most recent progress in the formulation and evaluation of electroweak boxes can be found in [46–48]. The denominator of the

parity-violating asymmetry given in Eq. (1) corresponds to the unpolarized cross section, which is dominated by QED corrections. The QED contributions up to NLO have been calculated previously in [49,50]. In the present work, we extend this analysis by incorporating electroweak corrections, applied consistently to both the numerator and denominator of the asymmetry.

The NLO and NNLO corrected amplitude squared ($|\mathcal{M}|^2$) is given below:

$$\begin{aligned} |\mathcal{M}_{\text{NLO}}^{0+1}|^2 &= |\mathcal{M}_{\text{tree}}|^2 + 2\Re(\mathcal{M}_{\text{tree}} \times \mathcal{M}_{1l}^\dagger), \\ |\mathcal{M}_{\text{NNLO}}^{0+1+2}|^2 &= |\mathcal{M}_{\text{NLO}}^{0+1}|^2 + |\mathcal{M}_{1l}|^2 + 2\Re(\mathcal{M}_{\text{tree}} \times \mathcal{M}_{2lr}^\dagger), \end{aligned} \quad (33)$$

where in Eq. (33), the superscripts 0,1, and 2 are used for up to the LO, NLO, and NNLO contributions in amplitude

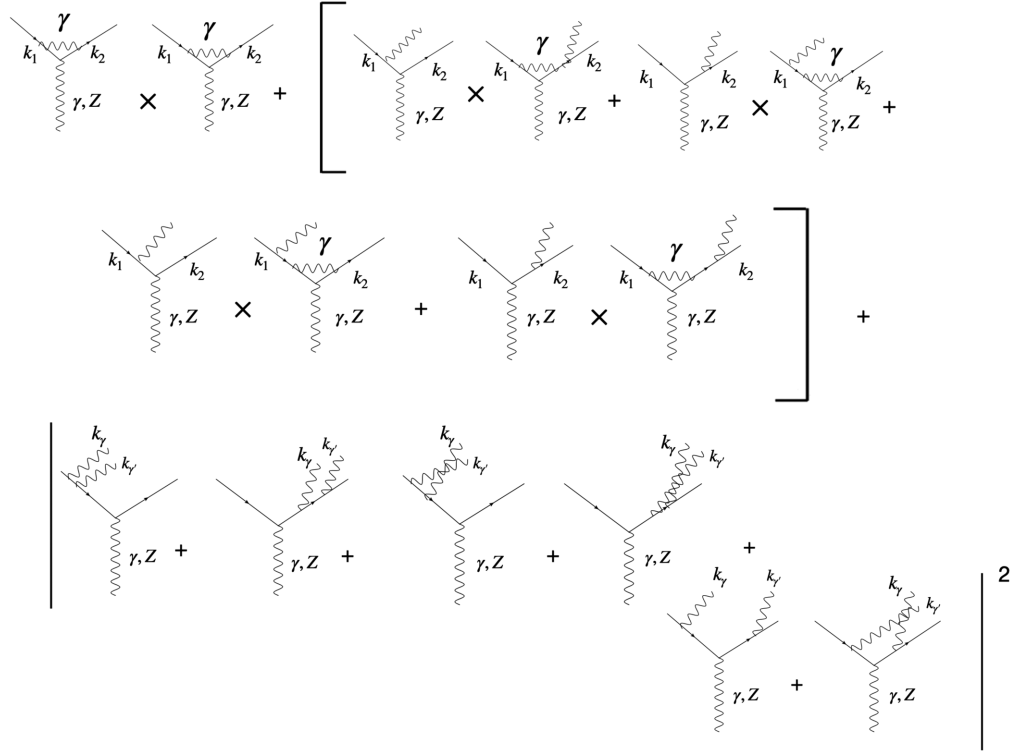


FIG. 18. SPB treatment for IR divergence in quadratic NNLO photon vertex-squared contribution. This includes the sum of NLO contribution arising from initial and final state photon emission and two photon emission graphs.

squared. The term $|\mathcal{M}_{1l}|^2$ represents the one-loop level squared amplitude which is the quadratic contribution. The term $2\Re(\mathcal{M}_{\text{tree}} \times \mathcal{M}_{2lr}^\dagger)$ is for the reducible two-loop interference term.

Our numerical results for elastic lp scattering using the kinematics of the experimental programs of Q_{weak} , P2, MOLLER, EIC, and MUSE are shown in Figs. 19–26. These graphs show the tree-level A_{PV} along with the corrections in A_{PV} at the NLO and NNLO levels. The SPB results with the emission of one and two photons are also added in these graphs to make them IR finite. These A_{PV} graphs are plotted versus both the momentum transfer $-q^2$ and the scattering angle (θ_{lab}) in the lab reference frame. In our calculations, we take the soft photon energy cut as $\Delta E = 0.05\sqrt{s}\text{GeV}$ with s being the center-of-mass energy squared, i.e., $s = E_{\text{CMS}}^2$.

A. Q_{weak} kinematics

In case of the Q_{weak} experiment, Fig. 19 shows our numerical results with the beam energy $E_{\text{beam}} = 1.16\text{ GeV}$. At this E_{beam} , the Q_{weak} measured value of A_{PV} at $\theta_{\text{lab}} = 7.9^\circ$ is $-226.5 \pm 7.3(\text{statistical}) \pm 5.8(\text{systematic})$ ppb [3]. Using these kinematics, our calculated values of the corrected asymmetries at the tree, NLO, and NNLO are given in Table I. We also calculate the A_{PV} correction percentage at NLO ($\delta_{A_{\text{PV}}}^1$) and NNLO ($\delta_{A_{\text{PV}}}^2$) (quadratic and two-loop

reducible) as given by the formulas

$$\begin{aligned} \text{up to NLO correction } (\delta_{A_{\text{PV}}}^1 \%) &= \left(\frac{A_{\text{PV}}^0 - A_{\text{PV}}^{0+1}}{A_{\text{PV}}^0} \right) \times 100, \\ \text{up to NNLO correction } (\delta_{A_{\text{PV}}}^2 \%) &= \left(\frac{A_{\text{PV}}^0 - A_{\text{PV}}^{0+1+2}}{A_{\text{PV}}^0} \right) \times 100. \end{aligned} \quad (34)$$

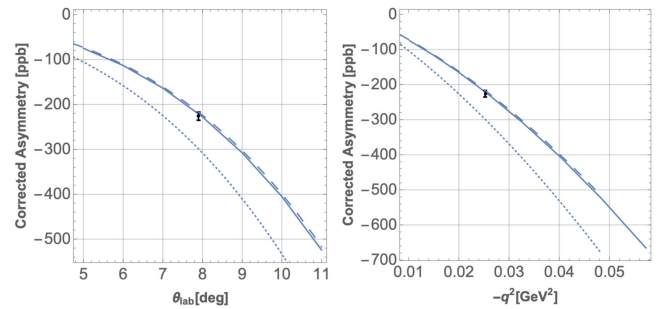


FIG. 19. Q_{weak} kinematics. Tree level (dotted line), NLO level (dashed line), and NNLO (quadratic+two-loop reducible) level (solid line) ep scattering correction asymmetry plotted versus θ_{lab} and $-q^2$. The Q_{weak} measured value at $\theta_{\text{lab}} = 7.9^\circ$ is shown by the point with error bars.

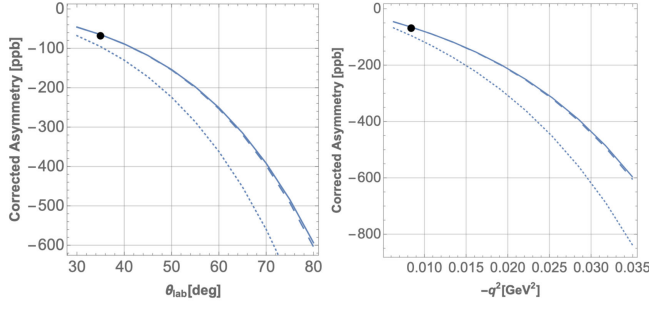


FIG. 20. P2 kinematics. Tree level (dotted line), NLO level (dashed line), and NNLO (quadratic+two-loop reducible) level (solid line) ep scattering correction asymmetry plotted versus θ_{lab} and $-q^2$. The P2 proposed value at $\theta_{lab} = 35^\circ$ is shown by the point.

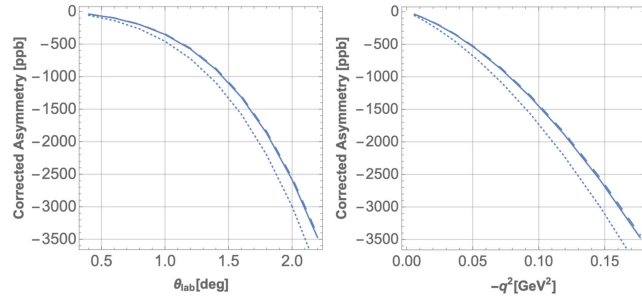


FIG. 21. MOLLER kinematics. Tree level (dotted line), NLO level (dashed line), and NNLO (quadratic+two-loop reducible) level (solid line) ep scattering correction asymmetry plotted versus θ_{lab} and $-q^2$. The NLO and NNLO lines are on top of each other in these plots. The difference between NLO and NNLO plots in terms of correction percentage is shown in Fig. 29.

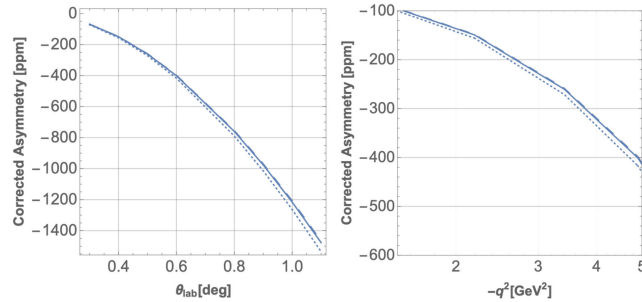


FIG. 22. EIC kinematics. Tree level (dotted line), NLO level (dashed line), and NNLO (quadratic+two-loop reducible) level (solid line) ep scattering correction asymmetry plotted versus θ_{lab} at $E_{CMS} = 20$ GeV and $-q^2$. The NLO and NNLO lines are on top of each other in these plots. The difference between NLO and NNLO plots in terms of correction percentage is shown in Fig. 30.

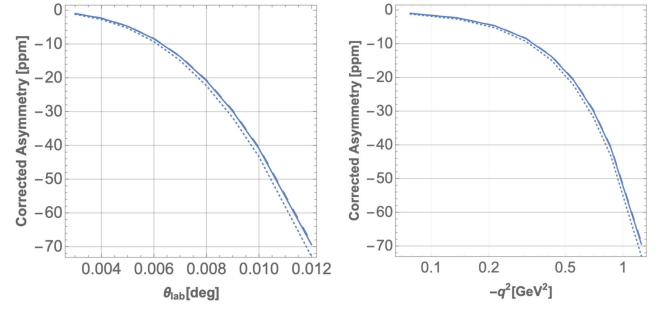


FIG. 23. EIC kinematics. Tree level (dotted line), NLO level (dashed line), and NNLO (quadratic + two loop reducible) level (solid line) ep scattering correction asymmetry plotted versus θ_{lab} at $E_{CM} = 100$ GeV and $-q^2$. The NLO and NNLO lines are on top of each other in these plots. The difference between NLO and NNLO plots in terms of correction percentage is shown in Fig. 31.

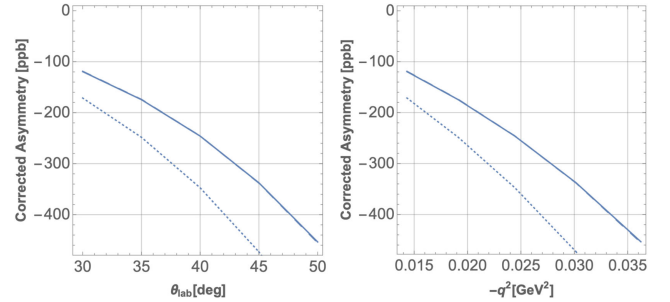


FIG. 24. MUSE (ep) scattering kinematics. Tree level (dotted line), NLO level (dashed line), and NNLO (quadratic+two-loop reducible) level (solid line) ep scattering correction asymmetry plotted versus θ_{lab} and $-q^2$ at $E_{lab} = 0.235$ GeV. The NLO and NNLO lines are on top of each other in these plots. The difference between NLO and NNLO plots in terms of correction percentage is shown in Fig. 33.

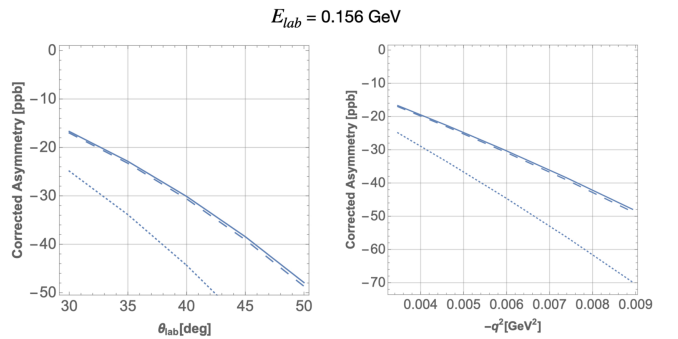


FIG. 25. MUSE (μp) scattering kinematics. Tree level (dotted line), NLO level (dashed line), and NNLO (quadratic+two-loop reducible) level (solid line) μp scattering correction asymmetry plotted versus θ_{lab} and $-q^2$ at $E_{lab} = 0.156$ GeV.

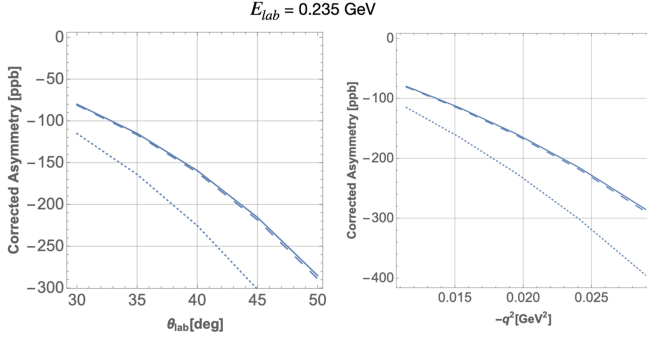


FIG. 26. MUSE (μp) scattering kinematics. Tree level (dotted line), NLO level (dashed line), and NNLO (quadratic+two-loop reducible) level (solid line) μp scattering correction asymmetry plotted versus θ_{lab} and $-q^2$ at $E_{\text{lab}} = 0.235$ GeV.

The NLO and NNLO A_{PV} correction percentages $\delta_{A_{\text{PV}}}^1\%$ and $\delta_{A_{\text{PV}}}^2\%$ for Q_{weak} kinematics at different θ_{lab} are given in Table II. These results are calculated in ppb.

The graphs for $\delta_{A_{\text{PV}}}^1\%$ and $\delta_{A_{\text{PV}}}^2\%$ plotted versus different $-q^2$ values in the case of Q_{weak} kinematics are shown in Fig. 27.

B. P2 kinematics

Using the experimental kinematics of the proposed P2 experiment, our corrected A_{PV} results at tree, NLO, and NNLO plotted versus $-q^2$ and θ_{lab} are shown in Fig. 20. This experiment will operate at a very low beam energy $E_{\text{beam}} = 155$ MeV, making it possible to measure the scattered particles at large scattering angles. The value of

TABLE I. Q_{weak} kinematics. Our calculated tree, one-loop, quadratic, reducible two-loop and total (quadratic + reducible two-loop) corrected A_{PV} at different scattering angles θ_{lab} . Here (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop corrected A_{PV} . These results are calculated in ppb.

$\theta_{\text{lab}}(\text{deg})$	Tree A_{PV} (ppb)	1-loop A_{PV} (ppb)	qud. A_{PV} (ppb)	2-loop red. A_{PV} (ppb)	Total A_{PV} (ppb)
5.0	-105.61	-73.06	-74.47	-73.61	-75.02
6.0	-158.15	-111.34	-112.48	-112.12	-113.27
7.0	-224.89	-160.99	-162.61	-162.04	-163.66
7.9	-298.90	-217.11	-219.89	-218.40	-221.19
8.0	-308.02	-224.08	-226.49	-225.41	-227.83
9.0	-409.93	-302.82	-306.06	-304.43	-307.68
10.0	-533.22	-399.59	-403.31	-401.49	-405.21
11.0	-680.63	-516.94	-521.81	-519.09	-523.97
12.0	-854.99	-657.45	-663.34	-659.82	-665.73
13.0	-1059.18	-823.79	-830.75	-826.33	-833.31
14.0	-1296.08	-1018.60	-1026.69	-1021.25	-1029.36
15.0	-1568.53	-1244.52	-1253.76	-1247.17	-1256.44

TABLE II. Q_{weak} kinematics. $\delta_{A_{\text{PV}}}^1\%$ and $\delta_{A_{\text{PV}}}^2\%$ at different scattering angles θ_{lab} . Here $\delta_{A_{\text{PV}}}^1\%$ and $\delta_{A_{\text{PV}}}^2\%$ are NLO and NNLO A_{PV} correction percentages as given by Eq. (34), whereas (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level correction%.

$\theta_{\text{lab}}(\text{deg})$	$\delta_{A_{\text{PV}}}^1$ (1 loop)%	$\delta_{A_{\text{PV}}}^2$ (qud.)%	$\delta_{A_{\text{PV}}}^2$ (2-loop red.)%	$\delta_{A_{\text{PV}}}^2$ (qud. + 2-loop red.)%
5.0	30.82	29.49	30.30	28.97
6.0	29.60	28.88	29.11	28.38
7.0	28.41	27.69	27.95	27.23
7.9	27.37	26.44	26.93	26.00
8.0	27.25	26.47	26.82	26.03
9.0	26.13	25.34	25.73	24.94
10.0	25.06	24.36	24.70	24.01
11.0	24.05	23.33	23.73	23.02
12.0	23.10	22.42	22.83	22.14
13.0	22.22	21.57	21.98	21.32
14.0	21.41	20.78	21.21	20.58
15.0	20.66	20.07	20.49	19.89

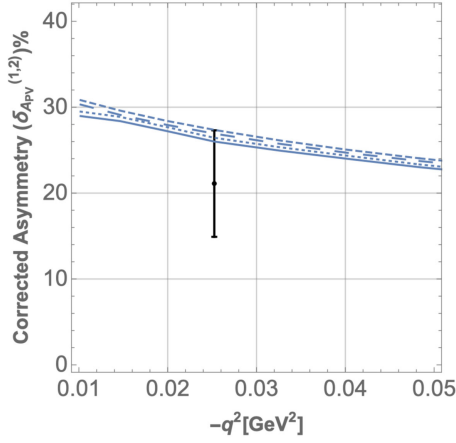


FIG. 27. Q_{weak} kinematics. Tree + NLO $\delta_{A_{PV}}^1$ % (dashed line), Tree + NLO + quadratic $\delta_{A_{PV}}^2$ % (dotted line), Tree+NLO+two-loop reducible $\delta_{A_{PV}}^2$ % (large-dashed line), and Tree+NLO+quadratic+two-loop reducible $\delta_{A_{PV}}^2$ % (solid line) plotted versus $-q^2$. The point with error bar represents the Q_{weak} measured corrected A_{PV} at $-q^2 = 0.025 \text{ GeV}^2$. Here $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % are NLO and NNLO A_{PV} correction percentages as given by Eq. (34).

A_{PV} proposed by the recent P2 Collaborations at $\theta_{\text{lab}} = 35^\circ$ is -67.34 ppb [6]. Our calculated values of A_{PV} at tree, NLO, and NNLO for this angle are given in Table III.

The NLO and NNLO A_{PV} correction percentages $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % for P2 kinematics are given in Table IV and shown graphically in Fig. 28.

C. MOLLER kinematics

The MOLLER experiment will measure the parity-violating asymmetry in the scattering of longitudinally polarized electrons off unpolarized electrons in a liquid hydrogen target. In this scenario ep scattering is a significant background process for which one needs to account. The asymmetry in ep elastic scattering is estimated to constitute

a background of the order 8% to the Møller measurement with a projected fractional systematic errors on the measurement of electron weak charge (Q_W^e) $\approx 0.3\%$ [46,51].

Our numerical results for this background (ep) scattering process using the MOLLER experimental kinematics at $E_{\text{lab}} = 11 \text{ GeV}$ are plotted in Fig. 21 versus different scattering angles θ_{lab} and momentum transfer squared ($-q^2$). The A_{PV} values at tree, one-loop, quadratic, two-loop reducible, and total (quadratic + two-loop reducible) with respect to $\theta_{\text{lab}} \sim (0.4^\circ - 2.2^\circ)$ pion [52] are giroposed by the MOLLER Collaboratven in Table V. These values are calculated in ppb.

The NLO and NNLO A_{PV} correction percentages $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % for MOLLER kinematics at different scattering angles (θ_{lab}) are given in Table VI and shown graphically in Fig. 29.

D. EIC kinematics

The EIC at the Brookhaven Laboratory aims to precisely measure the constituent quarks and gluons of the proton using a polarized electron-proton beam that will be steered into head-on collisions. This experiment is aimed to be conducted at the center-of-mass energies ranging from 20 GeV to 100 GeV and gradeable to $\sim 140 \text{ GeV}$ [8].

Using EIC kinematics and considering the center-of-mass energies 20 GeV and 100 GeV, we calculated the higher-order radiative corrections in A_{PV} up to NNLO (quadratic and reducible two-loop) via elastic polarized electron scattering with an unpolarized proton target. These higher-order calculations are a good background check in the case of the future EIC experiment and provide new constraints on the polarized elastic ep scattering. Some recent work related to left-right asymmetry at the EIC, including NLO electroweak radiative corrections can be found in [53].

The NLO and NNLO A_{PV} corrections, as well as correction percentages $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % for EIC kinematics

TABLE III. P2 kinematics: Our calculated tree, one-loop, quadratic, reducible two-loop and total (quadratic + reducible two-loop) corrected A_{PV} at different scattering angles θ_{lab} . Here (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level corrected A_{PV} . These results are calculated in ppb.

$\theta_{\text{lab}}(\text{deg})$	Tree A_{PV} (ppb)	1-loop A_{PV} (ppb)	qud.- A_{PV} (ppb)	2-loop red. A_{PV} (ppb)	Total A_{PV} (ppb)
30	-67.80	-45.98	-46.31	-45.70	-46.03
35	-95.61	-65.14	-65.56	-64.67	-65.09
40	-130.08	-89.04	-89.57	-88.30	-88.83
45	-172.40	-118.59	-119.23	-117.46	-118.10
50	-223.99	-154.91	-155.66	-153.25	-154.00
55	-286.52	-199.33	-200.19	-196.97	-197.82
60	-361.88	-253.39	-254.37	-250.12	-251.08
65	-452.19	-318.89	-319.95	-314.45	-315.49
70	-559.81	-397.75	-398.88	-391.86	-392.97
75	-687.19	-492.07	-493.26	-484.42	-485.58
80	-836.91	-604.07	-605.29	-594.31	-595.48

TABLE IV. P2 kinematics. $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % at different scattering angles θ_{lab} . Here $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % are NLO and NNLO A_{PV} correction percentages as given by Eq. (34), whereas (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level correction%.

$\theta_{lab}(deg)$	$\delta_{A_{PV}}^1$ (1 loop)%	$\delta_{A_{PV}}^2$ (qud.)%	$\delta_{A_{PV}}^2$ (2-loop red.)%	$\delta_{A_{PV}}^2$ (qud. + 2-loop red.)%
30	32.18	31.70	32.59	32.12
35	31.87	31.43	32.36	31.91
40	31.55	31.14	32.12	31.71
45	31.21	30.84	31.87	31.49
50	30.84	30.51	31.58	31.25
55	30.43	30.13	31.26	30.96
60	29.98	29.71	30.88	30.62
65	29.48	29.24	30.46	30.23
70	28.95	28.75	30.00	29.80
75	28.39	28.22	29.51	29.34
80	27.82	27.68	28.99	28.85

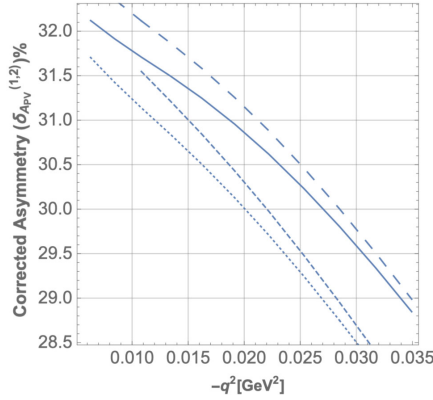


FIG. 28. P2 kinematics. Tree + NLO $\delta_{A_{PV}}^1$ % (dashed line), Tree + NLO + quadratic $\delta_{A_{PV}}^2$ % (dotted line), Tree + NLO + two-loop reducible $\delta_{A_{PV}}^2$ % (large-dashed line), and Tree + NLO + quadratic + two-loop reducible $\delta_{A_{PV}}^2$ % (solid line) plotted versus $-q^2$. Here $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % are NLO and NNLO A_{PV} correction percentages as given by Eq. (34).

at $E_{CMS} = 20$ GeV and 100 GeV with different scattering angles (θ_{lab}), are given in Tables VII–X and shown graphically in Figs. 30 and 31. These values are calculated in parts per million (ppm).

E. MUSE kinematics

MUSE is proposed to measure μp and ep scattering in the same experiment at the same time. This experiment has the potential to demonstrate whether the μp and ep interactions are consistent or different, and whether any difference results from beyond the standard model physics or two-photon exchange.

We used the proposed kinematics of the MUSE experiment with beam energy in lab reference frame $E_{lab} = 0.156$ GeV and 0.235 GeV and calculated up to NNLO electroweak A_{PV} corrections using elastic polarized μp and ep scattering. In the case of $E_{lab} = 0.156$ GeV, the MUSE ep scattering results are the same as given in Sec. VB, whereas at $E_{lab} = 0.235$ GeV, the A_{PV} results up to NNLO

TABLE V. MOLLER kinematics. Our calculated tree, one-loop, quadratic, reducible two-loop, and total (quadratic + reducible two-loop) corrected A_{PV} at different scattering angles θ_{lab} . Here (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level corrected A_{PV} . These results are calculated in ppb.

$\theta_{lab}(deg)$	Tree A_{PV} (ppb)	1-loop A_{PV} (ppb)	qud.- A_{PV} (ppb)	2-loop red. A_{PV} (ppb)	Total A_{PV} (ppb)
0.4	-58.10	-39.12	-39.85	-39.61	-40.34
0.6	-138.92	-97.26	-99.08	-98.42	-100.24
0.8	-267.03	-194.19	-197.38	-196.32	-199.51
1.0	-456.59	-344.12	-349.03	-347.55	-352.46
1.2	-724.82	-564.13	-571.08	-569.20	-576.16
1.4	-1091.22	-873.53	-882.84	-880.64	-889.97
1.6	-1576.93	-1293.15	-1305.16	-1302.72	-1314.76
1.8	-2203.99	-1844.67	-1859.70	-1857.16	-1872.23
2.0	-2994.61	-2549.88	-2568.26	-2565.79	-2584.25
2.2	-3970.56	-3430.12	-3452.20	-3450.04	-3472.22

TABLE VI. MOLLER kinematics. $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % at different scattering angles θ_{lab} . Here $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % are NLO and NNLO A_{PV} correction percentages as given by Eq. (34), whereas (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level correction %.

$\theta_{lab}(deg)$	$\delta_{A_{PV}}^1$ (1 loop)%	$\delta_{A_{PV}}^2$ (qud.)%	$\delta_{A_{PV}}^2$ (2-loop red.)%	$\delta_{A_{PV}}^2$ (qud. + 2-loop red.)%
0.4	32.67	31.41	31.82	30.56
0.6	29.99	28.68	29.15	27.85
0.8	27.28	26.08	26.48	25.28
1.0	24.63	23.56	23.88	22.81
1.2	22.17	21.21	21.47	20.51
1.4	19.95	19.09	19.29	18.44
1.6	17.99	17.23	17.39	16.63
1.8	16.30	15.62	15.74	15.05
2.0	14.85	14.24	14.32	13.70
2.2	13.61	13.06	13.11	12.55

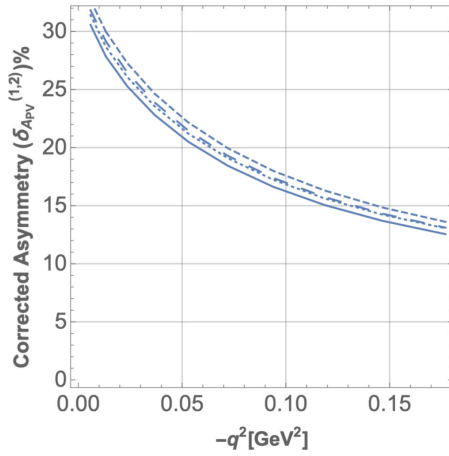


FIG. 29. MOLLER kinematics. Tree + NLO $\delta_{A_{PV}}^1$ % (dashed line), Tree + NLO + quadratic $\delta_{A_{PV}}^2$ % (dotted line), Tree+NLO +two-loop reducible $\delta_{A_{PV}}^2$ % (large-dashed line), and Tree+NLO +quadratic+two-loop reducible $\delta_{A_{PV}}^2$ % (solid line) plotted versus $-q^2$. Here $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % are NLO and NNLO A_{PV} correction percentages as given by Eq. (34).

are plotted in Fig. 24 versus different scattering angles θ_{lab} and momentum transfer $(-q^2)$. The MUSE μp scattering A_{PV} results with $E_{lab} = 0.156$ GeV and 0.235 GeV are plotted in Figs. 25 and 26 versus θ_{lab} and $(-q^2)$. The ep and μp scattering A_{PV} correction and correction % values with $E_{lab} = 0.156$ GeV and 0.235 GeV at tree, one-loop, quadratic, two-loop reducible, and total (quadratic + two-loop reducible) with respect to $\theta_{lab} \sim (30^0 - 50^0)$ proposed by the MUSE Collaboration [9,54] are given in Tables XI–XVI. Our results obtained using MUSE kinematics are relevant in case of longitudinally polarized lepton scattering with an unpolarized proton target. However, in the MUSE experiment the primary measurements are conducted with unpolarized lepton beams and an unpolarized liquid hydrogen proton target. The potential for using polarized beams and targets has been considered in theoretical studies related to the experiment’s kinematic region to separate electric and magnetic form factors or measure specific asymmetries. Muons naturally have some degree of longitudinal polarization when produced from pion decay, and theoretical calculations for MUSE kinematics take this into account.

TABLE VII. EIC kinematics at 20 GeV CMS energy. Our calculated tree, one-loop, quadratic, reducible two-loop, and total (quadratic + reducible two-loop) corrected A_{PV} at different scattering angles θ_{lab} . Here (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level corrected A_{PV} . These results are calculated in ppm.

$\theta_{lab}(deg)$	Tree A_{PV} (ppm)	1-loop A_{PV} (ppm)	qud.- A_{PV} (ppm)	2-loop red. A_{PV} (ppm)	Total A_{PV} (ppm)
0.3	−72.26	−68.23	−68.39	−68.56	−68.72
0.4	−155.92	−148.29	−148.57	−149.00	−149.29
0.5	−271.46	−258.96	−259.38	−260.22	−260.66
0.6	−416.71	−398.05	−398.62	−400.04	−400.66
0.8	−789.54	−754.79	−755.71	−758.77	−759.79
0.9	−1014.43	−969.80	−970.89	−975.03	−976.26
1.0	−1263.37	−1207.67	−1208.92	−1214.32	−1215.77
1.1	−1535.19	−1467.28	−1468.68	−1475.51	−1477.17

TABLE VIII. EIC kinematics for 20 GeV CMS energy. $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % at different scattering angles θ_{lab} . Here $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % are NLO and NNLO A_{PV} correction percentages as given by Eq. (34), whereas (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level correction %.

$\theta_{lab}(deg)$	$\delta_{A_{PV}}^1$ (1 loop)%	$\delta_{A_{PV}}^2$ (qud.)%	$\delta_{A_{PV}}^2$ (2-loop red.)%	$\delta_{A_{PV}}^2$ (qud. + 2-loop red.)%
0.3	5.57	5.35	5.12	4.89
0.4	4.89	4.71	4.44	4.25
0.5	4.61	4.45	4.14	3.98
0.6	4.48	4.34	3.99	3.85
0.8	4.40	4.28	3.89	3.77
0.9	4.39	4.29	3.88	3.76
1.0	4.41	4.31	3.88	3.77
1.1	4.42	4.33	3.89	3.78

TABLE IX. EIC kinematics at 100 GeV CMS energy. Our calculated tree, one-loop, quadratic, reducible two-loop, and total (quadratic + reducible two-loop) corrected A_{PV} at different scattering angles θ_{lab} . Here (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level corrected A_{PV} . These results are calculated in ppm.

$\theta_{lab}(deg)$	Tree A_{PV} (ppm)	1-loop A_{PV} (ppm)	qud. A_{PV} (ppm)	2-loop red. A_{PV} (ppm)	Total A_{PV} (ppm)
0.003	-1.20	-0.97	-0.98	-0.98	-0.99
0.004	-2.73	-2.32	-2.34	-2.34	-2.36
0.005	-5.32	-4.68	-4.71	-4.71	-4.74
0.006	-9.28	-8.36	-8.40	-8.41	-8.46
0.007	-14.89	-13.62	-13.68	-13.70	-13.76
0.008	-22.35	-20.67	-20.75	-20.79	-20.87
0.009	-31.82	-29.65	-29.76	-29.81	-29.92
0.010	-43.37	-40.66	-40.78	-40.87	-40.99
0.012	-72.93	-68.87	-69.05	-69.22	-69.40

TABLE X. EIC kinematics for 100 GeV CMS energy. $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % at different scattering angles θ_{lab} . Here $\delta_{A_{PV}}^1$ % and $\delta_{A_{PV}}^2$ % are NLO and NNLO A_{PV} correction percentages as given by Eq. (34), whereas (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level correction %.

$\theta_{lab}(deg)$	$\delta_{A_{PV}}^1$ (1 loop)%	$\delta_{A_{PV}}^2$ (qud.)%	$\delta_{A_{PV}}^2$ (2 loop red.)%	$\delta_{A_{PV}}^2$ (qud. + 2-loop red.)%
0.003	19.39	18.34	18.56	17.51
0.004	15.02	14.23	14.31	13.52
0.005	12.01	11.39	11.38	10.76
0.006	9.95	9.44	9.38	8.87
0.007	8.52	8.10	7.99	7.57
0.008	7.52	7.16	7.01	6.64
0.009	6.79	6.47	6.30	5.98
0.010	6.26	5.97	5.78	5.48
0.012	5.56	5.31	5.08	4.83

However, the primary experimental method in MUSE involves switching the beam charge polarity (positive and negative muons and electrons) rather than using a deliberately polarized beam to extract specific physics information related to two-photon exchange [55].

It has been shown that the effect of the lepton mass (m_l) is important and becomes significantly enhanced when the beam energy is of the order of m_l [56]. In our calculations, we keep the lepton mass (m_e , m_μ) explicitly. The graph showing the effects of m_l on the parity-violating asymmetry

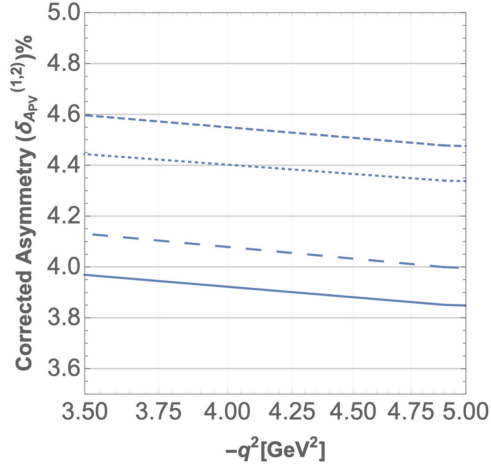


FIG. 30. EIC kinematics at $E_{\text{CMS}} = 20$ GeV. Tree + NLO $\delta_{A_{\text{PV}}}^1\%$ (dashed line), Tree + NLO + quadratic $\delta_{A_{\text{PV}}}^2\%$ (dotted line), Tree + NLO + two – loop reducible $\delta_{A_{\text{PV}}}^2\%$ (large-dashed line), and Tree+NLO+quadratic+two-loop reducible $\delta_{A_{\text{PV}}}^2\%$ (solid line) plotted versus $-q^2$. Here $\delta_{A_{\text{PV}}}^1\%$ and $\delta_{A_{\text{PV}}}^2\%$ are NLO and NNLO A_{PV} correction percentages as given by Eq. (34).

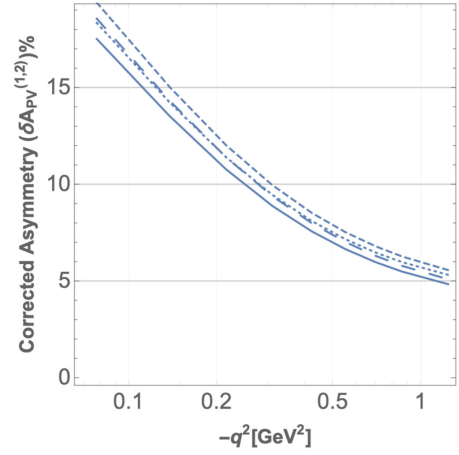


FIG. 31. EIC kinematics at $E_{\text{CMS}} = 100$ GeV. Tree + NLO $\delta_{A_{\text{PV}}}^1\%$ (dashed line), Tree + NLO + quadratic $\delta_{A_{\text{PV}}}^2\%$ (dotted line), Tree + NLO + two – loop reducible $\delta_{A_{\text{PV}}}^2\%$ (large-dashed line), and Tree + NLO + quadratic + two – loop reducible $\delta_{A_{\text{PV}}}^2\%$ (solid line) plotted versus $-q^2$. Here $\delta_{A_{\text{PV}}}^1\%$ and $\delta_{A_{\text{PV}}}^2\%$ are NLO and NNLO A_{PV} correction percentages as given by Eq. (34).

TABLE XI. MUSE kinematics for ep scattering at $E_{\text{lab}} = 0.235$ GeV. Our calculated tree, one-loop, quadratic, reducible two-loop, and total (quadratic + reducible two-loop) A_{PV} at different scattering angles θ_{lab} . Here (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level corrected A_{PV} . These results are calculated in ppb.

$\theta_{\text{lab}}(\text{deg})$	Tree A_{PV} (ppb)	1-loop A_{PV} (ppb)	qud.- A_{PV} (ppb)	2-loop red. A_{PV} (ppb)	Total A_{PV} (ppb)
30	-171.27	-118.89	-119.99	-118.36	-119.46
35	-248.25	-173.98	-175.45	-172.98	-174.44
40	-347.35	-245.81	-247.70	-244.06	-245.94
45	-473.05	-338.12	-340.46	-335.29	-337.62
50	-630.29	-455.11	-457.94	-450.77	-453.59

TABLE XII. MUSE (ep) scattering kinematics at $E_{\text{lab}} = 0.235$ GeV. $\delta_{A_{\text{PV}}}^1\%$ and $\delta_{A_{\text{PV}}}^2\%$ at different scattering angles θ_{lab} . Here $\delta_{A_{\text{PV}}}^1\%$ and $\delta_{A_{\text{PV}}}^2\%$ are NLO and NNLO A_{PV} correction percentages as given by Eq. (34), whereas (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level correction%.

$\theta_{\text{lab}}(\text{deg})$	$\delta_{A_{\text{PV}}}^1$ (1 loop)%	$\delta_{A_{\text{PV}}}^2$ (qud.)%	$\delta_{A_{\text{PV}}}^2$ (2-loop red.)%	$\delta_{A_{\text{PV}}}^2$ (qud. + 2-loop red.)%
30	30.59	29.94	30.89	30.25
35	29.92	29.32	30.32	29.73
40	29.23	28.69	29.74	29.19
45	28.52	28.03	29.12	28.63
50	27.79	27.35	28.48	28.04

TABLE XIII. MUSE kinematics for μp scattering at $E_{\text{lab}} = 0.156$ GeV. Our calculated tree, one-loop, quadratic, reducible two-loop, and total (quadratic + reducible two-loop) corrected A_{PV} at different scattering angles θ_{lab} . Here (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level corrected A_{PV} . These results are calculated in ppb.

$\theta_{\text{lab}}(\text{deg})$	Tree A_{PV} (ppb)	1-loop A_{PV} (ppb)	qud.- A_{PV} (ppb)	2-loop red. A_{PV} (ppb)	Total A_{PV} (ppb)
30	-24.89	-17.02	-16.88	-16.84	-16.69
35	-33.89	-23.29	-23.12	-23.04	-22.88
40	-44.34	-30.63	-30.43	-30.32	-30.12
45	-56.29	-39.07	-38.83	-38.68	-38.44
50	-69.82	-48.68	-48.38	-48.21	-47.91

TABLE XIV. MUSE kinematics for $E_{\text{lab}} = 0.156$ GeV. $\delta_{A_{\text{PV}}}^1$ % and $\delta_{A_{\text{PV}}}^2$ % at different scattering angles θ_{lab} . Here $\delta_{A_{\text{PV}}}^1$ % and $\delta_{A_{\text{PV}}}^2$ % are NLO and NNLO A_{PV} correction percentages as given by Eq. (34), whereas (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level correction %.

$\theta_{\text{lab}}(\text{deg})$	$\delta_{A_{\text{PV}}}^1$ (1 loop)%	$\delta_{A_{\text{PV}}}^2$ (qud.)%	$\delta_{A_{\text{PV}}}^2$ (2-loop red.)%	$\delta_{A_{\text{PV}}}^2$ (qud. + 2-loop red.)%
30	31.62	32.12	32.33	32.89
35	31.29	31.78	32.01	32.49
40	30.93	31.38	31.64	32.08
45	30.59	31.03	31.29	31.72
50	30.28	30.71	30.95	31.37

TABLE XV. MUSE kinematics for μp scattering at $E_{\text{lab}} = 0.235$ GeV. Our calculated tree, one-loop, quadratic, reducible two-loop, and total (quadratic + reducible two-loop) corrected A_{PV} at different scattering angles θ_{lab} . Here (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level corrected A_{PV} . These results are calculated in ppb.

$\theta_{\text{lab}}(\text{deg})$	Tree A_{PV} (ppb)	1-loop A_{PV} (ppb)	qud.- A_{PV} (ppb)	2-loop red. A_{PV} (ppb)	Total A_{PV} (ppb)
30	-115.15	-81.18	-80.77	-80.41	-80.01
35	-164.04	-116.66	-116.04	-115.61	-114.99
40	-225.46	-161.81	-160.91	-160.44	-159.54
45	-301.61	-218.54	-217.29	-216.78	-215.54
50	-394.93	-289.00	-287.34	-286.79	-285.14

TABLE XVI. MUSE kinematics for μp scattering at $E_{\text{lab}} = 0.235$ GeV. $\delta_{A_{\text{PV}}}^1$ % and $\delta_{A_{\text{PV}}}^2$ % at different scattering angles θ_{lab} . Here $\delta_{A_{\text{PV}}}^1$ % and $\delta_{A_{\text{PV}}}^2$ % are NLO and NNLO A_{PV} correction percentages as given by Eq. (34), whereas (qud.) represents the quadratic and (2-loop red.) is for reducible two-loop level correction %.

$\theta_{\text{lab}}(\text{deg})$	$\delta_{A_{\text{PV}}}^1$ (1 loop)%	$\delta_{A_{\text{PV}}}^2$ (qud.)%	$\delta_{A_{\text{PV}}}^2$ (2-loop red.)%	$\delta_{A_{\text{PV}}}^2$ (qud. + 2-loop red.)%
30	29.50	29.85	30.17	30.52
35	28.88	29.26	29.52	29.89
40	28.23	28.63	28.84	29.24
45	27.54	27.95	28.13	28.54
50	26.82	27.24	27.38	27.80

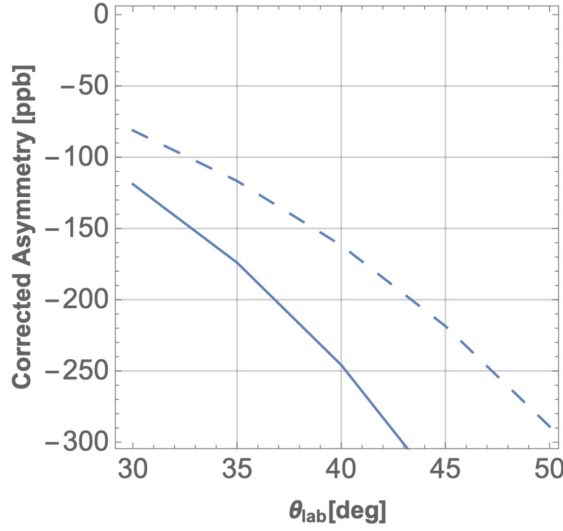


FIG. 32. Lepton mass effect using MUSE kinematics for ep and μp scattering at $E_{\text{lab}} = 0.235$ GeV. Tree + NLO μp scattering A_{PV} (large-dashed line) and Tree + NLO ep scattering A_{PV} (solid line) plotted versus θ_{lab} .

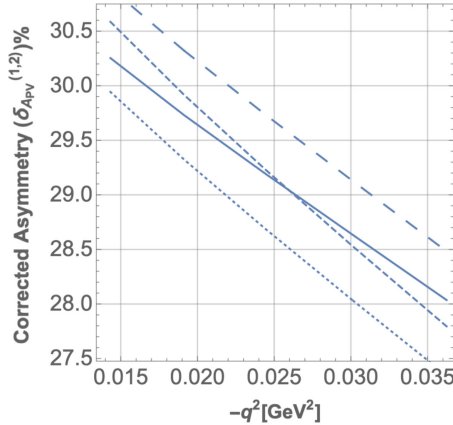


FIG. 33. MUSE kinematics for ep scattering at $E_{\text{lab}} = 0.235$ GeV. Tree + NLO δ^1_{APV} % (dashed line), Tree + NLO + quadratic δ^2_{APV} % (dotted line), Tree + NLO + two-loop reducible δ^2_{APV} % (large-dashed line), and Tree + NLO + quadratic + two-loop reducible δ^2_{APV} % (solid line) plotted versus $-q^2$. Here δ^1_{APV} % and δ^2_{APV} % are NLO and NNLO A_{PV} correction percentages as given by Eq. (34).

for MUSE kinematics is presented in Fig. 32. It would be an interesting possibility for future MUSE experimental setups to perform such measurements with a longitudinally polarized lepton beam.

The MUSE ep scattering A_{PV} correction percentage at $E_{\text{lab}} = 0.235$ GeV is shown graphically in Fig. 33.

The MUSE μp scattering A_{PV} correction percentages at $E_{\text{lab}} = 0.156$ GeV and $E_{\text{lab}} = 0.235$ GeV are shown graphically in Fig. 34.

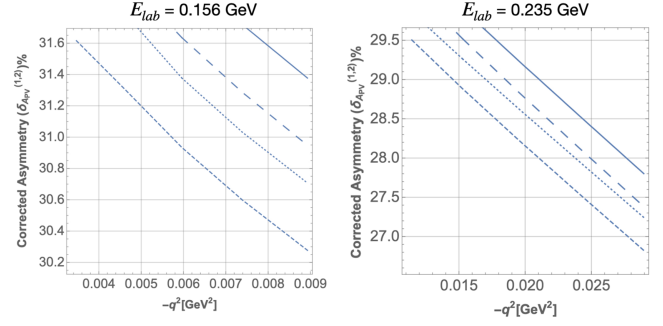


FIG. 34. MUSE kinematics for μp scattering at $E_{\text{lab}} = 0.156$ GeV and $E_{\text{lab}} = 0.235$ GeV. Tree + NLO δ^1_{APV} % (dashed line), Tree + NLO + quadratic δ^2_{APV} % (dotted line), Tree + NLO + two-loop reducible δ^2_{APV} % (large-dashed line), and Tree + NLO + quadratic + two-loop reducible δ^2_{APV} % (solid line) plotted versus $-q^2$. Here δ^1_{APV} % and δ^2_{APV} % are NLO and NNLO A_{PV} correction percentages as given by Eq. (34).

VI. CONCLUSIONS

We have developed various computational techniques in this work which have been checked with the higher-order QED corrections in A_{PV} . Our analytical and programming routines show considerable promise for extension and applications toward the experiments which are searching for physics beyond the Standard Model. Our numerical results up to NNLO (quadratic + reducible two-loop) level electroweak A_{PV} are in good agreement with the measured values by Q_{weak} experiment as well as predictions for the proposed future P2 experiment. The Q_{weak} measured value of A_{PV} at $\theta_{\text{lab}} = 7.9^\circ$ is -226.5 ± 7.3 (statistical) ± 5.8 (systematic) ppb [3], whereas our calculated A_{PV} up to NNLO is -221.46 ppb. Similarly, the P2 predicted value of A_{PV} at $\theta_{\text{lab}} = 35^\circ$ is -67.34 ppb [6], while our calculated A_{PV} at the same θ_{lab} is -65.09 ppb.

The theoretical predictions we make in the case of polarized ep and μp scattering using the kinematics of highly anticipated experiments MOLLER and MUSE, and programs at EIC will play an important role either directly or in the background studies for the search of physics beyond the Standard Model at the precision frontier. These calculations can further be improved by adding electroweak box diagrams. Since box diagrams are self-gauge invariant, we can calculate them separately, and so our next target is to investigate different approaches for the calculation of boxes. In the future, we would also like to further extend this work by considering a polarized proton target in the case of both elastic and inelastic scattering scenarios. This will be achieved by calculating a polarized hadronic tensor at higher momentum transfer.

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APPENDIX A: LEADING ORDER ELECTROWEAK LEPTONIC TENSOR STRUCTURE FUNCTIONS

DATA AVAILABILITY
The data are not publicly available. The data are available from the authors upon reasonable request.

The LO electroweak leptonic tensor structure functions with a polarized incoming electron having a helicity state $\beta = \pm 1$ are given below:

$$\begin{aligned}
l_1 &= \frac{\alpha\pi}{4c_W^3 s_W^3} (4q^2 c_W^2 s_W^2 (-1 + 4s_W^2) + c_W s_W (-2m_l^2 + q^2 + 4q^2 s_W^2) \\
&\quad + 2\beta m_l (c_W s_W - 4s_W^2 (c_W^2 + c_W s_W)) (s_1 \cdot k_2)), \\
l_2 &= \frac{\alpha\pi}{2c_W^2 s_W^2} (1 + 4c_W s_W (2c_W s_W + 4s_W^2 - 1) - 4s_W^2 (1 - 2s_W^2)), \\
l_3 &= \frac{\alpha\pi}{2c_W^2 s_W^2} (1 + 4c_W s_W (2c_W s_W + 4s_W^2 - 1) - 4s_W^2 (1 - 2s_W^2)), \\
l_4 &= \frac{2\beta\alpha m_l \pi}{c_W^3 s_W} (c_W s_W (2s_W^2 - 1) + c_W^2 (2c_W s_W + 4s_W^2 - 1)), \\
l_5 &= -\frac{\beta\alpha m_l \pi}{2c_W^3 s_W^3} (4c_W^2 s_W^2 (4s_W^2 - 1) + c_W s_W (1 + 4s_W^2)), \\
l_6 &= \frac{\alpha\pi}{2c_W^2 s_W^2} (1 - 4s_W (c_W + s_W)), \\
l_7 &= \frac{\beta\alpha m_l \pi}{2c_W^2 s_W^2} (4s_W (c_W + s_W) - 1), \\
l_8 &= \frac{\beta\alpha m_l \pi}{2c_W^2 s_W^2} (4s_W (c_W + s_W) - 1).
\end{aligned}$$

APPENDIX B: LEADING ORDER ELECTROWEAK HADRONIC TENSOR STRUCTURE FUNCTIONS

The LO unpolarized electroweak hadronic tensor structure functions are given as

$$\begin{aligned}
H_1^{VV} &= \frac{\alpha\pi C_2}{32c_W^2 s_W^2} \left(q^2 ((F_{2n}^\gamma)^2 + 2F_{2n}^\gamma (2 + F_{2p}^\gamma) \times (4s_W (c_W + s_W) - 1) + (2 + F_{2p}^\gamma)^2 (1 + 8s_W^2 + 8c_W s_W (4s_W^2 - 1))) \right. \\
&\quad \left. + 4G_A^2 (q^2 - 4m_p^2) \right), \\
H_2^{VV} &= \frac{\alpha\pi C_2}{128m_p^2 c_W^2 s_W^2} \left(4m_p^2 (8 + (F_{2n}^\gamma)^2 + 4F_{2p}^\gamma + (F_{2p}^\gamma)^2) + 8G_A^2 - 8c_W s_W (8 + 4F_{2p}^\gamma + (F_{2p}^\gamma)^2) + 8(8 + F_{2p}^\gamma (4 + F_{2p}^\gamma)) \right. \\
&\quad \times \left(2c_W (c_W + 2s_W) - 1 \right) s_W^2 + 16 \left(8 + F_{2p}^\gamma (4 + F_{2p}^\gamma) \right) s_W^2 + 2F_{2n}^\gamma (2 + F_{2p}^\gamma) (4s_W (c_W + s_W) - 1) \\
&\quad \left. - \left(q^2 (F_{2n}^\gamma - F_{2p}^\gamma) (F_{2n}^\gamma + F_{2p}^\gamma (8c_W s_W - 1)) + 8F_{2p}^\gamma (F_{2n}^\gamma + F_{2p}^\gamma (2c_W (c_W + 2s_W) - 1)) s_W^2 + 16(F_{2p}^\gamma)^2 s_W^2 \right) \right), \\
H_3^{VV} &= \frac{\alpha\pi C_2}{128m_p^2 c_W^2 s_W^2} \left(4m_p^2 (8 + (F_{2n}^\gamma)^2 + 4F_{2p}^\gamma + (F_{2p}^\gamma)^2) + 8G_A^2 - 8c_W s_W (8 + 4F_{2p}^\gamma + (F_{2p}^\gamma)^2) + 8(8 + F_{2p}^\gamma (4 + F_{2p}^\gamma)) \right. \\
&\quad \times \left(2c_W (c_W + 2s_W) - 1 \right) s_W^2 + 16 \left(8 + F_{2p}^\gamma (4 + F_{2p}^\gamma) \right) s_W^2 + 2F_{2n}^\gamma (2 + F_{2p}^\gamma) (4s_W (c_W + s_W) - 1) \\
&\quad \left. - \left(q^2 (F_{2n}^\gamma - F_{2p}^\gamma) (F_{2n}^\gamma + F_{2p}^\gamma (8c_W s_W - 1)) + 8F_{2p}^\gamma (F_{2n}^\gamma + F_{2p}^\gamma (2c_W (c_W + 2s_W) - 1)) s_W^2 + 16(F_{2p}^\gamma)^2 s_W^2 \right) \right), \\
H_4^{VV} &= -\frac{\alpha\pi C_2}{128m_p^2 c_W^2 s_W^2} \left((F_{2n}^\gamma)^2 (q^2 + 4m_p^2) + 2F_{2n}^\gamma (4s_W (c_W + s_W) - 1) (4m_p^2 (2 + F_{2p}^\gamma) + q^2 F_{2p}^\gamma) \right. \\
&\quad \left. + F_{2p}^\gamma (1 + 8s_W^2 + 8c_W s_W (4s_W^2 - 1)) (4m_p^2 (4 + F_{2p}^\gamma) + q^2 F_{2p}^\gamma) \right),
\end{aligned}$$

$$\begin{aligned}
H_5^{VV} &= -\frac{\alpha\pi C_2}{128m_p^2 c_W^2 s_W^2} ((F_{2n}^\gamma)^2 (q^2 + 4m_p^2) + 2F_{2n}^\gamma (4s_W(c_W + s_W) - 1)(4m_p^2(2 + F_{2p}^\gamma) + q^2 F_{2p}^\gamma) \\
&\quad + F_{2p}^\gamma (1 + 8s_W^2 + 8c_W s_W (4s_W^2 - 1))(4m_p^2(4 + F_{2p}^\gamma) + q^2 F_{2p}^\gamma)), \\
H_6^{VV} &= \frac{\alpha\pi C_2 G_A^2}{4c_W^2 s_W^2} (F_{2n}^\gamma + (2 + F_{2p}^\gamma)(4s_W(c_W + s_W) - 1)).
\end{aligned}$$

Here the terms $F_{(1,2)p}^\gamma$ and $F_{(1,2)n}^\gamma$ are the electric and magnetic form factors of the proton and neutron, respectively. The values of these form factors at $q^2 \rightarrow 0$ are given by

$$\begin{aligned}
F_{1p}^\gamma(q^2 = 0) &= 1, \\
F_{2p}^\gamma(q^2 = 0) &= 2.793 - 1, \\
F_{1n}^\gamma(q^2 = 0) &= 0, \\
F_{2n}^\gamma(q^2 = 0) &= -1.9147.
\end{aligned}$$

The structure of the form factors is represented by the term $C_i \equiv (\frac{\Lambda^2}{\Lambda^2 - q^2})^i$ with $\Lambda = \sqrt{0.83}m_p$. For simplicity, we used the dipole structure of the form factors given by $C_2 \equiv (\frac{\Lambda^2}{\Lambda^2 - q^2})^2$.

For higher-order (NLO and NNLO) electroweak leptonic and hadronic tensor structure functions are too cumbersome to show here, and we can provide the analytical expressions at the request from the authors.

APPENDIX C: ONE AND TWO PHOTON EXCHANGE SOFT BREMSSTRAHLUNG

The soft photon integral term $I(k_i, k_j)$ has the following form:

$$I(k_i, k_j) = \frac{4\pi\alpha}{(2\pi)^3} ((2m_l^2 - q^2)k_{i,j} - 2m_l^2 k_{i,i}), \quad (\text{C1})$$

where $k_{i,j}$ terms in Eq. (C1) are defined as

$$\begin{aligned}
k_{i,j} &= \frac{2\pi l_{i,j}}{l_{i,j}^2 m_i^2 - m_j^2} \left(\frac{1}{2} \ln \left[\frac{l_{i,j}^2 m_j^2}{m_j^2} \right] \ln \left[\frac{4\Delta E^2}{\lambda} \right] \right. \\
&\quad + \frac{1}{4} \left(\ln \left[\frac{E_i - p}{E_i + p} \right] \right)^2 - \frac{1}{4} \left(\ln \left[\frac{E_j - p}{E_j + p} \right] \right)^2 \\
&\quad + \text{Li}_2 \left[2, 1 - \frac{l_{i,j}}{v_{i,j}} (E_i + p) \right] + \text{Li}_2 \left[2, 1 - \frac{l_{i,j}}{v_{i,j}} (E_i - p) \right] \\
&\quad \left. - \text{Li}_2 \left[2, 1 - \frac{1}{v_{i,j}} (E_j + p) \right] - \text{Li}_2 \left[2, 1 - \frac{1}{v_{i,j}} (E_j - p) \right] \right),
\end{aligned}$$

where Li_2 represents the Spence function or dilogarithm whose properties are given in [45].

The rest of the terms are defined as

$$\begin{aligned}
m_{1,2}^2 &\equiv m_l^2, p^2 = \frac{(E_{\text{CMS}}^2 + m_p^2 - m_l^2)}{4E_{\text{CMS}}^2} - m_p^2, \\
E_{1,2} &= \sqrt{p^2 + m_l^2}, v_{i,j} = \frac{l_{i,j}^2 m_i^2 - m_j^2}{2(l_{i,j} E_i - E_j)}, \\
l_{1,1} &= 1, l_{1,2} = 1 - \frac{q^2}{2m_l^2} + \frac{\sqrt{q^4 - 4q^2 m_l^2}}{2m_l^2}.
\end{aligned}$$

The terms $E_{1,2}$ are the energies of the incoming and outgoing leptons. The term p^2 is the momentum squared in the spatial center-of-mass reference frame for incoming or outgoing lepton (proton) with E_{CMS} as the center-of-mass energy. The terms m_l and m_p represent the lepton and proton mass, respectively.

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