

$a_0(980)$ and $f_0(980)$ excitation in the $D^+ \rightarrow \pi^+ \eta \eta$ decayJing Song^{1,2,3,*} Yi-Yao Li^{4,5,6,3,†} Melahat Bayar^{7,3,‡} and Eulogio Oset^{3,8,§}¹*Center for Theoretical Physics, School of Physics and Optoelectronic Engineering, Hainan University, Haikou 570228, People's Republic of China*²*School of Physics, Beihang University, Beijing, 102206, People's Republic of China*³*Departamento de Física Teórica and IFIC, Centro Mixto Universidad de Valencia-CSIC Institutos de Investigación de Paterna, 46071 Valencia, Spain*⁴*State Key Laboratory of Nuclear Physics and Technology, Institute of Quantum Matter, South China Normal University, Guangzhou 510006, People's Republic of China*⁵*Key Laboratory of Atomic and Subatomic Structure and Quantum Control (MOE), Guangdong-Hong Kong Joint Laboratory of Quantum Matter, Guangzhou 510006, People's Republic of China*⁶*Guangdong Basic Research Center of Excellence for Structure and Fundamental Interactions of Matter, Guangdong Provincial Key Laboratory of Nuclear Science, Guangzhou 510006, People's Republic of China*⁷*Department of Physics, Kocaeli University, 41380, Izmit, Turkey*⁸*Department of Physics, Guangxi Normal University, Guilin 541004, People's Republic of China*

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We have made a thorough study of the $D^+ \rightarrow \pi^+ \eta \eta$ reaction, recently measured by the Beijing Spectrometer III Collaboration, which shows an abnormal strength at high invariant masses in the $\pi \eta$ mass distribution. We studied in detail the triangle mechanism and the $f_0(1370)$ excitation modes that have been suggested to explain this abnormal feature, and concluded that they are too small to have any important role in the solution to that problem. We have also studied other possible solutions, evaluating the contribution of excitations of other f_0 , a_0 and f_2 resonances, and reached the same conclusion. Unexpectedly, the solution to the problem is found considering the $f_0(980)$ excitation, with the $f_0(980)$ decaying to two η , which is tied to the $a_0(980)$ production, and well under control. At the same time, the consideration of the $f_0(980)$ excitation solves another nonreported problem, which is the $\eta \eta$ mass distribution that comes when only the $a_0(980)$ resonance is allowed to be excited, which produces a large deficiency at low invariant masses compared with experiments.

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I. INTRODUCTION

The BESIII (Beijing Spectrometer III) Collaboration has recently measured the $D^+ \rightarrow \pi^+ \eta \eta$ reaction [1] and observed a clear signal for the $a_0(980)$ in the $\pi^+ \eta$ mass distribution; however, with an abnormal shape, with a large contribution in the $\pi^+ \eta$ mass region above the $K \bar{K}$ threshold. They attributed this shape to the presence of a loop rescattering effect (see Fig. 1). The mechanism is the same

that sometimes leads to a triangle singularity (TS), although no claims of a TS were made in Ref. [1].

A test for a TS can be easily done by application of Eq. (18) of Ref. [2], and one can indeed see that for the nominal mass of the $K_0^*(1430)$ the diagram of Fig. 1 does not develop a TS, among other things because the mass of $K_0^*(1430)$ plus the kaon is about 1923 MeV, bigger than the mass of D^+ (1869 MeV). However, the $K_0^*(1430)$ has a large width, $\Gamma_{K_0^*} = 270 \pm 80$ MeV [3]. If we take $\tilde{M}_{K_0^*} = M_{K_0^*} - 0.4\Gamma_{K_0^*} \simeq 1322$ MeV, then we can see that Eq. (18) of Ref. [2] is fulfilled and we have a TS for this mass. This means that from a fraction of the mass distribution of the $K_0^*(1430)$, one does actually get a TS from the mechanism of Fig. 1, which could substantiate the claims of the authors of Ref. [1].

Shortly after the work of Ref. [1] was released, a theoretical paper appeared [4] in which, based on the nominal mass of the $K_0^*(1430)$, the TS was dismissed, and instead the shape of the $\pi^+ \eta$ mass distribution in Ref. [1] was attributed to the contribution of the $f_0(1370)$ state.

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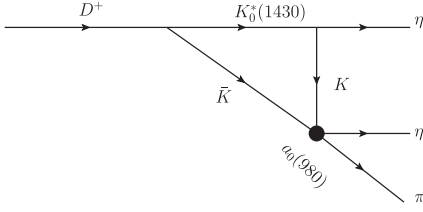


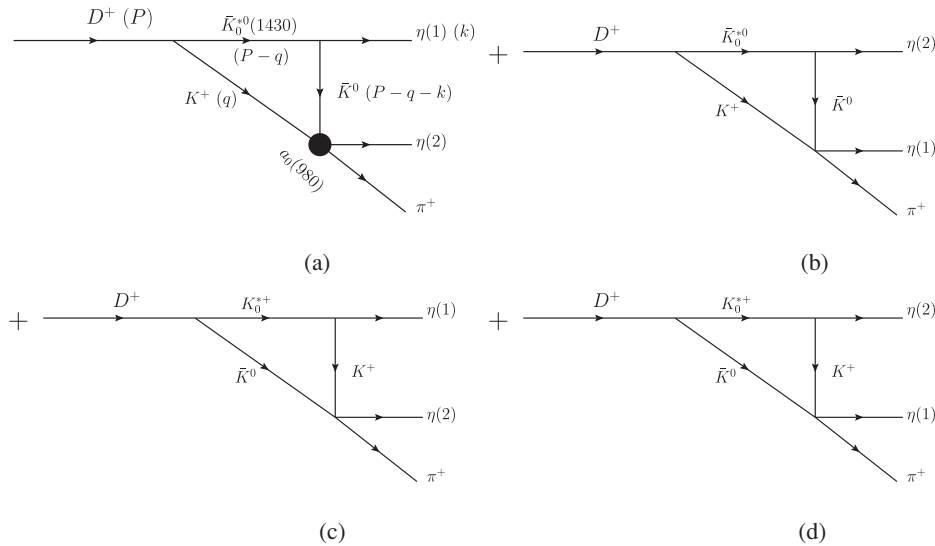
FIG. 1. Triangle mechanism suggested in Ref. [1].

The purpose of the present paper is to examine these ideas and present a quantitative contribution from these sources based on empirical information of different decays. Our first conclusion is that both contributions are small, of the order of 1% of the experimental branching ratio of the $D^+ \rightarrow \pi^+ \eta \eta$ reaction, hence, cannot be responsible for the observed shape of the $\pi^+ \eta$ mass distribution.

In view of these results, we have also studied the contribution from $f_0(1500)$, $f_2(1525)$, $a_0(1450)$, and concluded that they also provide contributions too small to explain the experimental shape. Surprisingly, we observed that linked to the $a_0(980)$ production, there is also excitation of the $f_0(980)$, decaying to $\eta\eta$, with a strength tied to that of the $a_0(980)$, such that there is no freedom in the relative weights of the two excitation mechanisms. We find that this mechanism naturally explains the large observed strength in the region of $\pi^+ \eta$ mass above the $K\bar{K}$ threshold, and also provides the contribution missing in the $\eta\eta$ mass distribution at low invariant masses.

II. TRIANGLE SINGULARITY CONTRIBUTION

Following the suggestion of the triangle mechanism of the BESIII work [1], we look at the contribution of diagrams in Fig. 2.

FIG. 2. Triangle diagrams through intermediate $\bar{K}_0^{*0} K^+$ or $K_0^{*+} \bar{K}^0$ production. The momenta of the particles are shown in parentheses. (a-d) correspond to different possible diagrams leading to the same final state.

The evaluation of the decay rate from the triangle mechanism of Fig. 2 can be done, providing not just a shape but also absolute numbers, using empirical information available in the literature. The $\bar{K}_0^{*0} K^+ \rightarrow \pi^+ \eta$ scattering amplitude is available from the chiral unitary approach [5]. We need the $D^+ \rightarrow \bar{K}_0^{*0}(1430) K^+$ coupling and the one for $\bar{K}_0^{*0}(1430) \rightarrow \bar{K}^0 \eta$ which we obtain from decay widths reported in the literature.

A. $D^+ \rightarrow \bar{K}_0^{*0}(1430) K^+$ and $K_0^{*+} \rightarrow K \eta$

In the Particle Data Group (PDG) [3] we find the branching ratio (BR) as follows:

$$\text{BR}[D^+ \rightarrow \bar{K}_0^{*0}(1430) K^+; \bar{K}_0^{*0} \rightarrow K^- \pi^+] = (1.82 \pm 0.35) \times 10^{-3}, \quad (1)$$

$$\text{BR}[K_0^{*+}(1430) \rightarrow K \pi] = (93 \pm 10)\%, \quad (2)$$

$$\text{BR}[K_0^{*+}(1430) \rightarrow K \eta] = (8.6_{-3.4}^{+2.7})\%. \quad (3)$$

Since $\bar{K}_0^{*0}$ has $I = 1/2, I_3 = 1/2$, using Clebsch-Gordan coefficients, the relative rate of $\bar{K}_0^{*0}$ decaying to $\pi^+ K^-$ is $\frac{2}{3}$ versus $\frac{1}{3}$ for $\pi^0 \bar{K}^0$. Thus, considering Eqs. (1) and (2), we find

$$\text{BR}[D^+ \rightarrow \bar{K}_0^{*0}(1430) K^+] = \frac{1.82}{\frac{2}{3} \times 0.93} \simeq (3.0 \pm 0.6) \times 10^{-3}. \quad (4)$$

By taking the experimental width [3]

$$\Gamma_{K_0^{*+}(1430)} = 270 \pm 80 \text{ MeV} \quad (5)$$

and using Eq. (3), we find

$$\Gamma_{\bar{K}_0^{*0}(1430) \rightarrow \bar{K}^0 \eta} = 270 \frac{8.6}{100} \text{ MeV} = 23.2 \pm 6.9 \text{ MeV}. \quad (6)$$

The decay width for $D^+ \rightarrow \bar{K}_0^{*0} K^+$ is given in terms of the $\bar{K}_0^{*0} K^+$ coupling, $g_{D^+, \bar{K}_0^* K}$, by

$$\Gamma_{D^+ \rightarrow \bar{K}_0^{*0} K^+} = \frac{1}{8\pi} \frac{1}{m_{D^+}^2} g_{D^+, \bar{K}_0^* K}^2 q, \quad (7)$$

with

$$q = \frac{\lambda^{1/2}(m_{D^+}^2, m_{\bar{K}_0^*}^2, m_{K^+}^2)}{2m_{D^+}} \Theta(m_{D^+} - m_{\bar{K}_0^*} - m_{K^+}). \quad (8)$$

The first thing we realize is that for the nominal mass of the K_0^* , this decay cannot occur. However, it is possible thanks to the large width of the K_0^* that gives rise to a wide K_0^* mass distribution. We can get the decay width through a convolution as

$$\begin{aligned} \Gamma_{D^+ \rightarrow \bar{K}_0^* K} &= \frac{1}{N} \int_{(m_{K_0^*} - 2\Gamma_{K_0^*})^2}^{(m_{K_0^*} + 2\Gamma_{K_0^*})^2} d\tilde{m}^2 \left(-\frac{1}{\pi} \right) \\ &\times \text{Im} \left[\frac{1}{\tilde{m}^2 - m_{K_0^*}^2 + im_{K_0^*} \Gamma_{K_0^*}} \right] \\ &\times \Gamma_{D^+ \rightarrow \bar{K}_0^* K}(m_{K_0^*} \equiv \tilde{m}), \end{aligned} \quad (9)$$

with a normalization factor

$$N = \int_{(m_{K_0^*} - 2\Gamma_{K_0^*})^2}^{(m_{K_0^*} + 2\Gamma_{K_0^*})^2} d\tilde{m}^2 \left(-\frac{1}{\pi} \right) \text{Im} \left[\frac{1}{\tilde{m}^2 - m_{K_0^*}^2 + im_{K_0^*} \Gamma_{K_0^*}} \right]. \quad (10)$$

By equating Eq. (9) to the experimental branching ratio of Eq. (4), we obtain

$$\frac{(g_{D^+, \bar{K}_0^* K})^2}{\Gamma_{D^+}} = (2584 \pm 517) \text{ MeV}. \quad (11)$$

On the other hand, using

$$\Gamma_{\bar{K}_0^*, \bar{K} \eta} = \frac{1}{8\pi} \frac{1}{m_{\bar{K}_0^*}^2} g_{\bar{K}_0^*, \bar{K} \eta}^2 q_\eta \quad (12)$$

with

$$q_\eta = \frac{\lambda^{1/2}(m_{\bar{K}_0^*}^2, m_{\bar{K}}^2, m_\eta^2)}{2m_{\bar{K}_0^*}} \quad (13)$$

and taking the experimental width of Eq. (6), we obtain

$$(g_{\bar{K}_0^*, \bar{K} \eta})^2 = (2.45 \pm 0.72) \times 10^6 \text{ MeV}^2. \quad (14)$$

B. The triangle mechanism width

We are now in a position to make a precise evaluation of the rate of D^+ decay through the triangle mechanism of Fig. 2. Next, we evaluate the amplitude for the mechanism of Fig. 2(a) as

$$\begin{aligned} -it_{TS} &= \int \frac{d^4 q}{(2\pi)^4} (-i) g_{D^+, \bar{K}_0^* K} (-i) g_{\bar{K}_0^*, \bar{K} \eta} (-i) t_{K^+ \bar{K}^0, \pi^+ \eta} \\ &\times \frac{i}{(P-q)^2 - m_{\bar{K}_0^*}^2 + i\epsilon} \frac{i}{q^2 - m_{K^+}^2 + i\epsilon} \frac{i}{(P-q-k)^2 - m_{\bar{K}^0}^2 + i\epsilon}. \end{aligned} \quad (15)$$

The evaluation can be extremely simplified if we take into account that the TS appears when all the intermediate particles in the loop are placed on shell [2,6]. This means that we can separate the positive and negative energy parts of a meson propagator as

$$\frac{1}{q^2 - m^2 + i\epsilon} = \frac{1}{2\omega(q)} \left(\frac{1}{q^0 - \omega(q) + i\epsilon} - \frac{1}{q^0 + \omega(q) - i\epsilon} \right), \quad (16)$$

with $\omega(q) = \sqrt{\vec{q}^2 + m^2}$, and keep only the positive energy part, which is the only one that can be placed on shell. Then, the amplitude of Eq. (15) can be written in the D^+ rest frame, where $\vec{P} = 0$, as

$$\begin{aligned} t_{TS} &= i g_{D^+, \bar{K}_0^* K} g_{\bar{K}_0^*, \bar{K} \eta} t_{K^+ \bar{K}^0, \pi^+ \eta} (M_{\text{inv}}(\pi^+ \eta)) \int \frac{d^4 q}{(2\pi)^4} \frac{1}{2\omega_{\bar{K}_0^*}(\vec{q})} \frac{1}{2\omega_{K^+}(\vec{q})} \frac{1}{2\omega_{\bar{K}^0}(\vec{q} + \vec{k})} \\ &\times \frac{1}{P^0 - q^0 - \omega_{\bar{K}_0^*}(\vec{q}) + i\epsilon} \frac{1}{q^0 - \omega_{K^+}(\vec{q}) + i\epsilon} \frac{1}{P^0 - q^0 - k^0 - \omega_{\bar{K}^0}(\vec{q} + \vec{k}) + i\epsilon}, \end{aligned} \quad (17)$$

where $\omega_i(p) = \sqrt{m_i^2 + \vec{p}^2}$. Next, we perform the q^0 integration using Cauchy's theorem and obtain

$$t_{\text{TS}} = g_{D^+, \bar{K}_0^* K} g_{\bar{K}_0^* \bar{K} \eta} t_{K^+ \bar{K}^0, \pi^+ \eta}(M_{\text{inv}}(\pi^+ \eta)) \int \frac{d^3 q}{(2\pi)^3} \frac{1}{2\omega_{K^+}(\vec{q})} \frac{1}{2\omega_{K^+}(\vec{q})} \frac{1}{2\omega_{\bar{K}^0}(\vec{q} + \vec{k})} \\ \times \frac{1}{P^0 - \omega_{K^+}(\vec{q}) - \omega_{K^+}(\vec{q}) + i\frac{\Gamma_{K^+}}{2}} \frac{1}{P^0 - \omega_{K^+}(\vec{q}) - k^0 - \omega_{\bar{K}^0}(\vec{q} + \vec{k}) + i\epsilon}, \quad (18)$$

where we have replaced $\omega_{K^+} - i\epsilon$ by $\omega_{K^+} - \frac{i\Gamma_{K^+}}{2}$ to account for the K^+ width. We have also introduced the factor $\Theta(q_{\text{max}} - |\vec{q}^*|)$, where q^* is the K^+ momentum in the rest frame of the $\pi\eta$ forming the $a_0(980)$ resonance, given by

$$\vec{q}^* = \left[\left(\frac{E_R}{M_{23}} - 1 \right) \frac{\vec{q} \cdot \vec{k}}{k^2} + \frac{q^0}{M_{23}} \right] \vec{k} + \vec{q}, \quad (19)$$

where $M_{23} \equiv M_{\text{inv}}(\pi^+ \eta)$, q_{max} is the regulator of the loop in the cutoff regularization used to obtain the $t_{K\bar{K}, \pi\eta}$ amplitudes in Ref. [7], k is the momentum of η_1 , $E_R = \sqrt{M_{23}^2 + \vec{k}^2}$, and $q^0 = \sqrt{m_{K^+}^2 + \vec{q}^2}$. The factor $\Theta(q_{\text{max}} - |\vec{q}^*|)$ appears because the calculated scattering amplitudes factorize as $t(\vec{q}, \vec{q}') = t\Theta(q_{\text{max}} - |\vec{q}|)\Theta(q_{\text{max}} - |\vec{q}'|)$ [8].

The final amplitude, properly symmetrized, corresponds to the sum of the diagrams of Fig. 2. First, we assume that Figs. 2(a) and 2(c) give the same contribution, hence we multiply by 2 the t matrix of Eq. (15), and then symmetrize the amplitudes, exchanging $\eta(1) \leftrightarrow \eta(2)$ to account for the diagrams of Figs. 2(b) and 2(d). The first two amplitudes depend on the invariant mass M_{23} labeling the states $\eta(1), \eta(2), \pi^+(3)$. The second pair of diagrams are a function of M_{13} . Then we use the formulas of the PDG to obtain the width,

$$\frac{d\Gamma}{dM_{12}dM_{23}} = \frac{1}{2} \frac{1}{(2\pi)^3} \frac{1}{32m_{D^+}^3} |2t_{\eta_1\eta_2} + 2t_{\eta_2\eta_1}|^2 \times 2M_{12}2M_{23}, \quad (20)$$

where the factor 1/2 comes from having two identical η in the final state.

Considering Eq. (41) and integrating Eq. (20) over M_{12} and M_{23} with the limits of Ref. [3], we have the total width coming from the triangle mechanism. If we integrate over M_{23} or M_{12} , we obtain the corresponding mass distributions.

Once the formalism for the triangle diagram is settled, we are in a good condition to test the contribution of other possible intermediate states, rather than the $K_0^*(1430)$. One might think that the $K(1460)$ state, with similar mass, could also play a role. However, the $K(1460)$ is a 0^- pseudoscalar meson, and the vertex $K(1460) \rightarrow \eta K$ is forbidden by angular momentum-parity conservation.

Hence, this state does not contribute to the triangle diagram. Next, we could consider the $0^+ K_0^*(700)$. The first thing is that this intermediate state makes the triangle mechanism of Fig. 1 to be very far away from developing a triangle singularity. Indeed, application of Eq. (18) of Ref. [2] shows that we are very far away from fulfilling the conditions of triangle singularity, even taking into account the large width of the $K_0^*(700)$, which tells us that the contribution of the loop is small. But there is also another powerful argument. Indeed, from the branching fraction of 6.8×10^{-4} for $D^+ \rightarrow K^+ \bar{K}_0^*(700)$ decay of the PDG [3], and using Eq. (7), we obtain $g_{D^+, K^+ \bar{K}_0^*(700)}^2 / \Gamma_{D^+} = 83.4$ MeV. Compared to $g_{D^+, \bar{K}_0^*(1430) K}^2 / \Gamma_{D^+}$ of Eq. (11), this number is 30 times smaller. Altogether, the $K_0^*(700)$ intermediate state does not have a chance to make any relevant contribution, compared to the one of the $K_0^*(1430)$. Using the same argument, an intermediate K instead of $K_0^*(1430)$ leads to $g_{D^+, K \bar{K}}^2 / \Gamma_{D^+} = 331$ MeV, still 1 order of magnitude smaller than the rate of Eq. (11), and the particles in the loop are very off shell because $K \rightarrow K\eta$ is largely forbidden kinematically. Once again, this mechanism is very small compared to the one of the intermediate $K_0^*(1430)$.

III. DECAY THROUGH HADRONIZATION OF $q\bar{q}$ COMPONENTS

The reactions $D^+ \rightarrow \pi^+ \eta \eta$ and $D^+ \rightarrow \pi^+ \pi^0 \eta$ were studied from the theoretical point of view in Ref. [9]. The process is single Cabibbo suppressed, and many diagrams contribute to the reaction. Some approximations were done, but the work predicted clear signals of $a_0(980)$ excitations prior to the BESIII experiment [1], which confirmed the expectations. We take the complete set of Feynman diagrams for the weak decay at the quark level described in Ref. [9] and implement the hadronization plus final state interaction in detail, with the aim of reproducing the experimental spectrum of the BESIII measurements.

The mechanisms considered in Ref. [9] are shown in Figs. 3 and 4 at the quark level. Figure 3 shows the two diagrams involved in external emission and Fig. 4 those involved in internal emission. In Figs. 5 and 6, we take these mechanisms and implement the hadronization of a pair of quarks to produce two mesons.

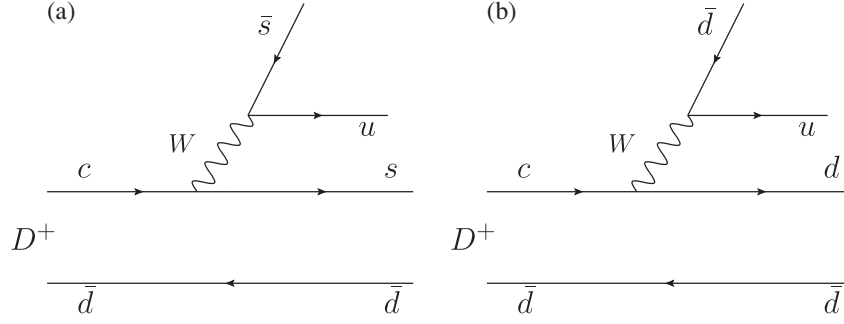


FIG. 3. Diagrams of external emission at the quark level: (a) Cabibbo suppressed $Wu\bar{s}$ vertex, (b) Cabibbo suppressed Wcd vertex.

The hadronization proceeds as follows: take, for instance, the diagram of Fig. 5(a). Then,

$$u\bar{s} \rightarrow \sum_i u\bar{q}_i q_i \bar{s} = P_{1i} P_{i3} \equiv (P^2)_{13}, \quad (21)$$

where P is the $q_i \bar{q}_j$ matrix, which we write in terms of physical meson states in Eq. (21):

$$P \equiv \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} & K^0 \\ K^- & \bar{K}^0 & -\frac{\eta}{\sqrt{3}} \end{pmatrix}, \quad (22)$$

where we have used the standard $\eta - \eta'$ mixing of Ref. [10] and omitted the η' , which does not play a role here. Similarly to Eq. (21), we can perform the hadronization for all the other diagrams with the following results:

$$\begin{aligned} \text{Fig.5(a): } (P^2)_{13} \bar{K}^0 &= \left[\left(\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} \right) K^+ + \pi^+ K^0 - \frac{K^+ \eta}{\sqrt{3}} \right] \bar{K}^0, \end{aligned} \quad (23)$$

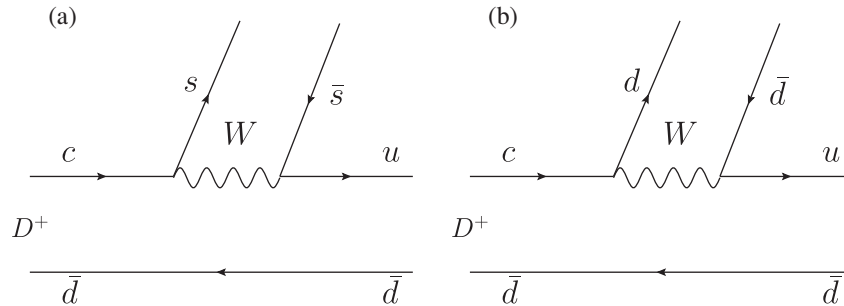


FIG. 4. Diagrams of internal emission at the quark level: (a) Cabibbo suppressed $Wu\bar{s}$ vertex, (b) Cabibbo suppressed Wcd vertex.

$$\begin{aligned} \text{Fig.5(c): } (P^2)_{12} \left(-\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} \right) &= \left[\left(\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} \right) \pi^+ + \pi^+ \left(-\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} \right) \right. \\ &\quad \left. + K^+ \bar{K}^0 \right] \left(-\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} \right). \end{aligned} \quad (24)$$

In Ref. [9], the $\frac{\eta}{\sqrt{3}} K^+$ terms canceled for Fig. 5(a) and the $\frac{\pi^0 \pi^+}{\sqrt{2}}$ terms canceled in Fig. 5(c). We correct this here because the order of the fields matters. Indeed, as can be obtained from Refs. [11–13] (see also Refs. [14,15]), the WPP Lagrangian goes as

$$W_\mu \langle [P, \partial_\mu P] \rangle, \quad (25)$$

and the terms ηK^+ , $(-K^+ \eta)$ go as $(\eta \partial_\mu K^+ - K^+ \partial_\mu \eta)$ and $(-)(K^+ \partial_\mu \eta - \eta \partial_\mu K^+)$, respectively, and hence do not cancel but add. On the other hand, the $\eta \pi^-$ and $\pi^+ \eta$ terms in Fig. 5(c) cancel rather than add. This said, we have for the sum of Figs. 5(a) and 5(c), as follows:

$$\begin{aligned} &\left(\frac{\pi^0}{\sqrt{2}} K^+ + \frac{2}{\sqrt{3}} \eta K^+ + \pi^+ K^0 \right) \bar{K}^0 - \pi^0 \pi^+ \pi^0 \\ &+ \sqrt{\frac{2}{3}} \pi^0 \pi^+ \eta - K^+ \bar{K}^0 \frac{\pi^0}{\sqrt{2}} + K^+ \bar{K}^0 \frac{\eta}{\sqrt{3}}. \end{aligned} \quad (26)$$

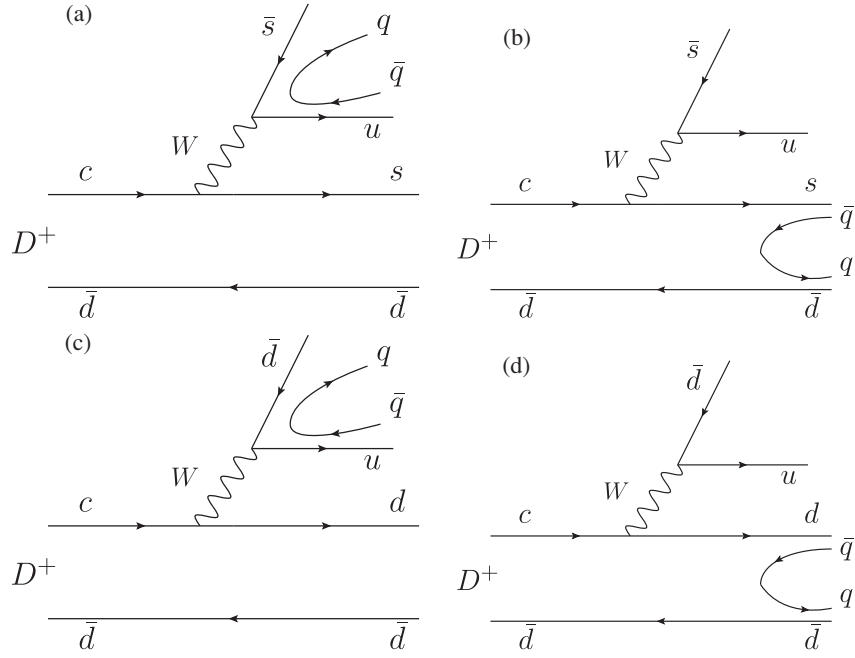


FIG. 5. Hadronization in the diagrams of Fig. 3. (a–d) correspond to different topologies for hadronization.

The Lagrangian of Eq. (25) has also the consequence of suppressing terms where the masses of the two mesons are equal or similar, since we have the contribution $p^\mu(1) - p^\mu(2)$, which is dominated by the $\mu = 0$ component, and then the difference of energies vanishes as an average. There is one term in Eq. (26) which could lead to $\pi^+\eta\eta$ in the final state, the $\eta K^+\bar{K}^0$ through $K^+\bar{K}^0 \rightarrow \pi^+\eta$ rescattering, but given the similar mass of the η and K^+ , we

also neglect this term. In the $(\pi^+K^0)\bar{K}^0$, we also have $(\omega_{\pi^+} - \omega_{K^0})$, which would also be zero in the SU(3) limit, but given the relatively large momenta of these particles in the phase space, this difference of energies is also small, and we will equally neglect this term. As a consequence, we get no contribution from the hadronization of the $W^\mu q\bar{q}$ vertex of external emission of Figs. 5(a) and 5(c). The rest of the terms are the same as in Ref. [9], as follows:

$$\text{Fig.5(b): } K^+(P^2)_{32} = K^+ \left[K^- \pi^+ + \bar{K}^0 \left(-\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}} \right) - \frac{\eta}{\sqrt{3}} \bar{K}^0 \right] = K^+ \left(K^- \pi^+ - \bar{K}^0 \frac{\pi^0}{\sqrt{2}} \right), \quad (27)$$

$$\text{Fig.5(d): } \pi^+(P^2)_{22} = \pi^+ \left(\pi^- \pi^+ + \frac{\pi^0 \pi^0}{2} + \frac{\eta\eta}{3} - \frac{\sqrt{2}}{\sqrt{3}} \pi^0 \eta + K^0 \bar{K}^0 \right). \quad (28)$$

From Fig. 5(d), we get the good term $\frac{\pi^+\eta\eta}{3}$. In addition, we can have $K^+K^- \rightarrow \eta\eta$, $K^0\bar{K}^0 \rightarrow \eta\eta$, which has isospin $I = 0$ and will be dominated by the $f_0(980)$. This term is not considered in Ref. [4]. We should also mention a difference with the work of Ref. [4]. In Ref. [4], the term $K^+\bar{K}^0\eta$ was obtained from the intermediate $q\bar{q} \equiv d\bar{d}$ in Fig. 5(b). However, as we can see in Eq. (27), this term is canceled by the contribution from $q\bar{q} \equiv s\bar{s}$, as follows:

$$\text{Fig.6(a): } (P^2)_{33}\pi^+ = \left(K^- K^+ + \bar{K}^0 K^0 + \frac{\eta\eta}{3} \right) \pi^+, \quad (29)$$

$$\text{Fig.6(c): } (P^2)_{22}\pi^+ = \left(\pi^- \pi^+ + \frac{\pi^0 \pi^0}{2} + \frac{\eta\eta}{3} - \frac{2}{\sqrt{6}} \pi^0 \eta + K^0 \bar{K}^0 \right) \pi^+. \quad (30)$$

We obtain again the term $\frac{\eta\eta\pi^+}{3}$ in both Fig. 6(a) and Fig. 6(c). Once again, $K^0\bar{K}^0 \rightarrow \eta\eta$ can lead to the final state through the tail of the $f_0(980)$ resonance, as follows:

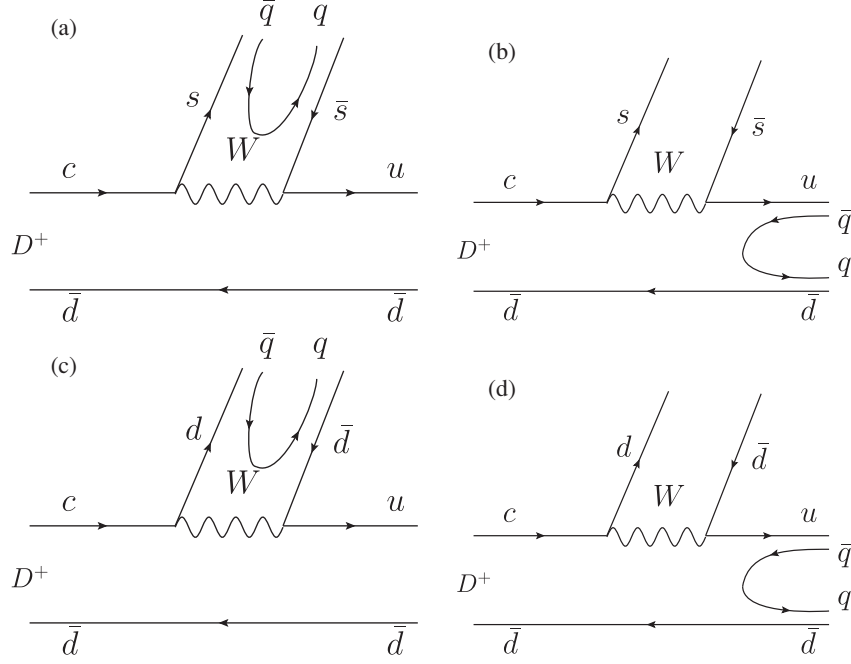


FIG. 6. Hadronization in the diagrams of Fig. 4. (a–d) correspond to different topologies for hadronization.

$$\text{Fig.6(b): } \left(-\frac{\eta}{\sqrt{3}}\right)(P^2)_{12} = \left(-\frac{\eta}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\eta\pi^+ + K^+\bar{K}^0\right), \quad (31)$$

$$\text{Fig.6(d): } \left(-\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}}\right)(P^2)_{12} = \left(-\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\eta\pi^+ + K^+\bar{K}^0\right). \quad (32)$$

We find that the $\eta\eta\pi^+$ terms, and the $K^+\bar{K}^0\eta$ terms which could lead to the good final state through $K^+\bar{K}^0 \rightarrow \pi^+\eta$, cancel in the sum of Eqs. (31) and (32).

We also note that in Ref. [4], the terms from internal emission have not been considered, because they are suppressed with respect to those of external emission. We keep these terms. We shall give a weight A to the terms in external emission and a weight B to those from internal emission, assuming approximately that $B/A \sim \frac{1}{N_c}$,

with $N_c = 3$, the number of colors. In the final state interaction, we shall keep the $\eta\pi^+ \rightarrow \eta\pi^+$ interaction, which leads to the $a_0(980)$ production, and $K^+K^- \rightarrow \eta\eta$, $K^0\bar{K}^0 \rightarrow \eta\eta$, $\pi^+\pi^- \rightarrow \eta\eta$, $\pi^0\pi^0 \rightarrow \eta\eta$, $\eta\eta \rightarrow \eta\eta$, which involve the $f_0(980)$ tail contribution.

All this said, we have the mechanisms for $D^+ \rightarrow \pi^+\eta\eta$ production depicted in Figs. 7 and 8.

With the weights A, B for external, internal emission, the valid terms that can lead to the final state are

$$\text{Fig.5(b): } AK^+K^-\pi^+, \quad (33)$$

$$\text{Fig.5(d): } A\frac{\eta\eta}{3}\pi^+; \quad AK^0\bar{K}^0\pi^+; \quad A\pi^+\pi^-\pi^+; \quad A\pi^+\pi^0\pi^0, \quad (34)$$

$$\text{Fig.6(a): } B\frac{\eta\eta}{3}\pi^+; \quad BK^+K^-\pi^+; \quad BK^0\bar{K}^0\pi^+, \quad (35)$$

$$\text{Fig.6(c): } B\frac{\eta\eta}{3}\pi^+; \quad BK^0\bar{K}^0\pi^+; \quad B\pi^+\pi^-\pi^+; \quad B\pi^+\pi^0\pi^0. \quad (36)$$

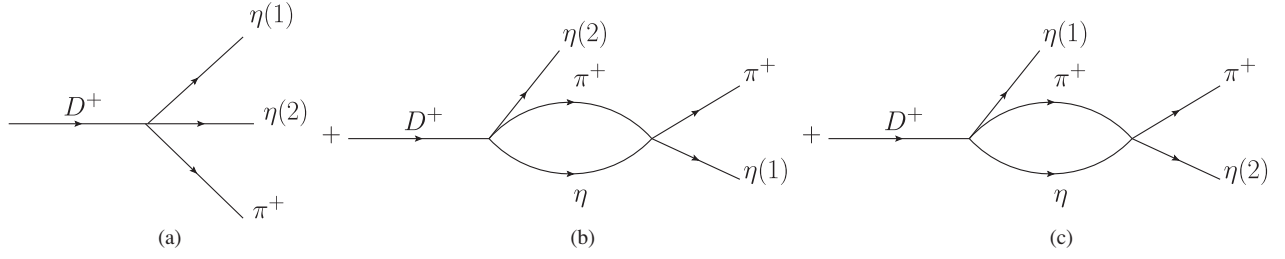


FIG. 7. Tree level and $\pi^+\eta$ rescattering diagrams leading to the $a_0(980)$. (a) tree level, (b), (c) different possibilities for $\pi^+\eta$ final state interaction

With these, we write the D^+ decay amplitudes corresponding to Figs. 7 and 8 as

$$t_{a_0} = \frac{2}{3}(A + 2B)(1 + G_{\pi^+\eta}(M_{\text{inv}}(\pi^+\eta 1))t_{\pi^+\eta,\pi^+\eta}(M_{\text{inv}}(\pi^+\eta 1)) + G_{\pi^+\eta}(M_{\text{inv}}(\pi^+\eta 2))t_{\pi^+\eta,\pi^+\eta}(M_{\text{inv}}(\pi^+\eta 2))), \quad (37)$$

the factor 2 in t_{a_0} for the two identical η fields, as follows:

$$\begin{aligned} t_{f_0} = & (A + B)\sqrt{2}(G_{K^+K^-}(M_{\text{inv}}(\eta 1\eta 2))t_{K^+K^-, \eta\eta}(M_{\text{inv}}(\eta 1\eta 2))) + (A + 2B)\sqrt{2}(G_{K^0\bar{K}^0}(M_{\text{inv}}(\eta 1\eta 2))t_{K^0\bar{K}^0, \eta\eta}(M_{\text{inv}}(\eta 1\eta 2))) \\ & + (A + B)\left(2G_{\pi^+\pi^-}(M_{\text{inv}}(\eta 1\eta 2))t_{\pi^+\pi^-, \eta\eta}(M_{\text{inv}}(\eta 1\eta 2))\sqrt{2} + 2\frac{1}{2}\frac{1}{2}G_{\pi^0\pi^0}(M_{\text{inv}}(\eta 1\eta 2))t_{\pi^0\pi^0, \eta\eta}(M_{\text{inv}}(\eta 1\eta 2))2\right) \\ & + \frac{2}{3}(A + 2B)\frac{1}{2}(G_{\eta\eta}(M_{\text{inv}}(\eta 1\eta 2))t_{\eta\eta, \eta\eta}(M_{\text{inv}}(\eta 1\eta 2))2), \end{aligned} \quad (38)$$

where the factor $\sqrt{2}$ is introduced because in the chiral unitary approach where the t_{ij} scattering matrices are evaluated, the states with identical particles are normalized with the unitary normalization, in this case $\frac{1}{\sqrt{2}}\eta\eta$ (also $\frac{1}{\sqrt{2}}\pi^0\pi^0$), and then the good normalization has to be restored at the end. For the same reason, the factor 2 is implemented to $t_{\pi^0\pi^0, \eta\eta}$ and $t_{\eta\eta, \eta\eta}$ and a factor $\frac{1}{2}$ in the $\pi^0\pi^0, \eta\eta$ loops is considered due to the identity of the two particles. We take all the amplitudes needed here from the work of Ref. [7], corresponding to the $\eta - \eta'$ mixing case [Eqs. (A.4) of Ref. [7]].

The G functions appearing in Eqs. (37) and (38) are the meson-meson loop functions, regularized with a cutoff

$$G^{(12)} = \int_{|\vec{q}| < q_{\text{max}}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{\omega_1(q) + \omega_2(q)}{2\omega_1(q)\omega_2(q)} \times \frac{1}{s - [\omega_1(q) + \omega_2(q)]^2 + i\epsilon}, \quad (39)$$

and we take $q_{\text{max}} = 650$ MeV, which reflects the range of the interaction in momentum space, from Ref. [7].

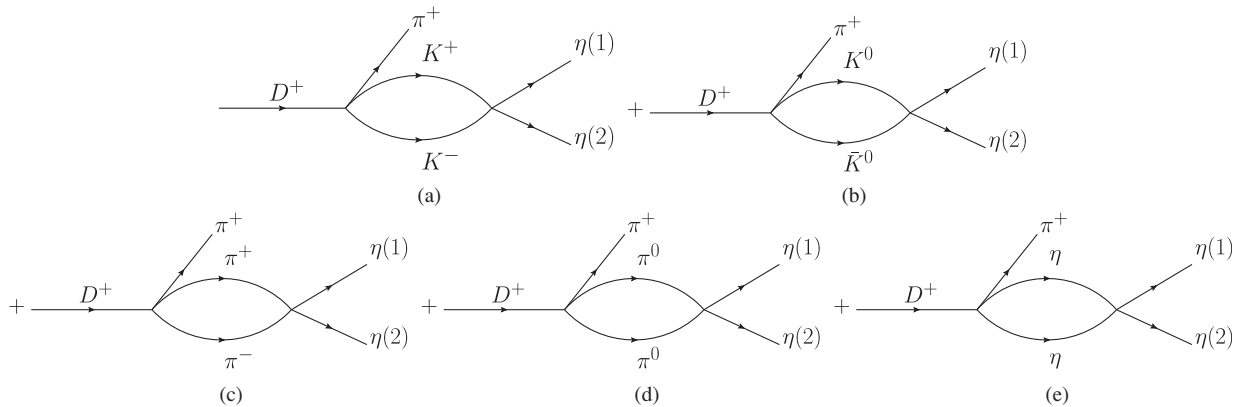


FIG. 8. Rescattering diagrams leading to the $f_0(980)$ production. (a–e), different possibilities for final state interaction.

The t_{a_0} and t_{f_0} contributions depend on different invariant masses. We evaluate the total decay width using the PDG formula [3]

$$\frac{d\Gamma}{dM_{12}M_{23}} = \frac{1}{2} \frac{1}{(2\pi)^3} \frac{1}{32m_{D^+}^3} |t_{a_0} + t_{f_0}|^2 \times 2M_{12}2M_{23}, \quad (40)$$

taking the particles $\eta(1), \eta(2), \pi^+(3)$, and the relationship

$$M_{12}^2 + M_{13}^2 + M_{23}^2 = m_{D^+}^2 + m_{\pi^+}^2 + 2m_{\eta}^2, \quad (41)$$

where the factor $\frac{1}{2}$ comes from having two identical $\eta\eta$ particles at the end. By integrating over one invariant mass with the limits of the PDG [3] and making permutation of the indices 1, 2, 3 in the formulas, one can obtain all the invariant mass distributions.

IV. $f_0(1370)$ CONTRIBUTION

In the work of Ref. [4], the contribution of the $f_0(1370)$ according to the diagram of Fig. 9 is suggested as a way to obtain the higher part of the $\pi^+\eta$ mass spectrum, and its strength is fitted to the experiment.

The $f_0(1370)$ is obtained as a molecular state of $\rho\rho$ in Ref. [16], and, as a consequence, it cannot decay into $\eta\eta$. Yet, in the work of Ref. [17], the approach of Ref. [16] is extended to SU(3), and the state gets a small component of $K^*\bar{K}^*$, which allows the decay into $\eta\eta$. Actually, this decay is reported in the PDG [3] as

$$\frac{\Gamma(\eta\eta)}{\Gamma(4\pi)} = (28 \pm 11) \times 10^{-3}; \quad (4.7 \pm 2) \times 10^{-3}, \quad (42)$$

where the first number corresponds to Ref. [18] and the second one to Ref. [19] and at the same time

$$\text{BR}(D^+ \rightarrow \pi^+ f_0(1370); \quad f_0 \rightarrow \pi^+ \pi^-) = (8 \pm 4) \times 10^{-5}. \quad (43)$$

Taking into account the $f_0 \rightarrow \pi^0 \pi^0$ decay width which, according to isospin, should be 1/2 that of $\pi^+ \pi^-$, we have

$$\text{BR}(D^+ \rightarrow \pi^+ f_0(1370); \quad f_0 \rightarrow \pi\pi) = (12 \pm 6) \times 10^{-5}. \quad (44)$$

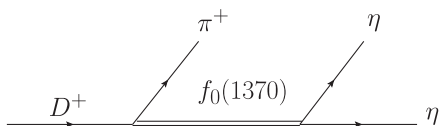


FIG. 9. $D^+ \rightarrow \pi^+ \eta\eta$ decay driven by the $f_0(1370)$ resonance.

The ratio $\Gamma_{\pi\pi}/\Gamma_{\text{tot}}$ in the PDG is contradictory. We then rely on theoretical findings. In Ref. [17], the $\pi\pi$ decay branching ratio is the largest. One obtains $\Gamma_{\text{tot}}(f_0(1370)) \sim 260$ MeV; $\Gamma_{\pi\pi}(f_0(1370)) \sim 210$ MeV. Hence, one finds a ratio

$$\frac{\Gamma_{\pi\pi}}{\Gamma_{\text{tot}}} \simeq 0.81. \quad (45)$$

And assuming that the width is exhausted by the 4π and 2π channels, one finds

$$\frac{\Gamma_{4\pi}}{\Gamma_{\pi\pi}} = 0.24. \quad (46)$$

This number is in agreement with the findings of Ref. [20], $\Gamma_{4\pi}/\Gamma_{\pi\pi} = 0.30 \pm 0.12$, which is also in line with the interval 0.10–0.25 claimed in the experiment of Ref. [21]. We shall take the result of Eq. (45) for our estimates. Taking into account all this information, we find finally, by taking the largest branching ratio of Eq. (42), as follows:

$$\text{BR}(D^+ \rightarrow \pi^+ f_0; f_0 \rightarrow \eta\eta) \approx \frac{12 \times 10^{-5}}{0.81 \times 0.24} \times 28 \times 10^{-3} = 1.7 \times 10^{-5}. \quad (47)$$

With the other branching ratio of Eq. (42), this number would be 2.8×10^{-6} . There are large uncertainties in this estimate, but we should compare this rate with the experimental branching ratio for $D^+ \rightarrow \pi^+ \eta\eta$ found in Ref. [1], as follows:

$$\text{BR}(D^+ \rightarrow \pi^+ \eta\eta) = (3.67 \pm 0.12 \pm 0.06) \times 10^{-3}. \quad (48)$$

We see that in the best of the cases, the rate in Eq. (47) about 200 times smaller than the experiment. Thus, we can safely ignore this mechanism.

V. CONTRIBUTION FROM

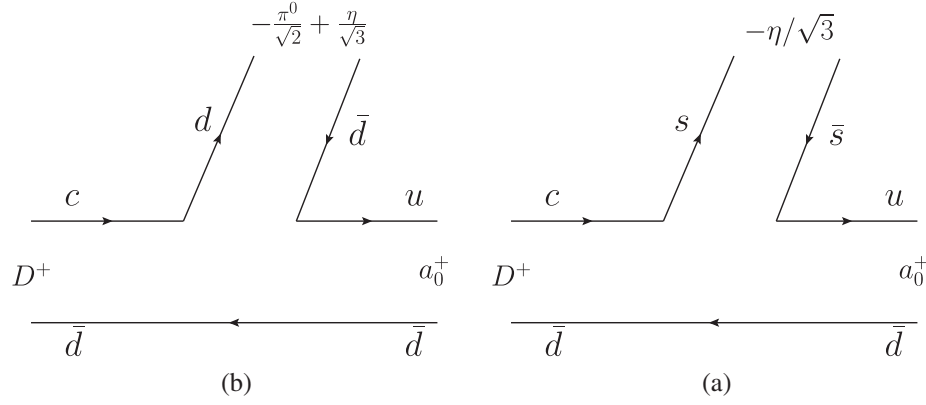
$$D^+ \rightarrow \eta a_0(1450); a_0 \rightarrow \pi^+ \eta$$

There is no $D^+ \rightarrow \eta a_0(1450)$ reported in the PDG, but there is, however, the decay mode $D^+ \rightarrow \pi^0 a_0(1450)$ with the following branching ratio:

$$\text{BR}(D^+ \rightarrow a_0(1450) \pi_0; a^+ \rightarrow \pi^+ \eta) = (1.4 \pm 0.6) \times 10^{-4}. \quad (49)$$

We can envisage the production in Fig. 10.

As we can see, in this Cabibbo suppressed mechanism, the $\pi^0 a_0^+$ can be produced, but the ηa_0^+ production cancels with the sum of these diagrams [note that the $a_0(980)$ would be produced through hadronization of the $u\bar{d}$ pair, but the cancellation occurs anyway].

FIG. 10. Diagrams for $D^+ \rightarrow \pi^0 a_0^+, \eta a_0^+$. (a), (b) two different diagrams of internal emission.

VI. CONTRIBUTIONS FROM

$D^+ \rightarrow f'_2(1525)\pi^+; f'_2(1525) \rightarrow \eta\eta$,
 $D^+ \rightarrow f_2(1270)\pi^+; f_2(1270) \rightarrow \eta\eta$,
 AND $D^+ \rightarrow f_0(1500)\pi^+; f_0(1500) \rightarrow \eta\eta$

In the PDG, there is no $D^+ \rightarrow f'_2(1525)\pi$ reported. However, one finds

$$\text{BR}(D^+ \rightarrow f_2(1270)\pi^+; f_2 \rightarrow \pi^+\pi^-) = (4.58 \pm 0.28) \times 10^{-4}. \quad (50)$$

The $f_2(1270)$ is obtained in Refs. [16] and [17] as a $\rho\rho$ bound state, decaying basically in $\pi\pi$, with a branching fraction of 84%. Since $f_2 \rightarrow \pi^0\pi^0$ has 1/2 the strength of $f_2 \rightarrow \pi^+\pi^-$, hence we have

$$\text{BR}(D^+ \rightarrow f_2(1270)\pi^+; f_2 \rightarrow \pi\pi) = (6.87 \pm 0.42) \times 10^{-4}, \quad (51)$$

and considering the 84% branching ratio of f_2 decaying to $\pi\pi$ we obtain

$$\text{BR}(D^+ \rightarrow f_2(1270)\pi^+) = (8.18 \pm 0.50) \times 10^{-4}. \quad (52)$$

From the molecular picture of Ref. [17], the $D^+ \rightarrow f_2(1270)\pi^+$ can proceed via external emission, as shown in Fig. 11.

The hadronization of $d\bar{d}$ in Fig. 11 to produce two vectors, which will give rise to $f_2(1270)$, is done again as

$$d\bar{d} \rightarrow \sum_i d\bar{q}_i q_i \bar{d} = \sum_i V_{2i} V_{i2} = (V^2)_{22}, \quad (53)$$

where now the $d\bar{q}_i$ becomes a vector meson. The $q_i \bar{q}_j$ matrix written in terms of physical vectors is given by

$$\begin{pmatrix} \frac{\rho_0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & \frac{-\rho_0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}} & K^{*0} \\ K^{*-} & \bar{K}^{*0} & \phi \end{pmatrix}, \quad (54)$$

and then

$$(V^2)_{22} = \rho^-\rho^+ + \left(-\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{2}}\right)^2 + K^{*0}\bar{K}^{*0}. \quad (55)$$

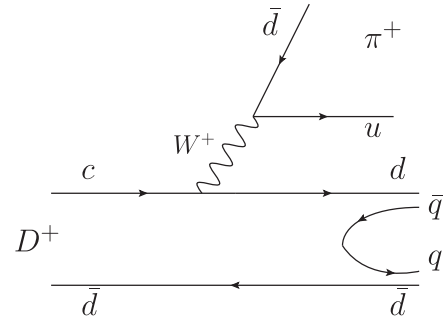
We see that we obtain a $\rho\rho$ component and a $K^*\bar{K}^*$ with the same strength. In Ref. [17], the $f_2(1270)$ is mostly $\rho\rho$ and the $f'_2(1525)$ mostly $K^*\bar{K}^*$. Within the picture of Ref. [17] for these two particles, we should expect similar results for $D^+ \rightarrow \pi^+ f_2(1270)$ and $D^+ \rightarrow \pi^+ f'_2(1525)$, and here we also assume

$$\text{BR}(D^+ \rightarrow \pi^+ f'_2(1525)) \simeq 8 \times 10^{-4}. \quad (56)$$

If now we take from the PDG that

$$\text{BR}(f'_2(1525) \rightarrow \eta\eta) = (10.3 \pm 2.2)\%, \quad (57)$$

then

FIG. 11. Mechanisms for $D^+ \rightarrow \pi^+ f_2(1270)$ through hadronization of the $d\bar{d}$ pair.

$$\text{BR}(D^+ \rightarrow \pi^+ f'_2(1525); f'_2 \rightarrow \eta\eta) \simeq 8 \times 10^{-5}. \quad (58)$$

This is bigger than the rate of $f_2(1270)$ production but still a factor 50 times smaller than the branching ratio of $D^+ \rightarrow \pi^+ \eta\eta$ of Eq. (48). The fact that the f'_2 decay requires the $\eta\eta$ in D -wave to conserve angular momentum and parity also softens the interference with the S -wave main mechanisms described so far. We have written the explicit amplitude for this process and added it coherently to t_{a_0} , and have seen that the correction introduced by the new mechanism is indeed negligible.

We have chosen resonances in the region of 1500 MeV because these are the potential candidates giving relevant signals in the Dalitz plot of $D^+ \rightarrow \pi^+ \eta\eta$. We have looked into $f_2(1270)$ and $f_0(1710)$ resonances, which are both of VV character in Ref. [17], and seen that they barely contribute in the phase space of the $D^+ \rightarrow \pi^+ \eta\eta$ reaction.

For instance, taking into account that [3]

$$\text{BR}(f_2(1270) \rightarrow \eta\eta) = 4 \times 10^{-3}, \quad (59)$$

and using Eq. (52), we find that

$$\text{BR}(D^+ \rightarrow \pi^+ f_2(1270); f_2(1270) \rightarrow \eta\eta) \simeq 3 \times 10^{-6}, \quad (60)$$

which is also negligible.

The mechanism $D^+ \rightarrow \pi^+ f_0(1500); f_0(1500) \rightarrow \eta\eta$ is also equally negligible. Indeed, taking

$$\begin{aligned} \text{BR}(D^+ \rightarrow \pi^+ f_0(1500); \\ f_0(1500) \rightarrow \pi^+ \pi^-) \simeq 1.1 \times 10^{-4}, \end{aligned} \quad (61)$$

together with

$$\text{BR}(f_0(1500) \rightarrow \eta\eta) \simeq 6 \times 10^{-2}, \quad (62)$$

$$\begin{aligned} \text{BR}(f_0(1500) \rightarrow \pi\pi) &= \frac{3}{2} \times \text{BR}(f_0(1500) \rightarrow \pi^+ \pi^-) \\ &\simeq 34.5 \times 10^{-2}, \end{aligned} \quad (63)$$

we obtain

$$\text{BR}(D^+ \rightarrow \pi^+ f_0(1500); f_0(1500) \rightarrow \eta\eta) \simeq 2.9 \times 10^{-5}, \quad (64)$$

again a negligible magnitude.

After this discussion, in the next section we discuss the results of the triangle singularity and the $a_0(980)$, $f_0(980)$ contributions.

VII. RESULTS

A. $a_0(980)$ contribution

In Fig. 12, we have the results that we obtain for the $D^+ \rightarrow \pi^+ \eta\eta$ mass distributions with the t_{a_0} amplitude of Eq. (37). We have taken $A = 1$ and $B = 1/3$.

We consider the $M_{\pi^+ \eta}$ distribution by looking at just one η , such that the integral over the mass of Figs. 12(a) and 12(b) are the same. When comparing with the data later, we will sum the $\pi^+ \eta(1)$ and $\pi^+ \eta(2)$ distributions. What we observe is that the shapes of the two mass distributions are very different than the experimental data shown in Fig. 15. The shape of $a_0(980)$ mass distribution resembles the shape seen in other experiments, and we note that there is a large discrepancy with the experimental results both at low and particularly at large $M_{\pi^+ \eta}$ masses. The other relevant feature in this figure is that the $\eta\eta$ mass distribution has a large peak around 1600 MeV in sheer discrepancy with the data. We shall see later that both problems are solved when we include the contribution of the $f_0(980)$ resonance, t_{f_0} of Eq. (38).

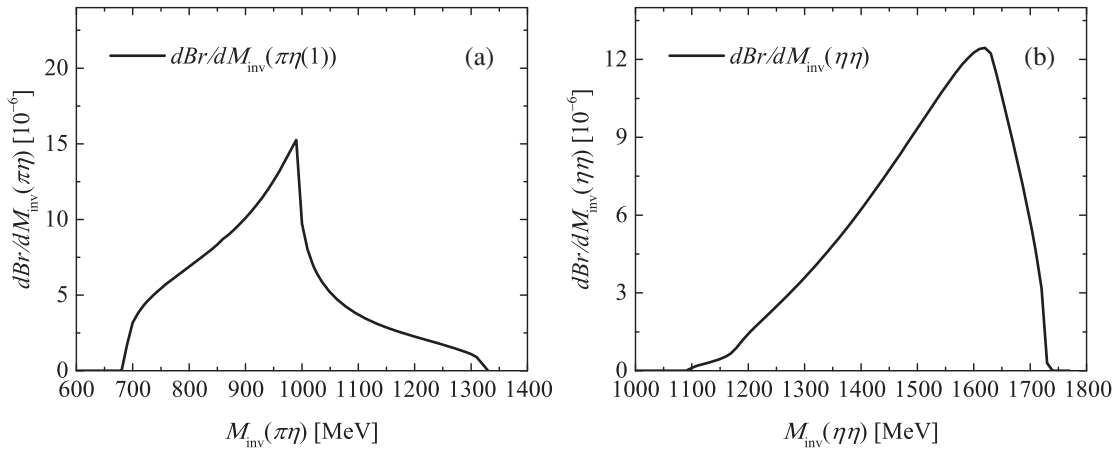


FIG. 12. (a) $\pi^+ \eta$ mass distribution, (b) $\eta\eta$ mass distribution considering only the $a_0(980)$ contribution.

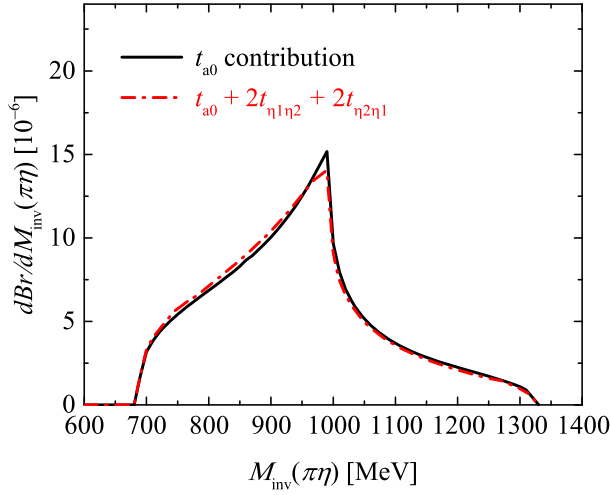


FIG. 13. Mass distribution of $\pi^+\eta$ with and without the triangle mechanism.

B. Triangle mechanism contribution

We have taken the formula of Eq. (20) and integrated over the two invariant masses and the TS amplitude, and we obtain the result

$$\text{BR}(D^+ \rightarrow \pi^+\eta\eta(TS)) = 6.16 \times 10^{-5}. \quad (65)$$

As we can see, the rate obtained, which could be calculated with absolute values, is very small compared to the experimental rate of Eq. (48).

We have also looked to see if the coherent sum of amplitudes could give some visible effect. For this we take $A + 2B$ of Eq. (37) such as to get the integrated branching ratio for $D^+ \rightarrow \pi^+\eta\eta$ with the t_{a_0} term alone, and then sum the amplitudes $2t_{\eta_1\eta_2} + 2t_{\eta_2\eta_1}$ of Eq. (20) and show the mass distribution, in Fig. 13.

As we can see, even summing coherently the $a_0(980)$ and the triangle diagram coherently, the effects of the TS mechanisms are negligible.

VIII. $f_0(980)$ CONTRIBUTION

The $f_0(980)$ amplitude comes from the $t_{K\bar{K},\eta\eta}, t_{\pi\pi,\eta\eta}, t_{\eta\eta,\eta\eta}$. Note that the same amplitudes automatically account for the $f_0(500)$ contribution. These amplitudes will provide a contribution to the $\eta\eta$ mass distribution starting from the $\eta\eta$ threshold. Yet, as one can see in Fig. 15, the experimental $\eta\eta$ mass distribution stretches up to 1700 MeV where the chiral unitary approach does not provide good amplitudes. When this happens, it is customary to cut the amplitudes and extrapolate them adiabatically, reducing them gradually as $M_{\text{inv}}(\eta\eta)$ grows up, as a realistic resonance amplitude would do. This is done in Refs. [22–24] using the formula

$$G(M_{\text{inv}})t_{ij}(M_{\text{inv}}) \rightarrow G(M_{\text{cut}})t_{ij}(M_{\text{cut}})e^{-\alpha(M_{\text{inv}}-M_{\text{cut}})}, \quad (66)$$

$$M_{\text{inv}} > M_{\text{cut}},$$

with values of $M_{\text{cut}} \sim 1050\text{--}1150$ MeV and $\alpha \sim 0.0040\text{--}0.0080$ MeV $^{-1}$. We will calculate the mass distributions within this range to evaluate uncertainties of our approach. Since the strength of the a_0 and f_0 are tied to the coefficients A and B , and interfere in the mass distributions, we should make the same cuts to the a_0 and f_0 amplitudes implementing Eq. (66) to the $\pi\eta$ and $\eta\eta$ invariant masses in the amplitudes, and this is done in the subsequent calculations.

In Fig. 14, we show the results for the $D^+ \rightarrow \pi^+\eta\eta$ decay considering only the t_{f_0} amplitude of Eq. (38), which represents the contribution from the $f_0(980)$ resonance. As in Fig. 12, the invariant mass in Figs. 14(a) and 14(b) are taken to be the same. The calculations are done using the same values of A and B in Fig. 12, $A = 1, B = 1/3$.

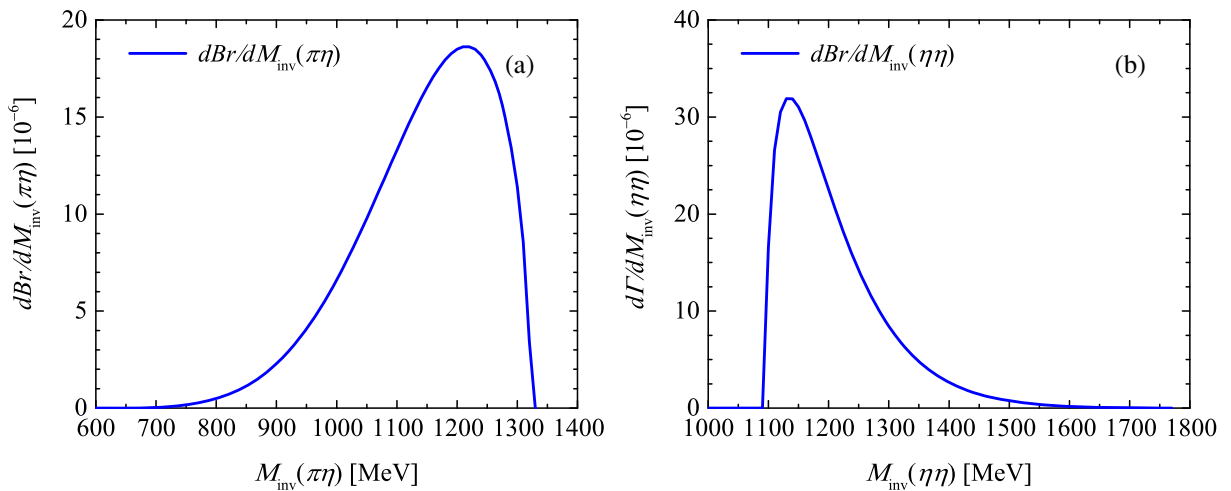


FIG. 14. (a) $\pi^+\eta$ mass distribution, (b) $\eta\eta$ mass distribution considering only the $f_0(980)$ contribution.

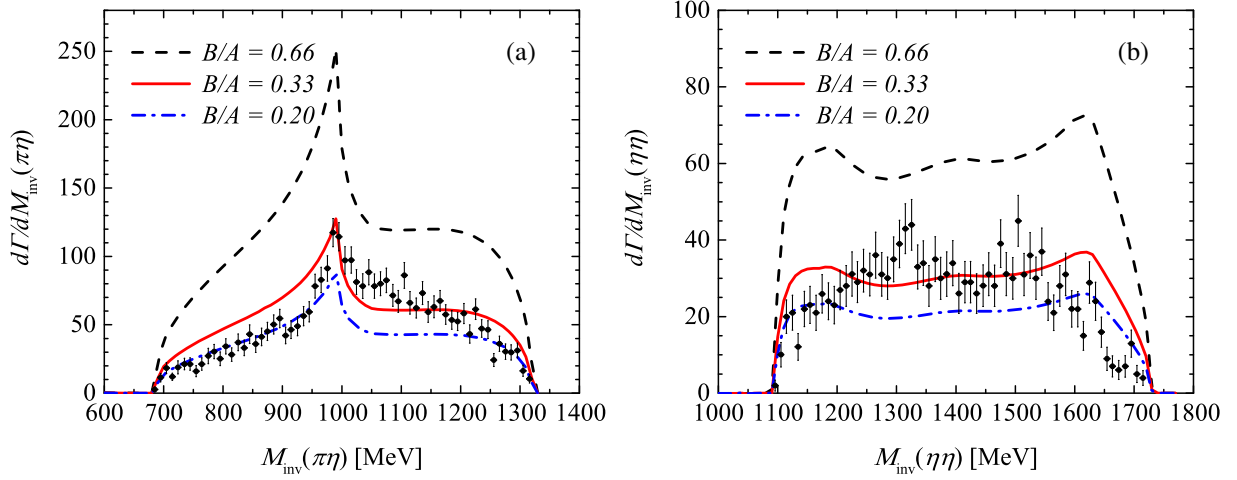


FIG. 15. (a) $M_{\pi^+\eta}$ and (b) $M_{\eta\eta}$ mass distributions obtained with $q_{\text{max}} = 650$ MeV, $M_{\text{cut}} = 1050$ MeV, $\alpha = 0.0060$ MeV $^{-1}$, and different B/A ratios. The y axis corresponds to events per 10 MeV of Ref. [1].

This allows us to compare the relative strengths of both mechanisms. The first thing that we realize is that the strength of the f_0 in the mass distributions is similar to that of the a_0 . As expected, the effect of the $f_0(980)$ is clearly seen in the $\eta\eta$ invariant mass distribution, producing a pronounced enhancement close to 1.0–1.1 GeV, which is most welcome in view of the experimental mass distribution. At the same time, it provides a sizeable contribution at large $\pi\eta$ invariant masses, again most welcome by the experiment.

In Fig. 15, we show the results adding t_{a_0} and t_{f_0} of Eqs. (37) and (38). Here, we need to differentiate between A and B . Hence, apart from the absolute normalization of the mass distribution, we need the ratio B/A . This ratio should be of the order of $1/N_c$ ($N_c = 3$, the number of the colors), but in order to make estimates of uncertainties we shall vary

$$B/A \in [0.2 - 0.66]. \quad (67)$$

We have also tried with B/A negative but the results are clearly bad, refusing this option.

First, we show the results in Fig. 15, taking as a reference $M_{\text{cut}} = 1050$ MeV, $\alpha = 0.0060$ MeV $^{-1}$, taking different values of B/A .

In Fig. 15, we have already summed the mass distributions $\pi^+\eta(1)$ and $\pi^+\eta(2)$ as in the experiment [1], such that the integrated $\pi^+\eta$ mass distribution has double strength than that of $\eta\eta$. The strength of A is adjusted to have an integrated mass distribution equal to that of the counts in the experiment for $B/A = 0.33$.

What we observe is that now we get a substantial strength in the $\pi^+\eta$ mass distribution at large $\pi^+\eta$ invariant masses, getting a shape roughly in agreement with the experiment. On the other hand, the $\eta\eta$ mass distribution is also tremendously improved, obtaining strength at lower $\eta\eta$ invariant masses that did not appear with the mechanism of the $a_0(980)$ alone.

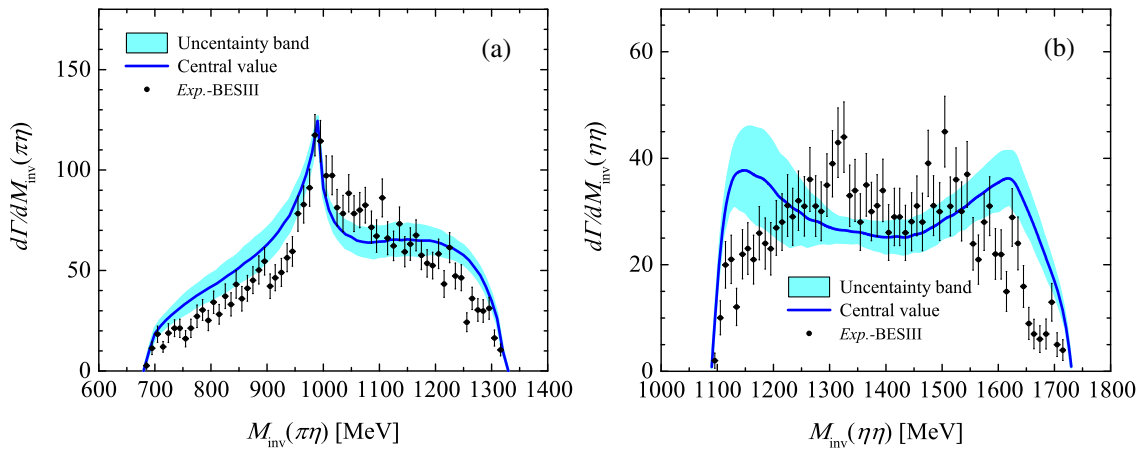


FIG. 16. Mass distributions of the $\pi\eta$ (a) and $\eta\eta$ systems (b) at $q_{\text{max}} = 650$ MeV, showing the uncertainty band for $M_{\text{cut}} = 1050$ –1150 MeV, $\alpha = 0.0040$ –0.0080 MeV $^{-1}$, and $B/A = 0.2$ –0.66. The y axis corresponds to events per 10 MeV of Ref. [1].

We should note that the two mechanisms of $a_0(980)$ and $f_0(980)$ production are linked in our approach. Up to an overall normalization, their relative strength is tied to the ratio of B/A , which should not be too far away from $1/3$. Then, Fig. 15 gives us an idea of the contribution that we have. In order to have a band of uncertainties of our results, we make a few runs of calculations choosing random numbers of B/A , M_{cut} , and α in the range

$$\begin{aligned}\alpha &\in [0.0040 - 0.0080] \text{ MeV}^{-1}, \\ M_{\text{cut}} &\in [1050 - 1150] \text{ MeV}, \\ B/A &\in [0.3 - 0.8].\end{aligned}\quad (68)$$

The results are shown in Fig. 16, but all of them have been normalized to the experimental integrated decay width. We remove 16% of the results in the lower and upper extremes to have 68% confidence level.

Considering the theoretical error band, the agreement with the data can be considered fair, as good as one could expect within the approximations done. Nevertheless, the obvious improvements obtained when considering the $f_0(980)$ contribution tell us unequivocally that the $f_0(980)$ excitation with decay to $\eta\eta$ has been essential to reproduce the experimental features, in particular the large strength in the $\pi^+\eta$ mass distribution at large invariant masses, which make the shape of the $a_0(980)$ very different from that in other experiments as noted in Ref. [1], and the large strength of the $\eta\eta$ mass distribution at low $\eta\eta$ invariant masses.

IX. CONCLUSIONS

We have made a thorough investigation of possible contributions to the $D^+ \rightarrow \pi^+\eta\eta$ decay. The motivation of the work stems from the recent experiment in BESIII [1], observing a $\pi^+\eta$ mass distribution very different than that observed in most experiments producing the $a_0(980)$. The large strength in the higher mass region of the $\pi^+\eta$ mass spectrum prompted the authors of the BESIII team to suggest that it should be due to a mechanism of production based on a triangle diagram, induced by the D^+ decay into $K_0^*(1430)$ and $\bar{K}$, with subsequent decay of $K_0^*(1430)$ into ηK and final fusion of the $K\bar{K}$ to produce the $a_0(980)$, which decays to $\pi^+\eta$. In a following work [4], the triangle mechanism was dismissed, because it does not develop a triangle singularity for the nominal mass of the $K_0^*(1430)$, and instead, the $D^+ \rightarrow \pi^+ f_0(1370)$; $f_0 \rightarrow \eta\eta$ mechanism was suggested to interpret the results. We have done a thorough study of all these suggested solutions and concluded that the triangle mechanism develops, indeed, a triangle singularity, taking into account the large width of the $K_0^*(1430)$ state. However, we were able to make a determination of the width in absolute terms, not just the shape, and found that its contribution to the process is 2 orders of magnitude smaller than the experimental branching ratio. Similarly, we also studied the $D^+ \rightarrow \pi^+ f_0(1370)$;

$f_0 \rightarrow \eta\eta$ mechanism suggested in Ref. [4] and could also determine its strength in absolute terms, finding also a contribution 2 orders of magnitude smaller than the experiment. We also studied the $D^+ \rightarrow \eta a_0(1450)$; $a_0 \rightarrow \eta\pi^+$, and found cancellations that make null its contribution. Similarly, we also studied the possible contributions of $D^+ \rightarrow \pi^+ f_2'(1525)$; $f_2' \rightarrow \eta\eta$, $D^+ \rightarrow f_2(1270)\pi^+$; $f_2(1270) \rightarrow \eta\eta$, and $D^+ \rightarrow f_0(1500)\pi^+$; $f_0(1500) \rightarrow \eta\eta$ and also found them extremely small.

Surprisingly, we found a natural solution to the problem by looking at the related, unavoidable contribution of the $f_0(980)$ excitation, which is linked to the $a_0(980)$ production. The $f_0(980)$ resonance, decaying to $\eta\eta$, is only operative through the tail of the resonance, but the weight of the amplitude is larger than that for $a_0(980)$ production and, at the end, the two mechanisms are competitive. The contribution of the $f_0(980)$ solves the problem of the large strength obtained at high invariant masses in the $\pi^+\eta$ mass distribution, and also fixes the problem of the small strength at low invariant masses of the $\eta\eta$ mass distribution, stemming from the $a_0(980)$ production.

The reaction studied also brings information on the nature of the scalar mesons, a subject that has attracted permanent attention [5,25–29], and has also links to the role that these resonances play in leptonic and semileptonic decays [30–32].

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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