

Covariant fractons and Weitzenböck torsion

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(Received 4 September 2025; accepted 15 October 2025; published 6 November 2025)

The relation between covariant fracton gauge theory and Møller-Hayashi-Shirafuji theory of gravity is investigated. The former is the gauge theory of a rank-two symmetric tensor with gauge symmetry given by the double derivative of a scalar parameter; the latter is the most general theory, whose action is quadratic in the Weitzenböck torsion. We show that the solutions of covariant fracton gauge theory describe a subsector of the space of solutions of Møller-Hayashi-Shirafuji theory, providing a new insight in the relation between covariant fractons and gravity, and elucidating the meaning of covariant fracton theory as a new type of gauge theory.

DOI: [10.1103/y426-6lkj](https://doi.org/10.1103/y426-6lkj)

I. INTRODUCTION AND OUTLOOK

In this paper we pursue the investigation on covariant fracton gauge theory and its properties, studying the relation between covariant fractons and teleparallel gravity.

Fracton phases, which are excitations characterized by restricted mobility due to the conservation of the dipole moment [1–3], were firstly introduced in condensed matter physics [4] but attracted interest in a broader context. In particular, covariant fracton gauge theory, introduced for the first time in [5], is the gauge theory of a rank-two symmetric tensor $h_{\mu\nu}^{(\text{fr})}$ with gauge symmetry given by the double derivative of a scalar parameter

$$\delta h_{\mu\nu}^{(\text{fr})} = \partial_\mu \partial_\nu \lambda. \quad (1.1)$$

Covariant fracton gauge theory elucidates the main properties of ordinary fracton excitations, in the context of the higher rank extensions of Maxwell theory [6], and it can be thought as an extension of linearized Einstein gravity, invariant with respect to longitudinal diffeomorphisms [7,8]. Recently, three-dimensional Chern-Simons-like models with covariant fracton gauge symmetry [9] and the analog of the electromagnetic duality in covariant fracton gauge theory [10] have been investigated.

The most general quadratic action in the derivatives of $h_{\mu\nu}^{(\text{fr})}$, invariant with respect to (1.1), can be written in terms

of the two possible scalar contractions of the gauge invariant fracton field strength [11]¹

$$f_{\mu\nu\rho}^{(\text{fr})} = \partial_\mu h_{\nu\rho}^{(\text{fr})} - \partial_\nu h_{\mu\rho}^{(\text{fr})}, \quad (1.2)$$

which is analogous to the field strength in electromagnetism. The action reads

$$S_{\text{fr}} = \int d^d x (\alpha f^{(\text{fr})\mu\nu\rho} f_{\mu\nu\rho}^{(\text{fr})} + \beta f^{(\text{fr})\mu\nu}{}_\nu f^{(\text{fr})\mu\rho}{}_\rho), \quad (1.3)$$

where α and β are arbitrary constants. Taking α and β such that $\frac{\beta}{\alpha} = -2$, and replacing the fracton gauge field with the perturbation of the metric in linearized gravity, one gets the Fierz-Pauli action [see Eq. (2.12)] [14], which is invariant under the full group of linearized diffeomorphisms.

Since the whole dynamics of covariant fracton gauge theory is encoded in the fracton field strength, one might ask which its gravitational analog is. We observe in this paper that the covariant fracton field strength is the analog of the linearized *Weitzenböck torsion* in gravity, when the linearized vielbein has vanishing totally anti-symmetric part. The first attempts of a torsion formulation of general relativity, usually known as teleparallel equivalent of general relativity, date back to Einstein [15], who unsuccessfully tried to unify gravitation and electromagnetism in four dimensions, using the 16 degrees of freedom of the vierbein, according to Cartan’s [16–19] and

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¹ $h_{\mu\nu}^{(\text{fr})}$ and $f_{\mu\nu\rho}^{(\text{fr})}$ are respectively denoted with $h_{\mu\nu}$ and $f_{\mu\nu\rho}$ in [11]. A formally equivalent tensor $f_{\mu\nu\rho}$ appeared previously in [12], where the theory of a partially massless spin-two field in de Sitter space is investigated. It appears also in [13] at the linear level.

Weitzenböck's [20] works. For the history and prehistory of the teleparallel gravity see for example [21,22]. In teleparallel gravity the dynamical variables are encoded in the torsion of a flat connection, known as the Weitzenböck connection:

$$W_{\mu}{}^e{}_{\nu} = e^e{}_a \partial_{\mu} e_{\nu}{}^a, \quad (1.4)$$

where $e_{\mu}{}^a$ is the vielbein, and $e^{\mu}{}_a$ its inverse. The Weitzenböck torsion, which is the antisymmetric part of the previous connection

$$T_{\mu}{}^e{}_{\nu} = W_{[\mu}{}^e{}_{\nu]} = e^e{}_a \partial_{[\mu} e_{\nu]}{}^a, \quad (1.5)$$

is a well-defined covariant tensor, which is very useful for its geometrical implications: namely, it was employed in the geometrization of gauge symmetries in the context of Scherk-Schwarz compactifications [23], and in generalized geometry [24–26]. Møller studied extensions of general relativity in teleparallel formulation, by considering the wider class of theories defined by the most general action quadratic in the Weitzenböck torsion [27–29]. At the same time, independently, the possible phenomenological implications of the model were first studied by Hayashi and Shirafuji [30]. The model can be also viewed as the gauge theory of translations [31]. The action for the theory considered by Møller and by Hayashi and Shirafuji, which we will call Møller-Hayashi-Shirafuji theory (MHS), reads

$$S_{\text{MHS}} = \frac{1}{2} \int d^d x e (\alpha_1 T^{\mu\nu\rho} T_{\mu\nu\rho} + \alpha_2 T^{\mu\nu\rho} T_{\mu\rho\nu} + \alpha_3 T^{\mu\nu}{}_{\nu} T_{\mu\rho}{}^{\rho}), \quad (1.6)$$

where $\alpha_1, \alpha_2, \alpha_3$ are arbitrary constants.

We will show that, when $\frac{\beta}{\alpha} \neq 2$, the solutions of the equations of motion of covariant fracton gauge theory can be viewed as the subsector in the space of solutions of the equations of motion of linearized MHS theory, identified by the vanishing antisymmetric part of the linearized vielbein, denoted with $b_{\mu\nu}$,

$$b_{\mu\nu} = 0. \quad (1.7)$$

The embedding of covariant fracton solutions in MHS theory is realized when the fracton gauge field $h_{\mu\nu}^{(\text{fr})}$ is identified with the symmetric part of the linearized vielbein, denoted with $h_{\mu\nu}$,

$$h_{\mu\nu}^{(\text{fr})} = h_{\mu\nu}, \quad (1.8)$$

and the constants α and β in the covariant fracton action (1.3) are identified with combinations of the constant α_1, α_2 , and α_3 in the MHS action (1.6),

$$\alpha = \frac{1}{4}(2\alpha_1 + \alpha_2), \quad \beta = \frac{1}{2}\alpha_3. \quad (1.9)$$

To show this, focusing on four dimensions for definiteness, we will compare the particle spectrum of both theories. Except for some particular cases, tuned by special values of the constants α_1, α_2 , and α_3 , such that the gauge invariance is enlarged, the spectrum of MHS theory in four dimensions consists in 6 degrees of freedom, given by a particle with helicity ± 2 , one with helicity ± 1 , and two scalar particles. The antisymmetric field $b_{\mu\nu}$ describes only a scalar particle (as expected, since a two-form gauge field in four dimensions is dual to a scalar field). The sector with vanishing $b_{\mu\nu}$ coincides with the particle spectrum of covariant fracton gauge theory, which indeed describes the remaining 5 propagating degrees of freedom [8], except for the particular cases in which the gauge symmetry is enlarged.

Linearized diffeomorphisms act on $b_{\mu\nu}$ as a one-form gauge transformation for a two-form field (usually known as Ramond-Kalb field [32]). The condition (1.7) is not diffeomorphism invariant, but it leaves a residual gauge symmetry when the gauge parameter is exact, that is, when it is a derivative. This is the case of longitudinal diffeomorphisms, so that the residual gauge symmetry is the fracton gauge symmetry. In a sense, covariant fracton solutions spontaneously break the diffeomorphism symmetry of MHS theory, preserving only the fracton gauge symmetry.

To elucidate this setting, we will take into account the Becchi-Rouet-Stora-Tyutin (BRST) formulation [33–36] of covariant fracton gauge theory. In the BRST setup, the gauge parameter is replaced by an anticommuting scalar field (fracton ghost) λ , and the gauge transformation is realized by a nilpotent differential operator s on the space of the fracton gauge field and of the ghost and their derivatives, evaluated in the same spacetime point. The local s -cohomology, which classifies the consistent anomalies, is generated by the undifferentiated ghost and by the fracton field strength and its derivatives, as in the analogous case of electromagnetism [11].

We will show that the BRST transformations of covariant fracton gauge theory can be conveniently formulated in an equivalent way, by including two BRST trivial doublets. In this formulation a symmetric rank-two tensor $h_{\mu\nu}$ and an antisymmetric one $b_{\mu\nu}$ are included. They can be respectively viewed as the symmetric and the antisymmetric parts of the linearized vielbein, but the BRST transformations do not involve any local Lorentz transformation: this is the case of Weitzenböck geometry, which is flat and it is invariant only under global Lorentz transformations [37]. Nevertheless, on the one side, the BRST transformations require the restriction of linearized diffeomorphisms to the “longitudinal” ones, bringing us to covariant fracton gauge transformations as they were first introduced [5]; on the

other side, $b_{\mu\nu}$ is equivalent to zero. This individuates the subsector of MHS theory in which covariant fracton gauge theory lives.

Covariant fractons were introduced in [5] only at linear level. In [38] it is shown that the theory is afflicted by instabilities, which in principle can be cured by considering suitable interactions. Going towards this direction, the result in this note can be viewed as a possible way to embed covariant fracton theory in a larger theory of gravity, whose nonlinear extension is automatically given by the extension of general relativity in teleparallel formulation, studied by Møller, Hayashi, and Shirafuji.

A possible nonlinear extension of covariant fracton gauge theory is also discussed in [13], where the Frölicher-Nijenhuis bracket is employed, acting on vector-valued differential forms and coinciding with the Lie derivative between vector fields when it acts on vectors [39–41]. The nonlinear extension of the fracton gauge transformation is realized by using a vector parameter, which is restricted to be the derivative of a scalar parameter only at the linear level; the gauge field is replaced by a vector-valued one-form, and a constraint is added in order to make it symmetric; a possible nonlinear equation of motion is postulated, without an action from which to deduce it; the linearization of such an equation is the equation of motion of covariant fracton gauge theory [with $\beta = 0$ in (1.3)].

It would be interesting to exploit the relation between covariant fractons and Weitzenböck torsion in understanding deeper this nonlinear extension. In particular, one could try to see if it is possible to relate the postulated nonlinear equation of motion with the equations of MHS theory, since it seems natural to identify the vector-valued one-form with the linearized vielbein.

The rest of the paper is organized as follows: in Sec. II covariant fracton theory is formulated in a way which makes manifest the relation with torsion formulation of gravity; in Secs. III, IV, and V torsion formulation of general relativity, the MHS theory, and its linearization are reviewed; in Secs. VI and VII the particle spectrum of covariant fracton and MHS theory are computed and compared, providing the embedding of covariant fractons in MHS theory; and finally, in Sec. VIII the equivalent BRST formulation is investigated. In Appendix A there is a summary of Weitzenböck geometry; in Appendix B a computation for BRST cohomology is shown; Appendix C is devoted to a comparison between different conventions; and in Appendix D there is a summary of polarization matrices.

II. COVARIANT FRACTON GAUGE THEORY IN “TORSIONLIKE FORMULATION”

In this section covariant fracton gauge theory [5] is formulated in a way in which its embedding in gravitational theories in torsion formulation will be natural.

Covariant fracton gauge theory is defined by a symmetric rank-two tensor $h_{\mu\nu}^{(\text{fr})}$ and by a scalar gauge parameter, promoted to an anticommuting ghost λ in BRST formalism [33–36]. The nilpotent BRST transformations are

$$sh_{\mu\nu}^{(\text{fr})} = \partial_\mu \partial_\nu \lambda, \quad s\lambda = 0. \quad (2.1)$$

One defines the fracton field strength

$$f_{\mu\nu\rho}^{(\text{fr})} := \partial_\mu h_{\nu\rho}^{(\text{fr})} - \partial_\nu h_{\mu\rho}^{(\text{fr})}, \quad (2.2)$$

which is antisymmetric in the first two indices, and whose totally antisymmetric part vanishes (algebraic Bianchi identity). Moreover, the fracton field strength is BRST invariant and it satisfies a differential Bianchi identity:

$$sf_{\mu\nu\rho}^{(\text{fr})} = 0, \quad (2.3a)$$

$$f_{\mu\nu\rho}^{(\text{fr})} + f_{\nu\rho\mu}^{(\text{fr})} + f_{\rho\mu\nu}^{(\text{fr})} = 0, \quad (2.3b)$$

$$\partial_\sigma f_{\mu\nu\rho}^{(\text{fr})} + \partial_\mu f_{\nu\rho\sigma}^{(\text{fr})} + \partial_\nu f_{\sigma\mu\rho}^{(\text{fr})} = 0. \quad (2.3c)$$

The most general invariant quadratic action in the derivative of $h_{\mu\nu}^{(\text{fr})}$ can be written in terms of the fracton field strength. There are only two independent quadratic scalar contractions of the fracton field strength, since it transforms in the mixed Young tableau (2,1).² So, the most general quadratic invariant action can be written as

$$\begin{aligned} S_{\text{fr}} &= \int d^d x \left(\alpha f^{(\text{fr})\mu\nu\rho} f_{\mu\nu\rho}^{(\text{fr})} + \beta f^{(\text{fr})\mu\nu} f_{\mu\nu}^{(\text{fr})\rho} \right) \\ &= \int d^d x \left(2\alpha \partial^\rho h^{(\text{fr})\mu\nu} \partial_\rho h_{\mu\nu}^{(\text{fr})} - 2 \left(\alpha - \frac{\beta}{2} \right) \partial^\nu h_{\mu\nu}^{(\text{fr})} \partial_\rho h^{(\text{fr})\mu\rho} \right. \\ &\quad \left. + \beta \partial_\mu h^{(\text{fr})\nu} \partial^\mu h^{(\text{fr})\rho} - 2\beta \partial_\mu h^{(\text{fr})\nu} \partial_\rho h^{(\text{fr})\mu\rho} \right), \end{aligned} \quad (2.4)$$

for arbitrary constants α and β . The corresponding equations of motion require the following symmetric tensor to vanish:

$$\begin{aligned} E_{\mu\nu}^{(\text{fr})} &:= -\frac{1}{2} \frac{\delta S_{\text{fr}}}{\delta h^{(\text{fr})\mu\nu}} = \alpha \partial_\rho f^{(\text{fr})\rho}{}_{(\mu\nu)} - \frac{\beta}{2} \partial_{(\mu} f^{(\text{fr})}{}_{\nu)\rho}{}^\rho \\ &\quad + \beta \eta_{\mu\nu} \partial_\rho f^{(\text{fr})\rho\sigma}{}_\sigma \end{aligned} \quad (2.5)$$

$$\begin{aligned} &= 2\alpha \partial^2 h_{\mu\nu}^{(\text{fr})} - \frac{2\alpha - \beta}{2} \partial_{(\mu} \partial^2 h_{\nu)\rho}^{(\text{fr})} - \beta \partial_\mu \partial_\nu h^{(\text{fr})\rho}{}_\rho \\ &\quad + \beta \eta_{\mu\nu} (\partial^2 h^{(\text{fr})\rho}{}_\rho - \partial_\rho \partial_\sigma h^{(\text{fr})\rho\sigma}) \end{aligned} \quad (2.6)$$

²We denote with $(\lambda_1, \dots, \lambda_n)$ the Young tableau with n rows and λ_i boxes in the i th row, $i = 1, \dots, n$. $(\lambda_1, \dots, \lambda_n)_t$ is the corresponding traceless part.

where the round brackets among indices denote symmetrization *without* any numerical coefficient.³ It is useful to define the following tensor:

$$\begin{aligned} R_{\mu\nu\varrho\sigma}^{(\text{fr})} &= \frac{1}{2}\partial_\sigma\partial_\mu h_{\nu\varrho}^{(\text{fr})} - \frac{1}{2}\partial_\sigma\partial_\nu h_{\mu\varrho}^{(\text{fr})} - \frac{1}{2}\partial_\varrho\partial_\mu h_{\nu\sigma}^{(\text{fr})} + \frac{1}{2}\partial_\varrho\partial_\nu h_{\mu\sigma}^{(\text{fr})} \\ &= \frac{1}{2}\partial_\sigma f_{\mu\nu\varrho}^{(\text{fr})} - \frac{1}{2}\partial_\varrho f_{\mu\nu\sigma}^{(\text{fr})}, \end{aligned} \quad (2.7)$$

which enjoys the same symmetries as the Riemann tensor, being antisymmetric in $\mu\nu$ and in $\varrho\sigma$ and symmetric in $\mu\nu \leftrightarrow \varrho\sigma$. The corresponding “Ricci tensor” and “Ricci scalar” respectively read⁴

$$R_{\mu\nu}^{(\text{fr})} = R^{(\text{fr})\varrho}{}_{\mu\varrho\nu} = -\frac{1}{4}\partial_\varrho f^{(\text{fr})\varrho}{}_{(\mu\nu)} - \frac{1}{4}\partial_{(\mu} f^{(\text{fr})}{}_{\nu)\varrho}{}^\varrho, \quad (2.8)$$

$$R^{(\text{fr})} = R^{(\text{fr})}{}^\mu{}_\mu = -\partial_\mu f^{(\text{fr})\mu\nu}{}_\nu. \quad (2.9)$$

Writing $\partial_{(\mu} f^{(\text{fr})}{}_{\nu)\varrho}{}^\varrho$ and $\partial_\mu f^{(\text{fr})\mu\nu}{}_\nu$ in terms of the “Ricci tensor” and of the “Ricci scalar” in the equations of motion (2.5), one recognizes the analog of the linearized Einstein tensor:

$$E_{\mu\nu}^{(\text{fr})} = 2\beta\left(R_{\mu\nu}^{(\text{fr})} - \frac{1}{2}R^{(\text{fr})}\eta_{\mu\nu}\right) + \left(\alpha + \frac{\beta}{2}\right)\partial_\varrho f^{(\text{fr})\varrho}{}_{(\mu\nu)}. \quad (2.10)$$

Indeed, if one replaces $h_{\mu\nu}^{(\text{fr})}$ in the action (2.4) with a rank-two symmetric tensor $h_{\mu\nu}$, thought as the perturbation of a metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and α and β are such that

$$\frac{\beta}{\alpha} = -2, \quad (2.11)$$

choosing the normalization $\alpha = \frac{1}{2}$, one recovers the massless Fierz-Pauli action [14]

$$\begin{aligned} S_{\text{FP}} &= \frac{1}{2} \int d^d x (\partial_\varrho h_{\mu\nu} \partial^\varrho h^{\mu\nu} - 2\partial^\nu h_{\mu\nu} \partial_\varrho h^{\mu\varrho} + 2\partial_\mu h_\nu{}^\nu \partial_\varrho h^{\mu\varrho} \\ &\quad - \partial_\mu h_\nu{}^\nu \partial^\mu h_\varrho{}^\varrho). \end{aligned} \quad (2.12)$$

Consistently, imposing the condition (2.11) in the equations of motion (2.10), the last term vanishes and one gets the linearized Einstein equations.

Using the equations of motion in the form (2.10), it is manifest that the first divergence is not identically zero, unless in the Fierz-Pauli case, since only the analog of the Einstein tensor is transverse:

³For example, $A_{(\mu\nu)} = A_{\mu\nu} + A_{\nu\mu}$. Similarly, $A_{[\mu\nu]} = A_{\mu\nu} - A_{\nu\mu}$.

⁴Consider that $\partial_{[\mu} f^{(\text{fr})}{}_{\nu]\varrho}{}^\varrho - \partial_\varrho f^{(\text{fr})\varrho}{}_{[\mu\nu]} = 0$ as a consequence of (2.3b)–(2.3c).

$$\partial^\mu E_{\mu\nu}^{(\text{fr})} = \left(\alpha + \frac{\beta}{2}\right)\partial^\mu \partial_\varrho f^{(\text{fr})\varrho}{}_{(\mu\nu)}. \quad (2.13)$$

Instead, one gets a transversality condition for all values of α and β by taking the second divergence, as a consequence of the invariance under the fracton gauge transformations (2.1),

$$\partial^\mu \partial^\nu E_{\mu\nu}^{(\text{fr})} = \left(\alpha + \frac{\beta}{2}\right)\partial^\mu \partial^\nu \partial_\varrho f_{\mu\nu\varrho}^{(\text{fr})} = 0, \quad (2.14)$$

which vanishes since the fracton field strength is antisymmetric in its first two indices.

III. REVIEW OF TELEPARALLEL GRAVITY

In the usual formulation of general relativity, one chooses a connection $\Gamma_\mu{}^\varrho{}_\nu$ compatible with the metric $g_{\mu\nu}$, such that the torsion vanishes and the curvature encodes the dynamical variables:

$$\nabla_\varrho(\Gamma)g_{\mu\nu} = \partial_\varrho g_{\mu\nu} - \Gamma_\varrho{}^\sigma{}_\mu g_{\sigma\nu} - \Gamma_\varrho{}^\sigma{}_\nu g_{\mu\sigma} = 0, \quad (3.1a)$$

$$T_\mu{}^\varrho{}_\nu(\Gamma) = \Gamma_\mu{}^\varrho{}_\nu - \Gamma_\nu{}^\varrho{}_\mu = 0, \quad (3.1b)$$

$$R_{\mu\nu}{}^\varrho{}_\sigma(\Gamma) = \partial_\mu \Gamma_\nu{}^\varrho{}_\sigma - \partial_\nu \Gamma_\mu{}^\varrho{}_\sigma + \Gamma_\mu{}^\varrho{}_\tau \Gamma_\nu{}^\tau{}_\sigma - \Gamma_\nu{}^\varrho{}_\tau \Gamma_\mu{}^\tau{}_\sigma. \quad (3.1c)$$

As known, there is a unique choice for such a connection, the Levi-Civita connection:

$$\Gamma_\mu{}^\varrho{}_\nu = \frac{1}{2}g^{\varrho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \quad (3.2)$$

In the following we will denote with ∇_μ the covariant derivative with respect to the Levi-Civita connection $\Gamma_\mu{}^\varrho{}_\nu$, and with $R^\varrho{}_{\sigma\mu\nu}$ the Riemann tensor.

There is an alternative but equivalent formulation of general relativity (at least classically and without boundary terms), known as the teleparallel equivalent of general relativity, in which one chooses a connection $W_\mu{}^\varrho{}_\nu$, compatible with the metric, such that the dynamical variables are encoded in the torsion and the curvature vanishes [15,20]:

$$\nabla_\varrho(W)g_{\mu\nu} = \partial_\varrho g_{\mu\nu} - W_\varrho{}^\sigma{}_\mu g_{\sigma\nu} - W_\varrho{}^\sigma{}_\nu g_{\mu\sigma} = 0, \quad (3.3a)$$

$$T_\mu{}^\varrho{}_\nu(W) = W_\mu{}^\varrho{}_\nu - W_\nu{}^\varrho{}_\mu, \quad (3.3b)$$

$$\begin{aligned} R_{\mu\nu}{}^\varrho{}_\sigma(W) &= \partial_\mu W_\nu{}^\varrho{}_\sigma - \partial_\nu W_\mu{}^\varrho{}_\sigma + W_\mu{}^\varrho{}_\tau W_\nu{}^\tau{}_\sigma \\ &\quad - W_\nu{}^\varrho{}_\tau W_\mu{}^\tau{}_\sigma = 0. \end{aligned} \quad (3.3c)$$

While in the usual formulation of general relativity the gravitational interaction is due to the curvature of

spacetime, in teleparallel formulation the gravitational interaction is mediated by a Lorentz-like force in a flat spacetime, as in Yang-Mills theories [42]. Indeed, while in the former case the action is linear in the curvature, in the latter one the action is quadratic in the torsion.

Denoting with $e_{\mu a}$ the components of the vielbein and with e^μ_a the components of the frame or inverse vielbein, it is possible to choose a connection $W_\mu{}^\rho{}_\nu$ such that the vielbein is parallel or covariantly constant, that is the following equation holds:

$$\partial_\mu e_\nu{}^a - W_\mu{}^\rho{}_\nu e_\rho{}^a = 0. \quad (3.4)$$

The unique connection solving the previous condition is the Weitzenböck connection [20]

$$W_\mu{}^\rho{}_\nu = e^\rho_a \partial_\mu e_\nu{}^a, \quad (3.5)$$

which consistently has vanishing curvature, since it is the Cartan-Maurer form out of the vielbein.⁵ Moreover, the condition (3.4) implies the compatibility condition (3.3a), since

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}. \quad (3.6)$$

In the following we will denote with D_μ the covariant derivative with respect to the Weitzenböck connection $W_\mu{}^\rho{}_\nu$, and with $T_\mu{}^\rho{}_\nu$ the Weitzenböck torsion,

$$T_\mu{}^\rho{}_\nu = e^\rho_a \partial_{[\mu} e_{\nu]}{}^a, \quad (3.7)$$

antisymmetric in the first and in the last index.

The Weitzenböck torsion satisfies the following Bianchi identity:

$$D_{[\mu} T_{\nu]}{}^\sigma{}_\rho + D_\rho T_\mu{}^\sigma{}_\nu - T_{[\mu}{}^{\sigma\tau} T_{\nu]\tau\rho} - T_{\mu\tau\nu} T_\rho{}^{\sigma\tau} = 0, \quad (3.8)$$

whose trace, which will be useful in the following, reads

$$D_{[\mu} T_{\nu]}{}^\rho{}_\rho + D_\rho T_\mu{}^\rho{}_\nu + T_{\mu\rho\nu} T^{\rho\sigma}{}_\sigma = 0. \quad (3.9)$$

Using the derivative of the vielbein determinant $\partial_\mu e = e W_\mu{}^\rho{}_\rho$, one can show that the following identity holds for the Weitzenböck covariant divergence, for any vector V^μ ,

$$e D_\mu V^\mu = \partial_\mu (e V^\mu) + e T_\mu{}^\mu{}_\nu V^\nu. \quad (3.10)$$

The relation between the Levi-Civita connection and the Weitzenböck is given by

⁵The spin connection corresponding to the Weitzenböck connection is zero (Weitzenböck condition). See Appendix A for more details.

$$\Gamma_\mu{}^\rho{}_\nu = W_\mu{}^\rho{}_\nu + K_\mu{}^\rho{}_\nu = W_\mu{}^\rho{}_\nu + \frac{1}{2}(T_{\mu\nu}{}^\rho + T_{\nu\mu}{}^\rho - T_\mu{}^\rho{}_\nu), \quad (3.11)$$

where the tensor $-K_\mu{}^\rho{}_\nu$ is the *contorsion*. Using the vanishing of the curvature of the Weitzenböck connection and the relation (3.11), one can write the Riemann tensor in terms of the contorsion in the following way:

$$R^\mu{}_{\nu\rho\sigma} = D_\rho K_\sigma{}^\mu{}_\nu - D_\sigma K_\rho{}^\mu{}_\nu + T_\rho{}^\tau{}_\sigma K_\tau{}^\mu{}_\nu + K_\rho{}^\mu{}_\tau K_\sigma{}^\tau{}_\nu - K_\sigma{}^\mu{}_\tau K_\rho{}^\tau{}_\nu. \quad (3.12)$$

Therefore, using the trace of the Bianchi identity (3.9), the Ricci tensor is

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = \frac{1}{2} D_\rho T_{(\mu\nu)}{}^\rho - \frac{1}{2} D_{(\mu} T_{\nu)}{}^\rho{}_\rho + \frac{1}{2} T_{(\mu\nu)\rho} T^{\rho\sigma}{}_\sigma + \frac{1}{4} T_{\rho\mu\sigma} T^{\rho\sigma}{}_\nu - \frac{1}{2} T_{\mu\rho\sigma} T_\nu{}^{\rho\sigma} - \frac{1}{2} T_{\mu\rho\sigma} T_\nu{}^{\sigma\rho}, \quad (3.13)$$

and the Ricci scalar is

$$R = R^\mu{}_\mu = -2 D_\mu T^{\mu\nu}{}_\nu - \frac{1}{4} T^{\mu\nu\rho} T_{\mu\nu\rho} - \frac{1}{2} T^{\mu\nu\rho} T_{\mu\rho\nu} - T_{\mu\nu}{}^\nu T^{\mu\rho}{}_\rho. \quad (3.14)$$

Finally, using the Weitzenböck covariant divergence (3.10), one gets the following identity:

$$e R = -\partial_\mu (2e T^{\mu\nu}{}_\nu) - e \left(\frac{1}{4} T^{\mu\nu\rho} T_{\mu\nu\rho} + \frac{1}{2} T^{\mu\nu\rho} T_{\mu\rho\nu} - T_{\mu\nu}{}^\nu T^{\mu\rho}{}_\rho \right), \quad (3.15)$$

which shows that, up to a boundary term, the Einstein-Hilbert action admits an equivalent formulation in terms of a linear combination of the quadratic scalar contractions of the Weitzenböck torsion [27–31,42]:

$$S_{\text{EH}} = -\frac{1}{2} \int d^d x e R = \frac{1}{2} \int d^d x e \left(\frac{1}{4} T^{\mu\nu\rho} T_{\mu\nu\rho} + \frac{1}{2} T^{\mu\nu\rho} T_{\mu\rho\nu} - T_{\mu\nu}{}^\nu T^{\mu\rho}{}_\rho \right), \quad (3.16)$$

where we set $\kappa = 8\pi G = 1$.⁶

⁶This proves the dynamical equivalence of the usual formulation of general relativity and of the teleparallel equivalent, since it is not necessary to assume the existence of a globally defined frame [43]. Although not manifest in teleparallel formulation, the precise combination in (3.16) is locally Lorentz invariant (see next Sections and [43]), and the inverse frame serves as a gauge field. Thus, the global extension of the frame is ensured by suitably choosing the gauge redundancy in the overlaps of the coordinates patches.

IV. REVIEW OF MØLLER-HAYASHI-SHIRAFUJI THEORY

The relation (3.16) shows that the Einstein-Hilbert action consists in a linear combination of three quadratic scalar contractions of the Weitzenböck torsion with precise relative coefficients. In [27–31] the theory defined by the same action with arbitrary coefficients is considered,

$$S_{\text{MHS}} = \frac{1}{2} \int d^d x e (\alpha_1 T^{\mu\nu\varrho} T_{\mu\nu\varrho} + \alpha_2 T^{\mu\nu\varrho} T_{\mu\varrho\nu} + \alpha_3 T^{\mu\nu}{}_{\nu} T_{\mu\varrho}{}^{\varrho}). \quad (4.1)$$

We call the theory described by this action Møller-Hayashi-Shirafuji theory.⁷

The peculiar combination picking the case of general relativity, which is

$$\alpha_1 = \frac{1}{4}, \quad \alpha_2 = \frac{1}{2}, \quad \alpha_3 = -1, \quad (4.2)$$

is captured by formally requiring the local Lorentz invariance, which indeed is the necessary requirement in the vielbein formulation of general relativity, beside diffeomorphism invariance, in order to recover the right number of degrees of freedom [44].⁸

The MHS action is the most general quadratic action in the Weitzenböck torsion, since there are only three possible independent quadratic scalar contractions of the torsion. Indeed, the torsion has three irreducible components⁹:

$$T_{\mu\varrho\nu} \rightarrow (1, 1) \otimes (1) = (1, 1, 1) \otimes (2, 1)_I \otimes (1). \quad (4.3)$$

(1,1,1) is the totally antisymmetric part $T_{[\mu\nu\varrho]}$, and (1) is the unique trace $T_{\mu\nu}{}^{\nu}$. Namely, the contraction out of two traces is $T_{\mu\nu}{}^{\nu} T^{\mu\varrho}{}_{\varrho}$; the contraction out of two totally antisymmetric tensors is $T^{[\mu\nu\varrho]} T_{[\mu\nu\varrho]}$, which is proportional to $T^{\mu\nu\varrho} T_{[\mu\nu\varrho]}$; the scalar contraction involving the remaining irreducible component can be replaced by $T^{\mu\nu\varrho} T_{\mu\varrho\nu}$; and the second contraction can be replaced by $T^{\mu\nu\varrho} T_{\mu\varrho\nu}$, since

$$T^{\mu\nu\varrho} T_{\mu\varrho\nu} = \frac{1}{2} T^{\mu\nu\varrho} T_{\mu\nu\varrho} - \frac{1}{4} T^{\mu\nu\varrho} T_{[\mu\nu\varrho]}. \quad (4.4)$$

Therefore, the three contractions in (4.1) can be chosen as an independent basis.

Using the variations

$$\delta e = -e e_{\mu}{}^a \delta e^{\mu}{}_a, \quad \delta T_{\mu}{}^{\varrho}{}_{\nu} = e_{[\mu}{}^a D_{\nu]} \delta e_a{}^{\varrho}, \quad (4.5)$$

one can compute the equations of motion of the three pieces in (4.1) [29]:

$$\frac{e_{\mu a}}{e} \frac{\delta}{\delta e^{\nu}{}_a} \int d^d x e T^{\alpha\beta\gamma} T_{\alpha\beta\gamma} = -g_{\mu\nu} T^{\alpha\beta\gamma} T_{\alpha\beta\gamma} - 2T_{\alpha\mu\beta} T^{\alpha}{}_{\nu}{}^{\beta} - 4T_{\mu\nu}{}^{\alpha} T_{\alpha\beta}{}^{\beta} + 4T_{\mu\alpha\beta} T_{\nu}{}^{\alpha\beta} - 4D_{\alpha} T_{\mu\nu}{}^{\alpha}, \quad (4.6a)$$

$$\frac{e_{\mu a}}{e} \frac{\delta}{\delta e^{\nu}{}_a} \int d^d x e T^{\alpha\beta\gamma} T_{\alpha\gamma\beta} = -g_{\mu\nu} T^{\alpha\beta\gamma} T_{\alpha\gamma\beta} - 2T_{\mu\alpha\nu} T^{\alpha\beta}{}_{\beta} + 2T_{\mu\alpha\beta} T_{\nu}{}^{\beta\alpha} - 2T_{\nu\mu\alpha} T^{\alpha\beta}{}_{\beta} - 2D_{\alpha} T_{\mu}{}^{\alpha}{}_{\nu} - 2D_{\alpha} T_{\nu\mu}{}^{\alpha}, \quad (4.6b)$$

$$\frac{e_{\mu a}}{e} \frac{\delta}{\delta e^{\nu}{}_a} \int d^d x e T^{\alpha\beta}{}_{\beta} T_{\alpha\gamma}{}^{\gamma} = g_{\mu\nu} T^{\alpha\beta}{}_{\beta} T_{\alpha\gamma}{}^{\gamma} + 2g_{\mu\nu} D_{\alpha} T^{\alpha\beta}{}_{\beta} - 2D_{\nu} T_{\mu\alpha}{}^{\alpha}. \quad (4.6c)$$

Therefore, the equations of motion $E_{\mu\nu} := \frac{e_{\mu a}}{e} \frac{\delta S_{\text{MHS}}}{\delta e^{\nu}{}_a} = 0$, decomposed in symmetric and antisymmetric part $E_{\mu\nu} = E_{\mu\nu}^{(\text{sym})} + E_{\mu\nu}^{(\text{anti})}$, with $E_{\mu\nu}^{(\text{sym})} = E_{\nu\mu}^{(\text{sym})}$, and $E_{\mu\nu}^{(\text{anti})} = -E_{\nu\mu}^{(\text{anti})}$, respectively read

$$E_{\mu\nu}^{(\text{sym})} = 2\alpha_1 T_{\mu\alpha\beta} T_{\nu}{}^{\alpha\beta} + \alpha_2 T_{\mu\alpha\beta} T_{\nu}{}^{\beta\alpha} - \alpha_1 T_{\alpha\mu\beta} T^{\alpha}{}_{\nu}{}^{\beta} - \left(\alpha_1 + \frac{\alpha_2}{2} \right) (T_{(\mu\nu)}{}^{\alpha} T_{\alpha\beta}{}^{\beta} + D_{\alpha} T_{(\mu\nu)}{}^{\alpha}) - \frac{\alpha_3}{2} D_{(\mu} T_{\nu)\alpha}{}^{\alpha} - \frac{1}{2} g_{\mu\nu} (\alpha_1 T^{\alpha\beta\gamma} T_{\alpha\beta\gamma} + \alpha_2 T^{\alpha\beta\gamma} T_{\alpha\gamma\beta} - \alpha_3 T^{\alpha\beta}{}_{\beta} T_{\alpha\gamma}{}^{\gamma} - 2\alpha_3 D_{\alpha} T^{\alpha\beta}{}_{\beta}), \quad (4.7a)$$

$$E_{\mu\nu}^{(\text{anti})} = \left(\frac{\alpha_2}{2} - \alpha_1 \right) (T_{[\mu\nu]}{}^{\alpha} T_{\alpha\beta}{}^{\beta} + D_{\alpha} T_{[\mu\nu]}{}^{\alpha}) - \left(\alpha_2 + \frac{\alpha_3}{2} \right) (T_{\mu\alpha\nu} T^{\alpha\beta}{}_{\beta} + D_{\alpha} T_{\mu}{}^{\alpha}{}_{\nu}), \quad (4.7b)$$

⁷Hayashi and Shirafuji called it new general relativity [30]. See Appendix C for a comparison with definitions and conventions in this paper and in [29] and in [30].

⁸Since in MHS theory local Lorentz invariance is spoiled; unless (4.2) are met, one has to assume the parallelizability of spacetime *a priori*, that is the existence of a frame which serves as a basis for the tangent bundle, which trivializes [30].

⁹See note 2 for the notation of Young tableaux.

where in the antisymmetric part we use the trace of the Bianchi identity (3.9). Using the choice (4.2), the antisymmetric equation is trivial (and this is the unique case) and the symmetric part, using (3.13) and (3.14), is recognized to be the Einstein equations. To make this manifest, use (3.13) and (3.14) to replace $D_{(\mu}T_{\nu)\alpha}{}^{\alpha}$ and $D_{\alpha}T^{\alpha\beta}{}_{\beta}$:

$$D_{(\mu}T_{\nu)\alpha}{}^{\alpha} = -2R_{\mu\nu} - T_{\mu\alpha\beta}T_{\nu}{}^{\alpha\beta} - T_{\mu\alpha\beta}T_{\nu}{}^{\beta\alpha} + \frac{1}{2}T_{\alpha\mu\beta}T^{\alpha}{}_{\nu}{}^{\beta} + T_{(\mu\nu)}{}^{\alpha}T_{\alpha\beta}{}^{\beta} + D_{\alpha}T_{(\mu\nu)}{}^{\alpha}, \quad (4.8)$$

$$D_{\alpha}T^{\alpha\beta}{}_{\beta} = -\frac{1}{2}R - \frac{1}{8}T^{\alpha\beta\gamma}T_{\alpha\beta\gamma} - \frac{1}{4}T^{\alpha\beta\gamma}T_{\alpha\gamma\beta} - \frac{1}{2}T^{\alpha\beta}{}_{\beta}T_{\alpha\gamma}{}^{\gamma}. \quad (4.9)$$

Moreover, it will be useful for the following to decompose the torsion in the components in transforming in the Young tableaux (2,1) and (1,1,1), the first one including the trace,

$$T_{\mu\nu\varrho} = f_{\mu\varrho\nu} + t_{\mu\nu\varrho}, \quad (4.10)$$

where¹⁰

$$f_{\mu\nu\varrho} = -f_{\nu\mu\varrho}, \quad f_{[\mu\nu\varrho]} = 0, \quad t_{\mu\nu\varrho} = \frac{1}{6}t_{[\mu\nu\varrho]}. \quad (4.11)$$

These symmetries imply that the following identities are satisfied:

$$f_{\alpha\beta\gamma}f^{\alpha\gamma\beta} - \frac{1}{2}f_{\alpha\beta\gamma}f^{\alpha\beta\gamma} = 0, \quad (4.12a)$$

$$f^{\alpha\beta}{}_{\beta}f_{\alpha\mu\nu} - f^{\alpha\beta}{}_{\beta}f_{\mu\nu\alpha} = 0, \quad (4.12b)$$

$$f_{\alpha\beta\gamma}t^{\alpha\beta\gamma} = 0, \quad (4.12c)$$

$$f_{(\mu}{}^{\alpha\beta}t_{\nu)\alpha\beta} + \frac{1}{2}f^{\alpha\beta}{}_{(\mu}t_{\nu)\alpha\beta} = 0, \quad (4.12d)$$

$$-\frac{1}{2}f_{\alpha\beta\mu}f^{\alpha\beta}{}_{\nu} + f_{\mu\alpha\beta}f_{\nu}{}^{\alpha\beta} - f_{\mu\alpha\beta}f_{\nu}{}^{\beta\alpha} = 0, \quad (4.12e)$$

$$f_{\alpha\beta\mu}f^{\alpha\beta}{}_{\nu} + f_{\alpha\beta(\mu}f_{\nu)}{}^{\alpha\beta} = 0. \quad (4.12f)$$

Using the replacement rules (4.8)–(4.9), the decomposition of the torsion (4.10), and the identities (4.12a)–(4.12f) in the equations of motion (4.7a) and (4.7b), one obtains

¹⁰The inverse relations read $f_{\mu\nu\varrho} = \frac{1}{3}(T_{\mu\nu\varrho} + T_{\varrho\mu\nu} - 2T_{\nu\varrho\mu})$ and $t_{\mu\nu\varrho} = \frac{1}{3}(T_{\mu\nu\varrho} + T_{\varrho\mu\nu} + T_{\nu\varrho\mu})$.

$$E_{\mu\nu}^{(\text{sym})} = \alpha_3 \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \right) - \left(4\alpha_1 - \alpha_2 + \frac{\alpha_3}{2} \right) f_{(\mu}{}^{\alpha\beta}t_{\nu)\alpha\beta} + \left(\alpha_1 - \alpha_2 - \frac{\alpha_3}{4} \right) \left(t_{\mu\alpha\beta}t_{\nu}{}^{\alpha\beta} - \frac{1}{2}g_{\mu\nu}t_{\alpha\beta\gamma}t^{\alpha\beta\gamma} \right) + (2\alpha_1 + \alpha_2 + \alpha_3) \left(\frac{1}{2}D_{\alpha}f^{\alpha}{}_{(\mu\nu)} + \frac{1}{2}f^{\alpha\beta}{}_{\beta}f_{\alpha(\mu\nu)} + f_{\mu\alpha\beta}f_{\nu}{}^{\beta\alpha} - \frac{1}{4}g_{\mu\nu}f_{\alpha\beta\gamma}f^{\alpha\beta\gamma} \right), \quad (4.13)$$

$$E_{\mu\nu}^{(\text{anti})} = \frac{1}{2}(2\alpha_1 + \alpha_2 + \alpha_3)(D_{\alpha}f^{\alpha}{}_{[\mu\nu]} + f^{\alpha\beta}{}_{\beta}f_{\alpha[\mu\nu]} + \left(-2\alpha_1 + 2\alpha_2 + \frac{\alpha_3}{2} \right) (D_{\alpha}t^{\alpha}{}_{\mu\nu} + f^{\alpha\beta}{}_{\beta}t_{\alpha\mu\nu})). \quad (4.14)$$

Using the choice (4.2), all the combinations of the constants $\alpha_1, \alpha_2, \alpha_3$ vanish, and the symmetric part gives the Einstein equations.

V. LINEARIZATION OF MØLLER-HAYASHI-SHIRAFUJI THEORY

The linearization of the Møller-Hayashi-Shirafuji theory [29,30] is obtained by defining

$$e_{\mu}{}^a = \delta_{\mu}^a + \varepsilon \bar{h}_{\mu}{}^a, \quad g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}, \quad (5.1)$$

where ε is infinitesimal. Moreover, defining

$$\bar{h}_{\mu}{}^{\nu} = \bar{h}_{\mu}{}^a \delta_a^{\nu}, \quad \bar{h}_{\mu\nu} = \bar{h}_{\mu}{}^e \eta_{e\nu} = \bar{h}_{\mu}{}^a \delta_a^e \eta_{e\nu} \quad (5.2)$$

and combining the linearization of $g_{\mu\nu}$ and $e_{\mu}{}^a$, we get that $h_{\mu\nu}$ is the symmetric part $\bar{h}_{\mu\nu} + \bar{h}_{\nu\mu}$. So, denoting with $b_{\mu\nu} = \bar{h}_{\mu\nu} - \bar{h}_{\nu\mu}$ the antisymmetric part, we can decompose

$$\bar{h}_{\mu\nu} = \frac{1}{2}(h_{\mu\nu} + b_{\mu\nu}). \quad (5.3)$$

Similarly for the inverse vielbein:

$$e^{\mu}{}_a = \delta_a^{\mu} - \varepsilon \bar{h}_a{}^{\mu}, \quad \bar{h}_a{}^{\nu} \delta_{\mu}^a = \bar{h}_{\mu}{}^a \delta_a^{\nu}, \quad \bar{h}_b{}^{\mu} \delta_{\nu}^b = \bar{h}_{\nu}{}^b \delta_b^{\mu}, \quad (5.4)$$

which are fixed in order to recover $e_{\mu}{}^a e^{\nu}{}_a = \delta_{\mu}^{\nu}$ and $e^{\mu}{}_a e^{\mu}{}_b = \delta_b^a$.

The linearized equations of motion are

$$\dot{E}_{\mu\nu}^{(\text{sym})} = \alpha_3 \left(\dot{R}_{\mu\nu} - \frac{1}{2}\dot{R}\eta_{\mu\nu} \right) + \frac{1}{2}(2\alpha_1 + \alpha_2 + \alpha_3)\partial_{\alpha}\dot{f}^{\alpha}{}_{(\mu\nu)}, \quad (5.5)$$

$$\dot{E}_{\mu\nu}^{(\text{anti})} = \left(-2\alpha_1 + 2\alpha_2 + \frac{\alpha_3}{2} \right) \partial_{\alpha}\dot{t}^{\alpha}{}_{\mu\nu} + \frac{1}{2}(2\alpha_1 + \alpha_2 + \alpha_3)\partial_{\alpha}\dot{f}^{\alpha}{}_{[\mu\nu]}, \quad (5.6)$$

where

$$\dot{T}_{\mu\nu Q} = \dot{f}_{\mu Q\nu} + \dot{t}_{\mu\nu Q} = \partial_\mu \bar{h}_{Q\nu} - \partial_Q \bar{h}_{\mu\nu} \quad (5.7)$$

$$= \frac{1}{2}(\partial_\mu h_{Q\nu} - \partial_Q h_{\mu\nu} + \partial_\mu b_{Q\nu} - \partial_Q b_{\mu\nu}), \quad (5.8)$$

$$\dot{f}_{\mu\nu Q} = \frac{1}{2}\partial_{[\mu} h_{\nu]Q} + \frac{1}{6}(\partial_{[\mu} b_{\nu]Q} + 2\partial_Q b_{\mu\nu}), \quad (5.9)$$

$$\dot{t}_{\mu\nu Q} = \frac{1}{6}\partial_{[\mu} b_{\nu]Q}, \quad (5.10)$$

$$\begin{aligned} \dot{R}_{\mu\nu} &= \frac{1}{2}\partial_Q \dot{T}_{(\mu\nu)}^Q - \frac{1}{2}\partial_{[\mu} \dot{T}_{\nu]}^Q = -\frac{1}{2}\partial_Q \dot{f}_{(\mu\nu)}^Q - \frac{1}{2}\partial_{(\mu} \dot{f}_{\nu)}^Q \\ &= -\frac{1}{2}\partial^2 h_{\mu\nu} + \frac{1}{2}\partial^\alpha \partial_{(\mu} h_{\nu)\alpha} - \frac{1}{2}\partial_\mu \partial_\nu h^\alpha{}_\alpha, \end{aligned} \quad (5.11)$$

$$\dot{R} = -2\partial_\mu \dot{T}^{\mu\nu}{}_\nu = -2\partial_\mu \dot{f}^{\mu\nu}{}_\nu = \partial_\mu \partial_\nu h^{\mu\nu} - \partial^2 h^\mu{}_\mu. \quad (5.12)$$

Since the MHS theory is formulated in a quadratic way, unlike general relativity in conventional formulation, the linearized equations of motion can be obtained directly by varying the action (4.1) with the torsion $T_{\mu\nu Q}$ replaced by the linearized one $\dot{T}_{\mu\nu Q}$:

$$\frac{1}{2} \int d^d x (\alpha_1 \dot{T}^{\mu\nu Q} \dot{T}_{\mu\nu Q} + \alpha_2 \dot{T}^{\mu\nu Q} \dot{T}_{\mu Q\nu} + \alpha_3 \dot{T}^{\mu\nu}{}_\nu \dot{T}_{\mu Q}^Q). \quad (5.13)$$

This action can be obtained working directly at the linear level in the following way. One has to look for the most general quadratic action in the derivatives of the perturbation $\bar{h}_{\mu\nu}$, invariant under linearized diffeomorphisms, which are encoded in the following BRST rules¹¹:

$$s\bar{h}_{\mu\nu} = \partial_\mu \xi_\nu, \quad s\xi^\mu = 0, \quad (5.14)$$

or equivalently, using (5.3),

$$sh_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad sb_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \quad s\xi^\mu = 0. \quad (5.15)$$

So, one has to study the local s -cohomology of ghost number zero on the space of $\bar{h}_{\mu\nu}$ and its derivatives, treated as independent variables (the so-called jet space). This is done in Appendix B. The result is that the s -cohomology is generated by the derivatives of the antisymmetrized combination $\partial_{[\mu} \bar{h}_{\nu]Q}$, which is the linearized torsion $\dot{T}_{\mu\nu Q}$. Thus, one has to consider the most general action quadratic

in $\dot{T}_{\mu\nu Q}$, which is a combination of the three independent quadratic scalar contractions $\dot{T}^{\mu\nu Q} \dot{T}_{\mu\nu Q}$, $\dot{T}^{\mu\nu Q} \dot{T}_{\mu Q\nu}$, $\dot{T}^{\mu\nu}{}_\nu \dot{T}_{\mu Q}^Q$ with arbitrary coefficients, and this is the action (5.13).

The case of general relativity (4.2) can be obtained directly by varying the massless Fierz-Pauli action (2.12), which is the most general quadratic action in the derivatives of the symmetric part $h_{\mu\nu}$, invariant under linearized diffeomorphisms

$$sh_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad s\xi^\mu = 0. \quad (5.16)$$

Indeed, the symmetric part is the only propagating part of $\bar{h}_{\mu\nu}$ in general relativity. If one starts with $\bar{h}_{\mu\nu}$, the anti-symmetric part $b_{\mu\nu}$ can be eliminated by modifying the BRST rules (5.15). One introduces a new BRST differential operator s_{FP} , adding an antisymmetric shift in the $b_{\mu\nu}$ transformation

$$\begin{aligned} s_{\text{FP}} h_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, & s_{\text{FP}} b_{\mu\nu} &= \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + 2\dot{\Omega}_{\nu\mu}, \\ s_{\text{FP}} \xi_\mu &= 0, & s_{\text{FP}} \dot{\Omega}_{\mu\nu} &= 0, \end{aligned} \quad (5.17)$$

or equivalently, putting the first two transformations together,

$$s_{\text{FP}} \bar{h}_{\mu\nu} = \partial_\mu \xi_\nu + \dot{\Omega}_{\nu\mu}, \quad s_{\text{FP}} \xi_\mu = 0, \quad s_{\text{FP}} \dot{\Omega}_{\mu\nu} = 0, \quad (5.18)$$

where $\dot{\Omega}_{\mu\nu} = -\dot{\Omega}_{\nu\mu}$ is anticommuting. Redefining $\tilde{\Omega}_{\nu\mu} =: \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + 2\dot{\Omega}_{\nu\mu}$, one realizes that the transformation of the antisymmetric part becomes $s_{\text{FP}} b_{\mu\nu} = \tilde{\Omega}_{\mu\nu}$, $s_{\text{FP}} \tilde{\Omega}_{\mu\nu} = 0$, so that $(b_{\mu\nu}, \tilde{\Omega}_{\nu\mu})$ is a trivial doublet, and the local s_{FP} cohomology does not depend on it [45,46]. On the other hand, the transformation of the symmetric part is the same as before $s_{\text{FP}} h_{\mu\nu} = sh_{\mu\nu}$. Therefore, the local s_{FP} cohomology on the space of $\{h_{\mu\nu}, \xi_\mu, b_{\mu\nu}, \tilde{\Omega}_{\mu\nu}\}$ and their derivatives is equivalent to the local s cohomology on the space of the starting field space $\{h_{\mu\nu}, \xi_\mu\}$ and derivatives.

The linearized torsion $\dot{T}_{\mu\nu Q}$ is not invariant under the modified transformations, but it transforms with respect to $\dot{\Omega}_{\mu\nu}$ as a gauge two-form with a spectator index

$$s_{\text{FP}} \dot{T}_{\mu\nu Q} = \partial_\mu \dot{\Omega}_{\nu Q} - \partial_Q \dot{\Omega}_{\nu\mu}. \quad (5.19)$$

Requiring the arbitrary combination in (5.13) to be invariant with respect to the above transformations, one selects the peculiar combination (4.2), which indeed gives the Einstein-Hilbert action in the nonlinear case. Consistently, the action (5.13) with coefficients (4.2) does not depend on the antisymmetric part $b_{\mu\nu}$ by integrating by parts.

¹¹They are the linearization of the BRST rule for the vielbein encoding the diffeomorphisms, which are generated by the Lie derivative: $se_\mu^a = \mathcal{L}_\xi e_\mu^a = \xi^\nu \partial_\nu e_\mu^a + e_\nu^a \partial_\mu \xi^\nu$, $s\xi^\mu = \frac{1}{2}\mathcal{L}_\xi \xi^\mu = \xi^\nu \partial_\nu \xi^\mu$, setting $\xi^\mu = \varepsilon^\mu$.

VI. PARTICLE SPECTRUM IN COVARIANT FRACTON GAUGE THEORY

The space of the solutions of covariant fraxon gauge theory $\mathcal{M}_{\text{fr}}$ (moduli space) is the submanifold in the space of fields $\{h_{\mu\nu}^{(\text{fr})}\}$, identified by Eqs. (2.5) or (2.10), modulo the gauge invariance,¹² which, as stated by the BRST rule in (2.1),¹³ amounts to linearized longitudinal diffeomorphisms,

$$\delta_{\text{fraxon}} h_{\mu\nu}^{(\text{fr})} = \partial_\mu \partial_\nu \lambda, \quad \forall \alpha, \beta. \quad (6.1)$$

This means that two solutions $h_{\mu\nu}^{(\text{fr})}(x)$ and $h_{\mu\nu}'^{(\text{fr})}(x)$ of the equations of motion, which differ by a fraxon gauge transformation, are identified in the moduli space.

When $\frac{\beta}{\alpha} = -2$, as in (2.11), the theory reduced to linearized Einstein gravity, and the gauge invariance is extended to all linearized diffeomorphisms:

$$\delta_{\text{diff}} h_{\mu\nu}^{(\text{fr})} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad \text{if } \frac{\beta}{\alpha} = -2. \quad (6.2)$$

There is also a case in which the fraxon theory is traceless, that is, it does not depend on $h^{(\text{fr})\rho}{}_\rho$, when $h_{\mu\nu}^{(\text{fr})}$ is decomposed in its traceless part and in its trace. One can check that this happens if and only if

$$\frac{\beta}{\alpha} = -\frac{2}{d-1}, \quad (6.3)$$

where α and β are the parameters introduced in (2.4). In that case, there is an additional gauge symmetry, whose effect is to shift the trace, corresponding to a linearized Weyl scaling of $h_{\mu\nu}^{(\text{fr})}$:

$$\delta_{\text{Weyl}} h_{\mu\nu}^{(\text{fr})} = 2\sigma \eta_{\mu\nu}, \quad \text{if } \frac{\beta}{\alpha} = -\frac{2}{d-1}. \quad (6.4)$$

Let us find the general solution of covariant fraxon gauge theory. For definiteness, we consider the four-dimensional case from now on. The problem was addressed in [8,38], where the particle content of the theory was studied: except for some special cases, the theory describes 5 propagating degrees of freedom, with helicities $0, \pm 1, \pm 2$ respectively. Working in Fourier space, p^μ denotes the momentum and $\tilde{h}_{\mu\nu}^{(\text{fr})}$ the Fourier transform of the $h_{\mu\nu}^{(\text{fr})}$. Equations (2.10) in Fourier space are

$$\begin{aligned} K_{\mu\nu}{}^{\rho\sigma}(p^2; \alpha, \beta) \tilde{h}_{\rho\sigma}^{(\text{fr})} &= 2\alpha p^2 \tilde{h}_{\mu\nu}^{(\text{fr})} - \frac{1}{2}(2\alpha - \beta) p_\alpha p_{(\mu} \tilde{h}_{\nu)}^{(\text{fr})\alpha} \\ &\quad - \beta p_\mu p_\nu \tilde{h}^{(\text{fr})} + \beta \eta_{\mu\nu} (p^2 \tilde{h}^{(\text{fr})} - p_\alpha p_\beta \tilde{h}^{(\text{fr})\alpha\beta}) = 0, \end{aligned} \quad (6.5)$$

where $K(p^2; \alpha, \beta)$, viewed as a 10×10 matrix, identifies a system of homogeneous linear equations in the 10 variables $\{\tilde{h}_{\mu\nu}^{(\text{fr})}\}$. Trivial or pure-gauge solutions are those which solve the system without any condition on the momentum, and they correspond to the gauge invariance of the theory. The number of independent trivial solutions is given by the dimension of the kernel of off-shell $K(p^2; \alpha, \beta)$ (null eigenspace, whose dimension is the number of variables decreased by the matrix rank). One can show that

$$\begin{aligned} \# \text{trivial solutions} &= \dim \ker K(p^2 \neq 0; \alpha, \beta) \\ &= \begin{cases} 6 & \text{if } \alpha = 0, \\ 4 & \text{if } \frac{\beta}{\alpha} = -2, \\ 2 & \text{if } \frac{\beta}{\alpha} = -\frac{2}{3}, \\ 1 & \text{otherwise.} \end{cases} \end{aligned} \quad (6.6)$$

$p_\mu p_\nu \lambda$ is always a trivial solution; in the linearized Einstein case $\frac{\beta}{\alpha} = -2$, this solution is encompassed in $p_{(\mu} \xi_{\nu)}$, which are four independent trivial solutions; in the traceless case $\frac{\beta}{\alpha} = -\frac{2}{3}$, besides $p_\mu p_\nu \lambda$, $2\eta_{\mu\nu} \sigma$ is trivial too. There is one peculiar case more, $\alpha = 0$, in which there is no kinetic term for the fraxon gauge field, so that the theory is expected to have no propagating degrees of freedom in this case [5].

In the massless case, the momentum can be canonically parametrized as $p^\mu = (1, 0, 0, 1)$, such that $p^2 = 0$. The dimension of the kernel in the massless case gives the number of independent solutions *without* taking into account the gauge redundancy. The result is

$$\dim \ker K(p^2 = 0, \alpha, \beta) = \begin{cases} 8 & \text{if } \frac{\beta}{\alpha} = 2, \\ 6 & \text{otherwise.} \end{cases} \quad (6.7)$$

The case $\frac{\beta}{\alpha} = 2$ is special, because the contraction $p^\alpha \tilde{h}_{\alpha\mu}^{(\text{fr})}$ disappears from Eq. (6.5). Subtracting the number of trivial solutions to the dimension of the kernel in the various cases, one gets the number of degrees of freedom of the theory:

$$\#_{\text{dof}} = \begin{cases} 6 - 6 = 0, & \text{if } \alpha = 0, \\ 6 - 4 = 2, & \text{if } \frac{\beta}{\alpha} = -2 \text{ (linearized gravity),} \\ 6 - 2 = 4, & \text{if } \frac{\beta}{\alpha} = -\frac{2}{3} \text{ (traceless limit),} \\ 8 - 1 = 7, & \text{if } \frac{\beta}{\alpha} = 2, \\ 6 - 1 = 5, & \text{otherwise,} \end{cases} \quad (6.8)$$

¹²The author thanks C. Imbimbo, who suggested how to expand this and the next section.

¹³In this section λ is a simple gauge parameter. We maintain the same notation of the corresponding ghost because this should not cause confusion.

where we see that, as expected, the theory has no propagating degrees of freedom when $\alpha = 0$. In the case of linearized Einstein gravity, the 2 degrees of freedom must describe the graviton, with helicities ± 2 . In the third case, we expect that the tracelessness condition eliminates a scalar particle, so that in the general case, which has 5 degrees of freedom, at least a scalar particle is included in the spectrum. Since the general case encompasses also the second, the graviton is also included. Only 2 degrees of freedom remain to be identified: they turn out to describe a particle with helicities ± 1 , as confirmed by the explicit solutions of the equations of motion. The 2 degrees of freedom more in the special case $\frac{\beta}{\alpha} = 2$ have again helicities ± 1 , as we will see in the end of this section.

Solving the equations of motion means finding a basis for the kernel in the massless case, eliminating the trivial solutions. In the general case, apart from the trivial solutions $p_\mu p_\nu \lambda$, the other five independent elements in the basis can be chosen to be equal to

$$\text{diag}\left(-1, -\frac{2\alpha + \beta}{2\beta}, -\frac{2\alpha + \beta}{2\beta}, 1\right), \quad \lambda_{\mu\nu}^{\pm 1}, \lambda_{\mu\nu}^{\pm 2},$$

$$\text{if } \beta \neq 0, \alpha \neq 0, \frac{\beta}{\alpha} = \pm 2, -\frac{2}{3}, \quad (6.9)$$

where $\lambda_{\mu\nu}^s$ are polarization matrices with helicity $s = \pm 1, \pm 2$ [the explicit expressions are in (D4b)–(D4c)], and the diagonal matrix has vanishing helicity. Therefore, in the general case, covariant fracton gauge theory describes particles with helicities $0, \pm 1, \pm 2$ [8]. If $\beta = 0$, the scalar solution in (6.9) is replaced by

$$\text{diag}(0, 1, 1, 0), \quad (6.10)$$

which again has vanishing helicity. If $\frac{\beta}{\alpha} = -2$, the trivial solutions $p_{(\mu} \xi_{\nu)}$ are completed to a basis for the kernel by $\lambda_{\mu\nu}^{\pm 2}$, as expected in linearized gravity. If $\frac{\beta}{\alpha} = -\frac{2}{3}$, the scalar solution in (6.9) becomes $\eta_{\mu\nu}$, since $-\frac{2\alpha + \beta}{2\beta} \rightarrow 1$, so it is trivial, and the theory describes only the modes $\pm 1, \pm 2$. Finally, in the special case $\frac{\beta}{\alpha} = 2$, the basis has eight independent elements: one of them is the trivial one $p_\mu p_\nu \lambda$; the remaining seven ones can be chosen as

$$\text{diag}(-1, -1, -1, 1), \quad \lambda_{\mu\nu}^{\pm 2},$$

$$\begin{pmatrix} 0 & -1 & \mp i & 0 \\ -1 & 0 & 0 & 0 \\ \mp i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \mp i \\ 0 & -1 & \mp i & 0 \end{pmatrix}, \quad (6.11)$$

where the diagonal solution has vanishing helicity, $\lambda_{\mu\nu}^{\pm 2}$ has helicities ± 2 , and the remaining two pairs of solutions have both helicities ± 1 . So, compared to the general case, there is one ± 1 helicity particle more.

As a last observation, consider that, the polarization matrices $\lambda_{\mu\nu}^s$ (D4a)–(D4c), in the general solution (6.9), are transversal in the massless case, meaning that

$$p^\mu \lambda_{\mu\nu}^s = 0, \quad s = \pm 1, \pm 2, \quad \text{if } p^\mu = (1, 0, 0, 1). \quad (6.12)$$

Instead, the scalar mode is longitudinal

$$p^\mu \text{diag}\left(-1, -\frac{2\alpha + \beta}{2\beta}, -\frac{2\alpha + \beta}{2\beta}, 1\right)_{\mu\nu} = p_\nu. \quad (6.13)$$

Nonetheless, this implies that the arbitrary solution $\tilde{h}_{\mu\nu}^{(\text{fr})}$ on-shell satisfies the condition

$$p^\alpha p_{[\mu} \tilde{h}_{\nu]\alpha}^{(\text{fr})} = 0 \quad \text{if } p^\mu = (1, 0, 0, 1), \quad (6.14)$$

which will be useful in the following.

VII. PARTICLE SPECTRUM OF MHS THEORY AND FRACTON EMBEDDING

Let us now find the particle spectrum for the MHS theory (see also [30]), following the same method as in the previous section. It will be convenient to redefine the constants α_1 and α_2 as

$$\tilde{\alpha}_1 = 2\alpha_1 + \alpha_2, \quad \tilde{\alpha}_2 = 2\alpha_1 - \alpha_2. \quad (7.1)$$

The moduli space $\mathcal{M}_{\text{MHS}}^{(\text{lin})}$ of the linearized MHS theory is the submanifold in the field space $\{h_{\mu\nu}, b_{\mu\nu}\}$ identified by the equations of motion (5.5)–(5.6), modulo the gauge invariance, which amounts to linearized diffeomorphisms acting both on $h_{\mu\nu}$ and $b_{\mu\nu}$,¹⁴

$$\delta_{\text{diff}} h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \quad \delta_{\text{diff}} b_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu,$$

$$\forall \tilde{\alpha}_1, \tilde{\alpha}_2, \alpha_3, \quad (7.2)$$

as in (5.15). As in covariant fracton gauge theory, there are some particular cases in which the gauge symmetry is enlarged. Linearized Einstein gravity is reproduced when the parameters are as in (4.2), or equivalently $\tilde{\alpha}_2 = 0$, $\frac{\alpha_3}{\tilde{\alpha}_1} = -1$, according to (7.1), up to an overall constant. In this case there is no dependence on $b_{\mu\nu}$, so it can be shifted away, and this is the effect of local Lorentz invariance, as previously discussed:

$$\delta_{\text{Lorentz}} h_{\mu\nu} = 0, \quad \delta_{\text{Lorentz}} b_{\mu\nu} = \Omega_{\mu\nu},$$

$$\text{if } \tilde{\alpha}_2 = 0, \quad \frac{\alpha_3}{\tilde{\alpha}_1} = -1. \quad (7.3)$$

¹⁴In this section ξ_μ , and $\Omega_{\mu\nu}$ are simple gauge parameters. We maintain the same notation of the corresponding ghosts (omitting the dot denoting the linearization) because this should not be cause confusion.

The traceless limit in MHS theory is reached if and only if

$$\frac{\alpha_3}{\tilde{\alpha}_1} = -\frac{1}{d-1}. \quad (7.4)$$

in which case there is no dependence on the trace h^e_e and the gauge invariance includes a Weyl scaling:

$$\delta_{\text{Weyl}} h_{\mu\nu} = 2\sigma\eta_{\mu\nu}, \quad \delta_{\text{Weyl}} b_{\mu\nu} = 0, \quad \text{if } \frac{\alpha_3}{\tilde{\alpha}_1} = -\frac{1}{d-1}. \quad (7.5)$$

A last limit case is when the theory does not depend on $h_{\mu\nu}$ at all. This happens when $\tilde{\alpha}_1 = \alpha_3 = 0$, in which case the symmetric equation (5.5) trivializes, and MHS theory reduces to the theory for a free two-form gauge field (Ramond-Kalb field RK [32]):

$$\delta_{\text{RK}} h_{\mu\nu} = \Delta_{\mu\nu}, \quad \delta_{\text{RK}} b_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \quad \text{if } \tilde{\alpha}_1 = \alpha_3 = 0, \quad (7.6)$$

where $\Delta_{\mu\nu}$ is any symmetric rank-two tensor, shifting $h_{\mu\nu}$.

In order to find the general solution of (5.5)–(5.6), we write them in Fourier space. Again, we consider from now on the four-dimensional case. The equations read

$$\begin{aligned} \frac{1}{2}\tilde{\alpha}_1 p^2 \tilde{h}_{\mu\nu} - \frac{1}{4}(\tilde{\alpha}_1 - \alpha_3) p_\alpha p_{(\mu} \tilde{h}_{\nu)}^\alpha - \frac{1}{2}\alpha_3 p_\mu p_\nu \tilde{h} \\ + \frac{1}{2}\alpha_3 \eta_{\mu\nu} (p^2 \tilde{h} - p_\alpha p_\beta \tilde{h}^{\alpha\beta}) + \frac{1}{4}(\tilde{\alpha}_1 + \alpha_3) p_\alpha p_{(\mu} \tilde{b}_{\nu)}^\alpha = 0, \end{aligned} \quad (7.7a)$$

$$\begin{aligned} \frac{1}{2}\tilde{\alpha}_2 p^2 \tilde{b}_{\mu\nu} - \frac{1}{4}(\tilde{\alpha}_1 - 2\tilde{\alpha}_2 + \tilde{\alpha}_3) p^\alpha p_{[\mu} \tilde{b}_{\nu]\alpha} \\ - \frac{1}{4}(\tilde{\alpha}_1 + \alpha_3) p^\alpha p_{[\mu} \tilde{h}_{\nu]\alpha} = 0. \end{aligned} \quad (7.7b)$$

This is a system of homogeneous linear equations in the $10 + 6$ variables $(\tilde{h}_{\mu\nu}, \tilde{b}_{\mu\nu})$.

Looking at the equations alone it is possible to deduce the embedding of the fracton moduli space in the MHS one. Indeed, one sees that, if¹⁵

$$\tilde{h}_{\mu\nu} = \tilde{h}_{\mu\nu}^{(\text{fr})}, \quad \tilde{b}_{\mu\nu} = 0, \quad \tilde{\alpha}_1 = 4\alpha, \quad \alpha_3 = 2\beta, \quad (7.8)$$

then the first equation (7.7a) reduces to the fracton equation (6.5), and the second equation (7.7b) reduces to the condition (6.14). This shows that the fracton moduli

space, if $\frac{\beta}{\alpha} \neq 2$, is contained in the subsector with vanishing $b_{\mu\nu}$ of the linearized MHS moduli space, upon using the identification in (7.8),

$$\mathcal{M}_{\text{fr}}\left(\frac{\beta}{\alpha} \neq 2\right) \subseteq \mathcal{M}_{\text{MHS}|b=0}^{(\text{lin})} \subset \mathcal{M}_{\text{MHS}}^{(\text{lin})}. \quad (7.9)$$

Observe that the condition $b_{\mu\nu} = 0$ in (7.8) breaks the gauge invariance (7.2), but it leaves a residual gauge freedom when ξ_μ is exact, that is, when it generates longitudinal diffeomorphisms $\xi_\mu \propto \partial_\mu \lambda$. This is consistently the fracton gauge invariance (6.1).

Actually, we can prove an even stronger statement: not only the fracton moduli space is included in the $b_{\mu\nu} = 0$ subsector, but it is also isomorphic to the same subsector. To see why, let us find the most general solution of the full equations (7.7a)–(7.7b). At the very end, we will see that, upon restricting in the $b_{\mu\nu} = 0$ subsector, the most general solutions coincides with the fracton one, if $\frac{\beta}{\alpha} \neq 2$.

A 16×16 matrix $K(p; \tilde{\alpha}_1, \tilde{\alpha}_2, \alpha_3)$ can be introduced, describing the system (7.7a)–(7.7b) in the variables $(\tilde{h}_{\mu\nu}, \tilde{b}_{\mu\nu})$, viewed as a vector with $10 + 6$ entries. Except for the special cases in which the gauge invariance is enlarged, the dimension of the kernel in the massless case is 10, and the number of trivial solutions is 4, encoded in $(\tilde{h}_{\mu\nu}, \tilde{b}_{\mu\nu}) = (p_{(\mu} \xi_{\nu)}, p_{[\mu} \xi_{\nu]})$. So, there are 6 degrees of freedom, which can be chosen to be equal to

$$\begin{aligned} \left(\text{diag}\left(-1, -\frac{\tilde{\alpha}_1 + \alpha_3}{2\alpha_3}, -\frac{\tilde{\alpha}_1 + \alpha_3}{2\alpha_3}, 1\right), O \right), \\ (\lambda_{\mu\nu}^{\pm 1}, O), \quad (\lambda_{\mu\nu}^{\pm 2}, O), \quad (O, \lambda_{\mu\nu}^0), \end{aligned} \quad (7.10)$$

where O is the 4×4 null matrix, and $\lambda_{\mu\nu}^s$ are the polarization matrices in (D4a)–(D4c), with helicities $s = 0, \pm 1, \pm 2$. Therefore, MHS theory describes particles with helicities $0, \pm 1, \pm 2$, encoded in the symmetric part $\tilde{h}_{\mu\nu}$, and a scalar particle, encoded in the antisymmetric part $\tilde{b}_{\mu\nu}$ —this is the expected result, since, as is well known, a two-form gauge field in four dimensions is dual to a scalar particle. Moreover, the subsector with vanishing antisymmetric part is the same as in covariant fracton gauge theory, when the special case $\frac{\beta}{\alpha} = 2$ is excluded, since, using the identification of the constants in (7.8), $\frac{\tilde{\alpha}_1 + \alpha_3}{2\alpha_3} = \frac{2\alpha + \beta}{2\beta}$, so that the first solution is the same as the first solution in (6.9). Therefore, the statement in (7.9) holds also in the stronger version

$$\mathcal{M}_{\text{fr}}\left(\frac{\beta}{\alpha} \neq 2\right) = \mathcal{M}_{\text{MHS}|b=0}^{(\text{lin})}. \quad (7.11)$$

VIII. BRST FORMULATION

In this concluding section BRST rules (2.1) defining covariant fracton gauge theory are formulated in an

¹⁵In direct space, this means that the linearized torsion $\hat{T}_{\mu\nu Q}$, corresponding to the choice (7.8), $\hat{T}_{\mu\nu Q} = \frac{1}{2}f_{\mu\nu Q}^{(\text{fr})}$. Moreover, there is no dependence in the projected equations on $\tilde{\alpha}_2$, and that the old constants α_1 and α_2 appear only in the combination $2\alpha_1 + \alpha_2 = \tilde{\alpha}_1$, which means that, had we varied the action with, say, $\alpha_2 = 0$, we would have obtained the same result. This is not surprising, since there are only two independent quadratic scalar contractions if the totally antisymmetric part of the torsion is vanishing.

equivalent way, off-shell realizing the embedding of covariant fractons in linearized MHS theory at the level of symmetries.

Consider two doublets $(B_{\mu\nu}, \tilde{b}_{\mu\nu})$ and $(r_\mu, \tilde{\xi}_\mu)$, with the following transformations:

$$sB_{\mu\nu} = \tilde{b}_{\mu\nu}, \quad s\tilde{b}_{\mu\nu} = 0, \quad (8.1a)$$

$$sr_\mu = \tilde{\xi}_\mu, \quad s\tilde{\xi}_\mu = 0, \quad (8.1b)$$

requiring that both $\tilde{b}_{\mu\nu}$ and r_μ are commuting with ghost number zero, so that $B_{\mu\nu}$ is anticommuting with ghost number -1 and $\tilde{\xi}_\mu$ is anticommuting with ghost number one. Then, using the doublet theorem [45,46], the local s cohomology on the space $\{h_{\mu\nu}^{(\text{fr})}, \lambda\}$ and derivatives is equivalent to the local s cohomology on the enlarged space including the two doublets $\{h_{\mu\nu}^{(\text{fr})}, \lambda, B_{\mu\nu}, \tilde{b}_{\mu\nu}, r_\mu, \tilde{\xi}_\mu\}$. If we redefine $\tilde{b}_{\mu\nu}$ and $\tilde{\xi}_\mu$ in this way

$$\tilde{b}_{\mu\nu} := b_{\mu\nu} - \partial_\mu r_\nu + \partial_\nu r_\mu, \quad \tilde{\xi}_\mu := \xi_\mu - \frac{1}{2}\partial_\mu \lambda, \quad (8.2)$$

for some $b_{\mu\nu}$ and ξ_μ , the last one being anticommuting, then the consistency of the BRST transformations implies that

$$sb_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \quad s\xi_\mu = 0. \quad (8.3)$$

Notice that

$$s(h_{\mu\nu}^{(\text{fr})} + \partial_\mu r_\nu + \partial_\nu r_\mu) = \partial_\mu \tilde{\xi}_\nu + \partial_\nu \tilde{\xi}_\mu \quad (8.4)$$

is the same transformation of the linearized metric $h_{\mu\nu}$ under linearized diffeomorphisms, with ghost $\tilde{\xi}_\mu$. So, upon defining

$$h_{\mu\nu} := h_{\mu\nu}^{(\text{fr})} + \partial_\mu r_\nu + \partial_\nu r_\mu, \quad (8.5)$$

the BRST rules (2.1) are equivalent to the transformations of linearized MHS theory (5.15), together with the transformation of a one-form field r_μ , whose ghost is the fracton ghost λ , shifted by the diffeomorphism ghost $\tilde{\xi}_\mu$, and with the transformation of a two-form field $B_{\mu\nu}$, whose transformation is the antisymmetrized derivatives of r_μ , shifted by $b_{\mu\nu}$:

$$\begin{cases} s\lambda = 0, \\ sh_{\mu\nu}^{(\text{fr})} = \partial_\mu \partial_\nu \lambda \end{cases} \Leftrightarrow \begin{cases} s\tilde{\xi}_\mu = 0, s\lambda = 0, \\ sh_{\mu\nu} = \partial_\mu \tilde{\xi}_\nu + \partial_\nu \tilde{\xi}_\mu, \\ sb_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \\ sr_\mu = \xi_\mu - \frac{1}{2}\partial_\mu \lambda, \\ sB_{\mu\nu} = b_{\mu\nu} - \partial_\mu r_\nu + \partial_\nu r_\mu. \end{cases} \quad (8.6)$$

The local s -cohomology on $\{h_{\mu\nu}^{(\text{fr})}, \lambda\}$ is equivalent to that on $\{B_{\mu\nu}, h_{\mu\nu}, b_{\mu\nu}, r_\mu, \tilde{\xi}_\mu, \lambda\}$. In other words, the doublet $(r_\mu, \tilde{\xi}_\mu)$ implements the condition “ $\tilde{\xi}_\mu = \frac{1}{2}\partial_\mu \lambda$,” which is implied by “ $r_\mu = 0$ ”; and the doublet $(B_{\mu\nu}, \tilde{b}_{\mu\nu})$ implements the condition “ $b_{\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu$,” which is implied by “ $B_{\mu\nu} = 0$.” In this sense, the covariant fracton gauge transformations can be thought as the restriction of linearized diffeomorphisms to the longitudinal ones, as they were firstly introduced [5], and covariant fracton gauge theory can be thought as the restriction of the linearized MHS theory in the case in which the antisymmetric part of the linearized vielbein is d-exact. As a consequence, the linearized torsion is cohomologically equivalent to the fracton field strength in this formulation. Indeed, the linearized vielbein reads

$$\bar{h}_{\mu\nu} := \frac{1}{2}(h_{\mu\nu} + b_{\mu\nu}) = \frac{1}{2}h_{\mu\nu}^{(\text{fr})} + \partial_\mu r_\nu + \frac{1}{2}sB_{\mu\nu}, \quad (8.7)$$

so that the linearized torsion becomes

$$\hat{T}_{\mu\nu\rho} = \partial_{[\mu} \bar{h}_{\nu]\rho} = \frac{1}{2}f_{\mu\nu\rho}^{(\text{fr})} + s\left(\frac{1}{2}\partial_{[\mu} B_{\nu]\rho}\right), \quad (8.8)$$

which, in particular, implies that the totally antisymmetric part of the torsion is cohomologically trivial

$$\hat{i}_{\mu\nu\rho} = s\left(\frac{1}{6}\partial_{[\mu} B_{\nu]\rho}\right). \quad (8.9)$$

This is the reason why the constants α_1 and α_2 appear only in the combination $2\alpha_1 + \alpha_2$ in embedding covariant fracton gauge theory in linearized MHS theory.

ACKNOWLEDGMENTS

The author is grateful to F. Fecit for encouragements and discussions, and he is deeply indebted with C. Imbimbo, who read the first draft of this work, pointing out a mistake and suggesting how improve it, stimulating the investigation in Secs. VI and VII.

DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: WEITZENBÖCK GEOMETRY WITHIN CARTAN FORMALISM

In the formalism introduced by Cartan, one considers the vielbein one-form $e^a = e_\mu^a dx^\mu$ and the spin connection one-form $\omega^a_b = \omega_\mu^a{}_b dx^\mu$, and defines the torsion two-form $T^a = \frac{1}{2}T_\mu^a{}_\nu dx^\mu dx^\nu$ and the curvature two-form $R^a_b = \frac{1}{2}R_\mu^a{}_b dx^\mu dx^\nu$ in the following way:

$$T^a(e, \omega) = de^a + \omega^a_b e^b, \quad (A1a)$$

$$R^a_b(\omega) = d\omega^a_b + \omega^a_c \omega^c_b. \quad (A1b)$$

$\omega_{\mu}^a{}_b$ serves as a frame connection (for flat indices). The basis or affine connection $\Gamma_{\mu}^e{}_{\nu}$ (for curved indices) is related to $\omega_{\mu}^a{}_b$ by the requirement that the full covariant derivative of the vielbein, both with respect to $\omega_{\mu}^a{}_b$ and with respect to $\Gamma_{\mu}^e{}_{\nu}$ (“second vielbein postulate”), vanishes¹⁶ [47]:

$$\partial_{\mu} e_{\nu}^a + \omega_{\mu}^a{}_b e_{\nu}^b - \Gamma_{\mu}^e{}_{\nu} e_{\rho}^a = 0, \quad (\text{A2})$$

which is solved by

$$\Gamma_{\mu}^e{}_{\nu}(e, \omega) = e_{\rho}^a \partial_{\mu} e_{\nu}^{\rho} + \omega_{\mu}^a{}_b e_{\nu}^b, \quad (\text{A3})$$

if the vielbein is invertible. If we define

$$T_{\mu}^e{}_{\nu}(\Gamma) = \Gamma_{[\mu}^e{}_{\nu]}, \quad (\text{A4a})$$

$$R^e{}_{\sigma\mu\nu}(\Gamma) = \partial_{[\mu} \Gamma_{\nu]}^e{}_{\sigma} + \Gamma_{[\mu}^e{}_{\alpha} \Gamma_{\nu]}^{\alpha}{}_{\sigma}, \quad (\text{A4b})$$

then they are related to the Cartan definitions (A1a)–(A1b) according to

$$T_{\mu}^e{}_{\nu}(\Gamma(e, \omega)) = T_{\mu}^a{}_b(e, \omega) e_{\nu}^b, \quad (\text{A5a})$$

$$R^e{}_{\sigma\mu\nu}(\Gamma(e, \omega)) = R_{\mu\nu}^a{}_b(e, \omega) e_{\sigma}^b. \quad (\text{A5b})$$

This follows straightforwardly by replacing (A3) in (A4a)–(A4b).

General relativity in usual formulation is recovered by choosing a torsionless spin connection. The spin connection appears in (A1a) algebraically, so that the torsionless condition can be solved uniquely for the spin connection in terms of the vielbein, assuming it to be invertible. Such a spin connection, which we denote with $\omega^{(\text{GR})}{}_{\mu}^a{}_b$, reads

$$\omega^{(\text{GR})}{}_{\mu}^a{}_b = \frac{1}{2} e^{\nu a} \partial_{[\mu} e_{\nu]b} - \frac{1}{2} e^{\nu}{}_b \partial_{[\mu} e_{\nu]}^a - \frac{1}{2} e^{\nu a} e_{\rho}^b e_{\mu}^c \partial_{[\nu} e_{\rho]c}. \quad (\text{A6})$$

The corresponding affine connection, according to (A3), which we denote by $\Gamma^{(\text{GR})}{}_{\mu}^e{}_{\nu}$, is the Levi-Civita connection:

$$\Gamma^{(\text{GR})}{}_{\mu}^e{}_{\nu} = \Gamma_{\mu}^e{}_{\nu}(e, \omega^{(\text{GR})}) = \frac{1}{2} g^{\alpha\beta} (\partial_{\mu} g_{\alpha\nu} + \partial_{\nu} g_{\mu\alpha} - \partial_{\alpha} g_{\mu\nu}). \quad (\text{A7})$$

Indeed, using $g_{\mu\nu} = e_{\mu}^a e_{\nu a}$ and (A6),

¹⁶This is equivalent to define the covariant derivative with respect to the affine connection $\nabla_{\mu}(\Gamma)$ in terms of the covariant derivative with respect to the spin connection $\nabla_{\mu}(\omega)$ in this way: $\nabla_{\mu}(\Gamma)(e^a{}_b V^b) = e^a{}_b \nabla_{\mu}(\omega) V^b$, for any V^a . Indeed on one side $\nabla_{\mu}(\Gamma)(e^a{}_b V^b) = \partial_{\mu} V^a + \Gamma_{\mu}^e{}_{\nu} e^{\nu a} V^{\nu}$; on the other $e^a{}_b \nabla_{\mu}(\omega) V^b = \partial_{\mu} V^a + e^a{}_b (\partial_{\mu} e_{\nu}^b + \omega_{\mu}^b{}_c e_{\nu}^c) V^{\nu}$, so that we obtain (A2).

$$\omega^{(\text{GR})}{}_{\mu}^a{}_b e^{\alpha}{}_a e_{\beta}^b = \frac{1}{2} g^{\alpha\beta} (\partial_{\mu} g_{\nu\beta} + \partial_{\beta} g_{\mu\nu} - \partial_{\nu} g_{\mu\beta}) - e^{\alpha}{}_a \partial_{\mu} e_{\beta}^a, \quad (\text{A8})$$

whence the result (A7) follows, by using (A3).

One could also look for a spin connection $\omega^{(\text{flat})}$ whose curvature is vanishing, instead of a torsionless spin connection. The flatness condition is uniquely solved by the Cartan-Maurer form out of an arbitrary Lorentz matrix $\lambda^a{}_b$:

$$\omega^{(\text{flat})}{}^a{}_b = (\lambda^{-1} d\lambda)^a{}_b = \lambda_c^a d\lambda^c{}_b. \quad (\text{A9})$$

The corresponding affine connection, according to (A3), is again a Cartan-Maurer form,¹⁷

$$\begin{aligned} \Gamma_{\mu}^e{}_{\nu}(e, \omega^{(\text{flat})}) &= e_{\rho}^a \partial_{\mu} e_{\nu}^{\rho} + \omega^{(\text{flat})}{}_{\mu}^a{}_b e_{\nu}^b e_{\rho}^a \\ &= (\lambda_c^b e_{\rho}^c) \partial_{\mu} (\lambda^c{}_a e_{\nu}^a), \end{aligned} \quad (\text{A10})$$

reflecting the fact that it is flat: $R(\Gamma(e, \omega^{(\text{flat})})) = 0$. So, starting from a globally defined frame $e^{\mu}{}_a$ in a parallelizable manifold, it is always possible to make a choice of λ such that the affine connection above is the Weitzenböck connection

$$W_{\mu}^e{}_{\nu} = \Gamma_{\mu}^e{}_{\nu}(e, \omega^{(\text{w})}) = e_{\rho}^a \partial_{\mu} e_{\nu}^{\rho}. \quad (\text{A11})$$

Instead, if we start from a frame and we identify $\Gamma(e, \omega^{(\text{flat})})$ with the Weitzenböck connection out of that frame, it means that the corresponding spin connection $\omega^{(\text{w})}$ is vanishing, fixing the λ matrix to be constant:

$$\omega^{(\text{w})} = 0. \quad (\text{A12})$$

This is called the Weitzenböck condition, which breaks the local Lorentz invariance, but it preserves the global Lorentz invariance.

A useful tensor is the contorsion, which is defined as the difference between an arbitrary affine connection and the Levi-Civita connection¹⁸

$$-K_{\mu}^e{}_{\nu}(\Gamma) := \Gamma_{\mu}^e{}_{\nu} - \Gamma^{(\text{GR})}{}_{\mu}^e{}_{\nu}, \quad (\text{A13})$$

which implies that $K_{\mu Q \nu} = K_{\mu}^{\sigma}{}_{\nu} g_{Q\sigma}$ is antisymmetric in the last two indices. Notice that the antisymmetric part in the first and in the last index is the torsion (A5):

$$-K_{[\mu}^e{}_{\nu]}(\Gamma) = T_{\mu}^e{}_{\nu}(\Gamma). \quad (\text{A14})$$

This expression can be solved for writing the contorsion in terms of the torsion

¹⁷The author thanks C. Imbimbo for pointing this out.

¹⁸Using (A3), we can also write the contorsion in terms of an arbitrary spin connection and the torsionless one: $-K_{\mu}^e{}_{\nu} = (\omega_{\mu}^a{}_b - \omega^{(\text{GR})}{}_{\mu}^a{}_b) e_{\nu}^b e_{\rho}^a$.

$$\begin{aligned}
K_{\mu Q \sigma} &= \frac{1}{2}(K_{\mu Q \nu} - K_{\nu Q \mu}) + \frac{1}{2}(K_{Q \mu \nu} - K_{\nu \mu Q}) + \frac{1}{2}(K_{Q \nu \mu} - K_{\mu \nu Q}) \\
&= -\frac{1}{2}(T_{\mu Q \nu} + T_{Q \mu \nu} + T_{Q \nu \mu}) = -\frac{1}{2}(T_{\mu[Q \nu]} + T_{Q \mu \nu}).
\end{aligned} \tag{A15}$$

We can relate the Levi-Civita connection, the Weitzenböck connection, and the Weitzenböck torsion in the following way. On one side, using (A13),

$$-K_{\mu}{}^e{}_{\nu}(W) = W_{\mu}{}^e{}_{\nu} - \Gamma^{(\text{GR})}{}_{\mu}{}^e{}_{\nu}. \tag{A16}$$

On the other, using (A15),

$$-K_{\mu}{}^e{}_{\nu}(W) = \frac{1}{2}(T_{\mu}{}^e{}_{\nu}(W) - T_{\mu \nu}{}^e(W) + T^e{}_{\mu \nu}(W)). \tag{A17}$$

Therefore,

$$\Gamma^{(\text{GR})}{}_{\mu}{}^e{}_{\nu} = W_{\mu}{}^e{}_{\nu} + \frac{1}{2}(T_{\mu}{}^e{}_{\nu}(W) - T_{\mu \nu}{}^e(W) + T^e{}_{\mu \nu}(W)). \tag{A18}$$

APPENDIX B: COMPUTATION OF A BRST COHOMOLOGY

Consider the following BRST transformations:

$$s\bar{h}_{\mu\nu} = \partial_{\mu}\bar{\xi}_{\nu}, \quad s\bar{\xi}_{\mu} = 0. \tag{B1}$$

$(\bar{h}_{\mu\nu}, \partial_{\mu}\bar{\xi}_{\nu})$ is a doublet, so it does not belong to the BRST cohomology. Since

$$\frac{1}{2}s\partial_{(Q}\bar{h}_{\mu)\nu} = \partial_Q\partial_{\mu}\bar{\xi}_{\nu}, \tag{B2a}$$

$$s\hat{T}_{Q\nu\mu} = 0, \tag{B2b}$$

where $\hat{T}_{Q\nu\mu} = \partial_{[Q}\bar{h}_{\mu]\nu}$, then $(\frac{1}{2}\partial_{(Q}\bar{h}_{\mu)\nu}, \partial_Q\partial_{\mu}\bar{\xi}_{\nu})$ is a doublet, whereas $\hat{T}_{Q\nu\mu}$ is in the cohomology. In general,

$$\frac{1}{(n+2)!}s\partial_{(\alpha_n}\dots\partial_{\alpha_1}\partial_Q\bar{h}_{\mu)\nu} = \partial_{\alpha_n}\dots\partial_{\alpha_1}\partial_Q\partial_{\mu}\bar{\xi}_{\nu}, \tag{B3a}$$

$$s\partial_{\alpha_n}\dots\partial_{\alpha_1}\hat{T}_{Q\nu\mu} = 0 \tag{B3b}$$

show that $(\frac{1}{(n+2)!}\partial_{(\alpha_n}\dots\partial_{\alpha_1}\partial_Q\bar{h}_{\mu)\nu}, \partial_{\alpha_n}\dots\partial_{\alpha_1}\partial_Q\partial_{\mu}\bar{\xi}_{\nu})$ is a doublet, for each n . This suggests that the cohomology is generated by $\bar{\xi}^{\mu}$, $\hat{T}_{Q\nu\mu}$ and by the derivatives of the last one:

$$H(s) = \{\bar{\xi}^{\mu}, \hat{T}_{Q\nu\mu}, \partial_{\alpha_1}\hat{T}_{Q\nu\mu}, \dots, \partial_{\alpha_n}\dots\partial_{\alpha_1}\hat{T}_{Q\nu\mu}, \dots\}. \tag{B4}$$

In order to show this, consider the decomposition $\partial_Q\bar{h}_{\mu\nu}$ in irreducible components,¹⁹

$$\begin{aligned}
(1) \otimes [(1) \oplus (2)_t \oplus (1, 1)] &= (1) \oplus [(1) \oplus (2, 1)_t \oplus (3)_t] \\
&\oplus [(1) \oplus (2, 1)_t \oplus (1, 1, 1)],
\end{aligned} \tag{B5}$$

which is realized by

$$\partial_Q\bar{h}_{\mu\nu} = \frac{1}{6}(F_{Q\mu\nu} + F_{Q\nu\mu} + G_{Q\mu\nu} + B_{\nu\mu Q} + H_{Q\mu\nu}), \tag{B6}$$

where

$$\begin{aligned}
F &\sim (1) \oplus (2, 1)_t, \quad G \sim (3)_t, \quad B \sim (1) \oplus (2, 1)_t, \\
H &\sim (1, 1, 1),
\end{aligned} \tag{B7}$$

and we choose F and B antisymmetric in the first two indices. Defining the symmetric and the antisymmetric part of $\bar{h}_{\mu\nu}$,

$$\begin{aligned}
\bar{h}_{\mu\nu} &= \frac{1}{2}(h_{\mu\nu} + b_{\mu\nu}), \quad \text{with} \quad h_{\mu\nu} = \bar{h}_{\mu\nu} + \bar{h}_{\nu\mu}, \\
b_{\mu\nu} &= \bar{h}_{\mu\nu} - \bar{h}_{\nu\mu},
\end{aligned} \tag{B8}$$

the previous tensor can be realized by

$$F_{\mu\nu Q} = \partial_{\mu}h_{\nu Q} - \partial_{\nu}h_{\mu Q}, \tag{B9a}$$

$$G_{\mu\nu Q} = \partial_{\mu}h_{\nu Q} + \partial_{\nu}h_{Q\mu} + \partial_Qh_{\mu\nu} = \frac{1}{2}\partial_{(\mu}h_{\nu)Q}, \tag{B9b}$$

$$B_{\mu\nu Q} = \partial_{\mu}b_{\nu Q} - \partial_{\nu}b_{\mu Q} - 2\partial_Qb_{\mu\nu}, \tag{B9c}$$

$$H_{\mu\nu Q} = \partial_{\mu}b_{\nu Q} + \partial_{\nu}b_{Q\mu} + \partial_Qb_{\mu\nu} = \frac{1}{2}\partial_{[\mu}b_{\nu]Q}. \tag{B9d}$$

Finally, if we decompose $\partial_{\mu}\bar{\xi}_{\nu}$,

$$\begin{aligned}
\partial_{\mu}\bar{\xi}_{\nu} &= \frac{1}{2}(S_{\nu\mu} + A_{\nu\mu}), \quad \text{with} \quad S_{\nu\mu} = \partial_{\mu}\bar{\xi}_{\nu} + \partial_{\nu}\bar{\xi}_{\mu}, \\
A_{\nu\mu} &= \partial_{\mu}\bar{\xi}_{\nu} - \partial_{\nu}\bar{\xi}_{\mu},
\end{aligned} \tag{B10}$$

then the BRST rules of the components of $\partial_Q\bar{h}_{\mu\nu}$ read

$$sF_{\mu\nu Q} = \partial_Q A_{\nu\mu}, \tag{B11a}$$

$$sG_{\mu\nu Q} = \partial_{\mu}S_{\nu Q} + \partial_{\nu}S_{Q\mu} + \partial_Q S_{\mu\nu}, \tag{B11b}$$

$$sB_{\mu\nu Q} = -3\partial_Q A_{\nu\mu}, \tag{B11c}$$

$$sH_{\mu\nu Q} = 0. \tag{B11d}$$

¹⁹See footnote 2.

Therefore, we recognize two independent invariants

$$s(3F_{\mu\nu Q} + B_{\mu\nu Q}) = 0, \quad sH_{\mu\nu Q} = 0, \quad (\text{B12})$$

and this means that the $\hat{T}_{\mu\nu Q}$ generates the cohomology, since it is a combination of the unique independent invariants:

$$\hat{T}_{\mu\nu Q} = \frac{1}{6}(3F_{\mu Q\nu} + B_{\mu Q\nu} + 2H_{\mu Q\nu}). \quad (\text{B13})$$

APPENDIX C: COMPARISON WITH OTHER CONVENTIONS

In [29] the following tensors are defined:

$$\gamma_{\lambda\nu\mu} = -K_{\mu\lambda\nu} = e_\lambda{}^a \nabla_\mu e_{\nu a}, \quad \varphi_\mu = \gamma^\nu{}_\mu = -e_\lambda{}^a \nabla_\mu e_{\nu a}. \quad (\text{C1})$$

The possible quadratic scalars contractions are chosen as

$$\varphi^\mu \varphi_\mu = T^{\mu\nu}{}_\nu T_{\mu Q}{}^Q, \quad (\text{C2a})$$

$$\gamma^{\mu\nu Q} \gamma_{\mu\nu Q} = \frac{3}{4} T^{\mu\nu Q} T_{\mu\nu Q} + \frac{1}{2} T^{\mu\nu Q} T_{\mu Q\nu}, \quad (\text{C2b})$$

$$\gamma^{\mu\nu Q} \gamma_{Q\nu\mu} = \frac{1}{4} T^{\mu\nu Q} T_{\mu\nu Q} + \frac{1}{2} T^{\mu\nu Q} T_{\mu Q\nu}, \quad (\text{C2c})$$

so that the action

$$S_{\text{Möller}} = \int d^4x e (\beta_1 \varphi^\mu \varphi_\mu + \beta_2 \gamma^{\mu\nu Q} \gamma_{\mu\nu Q} + \beta_3 \gamma^{\mu\nu Q} \gamma_{Q\nu\mu}) \quad (\text{C3})$$

is equivalent to (4.1), if

$$\beta_1 = \alpha_3, \quad \beta_2 = 2\alpha_1 - \alpha_2, \quad \beta_3 = 3\alpha_2 - 2\alpha_1, \quad (\text{C4})$$

and the Einstein-Hilbert action is recovered if

$$\beta_1 = -1, \quad \beta_2 = 0, \quad \beta_3 = 1. \quad (\text{C5})$$

In [30], the following tensors are defined in 4d:

$$T^{(\text{HS})\lambda}{}_{\mu\nu} = T_\nu{}^\lambda{}_\mu = e^\lambda{}_a \partial_{[\nu} e_{\mu]}{}^a, \quad (\text{C6a})$$

$$v_\mu = T^{(\text{HS})\lambda}{}_{\lambda\mu} = T_{\mu\lambda}{}^\lambda, \quad (\text{C6b})$$

$$a_\mu = \frac{1}{6} \varepsilon_{\mu\nu Q\sigma} T^{(\text{HS})\nu Q\sigma} = \frac{1}{6} \varepsilon_{\mu\nu Q\sigma} T^{\sigma\nu Q}, \quad (\text{C6c})$$

$$\begin{aligned} t_{\lambda\mu\nu}^{(\text{HS})} &= \frac{1}{2} (T^{(\text{HS})}{}_{\lambda\mu\nu} + T^{(\text{HS})}{}_{\mu\lambda\nu}) + \frac{1}{6} (g_{\nu\lambda} v_\mu + g_{\nu\mu} v_\lambda) - \frac{1}{3} g_{\lambda\mu} v_\nu \\ &= \frac{1}{2} (T_{\nu\lambda\mu} + T_{\nu\mu\lambda}) + \frac{1}{6} (g_{\nu\lambda} T_{\mu Q}{}^Q + g_{\nu\mu} T_{\lambda Q}{}^Q) - \frac{1}{3} g_{\lambda\mu} T_{\nu Q}{}^Q. \end{aligned} \quad (\text{C6d})$$

v_μ , a_μ and $t_{\lambda\mu\nu}$ are the irreducible components of the torsion in 4d, with a_μ (4d dual to) the totally antisymmetric part. In particular, $t_{\lambda\mu\nu}$ is traceless with vanishing totally antisymmetric part, and it is chosen to be symmetric in the first two

indices. The choice for a basis for the quadratic scalar contractions of the torsion is the following:

$$t^{(\text{HS})\lambda\mu\nu} t_{\lambda\mu\nu}^{(\text{HS})} = \frac{1}{2} T^{\lambda\mu\nu} T_{\lambda\mu\nu} + \frac{1}{2} T^{\lambda\mu\nu} T_{\lambda\nu\mu} - \frac{1}{2} T^{\lambda\mu}{}_\mu T_{\lambda\nu}{}^\nu, \quad (\text{C7a})$$

$$v^\mu v_\mu = T^{\lambda\mu}{}_\mu T_{\lambda\nu}{}^\nu, \quad (\text{C7b})$$

$$a^\mu a_\mu = \frac{1}{18} T^{\lambda\mu\nu} T_{\lambda\mu\nu} + \frac{1}{9} T^{\lambda\mu\nu} T_{\lambda\nu\mu}, \quad (\text{C7c})$$

so that the action

$$\begin{aligned} S_{\text{HS}} &= \int d^4x e (a_1 t^{(\text{HS})\lambda\mu\nu} t_{\lambda\mu\nu}^{(\text{HS})} + a_2 v^\mu v_\mu + a_3 a^\mu a_\mu) \\ &= \int d^4x e \left[\left(c_1 - \frac{1}{3} \right) t^{(\text{HS})\lambda\mu\nu} t_{\lambda\mu\nu}^{(\text{HS})} \right. \\ &\quad \left. + \left(c_2 + \frac{1}{3} \right) v^\mu v_\mu + \left(c_3 - \frac{3}{4} \right) a^\mu a_\mu \right] \end{aligned} \quad (\text{C8})$$

is equivalent to (4.1) if

$$a_1 = c_1 - \frac{1}{3} = \frac{2}{3} (2\alpha_1 + \alpha_2), \quad (\text{C9a})$$

$$a_2 = c_2 + \frac{1}{3} = \frac{1}{3} (2\alpha_1 + \alpha_2 + 3\alpha_3), \quad (\text{C9b})$$

$$a_3 = c_3 - \frac{3}{4} = 6(\alpha_1 - \alpha_2), \quad (\text{C9c})$$

and the Hilbert-Einstein action is obtained if

$$a_1 = -\frac{1}{3}, \quad a_2 = \frac{1}{3}, \quad a_3 = -\frac{3}{4}, \quad (\text{C10})$$

or equivalently,

$$c_1 = c_2 = c_3 = 0. \quad (\text{C11})$$

APPENDIX D: POLARIZATION MATRICES

See for example [47]. Choosing the generator of z -axis rotation as

$$J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{D1})$$

the polarization vectors with helicities ± 1 are

$$\varepsilon_\mu^\pm = \frac{1}{\sqrt{2}} (0, 1, \pm i, 0), \quad (\text{D2})$$

such that

$$J_z \varepsilon_\mu^\pm = \pm \varepsilon_\mu^\pm. \quad (\text{D3})$$

Define the polarization matrices as

$$\lambda_{\mu\nu}^0 = \varepsilon_{(\mu}^+ \varepsilon_{\nu)}^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{D4a})$$

$$\lambda_{\mu\nu}^{\pm 1} = \frac{1}{\sqrt{2}} p_{(\mu} \varepsilon_{\nu)}^{\pm} = \frac{1}{2} \begin{pmatrix} 0 & -1 & \mp i & 0 \\ -1 & 0 & 0 & 1 \\ \mp i & 0 & 0 & \pm i \\ 0 & 1 & \pm i & 0 \end{pmatrix}, \quad (\text{D4b})$$

$$\lambda_{\mu\nu}^{\pm 2} = \varepsilon_{(\mu}^{\pm} \varepsilon_{\nu)}^{\pm} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm i & 0 \\ 0 & \pm i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{D4c})$$

They are such that

$$[J_z, \lambda_{\mu\nu}^s] = s \lambda_{\mu\nu}^s. \quad (\text{D5})$$

Two useful formulas are also the following:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = (a - ib)\lambda^{+2} + (a + ib)\lambda^{-2},$$

$$\begin{pmatrix} 0 & -a & -b & 0 \\ -a & 0 & 0 & a \\ -b & 0 & 0 & b \\ 0 & a & b & 0 \end{pmatrix} = (a - ib)\lambda^{+1} + (a + ib)\lambda^{-1}. \quad (\text{D6})$$

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