

Long range energy-energy correlator at the LHC

Yuxun Guo^{1,*}, Xiaohui Liu^{2,3,†} and Feng Yuan^{1,4,‡}

¹*Nuclear Science Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*

²*Center of Advanced Quantum Studies, Department of Physics,
Beijing Normal University, Beijing, 100875, China*

³*Key Laboratory of Multi-scale Spin Physics, Ministry of Education,
Beijing Normal University, Beijing 100875, China*

⁴*Institute for Theoretical Physics, Universität Tübingen,
Auf der Morgenstelle 14, D-72076 Tübingen, Germany*



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We study the forward-backward azimuthal angular correlations of hadrons in association with multiparticle production in the central rapidity region in proton-proton collisions at the LHC. We apply the nucleon energy-energy correlator framework, where the spinning gluon distribution introduces a nontrivial $\cos(2\phi)$ asymmetries. We will demonstrate that the fundamental helicity structure of quantum chromodynamics (QCD) amplitudes predicts a unique power counting rule: $\cos(2\phi)$ asymmetry starts at $\mathcal{O}(\alpha_s^2)$ order for dijet, $\mathcal{O}(\alpha_s)$ for three jet and $\mathcal{O}(1)$ for four (and more) jet productions. Our results will help us to understand the longstanding puzzle of nearside ridge behavior observed in high multiplicity events of pp collisions at the LHC.

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I. INTRODUCTION

More than a decade ago, a long-range azimuthal angular correlation of $\cos(2\phi)$ between two hadrons, the so-called nearside ridge, was discovered in high multiplicity events in proton-proton (pp) collisions at the LHC, where the event generator, such as PYTHIA, has shown very small (close to zero) asymmetry [1]. This has stimulated tremendous experimental and theoretical efforts in the last decade, but the puzzle still stands [2–18]. In this paper, we investigate the long-range correlation from a different perspective, applying the nucleon energy-energy correlator (NEEC) [19–21] at the LHC. We will show that the spinning gluon distribution in this framework [21] leads to a nonzero $\cos(2\phi)$ azimuthal asymmetries in forward-backward energy correlators in pp collisions.

Previous studies of NEECs focus mainly on the deep inelastic scattering (DIS) [19–22] at the planned Electron-Ion Collider (EIC) [23–25]. In Ref. [26], we apply the concept of NEEC to the observable in pp collisions at the LHC and quite sizable azimuthal angular asymmetries are

found in Higgs boson and top quark pair productions. Here, one measures the energy-energy correlators along the beam directions of both incoming hadrons with polar angles $\theta_{a,b}$ and azimuthal angles $\phi_{a,b}$, respectively. The hard scattering, such as jet production shown in Fig. 1, happens in the central rapidity region. For small polar angles of θ_a and θ_b in opposite directions, their rapidity difference will be large, which corresponds to a long range correlation. The goal of the current paper is to investigate multijet production and demonstrate how $\cos(2\phi)$ asymmetries arise from the spinning gluon effects, where $\phi = \phi_a - \phi_b$.

The observables of interest are the forward-backward energy-energy correlators associated with multijet production in the central rapidity region. They are defined by measuring the energy deposits in the calorimeters in the forward and backward regions of the detector at large rapidities. The common feature between the nearside ridge in pp collisions [1] and the NEEC observables we proposed, i.e., $\cos(2\phi)$ azimuthal asymmetries of two particles with large rapidity difference, indicating that the same physics may be behind both phenomena. The difference is that the NEEC observable measures energy deposits whereas the nearside ridge counts the particles without energy weight. Moreover, jet production plays an important role in high multiplicity events in pp collisions, see, for example, Ref. [27]. Therefore, our calculation is relevant to understanding the underlying physics of the nearside ridge.

For jet production at the LHC, the gluon NEEC is the dominant contribution and can be defined as [21,26],

* Contact author: yuxunguo@lbl.gov

† Contact author: xiliu@bnu.edu.cn

‡ Contact author: fyuan@lbl.gov

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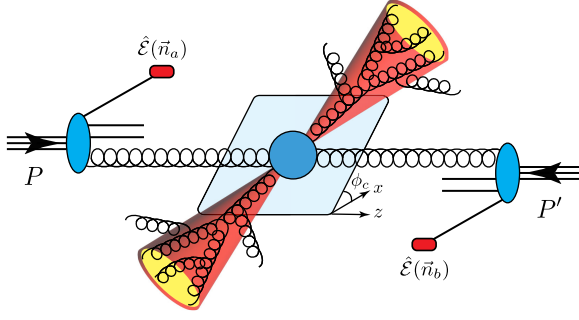


FIG. 1. Forward-backward energy-energy correlators in proton-proton collisions at the LHC where multijets are produced in the central rapidity region. Here, $\vec{n}_{a,b}$ denote the forward-backward positions for the energy-energy correlator detectors depending on both polar angles $\theta_{a,b}$ and azimuthal angles $\phi_{a,b}$, e.g., $\vec{n}_a = (\sin \theta_a \cos \phi_a, \sin \theta_a \sin \phi_a, \cos \theta_a)$. The spinning gluons will lead to a long range $\cos(2\phi)$ angular correlation where $\phi = \phi_a - \phi_b$. In addition, ϕ_c represents the azimuthal angle of the jet transverse momentum P_T in the dijet production, where an azimuthal asymmetry of $\cos(2\phi_a + 2\phi_b - 4\phi_c)$ can be generated from the spinning gluon as well, see, Fig. 4 below.

$$\begin{aligned}
 f_{g,\text{EEC}}^{\alpha\beta}(x, \vec{n}_a) &= \int \frac{dy^-}{2\pi x P^+} e^{-ixP^+ y^-} \\
 &\times \langle P | \mathcal{F}^{+\alpha}(y^-) \mathcal{L}^\dagger[\infty, y^-] \hat{\mathcal{E}}(\vec{n}_a) \\
 &\times \mathcal{L}[\infty, 0] \mathcal{F}^{+\beta}(0) | P \rangle \\
 &= (-g_T^{\alpha\beta}/2) f_{g,\text{EEC}} + h_T^{\alpha\beta} d_{g,\text{EEC}}, \quad (1)
 \end{aligned}$$

where x represents the momentum fraction of the nucleon momentum P carried by the gluon, $F^{\mu\nu}$ represents the gluonic field strength tensor and $\mathcal{L}$ for the gauge link in the adjoint representation for the gluon distribution. Because of the spin-1 feature, we have two independent gluon NEECs: the diagonal one $f_{g,\text{EEC}}$ projected from $g_T^{\alpha\beta} = g^{\alpha\beta} - (P^\alpha \bar{n}^\beta + \bar{n}^\alpha P^\beta)/\bar{n} \cdot P$ corresponds to the usual gluon distribution, whereas the off-diagonal one $d_{g,\text{EEC}}$ from $h_T^{\alpha\beta} = n_{a,T}^\alpha n_{a,T}^\beta / |n_{a,T}^2| + g_T^{\alpha\beta}/2$ represents the spinning gluon. Here, $\bar{n}$ is the light-cone vector conjugate to P in the $+z$ direction: $\bar{n} \cdot P = P^0 + P^z \equiv P^+$, whereas $n_{a,T}^\alpha$ is the transverse component of $n_a^\alpha = (0, \sin \theta_a \cos \phi_a, \sin \theta_a \sin \phi_a, \cos \theta_a)$, see the illustration of $\vec{n}_a$ in Fig. 1. A similar definition can be written for the gluon NEECs for the proton moving in $-\hat{z}$ direction with momentum P' and energy flow direction $n_b^\alpha = (0, \sin \theta_b \cos \phi_b, \sin \theta_b \sin \phi_b, \cos \theta_b)$. To generate a long-range correlation between $\vec{n}_a$ and $\vec{n}_b$, we need to couple two $d_{g,\text{EEC}}(\theta)$ from both incoming protons. In the following calculations, we follow the leading order perturbative results for $f_{g,\text{EEC}}$ and $d_{g,\text{EEC}}$ which are given by the unpolarized collinear PDFs $f_{q,g}$ convoluted with perturbative calculable Wilson coefficients. [20,21].

II. AZIMUTHAL $\cos(2\phi)$ ASYMMETRY FOR DIJET PRODUCTION

We start with two particle production in the central rapidity region with large transverse momentum P_T . The generic cross section measurement for forward-backward correlation takes the following form,

$$\begin{aligned}
 \Sigma^{\text{jet}}(Q^2; \theta_{a,b}, \phi) &= \sum_{ij} \int d\sigma^{\text{jet}}(Q^2) \frac{E_i E_j}{E_P E_P} \mathcal{F}(\phi; \vec{n}_{a,b}) \\
 &\times \delta(\vec{n}_a - \vec{n}_i) \delta(\vec{n}_b - \vec{n}_j) \\
 &= \int d\Omega [\sigma_0 + \sigma_2 \cos(2\phi)], \quad (2)
 \end{aligned}$$

where Q represents the hard momentum scale such as the jet energy, $\mathcal{F}(\phi; \vec{n}_{a,b})$ imposes the phase space measurement to construct ϕ and $\int d\Omega$ for the part related to multijet production in the central rapidity. In the above equation, σ_0 and σ_2 represent the azimuthal angle symmetric and asymmetric differential cross section contributions, respectively. Following the previous example of hard processes in proton-proton collisions [28], the factorization formula for σ_0 and σ_2 can be written as,

$$\sigma_0 = x_a f_{g,\text{EEC}}(x_a, \theta_a^2) x_b f_{g,\text{EEC}}(x_b, \theta_b^2) \hat{\sigma}_0 \quad (3)$$

$$\sigma_2 = x_a d_{g,\text{EEC}}(x_a, \theta_a^2) x_b d_{g,\text{EEC}}(x_b, \theta_b^2) \hat{\sigma}_2, \quad (4)$$

where $\hat{\sigma}_{0,2}$ represents the partonic cross sections for multijet productions and for simplicity we omit the superscript “jet”. We then define the $\cos(2\phi)$ asymmetries as $A_{\cos(2\phi)} \equiv \sigma_2/\sigma_0$. The final $\cos(2\phi)$ asymmetries depend on the NEEC gluon distributions, for which we will follow Ref. [26] and take the leading power contributions. The above factorization follows those for the NEEC observables in Refs. [19–22]. In the kinematic region of small- x , additional modifications for the NEEC distributions are needed, see, for example, Ref. [22]. We leave this study for a future publication.

The partonic cross section $\hat{\sigma}_0$ is the same as that for multijet production in unpolarized proton-proton collisions, whereas $\hat{\sigma}_2$ is more involved. Similar to those in Higgs boson and top quark pair productions [26], $\cos(2\phi)$ asymmetries come from interference between the double helicity flip amplitudes with positive helicities for both incoming gluons, see Fig. 2. This leads to a unique prediction as a result of the helicity structure of quantum chromodynamics (QCD) amplitudes [29,30]. Let us start with $\hat{\sigma}_2$ for dijet production in $gg \rightarrow gg$ channel,

$$\hat{\sigma}_2 \propto \sum_{\lambda_3 \lambda_4} \mathcal{A}(\pm, \pm, \lambda_3, \lambda_4) \mathcal{A}^*(\mp, \mp, \lambda_3, \lambda_4), \quad (5)$$

where the two incoming gluons have the same helicity but they are flipped in the complex conjugate of the amplitude.

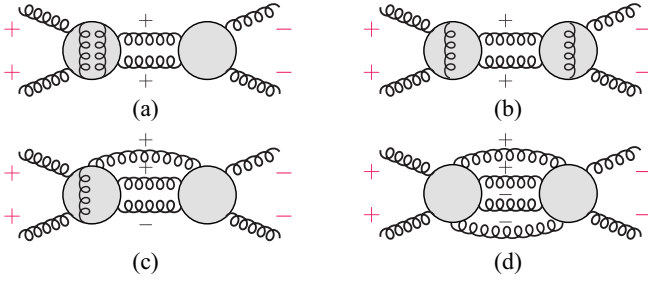


FIG. 2. Nonzero $\cos(2\phi)$ asymmetry comes from the interference between double helicity-flip amplitudes with positive helicity for both incoming gluons. Example of helicity amplitudes that contribute to $\cos(2\phi)$ asymmetries in dijet (order α_s^2) from (a, b), three jets (order α_s^3) from (c) and four jets (order 1) from (d) production in pp collisions. Cuts through the intermediate states are implied.

It is well known that the tree-level amplitudes vanish where either all the external gluons have the same helicity or only one of them has a different helicity [29,30]. Applying this rule to the $gg \rightarrow gg$ amplitudes, we have $\mathcal{A}^{(0)}(++++) = \mathcal{A}^{(0)}(+++-) = 0$. As a result, $\hat{\sigma}_2$ vanishes for any combination of λ_3 and λ_4 in Eq. (5). At one-loop order, although the above two amplitudes no longer vanish, there is no contribution from their interference with $\mathcal{A}^{(0)}$ due to orthogonal color structures. For example, nonvanishing $A^{(1)}(++++)$ has the following color structure [31–34]: $A^{(1)}(++++) \propto (\text{Tr}[T^a T^b T^c T^d] + \text{Tr}[T^a T^c T^d T^b] + \text{Tr}[T^a T^d T^b T^c])$, where a, b, c, d are color indices for the incoming and outgoing gluons, respectively. If we couple this with the color structure of the leading order amplitude which can be written as $A_s^{(0)} f_{abe} f_{cde} + A_t^{(0)} f_{ace} f_{bde} + A_u^{(0)} f_{ade} f_{bce}$, it does not yield any results. That means the $\cos(2\phi)$ vanishes at α_s order for dijet production too.

At NNLO, we will have the following helicity amplitude contributions: $A^{(1)}(++++)A^{(1)*}(-+--)$, $A^{(1)}(++++)A^{(1)*}(-++-)$, and $A^{(2)}(++++)A^{(0)*}(-+--)$. The first term is easy to derive because both amplitudes come from nonvanishing leading contribution at $\mathcal{O}(\alpha_s)$. On the other hand, the derivation of the second and third terms at NNLO is much more involved, due to nontrivial features in these amplitudes that lead to an emerging infrared (IR) divergence even though they vanish at NLO. As an important cross-check, we will show that these divergences will be canceled out by the real gluon radiations, and the final result is finite in the dimensional regulation with $D = 4 - 2\epsilon$.

To facilitate the IR cancellation between virtual and real contributions, we utilize the well established singular behavior of QCD amplitudes at two loop order [35,36]. They are factorized into color-independent jet function and color-correlated soft function [36],

$$\mathbf{I}^{(1)} \equiv \left[-\sum_i \left(\frac{\gamma_K^{[i](1)}}{2\epsilon^2} + \frac{\mathcal{G}_0^{[i](1)}}{\epsilon} \right) \mathbf{1} + \frac{\mathbf{\Gamma}^{(1)}}{\epsilon} \right] \left(-\frac{\mu^2}{s} \right)^\epsilon. \quad (6)$$

The first term associated with the identity matrix 1 is color-uncorrelated. It originates from the jet function of each parton in the process and thus sums over all partons i . The second term is collected by the so-called soft anomalous dimension matrix $\mathbf{\Gamma}^{(1)}$ which is color-correlated. It depends on the process, and can be generally written as [37],

$$\mathbf{\Gamma}^{(1)} = \frac{1}{2} \sum_i \sum_{j \neq i} \mathbf{T}_i \cdot \mathbf{T}_j \ln \left(\frac{-\mu^2}{s_{ij}} \right), \quad (7)$$

where $s_{ij} = (p_i + p_j)^2$ in all outgoing notation. With the above results, the IR divergence of the virtual contribution at NNLO can be obtained by sandwiching $\mathbf{I}^{(1)}$ between $A^{(1)}(++++)$ and $A^{(0)}(-+--)$. Although $\langle A^{(1)}(++++) | A^{(0)}(-+--)\rangle$ vanishes at $\epsilon = 0$, it contributes to an IR divergence at NNLO because of $1/\epsilon^2$ and $\mathbf{\Gamma}^{(1)}$ terms in $\mathbf{I}^{(1)}$,

$$\hat{\sigma}_2^{(2)v} = \frac{1}{\epsilon} (\hat{\sigma}_{2,J}^{(2)v} + \hat{\sigma}_{2,S}^{(2)v}) + \text{finite terms}. \quad (8)$$

Here, the soft and jet IR terms are defined as

$$\begin{aligned} \hat{\sigma}_{2,S}^{(2)v} &= \frac{2}{\mathcal{V}} \langle A^{(1)}(++++) | 2\text{Re}[\mathbf{\Gamma}^{(1)}] | A^{(0)}(-+--)\rangle + \text{H.c.}, \\ \hat{\sigma}_{2,J}^{(2)v} &= -\sum_i \gamma_K^{[i](1)} \hat{\sigma}_2^{(1)\epsilon} = -8C_A \hat{\sigma}_2^{(1)\epsilon}, \end{aligned} \quad (9)$$

respectively, where $\mathcal{V} = 4(N^2 - 1)^2$ represents the color and spin averaging factor and $\gamma_K^{[g](1)} = 2C_A$. Here and in the following, we expand the perturbative corrections in terms of $(\alpha_s/4\pi)^i$. In the above equation, we have used a shorthand notation $\hat{\sigma}_2^{(1)\epsilon}$ to represent the ϵ -term of NLO contribution, i.e., $\hat{\sigma}_2^{(1)} = \epsilon \sigma_2^{(1)\epsilon}$,

$$\epsilon \sigma_2^{(1)\epsilon} = \frac{2}{\mathcal{V}} \langle A^{(1)}(++++) | A^{(0)}(-+--)\rangle + \text{H.c.} \quad (10)$$

Now, we turn to the real contributions. To show the IR cancellation, we focus on soft gluon at low transverse momentum q_T . Their contributions can be derived with eikonal approximation and written as [38–43],

$$\hat{\sigma}_{2,S}^{(2)r} = \frac{2}{\mathcal{V}} \langle A^{(1)}(++++) | \overline{\mathcal{SS}} | A^{(0)}(-+--)\rangle + \text{H.c.} \quad (11)$$

The leading contributions in the limit of $q_T \ll P_T$ can be expressed, again, in terms $\mathbf{\Gamma}^{(1)}$ of (7) and 1,

$$\overline{SS} = \int [dq_T] 4\pi \left(\frac{1}{q_T^2} 2\text{Re}[\Gamma^{(1)}] + 4N \frac{1}{q_T^2} \ln \frac{\hat{s}}{q_T^2 R^2} \mathbf{1} \right) + \dots, \quad (12)$$

where R is the jet size and $\dots$ stands for terms that contribute to finite terms of $\hat{\sigma}_{2,S}^{(2)r}$. We have applied a narrow jet approximation to derive the R -dependence in the above soft gluon radiation [42,44]. To obtain the final result, we need to integrate out q_T with $\int [dq_T] \equiv \mu^{2\epsilon} \int^{q_0} d^d q_T / (2\pi)^d$, which leads to IR divergences,

$$\mu^{2\epsilon} \int^{q_0} \frac{d^d q_T}{(2\pi)^d} \frac{1}{q_T^2} = \frac{1}{4\pi} \left(-\frac{1}{\epsilon} + \ln \frac{q_0^2}{\mu^2} \right), \quad (13)$$

$$\mu^{2\epsilon} \int^{q_0} \frac{d^d q_T}{(2\pi)^d} \frac{1}{q_T^2} \ln \frac{\hat{s}}{q_T^2} = \frac{1}{4\pi} \left(\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\hat{s}}{\mu^2} + \dots \right), \quad (14)$$

where $d = 2 - 2\epsilon$. The soft gluon approximation is only valid in the kinematics of $q_T \ll P_T$. Therefore, we have introduced an upper bound q_0 in q_T integral. Additional IR divergences come from jet functions from the two gluon jets [42,44],

$$2\mathcal{J}_g = 2 \frac{\alpha_s N}{2\pi} \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left(2\beta_0 - \ln \frac{P_T^2 R^2}{\mu^2} \right) + \dots \right], \quad (15)$$

which is multiplied by $\epsilon \hat{\sigma}_2^{(1)\epsilon}$. With all these results, we show that the IR divergences are canceled out completely, in particular, $\hat{\sigma}_{2,S}^{(2)v}$ is canceled by the first term of Eq. (12), $\hat{\sigma}_{2,J}^{(2)v}$ by the second term of (12) and the jet function (15). After canceling all IR divergences and jet size R dependence, we arrive at the following result,

$$\sigma_2 = \int \frac{d\xi d\xi'}{\xi \xi'} x_a d_{g,\text{EEC}}(x'_a) x_b d_{g,\text{EEC}}(x'_b) \left\{ -\hat{\sigma}_{2,1}^{(2)}(x) \times [\mathcal{P}_{gg}^d(\xi) \delta(1-\xi') + \mathcal{P}_{gg}^d(\xi') \delta(1-\xi)] \right. \\ \left. + \delta(1-\xi) \delta(1-\xi') \left[\hat{\sigma}_{2,0}^{(2)}(x) \ln \left(\frac{\hat{s}}{q_0^2} \right) + \hat{\sigma}_{2,2}^{(2)}(x) \right] \right\}, \quad (16)$$

where $x = -t/s$, $x'_a = x_a/\xi$, $x'_b = x_b/\xi'$, $\mathcal{P}_{gg}^d(\xi) = \xi/(1-\xi)_+ + \beta_0 \delta(1-\xi)$ with $\beta_0 = 11/12 - N_f/18$. The above factorization result follows previous examples of NEEC calculations [19–21]. Here, $\hat{\sigma}_{2,1}^{(2)}$ represents collinear gluon radiation contribution from the incoming two gluons, $\hat{\sigma}_{2,0}^{(2)}$ for the anomalous soft gluon radiation generated from $\Gamma^{(1)}$ which has a logarithmic dependence on the upper bound of its transverse momentum as shown in (13), and $\hat{\sigma}_{2,2}^{(2)}$ for the remaining finite terms. Detailed derivation can be found in the Appendix. Here, we present the final results for $\hat{\sigma}_{2,i}^{(2)}$,

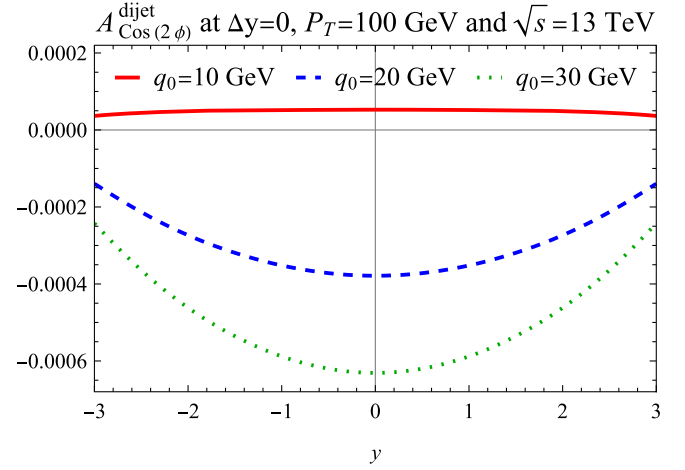


FIG. 3. Long range $\cos(2\phi)$ azimuthal asymmetries in nucleon EEC observable associated with dijet production as a function of the rapidity y of the two jets.

$$\hat{\sigma}_{2,0}^{(2)} = -(N^2 + 6)(N^2 - N_f) \left(\frac{\ln x}{\bar{x}} + \frac{\ln \bar{x}}{x} \right), \quad (17)$$

$$\hat{\sigma}_{2,1}^{(2)} = \frac{N^2(N_f - N)}{2} \left(\pi^2 - 2 \ln \bar{x} \ln x + \frac{1+x^2}{x^2} \ln^2 \bar{x} \right. \\ \left. + \frac{1+\bar{x}^2}{\bar{x}^2} \ln^2 x + \frac{4 \ln \bar{x}}{x} + \frac{4 \ln x}{\bar{x}} \right), \quad (18)$$

$$\hat{\sigma}_{2,2}^{(2)} = \frac{1}{24(x\bar{x})^2} \left\{ 12(N - N_f)^2 (1 - x\bar{x})^2 (2 - x\bar{x}) \right. \\ \left. + 5(N^2 - N_f)x\bar{x} \left[2(N^2 - N_f)x\bar{x}(x - \bar{x}) \ln \frac{x}{\bar{x}} \right. \right. \\ \left. \left. - \omega_1(x) \left(\ln^2 \frac{\bar{x}}{x} + \pi^2 \right) - 2N^2 \omega_2(x) \right] \right\} \\ - 15(N^2 - N_f) \frac{\ln x \ln \bar{x}}{x\bar{x}}, \quad (19)$$

where $\bar{x} = 1 - x$, $\omega_1(x) = 2N^2(1 + 2x\bar{x} - x^2\bar{x}^2) - N_f(x\bar{x} - 2x^2\bar{x}^2)$, and $\omega_2(x) = 4 \ln x \ln \bar{x} - x \ln^2 \bar{x} - \bar{x} \ln^2 x$.

The contribution from the real soft gluon radiation crucially depends on the upper bound q_0 , which can be regarded as a jet veto for the dijet events. To illustrate this dependence, three values of q_0 have been chosen in Fig. 3, where we show the numerical results for the $\cos(2\phi)$ asymmetries $A_{\cos(2\phi)}$ as functions of the rapidity y for the dijet for the kinematics at the LHC with $P_T = 100$ GeV. Another interesting feature of Eq. (16) is that there is a cancellation between the two dominant terms (the first two) because they have opposite signs. For some particular choice of q_0 , they cancel out each other completely. The experiment observation of this behavior will provide an important confirmation of our predictions.

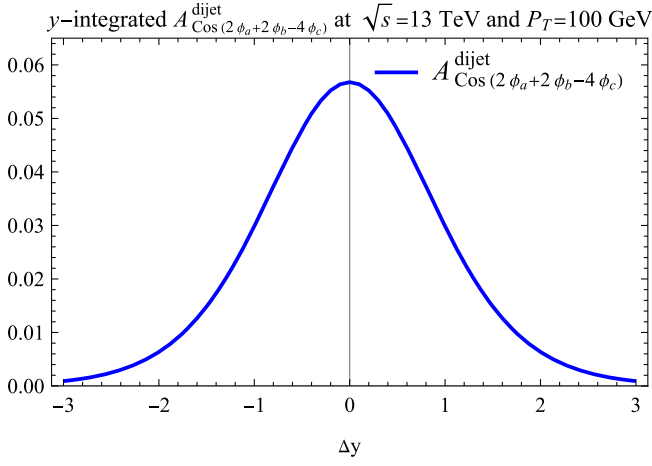


FIG. 4. $\cos(2\phi_a + 2\phi_b - 4\phi_c)$ azimuthal asymmetries associated with dijet production as a function of the rapidity difference between the two jets Δy .

The spinning gluon $d_{g,\text{EEC}}(x_a)$ can also contribute to azimuthal asymmetries which depend on the dijet production plane. This can be utilized as a cross-check of our derivations. In Fig. 4, we show the azimuthal correlations between ϕ_a , ϕ_b , and ϕ_c for the dijet production, where ϕ_c is the azimuthal angle for the jet transverse momentum. From our calculations, we find that the $\cos(2\phi_a + 2\phi_b - 4\phi_c)$ asymmetry is nonzero,

$$\begin{aligned} & \langle \cos(2\phi_a + 2\phi_b - 4\phi_c) \rangle_{\text{dijet}} \\ &= \frac{\int d\Omega d_{g,\text{EEC}}(x_a) d_{g,\text{EEC}}(x_b) \frac{9}{2} \frac{\hat{s}^2 + \hat{t}^2 + \hat{u}^2}{2\hat{s}^2}}{\int d\Omega f_{g,\text{EEC}}(x_a) f_{g,\text{EEC}}(x_b) \frac{9}{2} \frac{(\hat{s}^2 + \hat{t}^2 + \hat{u}^2)^3}{8\hat{s}^2 \hat{t}^2 \hat{u}^2}}. \end{aligned} \quad (20)$$

This asymmetry also comes from interference between double helicity flip amplitudes, but the incoming gluons have opposite helicities, i.e., $\mathcal{A}^{(0)}(+ - + -) \mathcal{A}^{(0)*}(- + + -)$ which is nonzero at the leading order. Numerically, $\langle \cos(2\phi_a + 2\phi_b - 4\phi_c) \rangle$ is about 10% for the typical kinematics at the LHC. It will vanish when we average over ϕ_c and will not contribute to the long range $\cos(2\phi)$ asymmetries discussed in previous examples. However, they come from the same physics, and it is important to cross-check this prediction.

III. DISCUSSION AND CONCLUSION

Before conclusion, we would like to comment on the azimuthal $\cos(2\phi)$ asymmetry in multiple jets production beyond dijet case. First, let us examine the case for three jet production. Applying a similar equation of (5) and the helicity amplitude rule [29,30], it is straightforward to show that $\hat{\sigma}_2$ vanishes at the tree-level for $gg \rightarrow ggg$. At one-loop order, it will receive contributions from the interference between $\mathcal{A}^{(1)}(+ + + -)$ and $\mathcal{A}^{(0)}(- - + +)$, see Fig. 2. The nontrivial color structures of these amplitudes

TABLE I. A brief summary of the power counting analysis for the long range azimuthal angular asymmetries of $\cos(2\phi)$ for jet production in proton-proton collisions.

Number of jets	2	3	≥ 4
$\langle \cos(2\phi) \rangle$ asymmetry	$\mathcal{O}(\alpha_s^2)$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(1)$

lead to a nonzero $\langle \cos(2\phi) \rangle$ for three jet production at α_s order.

For four jet production, $\cos(2\phi)$ asymmetry starts at the leading order, where $\hat{\sigma}_2$ comes from, e.g., $\mathcal{A}^{(0)}(+ + + + - -) \mathcal{A}^{(0)*}(- - + + - -)$ in Fig. 2 and these are the amplitudes that contribute to $\hat{\sigma}_0$ too. Therefore, the $\cos(2\phi)$ asymmetry is order 1. The same conclusion holds for five and more jets production. In Table I, we summarize this power counting result of $\cos(2\phi)$ for jet production.

The α_s hierarchy of $\langle \cos(2\phi) \rangle$ depending on the number of jets is a unique prediction from the helicity structure of parton scattering amplitudes in QCD. It is crucial to test this prediction in experiments, which will rely on explicit calculations for the three jet and four jet productions. We plan to carry out these calculations in the future. Nevertheless, the strong α_s dependence from Table I demonstrates a quantitative difference between 2, 3, and 4 jet events in pp collisions.

This observation has a significant implication for the nearside ridge phenomena. From the QCD point of view, the NEEC observables depend on the same partonic processes that may contribute to high multiplicity events. Therefore, the underlying spinning gluon effects should apply in both cases and contribute to a long range $\cos(2\phi)$ asymmetry. Of course, with energy-weighting the NEEC observable allows us to compute dijet production at NNLO. Extending it to two particle correlation associated with nearside ridge would require a two-loop calculation of $2 \rightarrow 4$ process, and similarly, one-loop for $2 \rightarrow 5$ and tree level for $2 \rightarrow 6$ processes, which is much more involved. An explicit calculation is needed to finally solve the nearside ridge puzzle and we hope to come back to this in the future. Meanwhile, we can apply a similar α_s power counting to make the following predictions: the long range $\cos(2\phi)$ asymmetry will be of order α_s^2 for the $2 \rightarrow 4$ process, α_s for the $2 \rightarrow 5$ process, and leading order for the $2 \rightarrow 6$ process. This is in qualitative agreement with the observed pattern for the nearside ridge, where the $\cos(2\phi)$ asymmetry increases with multiplicity [1], as a result of that higher multiplicity events favor a higher number of jets. This shall encourage further theoretical developments along this line.

In summary, we studied forward-backward azimuthal angular correlations in NEEC measurements in association with multijet production in the central rapidity region at the LHC. We demonstrated a peculiar prediction for multijet production, as a result of double helicity-flip for the $\cos(2\phi)$ asymmetry and the helicity structures of parton

scattering amplitudes in QCD, where $\langle \cos(2\phi) \rangle$ starts at $\mathcal{O}(\alpha_s^2)$ for dijet, $\mathcal{O}(\alpha_s)$ for three jet, and $\mathcal{O}(1)$ for four (and more) jet productions. Experimental verification of these predictions will be crucial to establish the NEEC physics at colliders and pave a method to understand the nearside ridge, a longstanding puzzle, observed in high multiplicity events in pp collisions at the LHC.

An important next step is to implement the spinning gluon idea in multijet production events in pp collisions. In our calculations, we have applied the energy weighting for the forward and backward hadrons. Two complementary approaches can be applied from our results. First, we can experimentally test the predictions presented in this paper, shown in Figs. 3 and 4. Second, one can apply the existing simulation codes for multigluon productions to estimate the $\cos(2\phi)$ asymmetries and compute their contributions to the nearside ridge in pp collisions. This can be either done through a modification of PYTHIA event generator or a combination of numeric simulations from various codes in the literature. Applying a similar idea in the final states [45–48] to high multiplicity events in e^+e^- annihilation process [49] will be highly desired as well.

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APPENDIX: DETAILED DERIVATION OF DIJET RESULT

In this Appendix, we provide a detailed derivation of our results presented in Eq. (16). As discussed in the main text, the nonzero $\cos(2\phi)$ asymmetry in dijet production only begins to appear at the NNLO, which consists of only three terms that has nonzero amplitudes

$$\begin{aligned} \hat{\sigma}_2^{(2)} = \frac{2}{\mathcal{V}} & [A^{(1)}(++)A^{(1)*}(-+ -) \\ & + A^{(1)}(++)A^{(1)*}(-+ +) \\ & + A^{(2)}(++)A^{(0)*}(-+ +) + \text{H.c.}], \end{aligned} \quad (\text{A1})$$

where the factor $\mathcal{V} = 4(N^2 - 1)^2$ averages over initial gluon colors and helicities and the extra factor of two accounts for the parity symmetric term by flipping the helicity of all the gluons, e.g., terms like $A^{(1)}(-+ -)A^{(1)*}(++ -)$. The first term is easy to derive, and the contribution is finite.

However, the last two terms contain infrared (IR) divergences, that shall be canceled by the real gluon radiations.

To show this more clearly, let us write down the NLO contribution $\hat{\sigma}_2^{(1)}$ explicitly,

$$\hat{\sigma}_2^{(1)} = \frac{2}{\mathcal{V}} \langle A^{(1)}(++++) | A^{(0)}(--++) \rangle + \text{H.c.} = \mathcal{O}(\epsilon), \quad (\text{A2})$$

which vanishes as $\epsilon \rightarrow 0$ with ϵ being the regulator of dimensional regularization. However, there are linear ϵ terms in $A^{(1)}(++++)$ which will potentially generate IR divergence with an extra gluon radiation at NNLO. Additionally, the extra gluon radiation will modify the color structures of the amplitudes, causing IR divergence as well at NNLO. The IR divergences caused by the two types of contributions from virtual gluon radiations must be canceled by their real counterparts, as we will show in the following part of this Appendix.

1. Virtual contributions to the dijet $\cos(2\phi)$ asymmetry at NNLO

To calculate the virtual contributions to the $\cos(2\phi)$ asymmetry at NNLO and their IR divergences, the two-loop amplitude of $A^{(2)}(++++)$ and one-loop $A^{(1)}(++-)$ are essential. Their IR structures are made obvious with the universal singular behavior of QCD amplitudes, that has been well established at two loop order based on the original development in [35]. Here we briefly introduce the notation and summarize the results therein for future convenience.

a. The universal singular behavior of QCD amplitudes

Following the notations in [35], one expands the amplitude as

$$|\mathcal{M}(\mu; \{p\})\rangle = 4\pi\alpha_s(\mu) \sum_{n=0} \left(\frac{\alpha_s(\mu)}{4\pi} \right)^n |\mathcal{M}^{(n)}(\mu; \{p\})\rangle, \quad (\text{A3})$$

where the $(\mu; \{p\})$ denotes the dependence of the amplitudes on the scale and momenta, which will be suppressed in the bracket notation $|\mathcal{M}\rangle$ for clarity hereafter. The bracket $|\mathcal{M}\rangle$ implies that they are vectors in color space, besides their helicity-dependence. It is convenient to write them in terms of a set of color-space basis as,

$$\mathcal{M}^{abcd}(\mu; \{p\}) = \sum_i \mathcal{C}_i^{abcd} \mathcal{M}_i(\mu; \{p\}), \quad (\text{A4})$$

such that $|\mathcal{M}\rangle = (\mathcal{M}_1, \mathcal{M}_2, \dots)$. Note that we consider the four gluons with momenta p_1, p_2, p_3 , and p_4 with color indices a, b, c , and d in all outgoing notation. Then the

amplitude square can be written

$$\langle \mathcal{M} | \mathcal{M} \rangle \equiv \sum_{abcd} \sum_{i,j} \mathcal{M}_i^*(\mu; \{p\}) \mathcal{C}_i^{abcd*} \mathcal{C}_j^{abcd} \mathcal{M}_j(\mu; \{p\}) \\ \times \sum_{i,j} \mathcal{M}_i^*(\mu; \{p\}) \mathcal{C} C_{ij} \mathcal{M}_j(\mu; \{p\}), \quad (\text{A5})$$

where $\mathcal{C} C_{ij} \equiv \sum_{abcd} \mathcal{C}_i^{abcd*} \mathcal{C}_j^{abcd}$ is the color space metric tensor.

With these color space notation, the one-loop and two-loop amplitudes can be written as,

$$|\mathcal{M}_m^{(1)}\rangle_{\text{R.S.}} = \mathbf{I}^{(1)}(\epsilon, \mu; \{p\}) |\mathcal{M}_m^{(0)}\rangle_{\text{R.S.}} + |\mathcal{M}_m^{(1)\text{fin}}\rangle_{\text{R.S.}}, \quad (\text{A6})$$

$$|\mathcal{M}_m^{(2)}\rangle_{\text{R.S.}} = \mathbf{I}^{(1)}(\epsilon, \mu; \{p\}) |\mathcal{M}_m^{(1)}\rangle_{\text{R.S.}} \\ + \mathbf{I}^{(2)}(\epsilon, \mu; \{p\}) |\mathcal{M}_m^{(0)}\rangle_{\text{R.S.}} \\ + |\mathcal{M}_m^{(2)\text{fin}}\rangle_{\text{R.S.}}, \quad (\text{A7})$$

where all IR divergence can be factorized into the universal structure $\mathbf{I}^{(1)}(\epsilon, \mu; \{p\})$ and $\mathbf{I}^{(2)}(\epsilon, \mu; \{p\})$. The bold $\mathbf{I}^{(1)}$ and $\mathbf{I}^{(2)}$ implies that they are generally matrices in color space whose explicit expressions can be found in [35]. The R.S. subscript indicates the scheme dependence of these amplitudes.

Later in [36], it was shown that these IR singularity can be further factorized into color-independent jet function and color-correlated soft function

$$\mathbf{I}^{(1)} \equiv \left[-\sum_i \left(\frac{\gamma_K^{[i](1)}}{2\epsilon^2} + \frac{\mathcal{G}_0^{[i](1)}}{\epsilon} \right) \mathbf{1} + \frac{\mathbf{\Gamma}^{(1)}}{\epsilon} \right] \left(-\frac{\mu^2}{s} \right)^\epsilon. \quad (\text{A8})$$

The first term associated with the identity matrix $\mathbf{I}$ is color-uncorrelated. It origins from the jet function of each parton in the process and thus sums over all partons i . The second term is collected by the so-called soft anomalous dimension matrix $\mathbf{\Gamma}^{(1)}$ which is color-correlated. It depends on the process, and can be generally written as [37],

$$\mathbf{\Gamma}^{(1)} = \frac{1}{2} \sum_i \sum_{j \neq i} \mathbf{T}_i \cdot \mathbf{T}_j \ln \left(\frac{-\mu^2}{s_{ij}} \right), \quad (\text{A9})$$

where $s_{ij} = (p_i + p_j)^2$.

b. The virtual contributions to the IR divergence

With the above result, the IR-divergence of the NNLO $\hat{\sigma}$ can be shown explicitly in terms of such universal IR structures in the amplitudes. Consider that the asymmetry $\hat{\sigma}_2$ can be expanded in the form of,

$$\hat{\sigma}_2 = (4\pi\alpha_S(\mu))^2 \sum_{n=0} \left(\frac{\alpha_S(\mu)}{4\pi} \right)^n \hat{\sigma}_2^{(n)}(\mu; \{p\}), \quad (\text{A10})$$

as well. At NLO, the $\hat{\sigma}_2^{(1)}$ is found to vanish up to $\mathcal{O}(\epsilon)$ terms due to the color space cancellation shown above

$$\hat{\sigma}_2^{(1)} = \frac{2}{V} \langle A^{(1)}(++++) | A^{(0)}(--++) \rangle + \text{H.c.} = \epsilon \hat{\sigma}_2^{(1)\epsilon}, \quad (\text{A11})$$

where the notation $\hat{\sigma}_2^{(1)\epsilon}$ is introduced that is of order $\mathcal{O}(\epsilon^0)$. Then the leading contribution to the dijet $\cos(2\phi)$ asymmetry will be from the NNLO $\hat{\sigma}_2^{(2)}$ as discussed above. Noting that since both $A^{(0)}(++++)$ and $A^{(0)}(--++)$ vanish and the amplitude $A^{(0)}(--++)$ is finite, the amplitude $A^{(1)}(++++)$ must be finite too and the IR divergences in Eq. (A1) can only be from the $A^{(1)*}(--++)$ in the second term and the $A^{(2)}(++++)$ in the third term. The IR divergences of neither of them involve $\mathbf{I}^{(2)}$ and therefore one has

$$\hat{\sigma}_2^{(2)} = \frac{4}{V} \langle A^{(1)}(++++) | \mathbf{I}^{(1)} | A^{(0)}(--++) \rangle + \text{H.c.} \\ + \text{finite terms}, \quad (\text{A12})$$

by substituting Eqs. (A6) and (A7) into Eq. (A1). Further putting in the expression of $\mathbf{I}^{(1)}$ in Eq. (A8), one has

$$\hat{\sigma}_2^{(2)} = \frac{1}{\epsilon} (\hat{\sigma}_{2,J}^{(2)} + \hat{\sigma}_{2,S}^{(2)}) + \text{finite terms}, \quad (\text{A13})$$

noting that the $\mathcal{G}_0^{[i](1)}$ term in $\mathbf{I}^{(1)}$ is only ϵ^{-1} divergent and thus does not contribute to the IR divergence since $\langle A^{(1)}(++++) | A^{(0)}(--++) \rangle$ is of order $\mathcal{O}(\epsilon)$. The remaining two terms in $\mathbf{I}^{(1)}$ contribute to IR divergence

$$\hat{\sigma}_{2,S}^{(2)} = \frac{4}{V} \langle A^{(1)}(++++) | \mathbf{\Gamma}^{(1)} | A^{(0)}(--++) \rangle + \text{H.c.}, \\ \hat{\sigma}_{2,J}^{(2)} = -\sum_i \gamma_K^{[i](1)} \hat{\sigma}_2^{(1)\epsilon} = -8C_A \hat{\sigma}_2^{(1)\epsilon}, \quad (\text{A14})$$

respectively, where in the second line, we use $\gamma_K^{[g](1)} = 2C_A$ and sum over the four gluons.

In the above expression, the $\hat{\sigma}_{2,S}^{(2)}$ term origins from the soft gluon radiation that change the color structure and thus break the color space cancelation, while the $\hat{\sigma}_{2,J}^{(2)}$ term origins from the incomplete color space cancelation at NLO at $\mathcal{O}(\epsilon)$, which turn into the $1/\epsilon$ divergence when combined with the $1/\epsilon^2$ in $\mathbf{I}^{(1)}$. The two terms can be separately evaluated. Utilizing the helicity amplitude in [50] and the soft anomalous dimension matrix in [51],

$$\mathbf{\Gamma}^{(1)} = -2 \begin{pmatrix} N(S+U) & 0 & 0 & (U-T) & 0 & (S-T) \\ 0 & N(S+T) & 0 & (T-U) & (S-U) & 0 \\ 0 & 0 & N(U+T) & 0 & (U-S) & (T-S) \\ 2(S-T) & 2(S-U) & 0 & 2NS & 0 & 0 \\ 0 & 2(T-U) & 2(T-S) & 0 & 2NT & 0 \\ 2(U-T) & 0 & 2(U-S) & 0 & 0 & 2NU \end{pmatrix}, \quad (\text{A15})$$

where

$$S = \ln\left(\frac{-\mu^2}{\hat{s}}\right), \quad U = \ln\left(\frac{-\mu^2}{\hat{u}}\right), \quad T = \ln\left(\frac{-\mu^2}{\hat{t}}\right). \quad (\text{A16})$$

one has

$$\hat{\sigma}_{2,S}^{(2)} = \frac{8N(30N + (6 + N^2)(N - N_f))}{3(N^2 - 1)} \left(\frac{\ln \bar{x}}{x} + \frac{\ln x}{\bar{x}} \right), \quad (\text{A17})$$

$$\hat{\sigma}_{2,J}^{(2)} = -8N\hat{\sigma}_2^{(1)\epsilon} = (-8N) \frac{N^2(N_f - N)}{3(N^2 - 1)} \left(\pi^2 - 2 \ln \bar{x} \ln x + \frac{1+x^2}{x^2} \ln^2 \bar{x} + \frac{1+\bar{x}^2}{\bar{x}^2} \ln^2 x + \frac{4 \ln \bar{x}}{x} + \frac{4 \ln x}{\bar{x}} \right), \quad (\text{A18})$$

where $x = -\hat{t}/\hat{s}$ and $\bar{x} = 1 - x$. We note that the above 6×6 matrix is written with respect to basis

$$\begin{aligned} C_1 &= \text{tr}[T^a T^b T^c T^d] + \text{tr}[T^a T^d T^c T^b], \\ C_2 &= \text{tr}[T^a T^b T^d T^c] + \text{tr}[T^a T^c T^d T^b], \\ C_3 &= \text{tr}[T^a T^c T^b T^d] + \text{tr}[T^a T^d T^b T^c], \\ C_4 &= \text{tr}[T^a T^b] \text{tr}[T^c T^d], \\ C_5 &= \text{tr}[T^a T^c] \text{tr}[T^b T^d], \\ C_6 &= \text{tr}[T^a T^d] \text{tr}[T^b T^c]. \end{aligned} \quad (\text{A19})$$

Such IR divergence must be canceled by the real soft radiation, as will be shown below.

2. Real contributions to the dijet $\cos(2\phi)$ asymmetry at NNLO

Real soft gluon radiation can be derived following that in Ref. [43]. In this section, we consider a slightly different notation such that the four gluons have initial momenta p_1 and p_2 , and final momenta k_1 and k_2 , with color indices a , b , c , and d to be consistent with those in Ref. [43]. The final results will be expressed with the standard 2-to-2 Mandelstam variables, without any confusion. The radiated soft gluon can attach to any of the external gluons. By choosing a particular gauge for the soft gluon, $\epsilon(k_g) \cdot p_2 = 0$, we can eliminate the gluon attachment to the p_2 line. Therefore, the soft gluon radiation amplitude can be written as [43],

$$\begin{aligned} \mathcal{S}_g^\mu([\dots]^{abcd}) &\equiv \frac{2k_1^\mu}{2k_1 \cdot k_g} f_{gcf}[\dots]^{abfd} \\ &+ \frac{2k_2^\mu}{2k_2 \cdot k_g} f_{gdf}[\dots]^{abcf} \\ &+ \frac{2p_1^\mu}{2p_1 \cdot k_g} f_{gaf}[\dots]^{fbcd}, \end{aligned} \quad (\text{A20})$$

where $[\dots]^{abcd}$ represents the amplitudes without emission and the associated color indices and g and μ represent the color and Lorentz indices of the emitted soft gluon. The helicity-dependence of the amplitudes is suppressed for clarity. The three terms correspond to the attachments to k_1 , k_2 , and p_1 , respectively. In the following, we will present the general structure of the real soft gluon radiation with a narrow jet approximation, and the corresponding contributions to the dijet $\cos(2\phi)$ asymmetry at NNLO.

a. The structure of the real soft gluon radiation

The contributions of the real soft gluon radiation to such processes can be written as the amplitude squared

$$\hat{\sigma}_{2,S}^{(2)r} = \frac{2}{V} \sum_{a,b,c,d,g} \mathcal{S}_g^{\mu*}([\dots]^{abcd}) \mathcal{S}_g^\nu([\dots]^{abcd}) \Gamma_{\mu\nu} + \text{H.c.}, \quad (\text{A21})$$

where $\Gamma^{\mu\nu}$ represents the polarization tensor for the radiated gluon in the $\epsilon(k_g) \cdot p_2 = 0$ gauge

$$\Gamma^{\mu\nu}(k_g) = \left(-g^{\mu\nu} + \frac{k_g^\mu p_2^\nu + k_g^\nu p_2^\mu}{k_g \cdot p_2} \right). \quad (\text{A22})$$

To simplify the presentation and also to make the comparison with the virtual contribution more obvious, we define the color-space matrix for the soft radiation $\overline{\mathcal{SS}}$ following the previous notation such that,

$$\hat{\sigma}_{2,S}^{(2)r} = \frac{2}{V} \langle A^{(1)}(++++)|\overline{\mathcal{SS}}|A^{(0)}(---++) \rangle + \text{H.c.} \quad (\text{A23})$$

The soft radiation matrix $\overline{\mathcal{SS}}$ can then be written accordingly as,

$$\begin{aligned} & \langle A^{(1)}(++++)|\overline{\mathcal{SS}}|A^{(0)}(---++) \rangle \\ &= \int [dq_T] \sum_{a,b,c,d,g} \mathcal{S}_g^{\mu*}(A^{(1)}(++++))^{abcd} \\ & \quad \times \mathcal{S}_g^\nu(A^{(0)}(---++))^{abcd} \Gamma_{\mu\nu}, \end{aligned} \quad (\text{A24})$$

where q_T represents the transverse momentum of the radiated soft gluon that needs to be integrated, assuming its longitudinal momentum fraction x_g has already been integrated as shown below.

Obviously, there exists a one-to-one correspondence between the soft radiation structure and the soft radiation matrix $\overline{\mathcal{SS}}$, independent of the amplitude $A^{(1)}(++++)$ or $A^{(0)}(---++)$, since both the left-hand side and right-hand side are bilinear in them. To show this, we schematically write the squared amplitude as

$$\begin{aligned} & \int [dq_T] \sum_{a,b,c,d,g} \mathcal{S}_g^{\mu*}([\dots]^{abcd}) \mathcal{S}_g^\nu([\dots]^{abcd}) \Gamma_{\mu\nu} \\ &= \sum_{a,b,c,d} [\dots]^{abcd*} \mathcal{SS}_{abcd} [\dots]^{abcd}, \end{aligned} \quad (\text{A25})$$

after contracting the Lorentz and color indices of the radiated gluon, where the $\mathcal{SS}_{abcd}$ represents the soft radiation operator that can be solved with this relation. Then the real soft radiation matrix $\overline{\mathcal{SS}}$ can be obtained by projecting $\mathcal{SS}_{abcd}$ onto the color basis defined before

$$\sum_k \mathcal{C}C_{ik} \overline{\mathcal{SS}}_{kj} = \sum_{a,b,c,d} \text{Tr}[\mathcal{C}_i^{abcd*}] \mathcal{SS}^{abcd} \text{Tr}[\mathcal{C}_j^{abcd}], \quad (\text{A26})$$

such that the matrix $\overline{\mathcal{SS}}$ carries all the structure of the soft radiation, independent of the amplitude $[\dots]^{abcd}$ and can be derived from $\mathcal{SS}_{abcd}$ with Eqs. (A20) and (A21) by simple contraction of indices.

For example, from the amplitude squared of the soft gluon radiation terms in the above, we have the contraction

$$\frac{2p_1^\mu}{2p_1 \cdot k_g} \frac{2p_1^\nu}{2p_1 \cdot k_g} \Gamma_{\mu\nu} = S_g(p_1, p_2), \quad (\text{A27})$$

$$\frac{2k_1^\mu}{2k_1 \cdot k_g} \frac{2k_1^\nu}{2k_1 \cdot k_g} \Gamma_{\mu\nu} = S_g(k_1, p_2), \quad (\text{A28})$$

$$\frac{2k_2^\mu}{2k_2 \cdot k_g} \frac{2k_2^\nu}{2k_2 \cdot k_g} \Gamma_{\mu\nu} = S_g(k_2, p_2), \quad (\text{A29})$$

where $S_g(p, q)$ is a short-handed notation: $S_g(p, q) \equiv \frac{2p \cdot q}{p \cdot k_g q \cdot k_g}$. Similarly, we derive the interferences between them,

$$\begin{aligned} & 2 \frac{2k_1^\mu}{2k_1 \cdot k_g} \frac{2p_1^\nu}{2p_1 \cdot k_g} \Gamma_{\mu\nu} \\ &= S_g(k_1, p_2) + S_g(p_1, p_2) - S_g(k_1, p_1), \\ & 2 \frac{2k_2^\mu}{2k_2 \cdot k_g} \frac{2p_1^\nu}{2p_1 \cdot k_g} \Gamma_{\mu\nu} \\ &= S_g(k_2, p_2) + S_g(p_1, p_2) - S_g(k_2, p_1), \\ & 2 \frac{2k_1^\mu}{2k_1 \cdot k_g} \frac{2k_2^\nu}{2k_2 \cdot k_g} \Gamma_{\mu\nu} \\ &= S_g(k_1, p_2) + S_g(k_2, p_2) - S_g(k_1, k_2). \end{aligned}$$

To evaluate these soft gluon radiation contributions at small transverse momentum q_T , we integrate out the phase space of longitudinal momentum fraction x_g . This integral normally generates large logs, e.g.,

$$\int_{x_{\min}} \frac{dx_g}{x_g} \Rightarrow \ln \frac{\hat{s}}{q_T^2}, \quad (\text{A30})$$

where $x_{\min} \sim q_T^2/\hat{s}$ is the lower limit from the kinematic constraint of the real gluon radiation. Performing these phase space integrals, we will obtain the following contributions,

$$\begin{aligned} S_g(p_1, p_2) &\Rightarrow \frac{1}{q_T^2} \left[2 \ln \frac{\hat{s}}{q_T^2} \right], \\ S_g(k_1, p_1) &\Rightarrow \frac{1}{q_T^2} \left[\ln \frac{\hat{s}}{q_T^2} + \ln \frac{\hat{t}}{\hat{u}} + \ln \frac{1}{R_1^2} + \frac{\epsilon}{2} \ln^2 \frac{1}{R_1^2} \right], \\ S_g(k_2, p_1) &\Rightarrow \frac{1}{q_T^2} \left[\ln \frac{\hat{s}}{q_T^2} + \ln \frac{\hat{u}}{\hat{t}} + \ln \frac{1}{R_2^2} + \frac{\epsilon}{2} \ln^2 \frac{1}{R_2^2} \right], \\ S_g(k_1, p_2) &\Rightarrow \frac{1}{q_T^2} \left[\ln \frac{\hat{s}}{q_T^2} + \ln \frac{\hat{u}}{\hat{t}} + \ln \frac{1}{R_1^2} + \frac{\epsilon}{2} \ln^2 \frac{1}{R_1^2} \right], \\ S_g(k_2, p_2) &\Rightarrow \frac{1}{q_T^2} \left[\ln \frac{\hat{s}}{q_T^2} + \ln \frac{\hat{t}}{\hat{u}} + \ln \frac{1}{R_2^2} + \frac{\epsilon}{2} \ln^2 \frac{1}{R_2^2} \right], \\ S_g(k_1, k_2) &\Rightarrow \frac{1}{q_T^2} \left[2 \ln \frac{\hat{s}^2}{\hat{t} \hat{u}} + \ln \frac{1}{R_1^2 R_2^2} + \frac{\epsilon}{2} \left(\ln^2 \frac{1}{R_2^2} \right. \right. \\ & \quad \left. \left. + \ln^2 \frac{1}{R_2^2} - 8 \ln \frac{\hat{s}}{-\hat{t}} \ln \frac{\hat{s}}{-\hat{u}} \right) \right], \end{aligned} \quad (\text{A31})$$

where a narrow jet approximation has been applied to derive the jet contribution with R_1 and R_2 to regulate the jet divergences associated with k_1 and k_2 for the final state particles, respectively.

Combining all the contracted squared amplitudes together and project them onto the color basis defined above, one finds that the $\overline{\mathcal{SS}}$ has a simple expression in terms of all the $S_g(p, q)$ s which eventually gives

$$\begin{aligned} \overline{\mathcal{SS}} = & \int [dq_T] \frac{4\pi}{q_T^2} \left[2\text{Re}[\Gamma^{(1)}] + 4N \ln \frac{\hat{s}\mu^4}{q_T^2 R_1 R_2 \hat{u}} \mathbf{1} \right] \\ & + \int [dq_T] \frac{4\pi\epsilon}{q_T^2} \left[4N(\ln^2(R_1) + \ln^2(R_2)) \mathbf{1} \right. \\ & \left. + \ln \frac{\hat{s}}{-\hat{t}} \ln \frac{\hat{s}}{-\hat{u}} \mathbf{L} \right], \end{aligned} \quad (\text{A32})$$

where the μ -dependence gets canceled between the two terms in the second line and $\mathbf{L}$ is a color space matrix that contains no IR divergence. It can be shown that after integrating over q_T , the IR divergence in $\overline{\mathcal{SS}}$ cancels the IR divergence of the virtual contribution in the previous section. The transverse integral is defined as $\int [dq_T] \equiv \mu^{2\epsilon} \int^{q_0} d^d q_T / (2\pi)^d$ with $d = 2 - 2\epsilon$ the dimension of transverse space and the following q_T -integrals are used:

$$\mu^{2\epsilon} \int^{q_0} \frac{d^d q_T}{(2\pi)^d} \frac{1}{q_T^2} = \frac{1}{4\pi} \left[-\frac{1}{\epsilon} + \ln \frac{q_0^2}{\mu^2} \right], \quad (\text{A33})$$

$$\begin{aligned} \mu^{2\epsilon} \int^{q_0} \frac{d^d q_T}{(2\pi)^d} \frac{1}{q_T^2} \ln \frac{\hat{s}}{q_T^2} = & \frac{1}{4\pi} \left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\hat{s}}{\mu^2} \right. \\ & \left. + \frac{1}{2} \left(\ln \frac{\hat{s}}{\mu^2} \right)^2 - \frac{1}{2} \left(\ln \frac{\hat{s}}{q_0^2} \right)^2 - \frac{\pi^2}{12} \right]. \end{aligned} \quad (\text{A34})$$

Performing the q_T -integral leads to the explicit expression of $\overline{\mathcal{SS}}$ that allows us to derive the real gluon contributions to the dijet $\cos(2\phi)$ asymmetry at NNLO.

b. Real gluon contributions to the dijet $\cos(2\phi)$ asymmetry

To derive the real soft gluon contributions, we write down explicitly the amplitudes in the following manner,

$$\begin{aligned} A^{(1)}(++++) = & \mathcal{F}_1 \mathcal{C}_1 + \mathcal{F}_2 \mathcal{C}_2 + \mathcal{F}_3 \mathcal{C}_3 + \mathcal{F}_4 \mathcal{C}_4 \\ & + \mathcal{F}_5 \mathcal{C}_5 + \mathcal{F}_6 \mathcal{C}_6, \end{aligned} \quad (\text{A35})$$

$$A^{(0)}(++--) = \mathcal{G}_1 \mathcal{C}_1 + \mathcal{G}_2 \mathcal{C}_2 + \mathcal{G}_3 \mathcal{C}_3. \quad (\text{A36})$$

where again the $\mathcal{C}_i$ corresponds to the color space basis. In the above expansion, $\mathcal{G}_i$ do not contain ϵ ,

$$\mathcal{G}_1 = \frac{4}{1-x}, \quad \mathcal{G}_2 = \frac{4}{x}, \quad \mathcal{G}_3 = -\frac{4}{x(1-x)}, \quad (\text{A37})$$

while $\mathcal{F}_i$ do have ϵ -expansion terms,

$$\mathcal{F}_{1,2,3} = \frac{4}{3} (N_f - N) + \epsilon \mathcal{F}_{1,2,3}^\epsilon(x) + \epsilon^2 \mathcal{F}_{1,2,3}^{\epsilon^2}(x), \quad (\text{A38})$$

$$\mathcal{F}_{4,5,6} = -8 + \epsilon \mathcal{F}_{4,5,6}^\epsilon(x) + \epsilon^2 \mathcal{F}_{4,5,6}^{\epsilon^2}(x). \quad (\text{A39})$$

With the above results, we can clearly check that $\langle A^{(1)}(++++) | A^{(0)}(++--) \rangle$ is proportional to $\mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3$ and vanishes at $\epsilon = 0$. However, because $\mathcal{F}_i^\epsilon$ are different, it does not vanish at $\mathcal{O}(\epsilon)$ as discussed above.

Explicitly contraction of the above matrix with the helicity amplitudes leads to

$$\begin{aligned} & \frac{\alpha_s}{2\pi^2} \frac{1}{q_T^2} \mathcal{V} \left\{ -\mathcal{N}_0 \left[\mathcal{G}_1 \ln \frac{1}{x} + \mathcal{G}_2 \ln \frac{1}{1-x} \right] + \epsilon \left[N \mathcal{M}_0^\epsilon \ln \frac{\hat{s}^2}{q_T^4 R_1^2 R_2^2} + 2N \left(\mathcal{M}_1^\epsilon \ln \frac{1}{x} + \mathcal{M}_2^\epsilon \ln \frac{1}{1-x} \right) \right. \right. \\ & \left. \left. + \mathcal{N}_0 (\mathcal{G}_1 + \mathcal{G}_2) \ln \frac{1}{x} \ln \frac{1}{1-x} \right] + \epsilon^2 N \mathcal{M}_0^{\epsilon^2} \ln \frac{\hat{s}^2}{q_T^4} + \dots \right\}, \end{aligned} \quad (\text{A40})$$

where $\mathcal{N}_0 = (N^2 - 1)(N^2 + 6)(N^2 - N_f)$ and

$$\mathcal{M}_0^\epsilon = N[3(\mathcal{F}_1^\epsilon \mathcal{G}_1 + \mathcal{F}_2^\epsilon \mathcal{G}_2 + \mathcal{F}_3^\epsilon \mathcal{G}_3) + (\mathcal{F}_4^\epsilon - \mathcal{F}_5^\epsilon) \mathcal{G}_1 + (\mathcal{F}_4^\epsilon - \mathcal{F}_6^\epsilon) \mathcal{G}_2], \quad (\text{A41})$$

$$\mathcal{M}_0^{\epsilon^2} = N[3(\mathcal{F}_1^{\epsilon^2} \mathcal{G}_1 + \mathcal{F}_2^{\epsilon^2} \mathcal{G}_2 + \mathcal{F}_3^{\epsilon^2} \mathcal{G}_3) + (\mathcal{F}_4^{\epsilon^2} - \mathcal{F}_5^{\epsilon^2}) \mathcal{G}_1 + (\mathcal{F}_4^{\epsilon^2} - \mathcal{F}_6^{\epsilon^2}) \mathcal{G}_2], \quad (\text{A42})$$

$$\mathcal{M}_1^\epsilon = \mathcal{G}_1(11\mathcal{F}_1^\epsilon + 2\mathcal{F}_2^\epsilon + 2\mathcal{F}_3^\epsilon + 6\mathcal{F}_4^\epsilon + 3\mathcal{F}_5^\epsilon + 6\mathcal{F}_6^\epsilon), \quad (\text{A43})$$

$$\mathcal{M}_2^\epsilon = \mathcal{G}_2(2\mathcal{F}_1^\epsilon + 11\mathcal{F}_2^\epsilon + 2\mathcal{F}_3^\epsilon + 6\mathcal{F}_4^\epsilon + 6\mathcal{F}_5^\epsilon + 3\mathcal{F}_6^\epsilon), \quad (\text{A44})$$

where we have applied $\mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 = 0$ to simplify the above expressions.

Meanwhile, there are collinear radiation contributions from the two incoming gluons. At low $q_T \ll P_T$, they read

$$\frac{\alpha_s}{2\pi^2} \frac{1}{q_T^2} \frac{4}{\mathcal{V}} e 2N \mathcal{M}_0^\epsilon \int \frac{d\xi}{\xi} \frac{d\xi'}{\xi'} x_a x_b d_{g,\text{EEC}}(x'_a) d_{g,\text{EEC}}(x'_b) [\mathcal{P}_{gg}^d(\xi) \delta(1 - \xi') + \mathcal{P}_{gg}^d(\xi') \delta(1 - \xi)], \quad (\text{A45})$$

where $x'_a = x_a/\xi$ and $x'_b = x_b/\xi'$, and $\mathcal{P}_{gg}^d(\xi) = \xi/(1 - \xi)_+ + \beta_0 \delta(1 - \xi)$. Putting the collinear radiation contributions together with the above real soft gluon contributions completes the real contributions to the asymmetry at NNLO.

3. Full dijet $\cos(2\phi)$ asymmetry at NNLO

With the real and virtual contributions calculated, it shall be obvious that the IR divergence cancels among them, which can be shown with the color-space matrices, even without the explicit amplitudes. It is interesting to note that some finite terms also completely cancel each between the real and virtual contributions, for example, those associated with $\mathcal{M}_{1,2}^\epsilon$, $\mathcal{M}_0^{\epsilon^2}$, and part of $\mathcal{M}_0^\epsilon$. The final finite terms contain three important contributions: one from the soft anomalous dimension matrix $\Gamma^{(1)}$, one from the collinear splitting, and the last one from the finite terms in the virtual contributions. These contributions are summarized in

Eq. (16) of the paper. The $\sigma_{2,2}^{(2)}$ is separated from the other terms that contain lns: $\sigma_2^{(2)} = \sigma_{2,0}^{(2)} \ln(\hat{s}/q_0^2) + \sigma_{2,1}^{(2)} \ln(\hat{s}/\mu^2) + \sigma_{2,2}^{(2)}$, where the $\ln(\hat{s}/\mu^2)$ in the second term vanishes when choosing $s = \mu^2$. Then the $\sigma_{2,2}^{(2)}$ can be further decomposed as,

$$\hat{\sigma}_{2,2}^{(2)} \equiv -\frac{8\mathcal{N}_1 N \ln x \ln \bar{x}}{3(N^2 - 1)x\bar{x}} + \hat{\sigma}_{2,2,1}^{(2)} + \hat{\sigma}_{2,2,2}^{(2)} + \hat{\sigma}_{2,2,3}^{(2)}, \quad (\text{A46})$$

where $\mathcal{N}_1 \equiv (N^2 + 36)N - (N^2 + 6)N_f$. In the above equation, $\hat{\sigma}_{2,2,1}^{(2)}$ comes from the finite term in the amplitude interference of $A^{(1)}(+++-)A^{(1)}(-+--)$, $\hat{\sigma}_{2,2,2}^{(2)}$ from $A^{(1)}(++++)A^{(1)}(--++)$, and $\hat{\sigma}_{2,2,3}^{(2)}$ from $A^{(2)}(++++)A^{(0)}(--++)$. The explicit expressions for those terms are

$$\hat{\sigma}_{2,2,1}^{(2)} = \frac{1}{9(N^2 - 1)} \left[4N^2(N - N_f)^2 \left(\frac{x}{\bar{x}^2} + \frac{\bar{x}}{x^2} - x\bar{x} \right) + 18 \left(N^2(N^2 + 12) - 2N(N^2 + 2)N_f + \left(N^2 + \frac{6}{N^2} - 2 \right) N_f^2 \right) \right], \quad (\text{A47})$$

$$\hat{\sigma}_{2,2,2}^{(2)} = \frac{-1}{N^2(N^2 - 1)} \left[N^3 \mathcal{N}_1 \left(\frac{4 \ln x \ln \bar{x}}{3x\bar{x}} + \pi^2 + \ln^2 \frac{\bar{x}}{x} \right) + \frac{\mathcal{N}_2}{3} \left(2 + 2 \ln \frac{\bar{x}}{x} (x - \bar{x}) + (1 - 2x\bar{x}) \left(\pi^2 + \ln^2 \frac{x}{\bar{x}} \right) \right) \right], \quad (\text{A48})$$

$$\hat{\sigma}_{2,2,3}^{(2)} = \frac{1}{9(N^2 - 1)} \left[4N^2(N - N_f)^2 \left(1 - \frac{1}{x\bar{x}} \right)^2 - \frac{6N\mathcal{N}_1}{x\bar{x}} (\pi^2 + x \ln^2 x + \bar{x} \ln^2 \bar{x}) \right], \quad (\text{A49})$$

where $\mathcal{N}_2 \equiv (N^4 - 6N^2 + 18)N_f^2 + (N^2 + 36)N^4 - 2N^3(N^2 + 6)N_f$. These terms in the decomposition of $\hat{\sigma}_{2,2}^{(2)}$ reproduce the Eq. (19) when added up with some substitution of $N = 3$ standing for the number of color.

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