

Radiation reaction force for scalar-tensor theories in effective-one-body formalism

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While most of the binary configurations in modified theories of gravity are studied under quasicircular orbit limit, eccentricity effects could play a significant role in future gravitational wave detections. We derive the gravitational radiation reaction force for nonspinning eccentric orbits within the effective-one-body (EOB) description for the massless scalar-tensor theories up to 1.5 post-Newtonian (PN) order. The effects in such theories start at $1/c^3$ and interestingly, the 1.5PN order effect is due to radiation reaction square effects, which is a conservative effect. The results derived here can be implemented in the quasicircular EOB-based waveform models to construct waveform templates for generic orbit binaries within massless scalar-tensor theories of gravity.

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I. INTRODUCTION

The detection of gravitational wave events by the Laser Interferometer Gravitational-Wave Observatory (LIGO) Collaboration [1] opened the possibility of studying the dynamics in the strong-gravity regime. Future gravitational wave events measured by the current [2–4] and next generation of detectors, such as the Einstein Telescope [5] and the Cosmic Explorer [6], will open the possibility of testing general relativity (GR) in the strong-gravity regime by constraining the parameters of alternative gravity theories.

The simplest extension of GR to consider scalar fields nonminimally coupled to gravity, called scalar-tensor (ST) theories, have been extensively studied in the literature both observationally [7–12], numerically [13–18], and theoretically [19–33], especially within the post-Newtonian (PN) formalism. The energy flux in scalar-tensor theories is known up to 1.5PN order for generic orbits (noncircular orbits without tides and spins) [31], and the angular momentum flux for inspiraling compact objects for

generic orbits up to 1.5PN order was recently computed in Ref. [34].

In addition to the post-Newtonian description, there has also been recent progress in constructing the effective-one-body (EOB) Hamiltonian from the two-body PN dynamics in scalar-tensor theories for nonspinning binaries [35–39] with leading-order tidal effects [37]. The EOB formalism [40,41] constructed by combining the information of the post-Newtonian theory and numerical relativity waveforms improves the waveform generation and hence is crucial to produce semianalytical accurate waveform templates to match filter against the data observed in the gravitational wave detectors. In the context of scalar-tensor theories, EOB-based models have always employed quasicircular radiation reaction force and waveforms. The aim of this paper is to extend the quasicircular radiation reaction force for the scalar-tensor theories to eccentric orbits up to 1.5PN order.

The paper is organized as follows: in Sec. II, we give a brief overview of the scalar-tensor theories, followed by the brief overview of the EOB description and the radiation reaction force (also called Abraham-Lorentz force) in Sec. III. In Sec. IV, we first derive a transformation between Arnowitt–Deser–Misner (ADM), harmonic, and EOB coordinate systems up to 2PN order, explore the arbitrariness (due to gauge choice) in the reactive terms at 1.5PN order using the method developed by Iyer and Will in Refs. [42,43], and then finally present the energy and angular momentum flux in EOB coordinates up to 2PN

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order. Finally, in Sec. V, we first discuss the approach used in this work and then compute the radiation reaction force up to 0.5PN order, followed by a discussion in Sec. VI.

In this work, we use the conventions of Refs. [34,37–39] to define the body-dependent scalar-tensor (ST) parameters (see Appendix A of Ref. [34] for details). Unless otherwise specified, we will also consider geometric units, such as $G = c = 1$.

II. BRIEF OVERVIEW: SCALAR-TENSOR

In this work, we consider the massless scalar-tensor theories for which the action is described by the coupling of the usual Einstein-Hilbert action with the scalar field φ as

$$S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} (R - 2g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi) + S_m[\Psi, \mathcal{A}(\varphi)^2 g_{\mu\nu}]. \quad (2.1)$$

Here, $g_{\mu\nu}$ is the Einstein metric, R is the Ricci scalar, Ψ collectively denotes the matter fields, $g \equiv \det(g_{\mu\nu})$, and G is the bare Newton's constant. The above action is expressed in the Einstein frame, which is obtained by a conformal transformation of the physical frame (Jordan frame) metric $\tilde{g}_{\mu\nu}$ as

$$\tilde{g}_{\mu\nu} = \mathcal{A}(\varphi)^2 g_{\mu\nu}, \quad (2.2)$$

to simplify the calculations, where $\mathcal{A}(\varphi)$ is called the coupling function. The scalar field couples minimally to Einstein frame metric and nonminimally to the matter fields; however, it is nonminimally coupled to the metric in the Jordan frame. The scalar-tensor theory is uniquely fixed when the function $\mathcal{A}(\varphi)$ is defined; general relativity is recovered when $\mathcal{A}(\varphi) = cst$. We adopt the conventions and notations of Damour and Esposito-Farese (DEF, hereafter) [19,44].

The field equations for the scalar-tensor theories in Einstein frame are derived in Ref. [19] and the coupling between the matter and the scalar field in the equations of motion is measured by the parameter

$$\alpha(\varphi) = \frac{\partial \ln \mathcal{A}}{\partial \varphi}. \quad (2.3)$$

The effective gravitational constant in scalar-tensor theories depends on the scalar field so that the compact object size and its internal gravity vary with the scalar field. Therefore, as suggested by Ref. [45], the compact, self-gravitating objects in scalar-tensor theories can be considered as point particles described by the mass function $m_J(\varphi)$, dependent on the value of scalar field at the position of particle. The matter action is then given by

$$S_m = - \sum_{J=A,B} \int \sqrt{-g_{\mu\nu}} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} m_J(\varphi), \quad (2.4)$$

where $m_J(\varphi)$ is the Einstein frame mass of body J , and λ is an affine parameter. Since $\tilde{g}_{\mu\nu} = \mathcal{A}(\varphi)^2 g_{\mu\nu}$, the Jordan frame mass is then defined as

$$\tilde{m}_J(\varphi) = \frac{m_J(\varphi)}{\mathcal{A}(\varphi)}. \quad (2.5)$$

The dimensionless body-dependent parameters that describe the scalar field effect in the Einstein frame up to the 2PN order are defined using the Einstein frame mass function $m_J(\varphi)$ following Refs. [19,44], i.e.,

$$\begin{aligned} \alpha_J &= \frac{d \ln m_J(\varphi)}{d\varphi}, \\ \beta_J &= \frac{d\alpha_J}{d\varphi}, \\ \beta'_J &= \frac{d\beta_J}{d\varphi}. \end{aligned} \quad (2.6)$$

The energy flux in Refs. [31,46,47] is expressed in Jordan frame equivalents of the above parameters. As we work in DEF conventions, we will make use of the relations given in Table I of Ref. [37] to perform the conversion of Jordan frame parameters to DEF ones.

III. BRIEF OVERVIEW: EOB AND RADIATION REACTION FORCE

In this section, before proceeding to the computations of the radiation reaction force in ST theory, we briefly review the EOB formalism proposed in Refs. [40,41] as a way to extend the validity of PN results beyond the weak-field and slow-motion regime by resumming the PN results. The three main features of the EOB approach are: description of the conservative (Hamiltonian) dynamics of a two-body system, expression for the radiation reaction effects, and the description of gravitational waves emitted by the coalescence of a compact binary.

The EOB approach maps the relative motion of two bodies with masses m_A and m_B into the motion of an effective particle of mass $\mu = m_A m_B / M$, where $M = m_A + m_B$, moving in an external metric. Throughout this paper, we describe the motion of the effective particle with rescaled polar coordinates (r, θ, ϕ) .

The conservative EOB dynamics is completely described by the following effective metric up to 2PN order:

$$ds_{\text{eff}}^2 = -A(r)c^2 dt_{\text{eff}}^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2), \quad (3.1)$$

where $A(r)$ and $B(r)$ are the two EOB potentials, t_{eff} is the coordinate time of the effective EOB metric, and r is the radial separation in EOB coordinates. In this work, we specialize to nonspinning systems and equatorial motions, i.e., set $\theta = \pi/2$. The generalization of the EOB formalism beyond 2PN order is attained by introducing the non-geodesic contributions to the effective metric (see, e.g., Ref. [48] for details).

The metric above, Eq. (3.1), can be used to obtain an EOB Hamiltonian, hereafter denoted by H , describing the conservative dynamics of generic binary orbits, both bound and unbound. The EOB Hamiltonian in the scalar-tensor theories for nonspinning binary is known up to 3PN order [35–39]. However, the current EOB realization of the radiation reaction force and the emitted waveform in scalar-tensor theories is limited to the case of quasicircular orbits. The main aim of our study is to extend the radiation reaction force along general orbits for nonspinning compact objects.

In GR, different approaches have been employed to study the gravitational radiation reaction. In this work, we will follow the *balance* approach developed in Refs. [42,43], i.e., we relate the energy and angular momentum losses of the system to the corresponding gravitational radiation at infinity. This approach was already used in Refs. [49,50] to derive the radiation reaction force for GR within the EOB formalism.

For nonprecessing binaries, when the motion is planar, Hamilton's equations read

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r}, & \dot{p}_r &= -\frac{\partial H}{\partial r} + \mathcal{F}_r, \\ \dot{\phi} &= \frac{\partial H}{\partial p_\phi}, & \dot{p}_\phi &= -\frac{\partial H}{\partial \phi} + \mathcal{F}_\phi, \end{aligned} \quad (3.2)$$

where the dot denotes differentiation with respect to time and radiation reaction forces $(\mathcal{F}_r, \mathcal{F}_\phi)$ [41,51] have been added to the conservative equations.

The force components $\mathcal{F}_\phi$ and $\mathcal{F}_r$ can be determined by computing the variation in the energy

$$E_{\text{system}} = H(r, p_r, p_\phi) - (m_A + m_B)c^2, \quad (3.3)$$

and the angular momentum

$$J_{\text{system}} = p_\phi, \quad (3.4)$$

of the binary system using Hamilton's equations, Eq. (3.2) above, and comparing them with the corresponding energy Φ_E and angular momentum Φ_J fluxes emitted in the form of gravitational waves at infinity. However, we cannot simply equate the energy and angular momentum losses of the system to $-\Phi_E$ and $-\Phi_J$, respectively, as the energy and angular momentum of the system have additional terms

called the Schott terms [52], say E_{Schott} and J_{Schott} , due to the system's interaction with the radiation field.

Using Hamilton's equations, and remembering H is independent of ϕ , we obtain that the time derivatives of the energy and angular momentum (radiation losses) of the system are

$$\begin{aligned} \dot{E}_{\text{system}} &= \dot{r}\mathcal{F}_r + \dot{\phi}\mathcal{F}_\phi, \\ J_{\text{system}} &= \mathcal{F}_\phi. \end{aligned} \quad (3.5)$$

The balance equation after including the Schott terms reads

$$\begin{aligned} \dot{E}_{\text{system}} + \dot{E}_{\text{Schott}} + \Phi_E &= 0, \\ J_{\text{system}} + J_{\text{Schott}} + \Phi_J &= 0. \end{aligned} \quad (3.6)$$

Combining Eqs. (3.6) and (3.5), we obtain the equations to solve for the two components of the force, as follows:

$$\begin{aligned} \mathcal{F}_r \dot{r} &= -\mathcal{F}_\phi \dot{\phi} - \dot{E}_{\text{Schott}} - \Phi_E, \\ \mathcal{F}_\phi &= -\dot{J}_{\text{Schott}} - \Phi_J. \end{aligned} \quad (3.7)$$

The far zone energy and angular momentum fluxes are gauge invariant but the radiation reaction forces $\mathcal{F}_r$ and $\mathcal{F}_\phi$, and the Schott terms could be gauge dependent quantities. The gauge freedom in the radiation reaction force at the leading order in GR, i.e., 2.5PN order, was first discussed by the authors of Refs. [42,43], who found that this gauge freedom is related to the freedom to change the coordinate system. Following this, the authors of Ref. [49] showed that the gauge freedom of the radiation reaction term at 2.5PN in GR is also related to gauge freedom in defining Schott terms, i.e., we can consider a vanishing J_{Schott} .

However, due to the presence of a nonvanishing dipole in scalar-tensor theories, in addition to the quadrupolar radiation, there is also a scalar dipolar radiation in scalar-tensor theories starting at a lower PN order. In Sec. IV A, we study the presence (or lack of) gauge freedom at 1.5PN order.

IV. COORDINATE SYSTEM TRANSFORMATIONS

In Refs. [31,34], the angular momentum flux and the energy flux for scalar-tensor theory are derived in the harmonic coordinate system. As the main aim of this paper is to derive the radiation reaction within the EOB formalism, we transform the fluxes into EOB coordinates. To do so, we use (i) the transformation between ADM and EOB coordinates derived in Refs. [35–37] and (ii) the transformation between ADM and harmonic coordinates derived in Refs. [35,37].

Here, we first define the family of fundamental scalars associated to each coordinate system, as follows:

$$\begin{aligned}
v_h^2 &= (\mathbf{v}_h \cdot \mathbf{v}_h), & \dot{r}_h^2 &= (\mathbf{n}_h \cdot \mathbf{v}_h)^2, & \frac{1}{r_h}, \\
v_a^2 &= (\mathbf{v}_a \cdot \mathbf{v}_a), & p_a^2 &= (\mathbf{p}_a \cdot \mathbf{p}_a), & \dot{r}_a^2 &= (\mathbf{n}_a \cdot \mathbf{v}_a)^2, \\
p_{r,a}^2 &= (\mathbf{n}_a \cdot \mathbf{p}_a)^2, & \frac{1}{r_a}, \\
\text{and} \\
p_e^2 &= (\mathbf{p}_e \cdot \mathbf{p}_e), & p_{r,e}^2 &= (\mathbf{n}_e \cdot \mathbf{p}_e)^2, & \frac{1}{r_e},
\end{aligned} \tag{4.1}$$

where the subscripts h , a , and e indicate the harmonic, ADM, and EOB coordinate systems, respectively. The boldfaced letters are used to denote (three-dimensional) spatial vectors. We will also use a tilde to define dimensionless variables, i.e.,

$$\tilde{\mathbf{p}}_X = \frac{\mathbf{p}_X}{\mu}, \quad \tilde{p}_{r,X}^2 = \left(\frac{p_{r,X}}{\mu} \right)^2, \tag{4.2}$$

for $X = \{h, a, e\}$.

A. ADM to harmonic

As the acceleration dependence in the Lagrangian starts at 2PN order, the transformation between the ADM and harmonic coordinate systems also starts at 2PN order. The explicit 2PN coordinate transformation is derived in Ref. [35] and the explicit 3PN coordinate transformation is derived in Ref. [37]. The transformation up to 2PN order is

$$\begin{aligned}
\mathbf{r}_h &= \mathbf{r}_a + \frac{1}{c^4} \left\{ \mathbf{r}_a \left[\frac{G_{AB}^2 M^2}{r_a^2} \left(\nu [f_{11} - f_{12} - f_{13} + f_{14}] + \frac{f_{13} m_1}{M} + \frac{f_{12} m_2}{M} \right) - \frac{G_{AB} M}{r_a} \nu \left(\frac{1}{8} + 3f_{10} + 3f_7 + 3\frac{m_1}{M} f_9 + 3\frac{m_2}{M} f_8 \right) \dot{r}_a^2 \right. \right. \\
&\quad \left. \left. - \frac{G_{AB} M}{r_a} \left(-\frac{7}{8} \nu - \frac{1}{2} \nu \gamma_{AB} + f_1 \nu - f_3 \nu - f_4 \nu + f_6 \nu - \frac{m_1}{M} f_6 - \frac{m_2}{M} f_1 + \frac{m_1}{M} \nu (f_4 + f_5 + f_6) + \frac{m_2}{M} \nu (f_1 + f_2 + f_3) \right) v_a^2 \right] \right. \\
&\quad \left. - \mathbf{v}_a \frac{G_{AB} M}{r_a} \nu \left[\frac{7}{4} + \gamma_{AB} + 2(f_1 + f_6) + 2\frac{m_1}{M} (f_5 - f_3) + 2\frac{m_2}{M} (f_2 - f_4) \right] r_a \dot{r}_a \right\} + \mathcal{O}\left(\frac{1}{c^5}\right), \\
\mathbf{v}_h &= \frac{d\mathbf{r}_h}{dt} = \mathbf{v}_a + \frac{1}{c^4} \left\{ \frac{G_{AB}^2 M^2}{r_a^2} \left[\mathbf{v}_a \left(\frac{7}{4} \nu + \nu (2f_1 + 2f_6 + f_{11} - f_{12} - f_{13} + f_{14}) + \frac{m_2}{M} (f_{12} + 2\nu f_2 - 2\nu f_4) \right. \right. \right. \\
&\quad \left. \left. + \frac{m_1}{M} (f_{13} + 2f_5 \nu - 2f_3 \nu) \right) + \frac{\mathbf{r}_a}{r_a} \dot{r}_a \left(-\frac{\nu}{4} - 2\nu (2f_1 + 3f_{10} - f_{11} + f_{12} + f_{13} - f_{14} - f_3 - f_4 + 2f_6 + 3f_7) \right. \right. \\
&\quad \left. \left. - \frac{m_1}{M} (2[f_{13} + f_6] - \nu [-2f_3 + 2f_4 + 4f_5 + 2f_6 + 6f_9]) - \frac{m_2}{M} (2[f_{12} + f_1] - \nu [-2f_4 + 2f_1 + 4f_2 + 2f_3 + 6f_8]) \right) \right] \\
&\quad \left. + \frac{G_{AB} M}{r_a} \left[\mathbf{v}_a \left(-\frac{7}{8} \nu + \nu [-3f_1 + f_3 + f_4 - 3f_6] + \frac{m_1}{M} [f_6 + \nu (2f_3 - f_4 - 3f_5 - f_6)] + \frac{m_2}{M} [f_1 + \nu (2f_4 - f_3 - 3f_2 - f_1)] \right) v_a^2 \right. \right. \\
&\quad \left. \left. + \mathbf{v}_a \nu \left(\frac{13}{8} + 2f_1 - 3f_{10} + 2f_6 - 3f_7 + \frac{m_1}{M} [-2f_3 + 2f_5 - 3f_9] + \frac{m_2}{M} [2f_2 - 2f_4 - 3f_8] \right) \dot{r}_a^2 + \frac{\mathbf{r}_a}{r_a} \nu \left(\frac{3}{8} + 9f_{10} + 9f_7 + 9\frac{m_1}{M} f_9 \right. \right. \right. \\
&\quad \left. \left. + 9\frac{m_2}{M} f_8 \right) \dot{r}_a^3 + \frac{\mathbf{r}_a}{r_a} \left(-\frac{9}{8} \nu + \nu [f_1 - 6f_{10} - f_3 - f_4 + f_6 - 6f_7] - \frac{m_1}{M} [f_6 - \nu (f_4 + f_5 + f_6 - 6f_9)] \right. \right. \\
&\quad \left. \left. - \frac{m_2}{M} [f_1 - \nu (f_1 + f_2 + f_3 - 6f_8)] \right) v_a^2 \dot{r}_a \right] \right\} + \mathcal{O}\left(\frac{1}{c^5}\right), \\
\frac{\mathbf{p}_a}{\mu} &= \mathbf{v}_a + \frac{1}{c^2} \left[\nu \frac{G_{AB} M}{r_a} \dot{r}_a \mathbf{n}_a + v_a^2 \frac{(1 - 3\nu)}{2} \mathbf{v}_a + (3 + \nu + 2\gamma_{AB}) \frac{G_{AB} M}{r_a} \mathbf{v}_a \right] + \frac{1}{c^4} \left\{ \frac{3}{8} (1 - 7\nu + 13\nu^2) v_a^4 \mathbf{v}_a \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{G_{AB}M}{2r_a} \left[\left(7 - 10\nu - 9\nu^2 - 8(f_1 + f_6)\nu + 4(1 - 2\nu)\gamma_{AB} + 8\frac{m_1}{M}(f_6 + [f_3 - f_5 - f_6]\nu) + 8\frac{m_2}{M}(f_1 + [f_4 - f_1 - f_2]\nu) \right) v_a^2 \right. \\
& + \left([1 - 5\nu + 4f_1 - 12f_{10} + 4f_6 - 12f_7]\nu + 4\frac{m_1}{M}[3f_{10}(1 - \nu) - f_6 + (f_5 + f_6 - f_3 - 3f_9)\nu] + 4\frac{m_2}{M}[3f_7(1 - \nu) - f_1 \right. \\
& + (f_1 + f_2 - f_4 - 3f_8)\nu] \Big) \dot{r}_a^2 \Big] \mathbf{v}_a + \frac{G_{AB}M}{2r_a} \dot{r}_a \left[\left([1 + 4f_1 + 4f_6 - 12f_7 - 12f_{10} - 5\nu]\nu + 4\frac{m_1}{M}[3f_{10}(1 - \nu) - f_6(1 - \nu) \right. \right. \\
& + (f_5 - f_3 - 3f_9)\nu] + 4\frac{m_2}{M}[3f_7(1 - \nu) - f_1(1 - \nu) + (f_2 - f_4 - 3f_8)\nu] \Big) v_a^2 + \left(24f_{10}\nu + 24f_7\nu + 3\nu^2 \right. \\
& + 24\frac{m_1}{M}[-f_{10} + (f_9 + f_{10})\nu] + 24\frac{m_2}{M}[-f_7 + (f_7 + f_8)\nu] \Big) \dot{r}_a^2 \Big] \mathbf{n}_a + \frac{G_{AB}^2M^2}{r_a^2} \left[\frac{7}{2} - \frac{17}{2}\nu + \nu^2 + \frac{11}{4}\gamma_{AB}^2 - 5\gamma_{AB}(-1 + \nu) \right. \\
& + \langle \beta \rangle (1 - \nu) + 2\nu\beta_+ - \langle \delta \rangle + 2(f_1 - f_{12} - f_{13} + f_6)\nu + 2\frac{m_1}{M}(f_{13} + f_6[-1 + \nu] + [f_{11} + f_{14} - f_3 + f_5]\nu) \\
& + 2\frac{m_2}{M}(f_{12} + f_1[-1 + \nu] + [f_{11} + f_{14} - f_4 + f_2]\nu) \Big] \mathbf{v}_a + \frac{G_{AB}^2M^2}{4r_a^2} \dot{r}_a \left[8(7 + 5\gamma_{AB} - \beta_- - 3\beta_+ + 2[f_1 + f_{12} + f_{13} + f_6] \right. \\
& + 3[f_{10} + f_7]\nu + 12\nu^2 + 4\langle \delta \rangle + (2 + \gamma_{AB})^2 + 8\frac{m_1}{M}(-3f_{10} - 2[f_{13} + f_6] + [3_{10} - 2(f_{11} + f_{14} + f_3 - f_5 - f_6) + 3f_9]\nu) \\
& + 8\frac{m_2}{M}(-3f_7 - 2[f_{12} + f_1] + [3_7 - 2(f_{11} + f_{14} + f_4 - f_2 - f_1) + 3f_8]\nu) \Big] \mathbf{n}_a \Big\} + \mathcal{O}\left(\frac{1}{c^6}\right), \\
\mathbf{v}_a = & \tilde{\mathbf{p}}_a + \frac{1}{c^2} \left(-\nu \frac{G_{AB}M}{r_a} \tilde{p}_{r,a} \mathbf{n}_a - \tilde{p}_a^2 \frac{(1 - 3\nu)}{2} \tilde{\mathbf{p}}_a - (3 + \nu + 2\gamma_{AB}) \frac{G_{AB}M}{r_a} \tilde{\mathbf{p}}_a \right) + \frac{1}{c^4} \left\{ \frac{3}{8} (1 - 5\nu + 5\nu^2) \tilde{p}_a^4 \tilde{\mathbf{p}}_a \right. \\
& + \frac{G_{AB}M}{r_a} \left[\frac{3}{8} \left(\nu - 4\nu^2 - 8f_{10}\nu - 8f_7\nu - \frac{m_1}{M}[8f_{10}(-1 + \nu) + 8f_9\nu] - \frac{m_2}{M}[8f_7(-1 + \nu) + 8f_8\nu] \right) \tilde{\mathbf{p}}_{r,a}^2 - \frac{1}{8} \left(5\nu + 4\nu^2 \right. \right. \\
& + 8f_1\nu + 8f_6\nu + \frac{m_1}{M}[8f_6(-1 + \nu) + (-8f_3 + 8f_5)\nu] + \frac{m_2}{M}[8f_1(-1 + \nu) + (-8f_4 + 8f_2)\nu] \Big) \tilde{p}_a^2 \Big] \tilde{\mathbf{p}}_{r,a} \mathbf{n}_a \\
& + \frac{G_{AB}M}{r_a} \left[\frac{1}{8} \left(17\nu - 4\nu^2 + 24[f_7 + f_{10}]\nu + 24\frac{m_1}{M}[f_9\nu + (-1 + \nu)f_{10}] + 24\frac{m_2}{M}[f_8\nu + (-1 + \nu)f_7] \right) \tilde{p}_{r,a}^2 + \frac{1}{8} \left(20 - 95\nu \right. \right. \\
& - 12\nu^2 + \gamma_{AB}[16 - 64\nu] + 8f_1\nu + 8f_6\nu + 8\frac{m_1}{M}[f_5\nu - f_3\nu + (-1 + \nu)f_6] + 8\frac{m_2}{M}[f_2\nu - f_4\nu + (-1 + \nu)f_1] \Big) \tilde{p}_a^2 \Big] \tilde{\mathbf{p}}_a \\
& - \frac{G_{AB}^2M^2}{4r_a^2} \left[\left(4 + 31\nu + 4\gamma_{AB}[1 + 6\nu] + \gamma_{AB}^2 - 16\langle \beta \rangle \nu - 8[\beta_+ + X_{AB}\beta_-]\nu + 4\langle \delta \rangle + 8[f_{12} + f_{13} - f_{11} - f_{14}]\nu - 8\frac{m_1}{M}f_{13} \right. \right. \\
& - 8\frac{m_2}{M}f_{12} \Big) \tilde{p}_{r,a} \mathbf{n}_a - \left(22 + 65\nu + 5\gamma_{AB}^2 + 4\gamma_{AB}[7 + 9\nu] - 4\langle \beta \rangle[1 + \nu] - 8\nu X_{AB}\beta_- + 4\langle \delta \rangle - 4[f_{11} - f_{12} - f_{13} + f_{14}]\nu \right. \\
& \left. \left. - 4\frac{m_1}{M}f_{13} - 4\frac{m_2}{M}f_{12} \right) \tilde{\mathbf{p}}_a \right] \Big\} + \mathcal{O}\left(\frac{1}{c^6}\right), \\
\mathbf{n}_h = & \mathbf{n}_a + \frac{1}{c^4} \left\{ \nu \frac{G_{AB}M}{r_a} \left[\frac{7}{4} + \gamma_{AB} + \frac{f_1}{2} + \frac{f_6}{2} + \frac{m_1}{M}(-2f_3 + 2f_5) + \frac{m_2}{M}(2f_2 - 2f_4) \right] (\mathbf{n}_a \dot{r}_a^2 - \mathbf{v}_a \dot{r}_a) \right\} + \mathcal{O}\left(\frac{1}{c^5}\right). \tag{4.3}
\end{aligned}$$

Therefore, the explicit transformation of the fundamental scalar quantities from the harmonic coordinate system to the ADM coordinate system up to 2PN order is

$$\begin{aligned}
v_h^2 = & \tilde{p}_a^2 + \frac{1}{c^2} \left(-2\nu \frac{G_{AB}M}{r_a} \tilde{p}_{r,a}^2 - (1 - 3\nu) \tilde{p}_a^4 - 2(3 + \nu + 2\gamma_{AB}) \frac{G_{AB}M}{r_a} \tilde{p}_a^2 \right) + \frac{1}{c^4} \left\{ \frac{1}{4} (4 - 21\nu + 24\nu^2) \tilde{p}_a^6 \right. \\
& + \frac{G_{AB}M}{r_a} \left[\left(8 - 6\nu^2 + \gamma_{AB}[6 - 22\nu] - \frac{127}{4}\nu + 2[f_1 + f_6] + 2\frac{m_1}{M}[f_6(-1 + \nu) - f_3 + f_5] + 2\frac{m_2}{M}[f_1(-1 + \nu) - f_4 + f_2] \right) \tilde{p}_a^4 \right. \\
& + \left(4\nu - 5\nu^2 + 6f_{10}\nu - 2f_6\nu - 2[f_1 + f_3 + f_4 - f_7]\nu + 2\frac{m_1}{M}[f_6 + 3f_{10}(-1 + \nu) - f_6\nu + (2f_3 + f_4 - f_5 + 3f_9)\nu] \right.
\end{aligned}$$

$$\begin{aligned}
& + 2\frac{m_2}{M}[f_1 + 3f_7(-1 + \nu) - f_1\nu + (2f_4 + f_3 - f_2 + 3f_8)\nu] \tilde{p}_{r,a}^2 \tilde{p}_a^2 - \frac{3}{4} \left(-\nu + 4\nu^2 + 8[f_{10} + f_7]\nu + 8\frac{m_1}{M}[f_9\nu + f_{10}(\nu - 1)] \right. \\
& + 8\frac{m_2}{M}[f_8\nu + f_7(\nu - 1)] \tilde{p}_{r,a}^4 \left. + \frac{G_{AB}^2 M^2}{r_a^2} \left[\left(-2 - \frac{\gamma_{AB}^2}{2} - 2\langle\delta\rangle - 2\gamma_{AB}(1 + 4\nu) + \nu \left[-\frac{19}{2} + 3\nu + 4\beta_+ - 4\beta_- X_{AB} + 8\langle\beta\rangle \right. \right. \right. \right. \\
& + 4f_{11} - 4f_{12} - 4f_{13} + 4f_{14} \left. \right] \nu + 4\frac{m_1}{M}f_{13} + 4\frac{m_2}{M}f_{12} \tilde{p}_{r,a}^2 + \left(20 + \frac{13}{2}\gamma_{AB}^2 + 2\langle\delta\rangle + \frac{77}{2}\nu + 4\nu\beta_- X_{AB} - 2\nu\beta_+ + \nu^2 \right. \\
& - 2\langle\beta\rangle[1 + \nu] + \gamma_{AB}[26 + 22\nu] - 2f_{11}\nu + 2f_{12}\nu + 2f_{13}\nu - 2f_{14}\nu - 2\frac{m_1}{M}f_{13} - 2\frac{m_2}{M}f_{12} \left. \right) \tilde{p}_a^2 \left. \right] \left. \right\} + \mathcal{O}\left(\frac{1}{c^6}\right), \\
\dot{r}_h^2 = & \tilde{p}_{r,a}^2 + \frac{1}{c^2} \left(-2\nu \frac{G_{AB}M}{r_a} \tilde{p}_{r,a}^2 - \tilde{p}_a^2(1 - 3\nu) \tilde{p}_{r,a}^2 - 2(3 + \nu + 2\gamma_{AB}) \frac{G_{AB}M}{r_a} \tilde{p}_{r,a}^2 \right) + \frac{1}{c^4} \left\{ \frac{1}{4}(4 - 21\nu + 24\nu^2) \tilde{p}_{r,a}^2 \tilde{p}_a^4 \right. \\
& - \frac{G_{AB}M}{2r_a} \left[-16 + 71\nu + 20\nu^2 + 4\nu\gamma_{AB} + 4\gamma_{AB}(-3 + 11\nu) + 8(f_1 + f_6)\nu + 8\frac{m_1}{M}(-f_6 + [-f_3 + f_5 + f_6]\nu) \right. \\
& + 8\frac{m_2}{M}(-f_1 + [-f_4 + f_1 + f_2]\nu) \left. \right] \tilde{p}_{r,a}^2 \tilde{p}_a^2 + \frac{G_{AB}M}{2r_a} \left[17\nu + 4\nu\gamma_{AB} - 8\nu^2 + 8[f_1 + f_6]\nu + 8\frac{m_1}{M}(-f_6 + [-f_3 + f_4 + f_5]\nu) \right. \\
& + 8\frac{m_2}{M}(-f_1 + [-f_4 + f_2 + f_1]\nu) \left. \right] \tilde{p}_{r,a}^4 + \frac{G_{AB}^2 M^2}{r_a^2} \left[(29 + 14\gamma_{AB} + 6\langle\beta\rangle) + 2\beta_+ + 2f_{11} - 2f_{12} - 2f_{13} + 2f_{14} \right] \nu + 4\nu^2 - 2\langle\beta\rangle \\
& + 6(1 + \gamma_{AB})(3 + \gamma_{AB}) + 2\frac{m_1}{M}f_{13} + 2\frac{m_2}{M}f_{12} \left. \right] \left. \right\} + \mathcal{O}\left(\frac{1}{c^4}\right), \\
\frac{1}{r_h} = & \frac{1}{r_a} \left\{ 1 + \frac{1}{c^4} \left[\frac{G_{AB}M}{8r_a} \left([-7 - 4\gamma_{AB} + 8f_1 - 8f_3 - 8f_4 + 8f_6]\nu + 8\frac{m_1}{M}[-f_6 + (f_4 + f_5 + f_6)\nu] + 8\frac{m_2}{M}[-f_1 \right. \right. \right. \right. \\
& + (f_1 + f_2 + f_3)\nu] \left. \right) \tilde{p}_a^2 + \frac{G_{AB}M}{8r_a} \left([15 + 8\gamma_{AB} + 16f_1 + 24f_{10} + 16f_6 + 24f_7]\nu + 8\frac{m_1}{M}[(2f_6 + 3f_{10})(\nu - 1) \right. \\
& + (2f_5 - 2f_3 + 3f_9)\nu] + 8\frac{m_2}{M}[(2f_1 + 3f_7)(\nu - 1) + (2f_2 - 2f_4 + 3f_8)\nu] \left. \right) \tilde{p}_{r,a}^2 \left. \right] - \frac{G_{AB}^2 M^2}{r_a^2} \left[(f_{11} - f_{12} - f_{13} + f_{14})\nu \right. \\
& + \frac{m_1}{M}f_{13} + \frac{m_2}{M}f_{12} \left. \right] \left. \right\} + \mathcal{O}\left(\frac{1}{c^6}\right). \tag{4.4}
\end{aligned}$$

Here, f_i 's are the parameters of the total derivative function f entering at 2PN order [35] and

$$\begin{aligned}
\langle\tilde{\beta}\rangle & \equiv -X_{AB}\beta_- + \beta_+, \\
\langle\delta\rangle & \equiv X_{AB}\delta_- + \delta_+,
\end{aligned}$$

with $X_{AB} \equiv (m_A - m_B)/M$ and the $\pm$ subscript denoting the symmetric and antisymmetric parts of the ST parameters, e.g., $x_{\pm} \equiv (x_A \pm x_B)/2$ [37]. From the above transformations, we verify that the harmonic to ADM coordinate transformation of scalar-tensor theory reduces to the coordinate transformation of GR in the GR limit [49].

Until now, we only computed the transformation involving the conservative orders up to 2PN order. However, there still remains the possible transformation (gauge choice) in the radiation reaction terms which enter at 1.5PN order due to the scalar dipolar radiation in the scalar-tensor theory. The lowest order, i.e., 1.5PN order dissipative relative acceleration term in harmonic coordinates is [31]

$$\mathbf{a}_h^{1.5\text{PN}} = \frac{4\zeta S_-^2}{3c^3 r_h} \frac{G_{AB}^2 M^2 \nu}{r_h^2} [3\dot{r}_h \mathbf{n}_h - \mathbf{v}_h]. \tag{4.5}$$

Now, to investigate the alternative gauge choices for radiation reaction at the leading order, we first write the general form of the radiation reaction term for the binary system without spins and tidal effects using the fact that the term: (i) must be a correction to the Newtonian acceleration implying that it should be proportional to $(GM)/r^2$, (ii) should vanish in the test-mass limit, i.e., should be proportional to symmetric mass-ratio ν , (iii) must be nonlinear in Newton's constant as it is related to the emission of gravitational radiation [i.e., should contain an extra factor of $(GM)/r$], and finally, (iv) must be odd in velocities, i.e., dissipative. As we are working in scalar-tensor theories, Newton's constant is replaced by the coupling constant G_{AB} of the theory. Therefore, the general radiation reaction term at the leading order reads

$$\mathbf{a}_{1.5\text{PN}} = \frac{G_{AB}^2 M^2}{r^3} \nu [A_{3/2} i\mathbf{n} + B_{3/2} \mathbf{v}], \tag{4.6}$$

where $\mathbf{v}$, $\mathbf{n}$ denote the relative velocity and separation vector. The values for the parameters $A_{3/2}$ and $B_{3/2}$ in harmonic coordinates can be read from Eq. (4.5). The

goal of our investigation would now be to determine these two arbitrary parameters in ADM coordinates, for which it will be sufficient to investigate for gauge freedom in defining radiation reaction terms.

For this, we take the post-Newtonian expressions for the energy and angular momentum, $\tilde{E} \equiv E/\mu$ and $\tilde{J} \equiv J/\mu$, and calculate $d\tilde{E}/dt$ and $d\tilde{J}/dt$ using 1PN equations of motion supplemented with 1.5PN radiation reaction of Eq. (4.6). The angular momentum and energy are conserved up to 1PN order but the presence of dissipative effects at 1.5PN order leads to non-vanishing values of $d\tilde{E}/dt$ and $d\tilde{J}/dt$, and hence we have the freedom to add arbitrary terms to $\tilde{E}$ and $\tilde{J}$ at 1.5PN order (see Ref. [43] for a similar analysis). However, the term should be such that it is proportional to ν (vanishes in the test-mass limit), $(GM)/r$ (non-linear effects for gravitational waves emission), and odd in $\dot{r}$ (dissipative effects). Therefore, the possible terms for $\tilde{E}$ and $\tilde{J}$ reduce to

$$\begin{aligned}\tilde{E}^* &\equiv \tilde{E}_N + \nu \dot{r} \left(\frac{G_{AB}M}{r} \right)^2 \alpha_{3/2}, \\ \tilde{J}^* &\equiv \tilde{J}_N + \nu \dot{r} \tilde{J}_N \left(\frac{G_{AB}M}{r} \right) \beta_{3/2},\end{aligned}\quad (4.7)$$

where we use $*$ to distinguish the arbitrary terms entering at 1.5PN order from the conserved quantities up to 1PN order, $\tilde{E}_N$ and $\tilde{J}_N = \mathbf{x} \times \mathbf{v}$ are the Newtonian order contribution to the energy and angular momentum, and $\alpha_{3/2}$ and $\beta_{3/2}$ are the arbitrary parameters entering at 1.5PN order. Therefore, the energy and angular momentum orbital loss reads

$$\begin{aligned}\frac{d\tilde{E}^*}{dt} &= \mathbf{v} \cdot \mathbf{a}_{1.5\text{PN}} \\ &\quad + \nu \alpha_{3/2} \left[\ddot{r} \left(\frac{G_{AB}M}{r} \right)^2 - 2 \frac{\dot{r}^2}{r} \left(\frac{G_{AB}M}{r} \right)^2 \right], \\ \frac{d\tilde{J}^*}{dt} &= \mathbf{x} \times \mathbf{a}_{1.5\text{PN}} + \nu \beta_{3/2} \tilde{J}_N \left[\frac{G_{AB}M}{r} \ddot{r} - \frac{G_{AB}M}{r} \frac{\dot{r}^2}{r} \right].\end{aligned}\quad (4.8)$$

Now, in order to find the expression of the derivatives, we use the Newtonian order equations of motion, i.e.,

$$\ddot{r} = -\frac{G_{AB}M}{r^2} + \frac{v^2}{r} - \frac{\dot{r}^2}{r}, \quad (4.9)$$

and acceleration at 1.5PN order from Eq. (4.6). Hence, the explicit expression of the energy and angular momentum loss reads

$$\begin{aligned}\frac{d\tilde{E}^*}{dt} &= \left(\frac{G_{AB}M}{r} \right)^2 \frac{\nu}{r} \left[A_{3/2} \dot{r}^2 + B_{3/2} v^2 \right. \\ &\quad \left. + \alpha_{3/2} \left(v^2 - \frac{G_{AB}M}{r} - 3\dot{r}^2 \right) \right], \\ \frac{d\tilde{J}^*}{dt} &= \frac{G_{AB}M}{r^2} \frac{\nu \tilde{J}_N}{r} \beta_{3/2} \left(v^2 - \frac{G_{AB}M}{r} - 2\dot{r}^2 \right) \\ &\quad + \left(\frac{G_{AB}M}{r} \right)^2 \frac{\nu \tilde{J}_N}{r} B_{3/2}.\end{aligned}\quad (4.10)$$

We now write the energy and angular momentum balance equations, i.e., we equate the orbital loss of energy and angular momentum to the far-zone fluxes:

$$\begin{aligned}\frac{d\tilde{E}^*}{dt} &= - \left(\frac{d\tilde{E}}{dt} \right)_{\text{far-zone}}, \\ \frac{d\tilde{J}^*}{dt} &= - \left(\frac{d\tilde{J}}{dt} \right)_{\text{far-zone}},\end{aligned}\quad (4.11)$$

where the far-zone fluxes are

$$\begin{aligned}\left(\frac{d\tilde{E}}{dt} \right)_{\text{far-zone}} &= \frac{4}{3} \zeta \mathcal{S}_-^2 \frac{\nu}{r} \left(\frac{G_{AB}M}{r} \right)^3, \\ \left(\frac{d\tilde{J}}{dt} \right)_{\text{far-zone}} &= \frac{4}{3} \zeta \mathcal{S}_-^2 \frac{\nu}{r} \left(\frac{G_{AB}M}{r} \right)^2 \tilde{J}_N.\end{aligned}\quad (4.12)$$

Comparing Eqs. (4.10) and (4.12) term by term, we see that all the four arbitrary parameters: $A_{3/2}$, $B_{3/2}$, $\alpha_{3/2}$, and $\beta_{3/2}$ are uniquely determined, namely

$$\begin{aligned}A_{3/2} &= 4\zeta \mathcal{S}_-^2, & \alpha_{3/2} &= \frac{4}{3} \zeta \mathcal{S}_-^2, \\ B_{3/2} &= -\frac{4}{3} \zeta \mathcal{S}_-^2, & \beta_{3/2} &= 0.\end{aligned}\quad (4.13)$$

The far-zone energy and angular momentum fluxes expressed in terms of the radiative multipole moments at infinity are gauge invariant, i.e., it is independent of the coordinate system that preserves the asymptotic flatness. Thus, the unique determination of the arbitrary coefficients implies that there is no residual gauge freedom at 1.5PN order in scalar-tensor theories, in contrast to the residual gauge freedom at 2.5PN order in GR (see Refs. [42,43]). Furthermore, this implies that there is no coordinate transformation between ADM and harmonic coordinates at 1.5PN order, and hence no coordinate gauge freedom in the radiation reaction force.

B. ADM to EOB

In this section, we derive the transformation of the fundamental scalars from ADM coordinates to EOB coordinates up to 2PN order. The ADM to EOB canonical

transformation up to 3PN order is derived in Refs. [35,37]. For the purpose of this work, we consider transformation only up to 2PN order.

The explicit transformation of the fundamental scalar quantities in the ADM coordinate system to the EOB coordinate system reads

$$\begin{aligned}
\frac{1}{r_a} &= \frac{1}{r_e} + \frac{1}{c^2} \left\{ \frac{1}{r_e} \left[-\frac{\nu}{2} \tilde{p}_e^2 + \frac{G_{AB}M}{r_e} \left(1 + \gamma_{AB} + \frac{\nu}{2} \right) \right] - \frac{\nu}{r_e} \tilde{p}_{r,e}^2 \right\} + \frac{1}{r_e c^4} \left\{ [\nu + 3\nu^2] \frac{\tilde{p}_e^4}{8} + \frac{3\nu^2}{2} \tilde{p}_{r,e}^4 + \frac{\nu}{2} \tilde{p}_e^2 \tilde{p}_{r,e}^2 \right. \\
&\quad + \frac{G_{AB}M}{r_e} \left[\left(-\frac{7}{8} \nu^2 - f_6 \nu - f_1 \nu + \frac{m_1}{M} [f_6 + (f_3 - f_5 - f_6 \nu)] + \frac{m_2}{M} [f_1 + (f_4 - f_1 - f_2 \nu)] \right) \tilde{p}_e^2 - \left(-3\nu + \frac{5}{8} \nu^2 - 3\gamma_{AB} \nu \right. \right. \\
&\quad + 2[f_1 + f_6] \nu + 3[f_{10} + f_7] \nu + \frac{m_1}{M} [(3f_{10} + 2f_6)(\nu - 1) + (3f_9 + 2f_5 - 2f_3) \nu] + \frac{m_2}{M} [(3f_7 + 2f_1)(\nu - 1) \\
&\quad + (3f_8 + 2f_2 - 2f_4) \nu] \tilde{p}_{r,e}^2 \left. \right] + \frac{G_{AB}^2 M^2}{r_e^2} \left[1 - \frac{15}{4} \nu + \frac{\nu^2}{2} + \gamma_{AB} \left(\frac{5}{2} - 2\nu \right) + \frac{11}{8} \gamma_{AB}^2 + \langle \beta \rangle + \frac{\nu}{2} \beta_+ - \frac{1}{2} \langle \delta \rangle \right. \\
&\quad \left. \left. + (f_{11} - f_{12} - f_{13} + f_{14}) \nu + \frac{m_1}{M} f_{13} + \frac{m_2}{M} f_{12} \right] \right\} + \mathcal{O}\left(\frac{1}{c^6}\right), \\
p_{r,a}^2 &= p_{r,e}^2 - \frac{1}{c^2} [\nu p_{r,e}^2 (2\tilde{p}_{r,e}^2 - \tilde{p}_e^2)] + \frac{p_{r,e}^2}{c^4} \left\{ 4\nu^2 \tilde{p}_{r,e}^4 - \frac{3\nu}{4} \tilde{p}_e^4 + \frac{G_{AB}^2 M^2}{r_e^2} \left[-\frac{19}{2} \nu + \frac{1}{2} \nu^2 + \gamma_{AB} (1 - 6\nu) + \frac{3}{4} \gamma_{AB}^2 + 2\langle \beta \rangle + \nu \beta_+ \right. \right. \\
&\quad - \langle \delta \rangle + 2(f_{11} - f_{12} - f_{13} + f_{14}) \nu - 2\frac{m_1}{M} f_{13} - 2\frac{m_2}{M} f_{12} \left. \right] + \nu(1 - 3\nu) \tilde{p}_e^2 \tilde{p}_{r,e}^2 + \frac{G_{AB}M}{r_e} (\tilde{p}_{r,e}^2 - \tilde{p}_e^2) \left[-4(-1 + f_1 + f_6 - \gamma_{AB}) \nu \right. \\
&\quad \left. \left. - \frac{3}{2} \nu^2 + 4\frac{m_1}{M} (f_6 - [f_5 + f_6 - f_3] \nu) + 4\frac{m_2}{M} (f_1 - [f_1 + f_2 - f_4] \nu) \right] \right\} + \mathcal{O}\left(\frac{1}{c^6}\right), \\
p_a^2 &= p_e^2 + \frac{1}{c^2} \left[\frac{G_{AB}M}{r_e} (p_e^2 - p_{r,e}^2) (2 + \nu + 2\gamma_{AB}) - \nu p_e^2 \tilde{p}_e^2 \right] + \frac{1}{c^4} \left\{ \left[\frac{1}{4} \nu + \nu^2 \right] p_e^2 \tilde{p}_e^4 + \frac{G_{AB}M}{4r_e} (p_{r,e}^2 - p_e^2) \left[\tilde{p}_e^2 \left(4\nu + 9\nu^2 + 4\gamma_{AB} \right. \right. \right. \\
&\quad + 8[f_1 + f_6] \nu + 8\frac{m_1}{M} [f_6(\nu - 1) - f_3 \nu + f_5 \nu] + 8\frac{m_2}{M} [f_1(\nu - 1) - f_4 \nu + f_2 \nu] \left. \right] + \tilde{p}_{r,e}^2 \left(3\nu^2 + 24[f_{10} + f_7] \nu \right. \\
&\quad + 24\frac{m_1}{M} [f_9 \nu + f_{10}(\nu - 1)] + 24\frac{m_2}{M} [f_8 \nu + f_7(\nu - 1)] \left. \right] + \frac{G_{AB}^2 M^2}{4r_e^2} p_e^2 \left[12 - 26\nu + 5\nu^2 + 4\gamma_{AB} (7 - 3\nu) + 15\gamma_{AB}^2 + 8\langle \beta \rangle \right. \\
&\quad + 4\nu \beta_+ - 4\langle \delta \rangle + 8\nu(f_{11} - f_{12} - f_{13} + f_{14}) + 8\frac{m_1}{M} f_{13} + 8\frac{m_2}{M} f_{12} \left. \right] + \frac{G_{AB}^2 M^2}{4r_e^2} p_{r,e}^2 \left[-12 + 64\nu - 7\nu^2 - 32\gamma_{AB} + 36\gamma_{AB} \nu \right. \\
&\quad \left. \left. - 18\gamma_{AB}^2 - 16\langle \beta \rangle - 8\nu \beta_+ + 8\langle \delta \rangle - 16\nu(f_{11} + f_{14} - f_{12} - f_{13}) - 16\frac{m_1}{M} f_{13} - 16\frac{m_2}{M} f_{12} \right] \right\} + \mathcal{O}\left(\frac{1}{c^6}\right). \tag{4.14}
\end{aligned}$$

C. Harmonic to EOB

As we want to transform the harmonic coordinate angular momentum flux and energy flux presented in Refs. [31,34,46,47], we combine the two transformation given in Secs. IV A–IV B to obtain the transformation from harmonic coordinates to EOB coordinates. The explicit expression of the transformation up to 2PN order reads

$$\begin{aligned}
i_h^2 &= \tilde{p}_{r,e}^2 + \frac{1}{c^2} \left[-2\nu \tilde{p}_{r,e}^4 + (4\nu - 1) \tilde{p}_e^2 \tilde{p}_{r,e}^2 - 2(3 + 2\nu + 2\gamma_{AB}) \frac{G_{AB}M}{r_e} \tilde{p}_{r,e}^2 \right] + \frac{1}{c^4} \left[4\nu^2 \tilde{p}_{r,e}^6 \right. \\
&\quad + (3\nu - 9\nu^2) \tilde{p}_{r,e}^4 \tilde{p}_e^2 + (1 - 6\nu + 6\nu^2) \tilde{p}_{r,e}^2 \tilde{p}_e^4 + \left(2 + \frac{51}{2} \nu + \frac{7}{2} \nu^2 + 2\gamma_{AB} + 12\gamma_{AB} \nu \right) \frac{G_{AB}M}{r_e} \tilde{p}_{r,e}^4 \\
&\quad + \left(6 - \frac{75}{2} \nu - \frac{15}{2} \nu^2 + 4\gamma_{AB} - 24\gamma_{AB} \nu \right) \frac{G_{AB}M}{r_e} \tilde{p}_{r,e}^2 \tilde{p}_e^2 \\
&\quad \left. + \left(12 + \frac{63}{2} \nu + \frac{3}{2} \nu^2 + 13\gamma_{AB} + \frac{5}{4} \gamma_{AB}^2 + 14\gamma_{AB} \nu + 3\langle \beta \rangle (-1 + \nu) + \beta_+ \nu + \langle \delta \rangle \right) \frac{G_{AB}^2 M^2}{r_e^2} \tilde{p}_{r,e}^2 \right] + \mathcal{O}\left(\frac{1}{c^5}\right),
\end{aligned}$$

$$\begin{aligned}
v_h^2 = & \tilde{p}_e^2 + \frac{1}{c^2} \left[(-1 + 2\nu) \tilde{p}_e^4 + \frac{G_{AB}M}{r_e} (\tilde{p}_{r,e}^2 (-2 - 3\nu - 2\gamma_{AB}) - \tilde{p}_e^2 (4 + \nu + 2\gamma_{AB})) \right] + \frac{1}{c^4} \left\{ (1 - 3\nu + \nu^2) \tilde{p}_e^6 \right. \\
& + \frac{G_{AB}M}{r_e} \left[\left(4 - \frac{55}{4}\nu + \frac{3}{4}\nu^2 + 2\gamma_{AB} - 5\gamma_{AB}\nu \right) \tilde{p}_e^4 + \left(4 + \nu - \frac{17}{2}\nu^2 + 4\gamma_{AB} - 7\gamma_{AB}\nu \right) \tilde{p}_e^2 \tilde{p}_{r,e}^2 \right] \\
& + \frac{G_{AB}^2 M^2}{r_e^2} \left[\left(5 + 17\nu - \frac{3}{4}\nu^2 + \gamma_{AB}[3 + 7\nu] - \frac{7}{4}\gamma_{AB}^2 + 2\langle\beta\rangle - \beta_+ \right) \tilde{p}_e^2 \right. \\
& + \left. \left(7 + \frac{29}{2}\nu + \frac{9}{4}\nu^2 + \gamma_{AB}[10 + 7\nu] + 3\gamma_{AB}^2 + 2\langle\beta\rangle[\nu - 2] + 2\beta_+\nu + \langle\delta\rangle \right) \tilde{p}_{r,e}^2 \right] \left. \right\} + \mathcal{O}\left(\frac{1}{c^5}\right), \\
\frac{1}{r_h} = & \frac{1}{r_e} + \frac{1}{c^2} \left[-\frac{\nu}{2r_e} \tilde{p}_e^2 + \frac{G_{AB}M}{r_e^2} \left(1 + \gamma_{AB} + \frac{\nu}{2} \right) - \frac{\nu}{r_e} \tilde{p}_{r,e}^2 \right] + \frac{1}{c^4} \left[(\nu + 3\nu^2) \frac{\tilde{p}_e^4}{8r_e} + \nu \frac{\tilde{p}_e^2 \tilde{p}_{r,e}^2}{2r_e} + 3\nu^2 \frac{\tilde{p}_{r,e}^4}{2r_e} \right. \\
& - \nu \frac{G_{AB}M}{8r_e^2} ([-39 + 5\nu - 32\gamma_{AB}] \tilde{p}_{r,e}^2 + [7 + 7\nu + 4\gamma_{AB}] \tilde{p}_e^2) + \frac{G_{AB}^2 M^2}{r_e^3} \left(1 - \frac{15}{4}\nu + \frac{\nu^2}{2} + \frac{[5 - 4\nu]}{2} \gamma_{AB} + \frac{11}{8} \gamma_{AB}^2 \right. \\
& + \left. \langle\beta\rangle + \beta_+\nu - \frac{\langle\delta\rangle}{2} \right) \left. \right] + \mathcal{O}\left(\frac{1}{c^5}\right). \tag{4.15}
\end{aligned}$$

We use the above derivations of the coordinate transformation from harmonic to EOB coordinate systems to express the energy and angular momentum flux in EOB coordinates [31,34,46,47]. The energy flux in EOB coordinates is expressed as

$$\phi_E^e = G_{AB} c^3 \phi_E, \tag{4.16}$$

where ϕ_E^e is the energy flux in the EOB coordinate system and ϕ_E is the physical energy flux expressed in harmonic coordinates. Similarly, the angular momentum flux in EOB coordinates can be expressed as

$$\phi_J^e = c^3 \frac{\phi_J}{M}, \tag{4.17}$$

where ϕ_J^e is the angular momentum flux in the EOB coordinate system and ϕ_J is the physical angular momentum flux expressed in harmonic coordinates. The explicit expression of EOB coordinate energy and angular momentum flux up to 2PN is given in Supplemental Material [53].

V. RADIATION REACTION

In this section, we use the EOB coordinate energy and angular momentum flux to derive the generic orbit radiation reaction force within the EOB formalism for nonspinning systems. Since from now on we will only work in EOB coordinates, we will drop the e subscript and tilde to define the dimensionless variables and denote EOB phase space coordinates with $(r, \varphi, p_r, p_\varphi)$.

Now, let us remember that there are two contributions to the fluxes in scalar-tensor theory, i.e., a scalar contribution in addition to the usual tensorial one. In this work, we will compute the radiation reaction force due to both the scalar and tensorial contributions up to 1.5PN order.

Numerically, in most regions of the parameter space the scalar fluxes are small contributions to the total energy or angular momentum fluxes.

Radiation reaction in the EOB framework was first computed in Ref. [41] for quasicircular inspirals in GR. The arbitrary gauge freedom discussed in Refs. [42,43] in the radiation reaction force was utilized in Ref. [41] to find a freedom in relating the angular ($\mathcal{F}_\phi$) and radial components ($\mathcal{F}_r$) of the radiation reaction force. However, as shown above, there is no gauge freedom at 1.5PN order in scalar-tensor theories, which implies that the quasicircular orbit relations of GR do not hold in ST theories at 1.5PN order.

GR relations were later used to compute the generic orbit radiation reaction force in Ref. [49] (see also Ref. [50]). We start with the approach of Ref. [50] in GR to derive radiation reaction forces that satisfy

$$\begin{aligned}
\mathcal{F}_\phi^{\text{qc}} &= -\frac{\Phi_E^{\text{qc}}}{\Omega}, \\
\mathcal{F}_r^{\text{qc}} &= \mathcal{F}_\phi^{\text{qc}} \frac{p_r}{p_\phi} = -\frac{\Phi_E^{\text{qc}} p_r}{\Omega p_\phi}, \tag{5.1}
\end{aligned}$$

for circular orbits, where Ω is the orbital frequency. This formulation then allows the generic orbit force to be factorized as

$$\begin{aligned}
\mathcal{F}_r &= \mathcal{F}_r^{\text{qc}} \mathcal{F}_r^{\text{ecc}}, \\
\mathcal{F}_\phi &= \mathcal{F}_\phi^{\text{qc}} \mathcal{F}_\phi^{\text{ecc}}, \tag{5.2}
\end{aligned}$$

where $\mathcal{F}_X^{\text{ecc}}$ now denote the eccentric fluxes.

To derive the radiation reaction force in the EOB coordinates using the above balance equations, we use the energy flux and angular momentum flux in the EOB coordinates and write a generic ansatz with the unknown

coefficients for the Schott terms. Then, we calculate the unknown coefficients in the Schott terms by requiring that the force be reduced to the radiation reaction force in the limit of quasicircular orbits, as follows:

$$\begin{aligned}\mathcal{F}_\phi &= -\Phi_J + \mathcal{O}(p_r^2) + \mathcal{O}(\dot{p}_r), \\ \frac{\mathcal{F}_r p_\phi}{\mathcal{F}_\phi p_r} &= 1 + \mathcal{O}(p_r^2),\end{aligned}\quad (5.3)$$

where the radial momenta p_r and its first derivative $\dot{p}_r$ are zero for circular orbits. As discussed previously, we will show that Eq. (5.3), which are valid in GR, cannot be upheld in ST theories and therefore must be relaxed.

A. Ansatz for Schott terms

The general ansatz for the Schott terms is dictated by the fact that these terms enter due to the interaction with the radiation field, i.e., these are radiation effects. Therefore, we follow the postulates of constructing the radiation reaction acceleration given in Eq. (4.6) and additionally requiring that the angular momentum remains a pseudo-vector to construct the general ansatz. This implies that the Schott energy and angular momentum must be proportional to the odd powers in p_r , m/r for the nonlinear effects and ν^2 for vanishing radiation effects in the test-mass limit.

However, before expressing the general form of the ansatz for Schott terms, we recall that due to the presence of the scalar field (and hence, the dipolar radiation due to the scalar field) in scalar-tensor theories, the contribution to flux starts one PN order before their GR equivalent, i.e., at -1 PN order. In order to account for this information for the construction of a general ansatz, we first explicitly compute both the energy and the angular momentum Schott terms at -1 PN order. To do this, we consider an analogy with the dipolar radiation due to an electric charge in electrodynamics. The radiation reaction force for a slowly moving point particle with charge e is [54]

$$F_{\text{rad}}^{\text{ED}} = \frac{2e^2}{3c^3} \dot{a}, \quad (5.4)$$

where a is the acceleration and the superscript ED specifies electrodynamics. In terms of the multipole moments, the above equation can then be expressed as

$$F_i^{\text{rad,ED}} = \frac{2e}{3c^3} I_i^{(3)}, \quad (5.5)$$

where $I_i^{(3)}$ is the electric dipole moment. This implies that the radiation reaction force for a point particle moving in a scalar field must also be proportional to the third-time derivative of the scalar dipole moment,

$$F_i^{\text{rad}} = C_s I_i^{(3)}, \quad (5.6)$$

where $I_i^s = \frac{4\alpha_{\text{AB}}(\alpha_B^0 - \alpha_A^0)\zeta M\nu}{(1-\zeta)} r_i = S_{\text{AB}} r_i$ is the scalar dipole moment and C_s is some constant of proportionality. Therefore, the energy and angular momentum radiated from the system due to the scalar dipolar radiation are

$$\begin{aligned}\dot{E}_{\text{system}} &= F_i^{\text{rad}} v^i, \\ \dot{J}_{\text{system}} &= \epsilon^{ijk} r_i F_j^{\text{rad}},\end{aligned}\quad (5.7)$$

where r_i and v_i are the position and the velocity of the particle, respectively. Using Eq. (5.6) in the above equations, we obtain

$$\begin{aligned}\dot{E}_{\text{system}} &= C_s I_i^{s(3)} v^i \\ &= \frac{C_s}{S_{\text{AB}}} I_i^{s(3)} \dot{I}_i^s,\end{aligned}\quad (5.8)$$

and

$$\begin{aligned}\dot{J}_{\text{system}} &= \epsilon^{ijk} C_s I_j^{s(3)} r_i \\ &= \frac{C_s}{S_{\text{AB}}} \epsilon^{ijk} I_i^s I_j^{s(3)}.\end{aligned}\quad (5.9)$$

The radiated energy given in Eq. (5.8) can be expressed in terms of the total time derivative term as

$$\dot{E}_{\text{system}} = \frac{C_s}{S_{\text{AB}}} \left[\frac{d}{dt} (\dot{I}_i^s \dot{I}_i^s) - \ddot{I}_i^s \dot{I}_i^s \right]. \quad (5.10)$$

Now, using the above equation and the balance law equation given in Eq. (3.6), we find that the leading-order energy Schott term, $E_{(\text{schott})}$, due to the scalar dipolar radiation is

$$E_{(\text{schott})} = \frac{C_s}{S_{\text{AB}}} \ddot{I}_i^s \dot{I}_i^s. \quad (5.11)$$

Similarly, the radiated angular momentum given in Eq. (5.9) can be expressed as

$$\dot{J}_{\text{system}} = \frac{C_s}{S_{\text{AB}}} \epsilon^{ijk} \left[\frac{d}{dt} (I_i^s \dot{I}_j^s) - \dot{I}_i^s \ddot{I}_j^s \right], \quad (5.12)$$

and the leading-order angular momentum Schott term, $J_{(\text{schott})}$, due to the scalar dipolar radiation, is

$$J_{(\text{schott})} = \frac{C_s}{S_{\text{AB}}} \epsilon^{ijk} I_i^s \ddot{I}_j^s. \quad (5.13)$$

Using the definition of the scalar dipole moment and the Newtonian order equations of motion in Eqs. (5.11) and (5.13), we obtain that for the dipolar radiation there is no angular momentum contribution to the Schott term, i.e., $J_{(\text{schott})} = 0$, whereas the energy Schott term is nonzero.

Furthermore, we note that the energy and the angular momentum flux at 0.5PN order (the radiation reaction square effects) is a total derivative term, implying that these can be used to define a new conserved quantity. Therefore, these terms must not contribute in the radiation reaction force.

After considering the above two observations, the ansatz for the Schott terms we use is

$$\mathcal{J}_{\text{Schott}} = \frac{\nu^2 p_\phi p_r}{r^2} \left[\frac{\xi_1}{c^2} + \frac{1}{c^3} \frac{\xi_2}{p_r r} \right], \quad (5.14)$$

$$E_{\text{Schott}} = \frac{\nu^2 p_r}{r^2} \left[\psi_0 + \frac{1}{c^2} \left(\psi_1 p_r^2 + \psi_2 p^2 + \frac{\psi_3}{r} \right) + \frac{1}{c^3} \frac{\psi_4}{p_r r^2} \right]. \quad (5.15)$$

B. Generic orbit radiation reaction force

Using the fluxes and Schott terms, the radiation reaction force is calculated using the balance equation and the radiation reaction condition in the circular orbit limit.

1. -1PN order

As mentioned previously, scalar-tensor contributions start one PN order before their GR equivalents. We also noted how there is no gauge freedom in the radiation reaction term at 1.5PN order between the harmonic and ADM coordinate systems.

At this order, the radiation reaction fluxes read

$$\begin{aligned} \mathcal{F}_\phi &= -\frac{4}{3} \nu^2 \zeta' S_-^2 p_\phi u^3 + O\left(\frac{1}{c^2}\right), \\ \mathcal{F}_r &= \frac{4}{3} \nu^2 u^3 \left[\frac{3}{2} \psi_0 p_r + \left(\zeta' S_-^2 - \frac{3}{4} \psi_0 \right) \frac{p_\phi^2 u^2}{p_r} \right. \\ &\quad \left. - \left(\zeta' S_-^2 - \frac{3}{4} \psi_0 \right) \frac{u}{p_r} \right] + O\left(\frac{1}{c^2}\right), \end{aligned} \quad (5.16)$$

where we defined $u \equiv 1/r$ and p_ϕ is such that $p^2 = p_r^2 + p_\phi^2 u^2$.

As mentioned previously, the unknown parameters entering the ansatz of the Schott terms are determined by satisfying the circular orbit limit, which requires that the radial flux in the above equation must not contain negative powers of the radial momentum p_r . This leads to the solution,

$$\psi_0 = \frac{4}{3} \zeta' S_-^2, \quad (5.17)$$

so that

$$\mathcal{F}_r = \frac{8}{3} \nu^2 \zeta' S_-^2 p_r u^3 + O\left(\frac{1}{c^2}\right). \quad (5.18)$$

At this level, it is easy to see we cannot impose $(\mathcal{F}_r p_\phi)/(\mathcal{F}_\phi p_r) = 1$. Instead we get

$$\frac{\mathcal{F}_r p_\phi}{\mathcal{F}_\phi p_r} = -2. \quad (5.19)$$

From here on, we will use the conditions

$$\begin{aligned} \mathcal{F}_\phi &= -\Phi_J + \mathcal{O}(p_r^2) + \mathcal{O}(\dot{p}_r), \\ \frac{\mathcal{F}_r p_\phi}{\mathcal{F}_\phi p_r} &= 1 + \mathcal{O}(p_r^2), \end{aligned} \quad (5.20)$$

in the limit of quasicircular orbits for scalar-tensor theories.

2. Newtonian order

At Newtonian order, we follow the same steps as above, i.e., impose that the radial momentum contains only positive powers of p_r , in addition to Eq. (5.20) to determine all four free parameters, as

$$\begin{aligned} \xi_1 &= \frac{8}{15} (2 + \gamma_{\text{AB}}) + \frac{4}{15} \zeta' [14 S_+^2 - S_+ (3 S_- X_{\text{AB}} + 80 \theta_+) + 10 S_- (\theta_- + X_{\text{AB}} \theta_+) - S_-^2 (6 - 5 \langle \bar{\beta} \rangle) + 5 \gamma_{\text{AB}} + \nu) + 120 \theta_+^2], \\ \psi_1 &= -\frac{4}{3} (2 + \gamma_{\text{AB}}) - \frac{2}{3} \zeta' [6 S_+^2 - 32 \theta_+ S_+ + 8 \theta_- S_- - S_-^2 (3 - 2 \langle \bar{\beta} \rangle) + 2 \gamma_{\text{AB}} + 40 \theta_+^2], \\ \psi_2 &= \frac{32}{15} (2 + \gamma_{\text{AB}}) + \frac{2}{15} [32 S_+^2 - 160 \theta_+ S_+ + 40 \theta_- S_- - S_-^2 (2 - 10 \langle \bar{\beta} \rangle) + 15 \nu) + 240 \theta_+^2], \\ \psi_3 &= -\frac{4}{15} S_- \zeta' [4 S_+ X_{\text{AB}} + S_- (31 + 10 \langle \bar{\beta} \rangle + 10 \gamma_{\text{AB}} - 7 \nu)]. \end{aligned} \quad (5.21)$$

At this order, the final forms for the radiation reaction forces are

$$\mathcal{F}_\phi = -\frac{4}{3} \nu^2 \zeta' S_-^2 p_\phi u^3 + \frac{1}{c^2} \mathcal{F}_\phi^{\text{N}} + O\left(\frac{1}{c^3}\right), \quad \mathcal{F}_r = \frac{8}{3} \nu^2 \zeta' S_-^2 p_r u^3 + \frac{1}{c^2} \mathcal{F}_r^{\text{N}} + O\left(\frac{1}{c^3}\right), \quad (5.22)$$

with

$$\begin{aligned}
\mathcal{F}_\varphi^N &= \frac{2}{15} \nu^2 p_\varphi u^3 \{ [-16(2 + \bar{\gamma}_{AB}) - \zeta'(32\mathcal{S}_+^2 - 160\theta_+ \mathcal{S}_+ + 40\theta_- \mathcal{S}_- - \mathcal{S}_-^2(2 - 10\langle\bar{\beta}\rangle + 15\nu) + 240\theta_+^2)] p^2 \\
&\quad + 15[2(2 + \bar{\gamma}_{AB}) + \zeta'(6\mathcal{S}_+^2 - 32\theta_+ \mathcal{S}_+ + 8\theta_- \mathcal{S}_- + \mathcal{S}_-^2(-3 + 2\langle\bar{\beta}\rangle - 2\bar{\gamma}_{AB}) + 48\theta_+^2)] p_r^2 - [8(2 + \bar{\gamma}_{AB}) \\
&\quad - \zeta'(24\mathcal{S}_+^2 - 160\theta_+ \mathcal{S}_+ + 40X_{AB}\theta_+ \mathcal{S}_- + \mathcal{S}_-^2(41 + 30\langle\bar{\beta}\rangle - 10\nu) + 240\theta_+^2)] u \}, \\
\mathcal{F}_r^N &= \frac{4}{15} \nu^2 p_r u^3 \{ [16(2 + \bar{\gamma}_{AB}) + \zeta'(32\mathcal{S}_+^2 - 160\theta_+ \mathcal{S}_+ + 40\theta_- \mathcal{S}_- - \mathcal{S}_-^2(2 - 10\langle\bar{\beta}\rangle + 15\nu) + 240\theta_+^2)] p^2 \\
&\quad - 5[2(2 + \bar{\gamma}_{AB}) + \zeta'(6\mathcal{S}_+^2 - 32\theta_+ \mathcal{S}_+ + 8\theta_- \mathcal{S}_- + \mathcal{S}_-^2(-3 + 2\langle\bar{\beta}\rangle - 2\bar{\gamma}_{AB}) + 48\theta_+^2)] p_r^2 \\
&\quad + [8(2 + \bar{\gamma}_{AB}) - \zeta'(24\mathcal{S}_+^2 - 160\theta_+ \mathcal{S}_+ + 40X_{AB}\theta_+ \mathcal{S}_- + \mathcal{S}_-^2(41 + 30\langle\bar{\beta}\rangle - 10\nu) + 240\theta_+^2)] u \}. \tag{5.23}
\end{aligned}$$

3. 0.5PN

We follow the same steps as previously to determine the unknowns entering at 0.5PN order. However, instead of requiring that $1/p_r$ term in $\mathcal{F}_r$ vanishes, we now impose that the $\mathcal{O}(p_r^0)$ vanishes (as $\mathcal{F}_r$ vanishes in the quasicircular orbit limit). This is because, at the 0.5PN order, the leading contribution is of order $\mathcal{O}(p_r^0)$, unlike -1 PN and Newtonian order, where such terms disappear and the leading contribution is $\mathcal{O}(1/p_r)$. This, along with the condition $(\mathcal{F}_r p_\phi)/(\mathcal{F}_\phi p_r) = -2$, leads to the solution,

$$\begin{aligned}
\psi_4 &= -\frac{2}{9} \mathcal{S}_-^2 \zeta' [4\mathcal{S}_- (\mathcal{S}_- + X_{AB} \mathcal{S}_+) \zeta' - \bar{\gamma}_{AB}], \\
\xi_2 &= -\frac{2}{9} \mathcal{S}_-^2 \zeta' [4\mathcal{S}_- (\mathcal{S}_- + X_{AB} \mathcal{S}_+) \zeta' - \bar{\gamma}_{AB}]. \tag{5.24}
\end{aligned}$$

At this order, the radiation reaction force vanishes, consistent with the fact that at 0.5PN order, the energy and the angular momentum flux are total derivatives, implying that these can be used to define a new conserved quantity. Hence, the radiation reaction force up to 0.5PN reads

$$\begin{aligned}
\mathcal{F}_\varphi &= -\frac{4}{3} \nu^2 \zeta' \mathcal{S}_-^2 p_\varphi u^3 + \frac{1}{c^2} \mathcal{F}_\varphi^N + \mathcal{O}\left(\frac{1}{c^4}\right), \\
\mathcal{F}_r &= \frac{8}{3} \nu^2 \zeta' \mathcal{S}_-^2 p_r u^3 + \frac{1}{c^2} \mathcal{F}_r^N + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{5.25}
\end{aligned}$$

with $\mathcal{F}_\varphi^N$ and $\mathcal{F}_r^N$ are given in Eq. (5.23).

VI. DISCUSSIONS

Recently there has been a significant effort to extend EOB waveform models to incorporate eccentricity effects in GR [49,55–57]. In this paper, we extend the quasicircular orbit approximation results of the radiation reaction force in massless scalar-tensor theories to eccentric orbits for non-spinning black hole binaries within the EOB framework.

In particular, we first derive the coordinate transformations required to map the energy and angular momentum

fluxes from harmonic coordinates to EOB coordinates up to 2PN order by deriving the transformation from the harmonic to the ADM coordinate system and from the ADM to the EOB coordinate system. As the radiation reaction effects in scalar-tensor theories start one PN order before (1.5PN order) in contrast to GR, we discuss the coordinate transformation from harmonic coordinates to ADM-like coordinates by investigating the gauge freedom in the radiation reaction term following Ref. [42,43]. Unlike in GR [42,43], we find that there is no gauge freedom in the radiation reaction effects at 1.5PN order for scalar-tensor theories, implying that there is no coordinate transformation between the harmonic and ADM-like coordinate systems at 1.5PN order. We also verified that the coordinate transformation between harmonic, ADM, and EOB coordinate systems are in agreement with existing PN-based results in the GR limit.

We hence made use of 2PN-order coordinate transformations between harmonic, ADM, and EOB frames to translate the harmonic coordinate energy and angular momentum fluxes derived in Refs. [31,34] into the EOB coordinate system. Next, we construct the general ansatz of the Schott terms by considering an analogy of the scalar dipolar radiation with the dipolar radiation due to an electric charge in electrodynamics and found that there is no angular momentum contribution to the Schott term, i.e., $J_{(\text{schott})} = 0$ at the leading order (-1 PN order with respect to GR) in scalar-tensor theories. Using these results, we derive the radiation reaction force for generic orbits up to relative 1.5PN order, i.e., -1 PN, Newtonian and 0.5PN order, where we find that at the 0.5PN order, the energy and the angular momentum flux are total derivatives, suggesting that the radiation reaction force vanishes at 0.5PN order.

The results presented in this paper will be useful when building accurate gravitational wave models that take into account multiple effects, such as eccentricity and ST corrections. Such models could then be employed in tests of GR in order to avoid possible biases due to the omission of one such effect.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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